Revisiting deformations of truncated Barsotti-Tate groups

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In the spring of 1971, in his course at the Collège de France, Grothendieck proved the existence of infinitesimal liftings of truncated Barsotti-Tate groups. This implies the smoothness of the corresponding stack. Grothendieck's proof was given in [7]. The purpose of this talk is to review the main statements, in the hope that today's tools could perhaps suggest simpler approaches and provide further results and questions.

We fix a prime p.

1. The main statement.

Definition 1.1. Let S be a scheme and let n be an integer $\geqslant 1$. A BT_n on S (*truncated Barsotti-Tate group of level* n)[1] is a finite, locally free commutative group scheme G on S, which is annihilated by p^n, is flat over $\mathbb{Z}/p^n$ as an fpqc sheaf of $\mathbb{Z}/p^n$-modules on S, and which, for $n = 1$, satisfies the additional condition $\mathrm{Ker}(V) = \mathrm{Im}(F)$, where $F : G_0 \to G_0^{(1)}$ (resp. $V : G_0^{(1)} \to G_0$) is the Frobenius (resp. Verschiebung) homomorphism of $G_0 := G \times_S S_0$ where $S_0 := V(p) \subset S$. [2]

The flatness of G over $\mathbb{Z}/p^n$ is equivalent to the exactness of the sequence $G \overset{p^{n-1}}{\to} G \overset{p}{\to} G$ (see [5], III 2.2) for more equivalences). On the other hand, the condition $\mathrm{Ker}(V) = \mathrm{Im}(F)$ is equivalent to $\mathrm{Ker}(F) = \mathrm{Im}(V)$. As epimorphisms of affine commutative group schemes over a field are faithfully flat, these conditions can be checked on the fibers.

If G is a sheaf of abelian groups, we use Grothendieck's notation $G(n)$ to denote the kernel of the multiplication by p^n (often denoted $G[p^n]$ nowadays).

If G is a BT_n on S, for $1 \leqslant n' \leqslant n$, $G(n')$ is a $\mathrm{BT}_{n'}$. The rank of $G(1)$ is of the form p^h, where h is called the *height* of G, denoted $\mathrm{ht}(G)$ and the rank of G is p^{nh}.

Theorem 1.2. (Grothendieck) ([7] Th. 4.4, (a)). Let $i : S \to S'$ be a nilimmersion, with S' affine. Let $n \geqslant 1$ and let G be a BT_n on S. Then:

(i) there exists a BT_n G' on S' such that $G'_S = G$;

(ii) for $1 \leqslant n' \leqslant n$, if $\mathrm{Def}(-, i)$ denotes the set of isomorphism classes of deformations from S to S', the map

$$\mathrm{Def}(G, i) \to \mathrm{Def}(G(n'), i)$$

[1] *échelon* n in Grothendieck's terminology in [7].

[2] We write $X^{(1)}$ instead of $X^{(p)}$ for the pull-back of an S_0-scheme X by the Frobenius F_{S_0} of S_0.

is surjective.

This implies that, for fixed $n \geqslant 1$, $h \geqslant 1$, the $\mathrm{Spec}(\mathbb{Z}_{(p)})$-stack $\mathrm{BT}_{n,h}$ of BT_n's of height h ([11] 1.8 (b)) is smooth, and the map $\mathrm{BT}_{n,h} \to \mathrm{BT}_{n',h}$, $G \mapsto G(n')$ is smooth.

We will discuss corollaries, complements, and applications of Th. 1.2 in the next section.

The proof relies on:

• The obstruction theory of [6]
• Key facts on the differential and cohomological structure of BT_n's, due to Grothendieck.

Standard dévissages reduce the proof of 1.2 to the case where S' is artinian local, with maximal ideal $\mathfrak{m}$, perfect residue field k of characteristic p, and the ideal J of i is annihilated by $\mathfrak{m}$, so that i inserts itself into

$$S_0 = \mathrm{Spec}(k) \hookrightarrow S \overset{i}{\hookrightarrow} S'. \tag{1.2.1}$$

Though the obstruction theory of [6] provides a nice cohomological formula for the obstruction $o(G)$ to deforming G over S' ([7], Prop. 3.2, and p. 173), namely

$$o(G) \in \mathrm{Ext}^2_{\mathbb{Z}}(G_0, \mathrm{t}_{G_0} \otimes_k J)$$

(where t_{G_0} is the Lie algebra of $G_0 := G_{S_0}$), it turns out to be out of reach to prove directly that $o(G)$ vanishes. Instead, Grothendieck's proof of this vanishing uses auxiliary and complementary results which have to be established beforehand. They involve the notion of BT (Barsotti-Tate) group.

Definition 1.3. Let S be a scheme. A BT group (or BT) over S is an abelian fpqc sheaf G on S satisfying the following conditions:
(i) G is of p^∞-torsion, i.e., $G = \varinjlim_{n \in \mathbb{N}} G(n)$,
(ii) G is p-divisible, i.e., $p : G \to G$ is an epimorphism,
(iii) $G(1)$ is finite, locally free over S.

If G is a BT over S, then, for all $n \geqslant 1$, $G(n)$ is a BT_n ([7], 1.6).

The two crucial results needed in the proof of Th. 1.2 are the following:

Lemma 1.4. ([7], 1.7) (Gabber-Ekedahl). Let k be a perfect field of characteristic p, and let G be a BT_n over $S = \mathrm{Spec}(k)$. Then there exists a BT H over S such that $G = H(n)$.

The (elementary) proof uses (classical) Dieudonné theory and exploits the condition $\mathrm{Ker}(V) = \mathrm{Im}(F)$ on $G(1)$.

The next statement is the core of the matter:

Proposition 1.5. ([7], 4.5.1) Suppose $i : S \hookrightarrow S'$ of ideal $J = \mathrm{Ker}(\mathcal{O}_{S'} \to \mathcal{O}_S)$ inserts itself into

$$S_0 \hookrightarrow S \xrightarrow{i} S',$$

where $\mathcal{O}_{S_0} = \mathcal{O}_{S'}/K$, with $K \supset J$, $JK = 0$, and $p\mathcal{O}_{S_0} = 0$. Let H be a BT over S. Then:

(i) For any $n \geqslant 1$, there exists a BT_n G'_n over S' extending $G_n := H(n)$;

(ii) for any $n \geqslant 1$ and any BT_n G'_n on S' extending G_n, there exists a BT H' on S' extending H such that $G'_n = H'(n)$. In fact, denoting by $\mathrm{Def}(-, i)$ the set of isomorphism classes of deformations from S to S', the restriction map

$$\mathrm{Def}(H(n+1), i) \to \mathrm{Def}(H(n), i)$$

is bijective.

1.6. Let us show that Lemma 1.4 and Proposition 1.5 imply the special case (1.2.1) of Th. 1.2 (and, consequently, Th. 1.2 in general). By Proposition 1.5 (i) it suffices to show that there exists a BT H over S such that $G = H(n)$. The scheme S is artinian local with maximal ideal $\mathfrak{n}$ $(= \mathfrak{m}/J)$. For $r \geqslant 0$ let $S_r := \mathrm{Spec}(A/\mathfrak{n}^{r+1})$, so that $S = S_N$ for some $N \geqslant 0$. One shows the existence of H by induction on r. For $r = 0$, by Lemma 1.4 there exists a BT H_0 on $S_0 = \mathrm{Spec}(k)$ such that $G_0 = H_0(n)$ (where $G_0 := G_k$). Assume a BT H_r/S_r constructed, with $H_r(n) = G|S_r$. Then, by Lemma 1.5 (ii) applied to $(S_0 \hookrightarrow S_r \hookrightarrow S_{r+1}, H_r = G|S_r)$, one finds a BT H_{r+1}/S_{r+1} such that $H_{r+1}|S_r = H_r$ and $H_{r+1}(n) = G|S_{r+1}$.

1.7. *Co-Lie and Lie complexes.* For the proof of Prop. 1.5 we need the notion of Lie and co-Lie complexes of truncated BT's. Let S be a scheme and let G be an S-group scheme which is flat and locally of finite presentation over S. Let $e : S \to G$ be the unit section. We denote by

$$\mathrm{coLie}_G := e^* L\Omega^1_G = \ell_G \in D(S, \mathcal{O}_S)$$

the co-Lie complex of G. It is of perfect amplitude in $[-1, 0]$. We denote its dual by

$$\mathrm{Lie}_G = \check{\ell}_G := R\mathcal{H}om(\ell_G, \mathcal{O}_S).$$

We'll use Grothendieck notation

$$\omega_G := H^0(\mathrm{coLie}_G),\ \mathrm{n}_G := H^{-1}(\mathrm{coLie}_G),\ \mathrm{t}_G := H^0(\mathrm{Lie}_G),\ \nu_G := H^1(\mathrm{Lie}_G).$$

The product map $m : G \times_S G \to S$ induces the diagonal map $\mathrm{coLie}_G \to \mathrm{coLie}_{G \times G} = \mathrm{coLie}_G \oplus \mathrm{coLie}_G$ (resp. the sum map $\mathrm{Lie}_{G\times G} = \mathrm{Lie}_G \oplus \mathrm{Lie}_G \to$

Lie_G). The antipodism $G \to G, g \mapsto g^{-1}$ induces $-\mathrm{Id}$ on both coLie_G and Lie_G. In particular, when G is commutative, for $n \in \mathbb{Z}$, the morphism $n\mathrm{Id}_G$ induces $n\mathrm{Id}$ on both coLie_G and Lie_G.[3] Thus, if G is annihilated by some $n \geqslant 1$, the same holds for coLie_G and Lie_G, but, in fact, these complexes enjoy a richer structure (see 1.9).

It is difficult, in general, to calculate the co-Lie complex of G. When G is commutative, finite and locally free, with Cartier dual G^*, the following formula, due to Grothendieck ([9], 14.1), is of critical use: for any quasi-coherent $\mathcal{O}_S$-module M, there exists a canonical, functorial isomorphism

$$R\mathcal{H}om_{\mathcal{O}_S}(\mathrm{coLie}_G, M) \xrightarrow{\sim} \tau^{\leqslant 1} R\mathcal{H}om_{\mathbb{Z}}(G^*, M). \tag{1.7.1}$$

For $M = \mathcal{O}_S$, this formula can be viewed as an infinitesimal form of the biduality isomorphism $G \xrightarrow{\sim} \mathcal{H}om(G^*, \mathbb{G}_m)$.

The proof of Prop. 1.5 relies on:

(a) the obstruction theory of [6];

(b) a number of differential and cohomological properties of truncated BT's, due to Grothendieck, that we list here. We will recall the main results of (a) afterwards.

Lemma 1.8. Let S be a scheme such that $p\mathcal{O}_S = 0$. Let G be a BT_n on S ($n \geqslant 1$).

(1) The $\mathcal{O}_S$-modules ω_G, n_G, t_G, ν_G are locally free of finite type, of the same rank, equal by definition to the *dimension* of G, denoted $\dim(G)$. If G^* is the Cartier dual of G, one has

$$\dim(G) + \dim(G^*) = \mathrm{ht}(G) = \mathrm{ht}(G^*).$$

(2) For $1 \leqslant n' \leqslant n$, the inclusion $G(n') \hookrightarrow G$ induces isomorphisms $\omega_G \xrightarrow{\sim} \omega_{G(n')}$, $\mathrm{t}_{G(n')} \xrightarrow{\sim} \mathrm{t}_G$, and, for $n > n'$, the zero morphisms $\mathrm{n}_G \to \mathrm{n}_{G(n')}$, $\nu_{G(n')} \to \nu_G$.

The epimorphism $p^{n-n'} : G \twoheadrightarrow G(n')$ induces isomorphisms $\mathrm{n}_{G(n')} \xrightarrow{\sim} G$, $\nu_{G(n')} \xrightarrow{\sim} \nu_G$, and, for $n > n'$ the zero morphisms $\omega_{G(n')} \to \omega_G$, $\mathrm{t}_G \to \mathrm{t}_{G(n')}$.

(3) If M is a quasi-coherent $\mathcal{O}_S$-module, for all $i \in \mathbf{Z}$, the sheaves $\mathcal{E}xt^i_{\mathbb{Z}/p^n}(G, M)$ and $\mathcal{E}xt^i_{\mathbb{Z}}(G, M)$ are quasi-coherent, and the canonical morphism

$$\mathcal{E}xt^2_{\mathbb{Z}/p^n}(G, M) \to \mathcal{E}xt^2_{\mathbb{Z}}(G, M)$$

[3] *Proof* (Gabber). The first assertion uses only the fact that the compositions

$$G \overset{(\mathrm{Id},e)}{\to} G \times_S G \overset{m}{\to} G, \; G \overset{(e,\mathrm{Id})}{\to} G \times_S G \overset{m}{\to} G$$

are the identity maps. The other assertions follow.

is an isomorphism. In particular, if S is affine,

$$\mathrm{Ext}^2_{\mathbb{Z}/p^n}(G, M) \to \mathrm{Ext}^2_{\mathbb{Z}}(G, M)$$

is an isomorphism.

(4) If S is affine, the natural maps

$$\mathrm{Ext}^2_{\mathbb{Z}/p^n}(G, \mathrm{t}_G \otimes M) \to \mathrm{Ext}^2_{\mathbb{Z}/p^n}(G, \mathrm{Lie}_G \otimes^L M)$$

and

$$\mathrm{Ext}^1_{\mathbb{Z}/p^n}(G, \mathrm{Lie}_G \otimes^L M) \to \mathrm{Hom}_{\mathbb{Z}/p^n}(G, \nu_G \otimes^L M)$$

are isomorphisms (and in addition, $\mathrm{Hom}_{Z/p^n}(G, \nu_G \otimes^L M) \xrightarrow{\sim} \mathrm{t}_{G^*} \otimes \nu_G \otimes M$ by (1.7.1)).

(5) For $1 \leqslant n' < n$, the inclusion $G(n') \subset G$ induces the zero map

$$\mathrm{Ext}^2_{\mathbb{Z}}(G, M) \to \mathrm{Ext}^2_{\mathbb{Z}}(G(n'), M).$$

The proofs of (1), (2) are essentially in [10]. The quasi-coherence assertion about the $\mathcal{E}xt^i$ in (3) follows from their calculation as spatial cohomology ([6], VI, 11.5.3.11). The remaining assertions in (3), (4), and (5) are delicate. They make an essential use of Grothendieck's formula (1.7.1). Lemma 1.8 only lists the ingredients used in the proof of 1.5. There are more general statements in ([7], section 2). In particular, one can assume only $p^N\mathcal{O}_S = 0$ provided that we have $n \geqslant N$ (without this condition, it is no longer true that ω_G and ν_G are locally free, as the case of μ_p over $\mathbb{Z}/p^2$ already shows).

1.9. We now recall the obstruction results needed for the proof of Prop. 1.5. Let A be a commutative ring (in practice, $A = \mathbb{Z}$ or $\mathbb{Z}/p^n$. Consider closed immersions

$$S_0 \hookrightarrow S \hookrightarrow S'$$

defined by ideals $J \subset K$, with $JK = 0$. Let G be a scheme in A-modules, flat and locally of finite presentation over S. The co-Lie complex coLie_G (resp. Lie complex Lie_G) in $D(S, \mathcal{O}_S)$ can be upgraded into an object of $D^{[-1,0]}(S, A \otimes^L_{\mathbb{Z}} \mathcal{O}_S)$ (where $A \otimes^L_{\mathbb{Z}} \mathcal{O}_S$ is considered as an animated ring on S ([2], 5.1.3 (3))), which category, in this case, can be identified to the corresponding derived category of differential graded modules over the differential graded ring $A \otimes^L_{\mathbb{Z}} \mathcal{O}_S$ considered in ([6], VII 4.1.4). Let $G_0 := G \times_S S_0$.Then ([7], Prop. 3.2):

(a) There exists an obstruction

$$o(G) \in \mathrm{Ext}^2_A(G_0, \mathrm{Lie}_{G_0} \otimes^L J)$$

whose vanishing is necessary and sufficient for the existence of a scheme in A-modules G', flat and locally of finite presentation over S', extending G. This obstruction has the following functorial property: if $u : F \to G$ is a morphism of schemes in A-modules (flat and locally of finite presentation over S), then

$$u_0^* o(G) = \mathrm{Lie}_{u_0} o(F) \in \mathrm{Ext}^2_A(F_0, \mathrm{Lie}_{G_0} \otimes^L J),$$

where

$$\mathrm{Ext}^2_A(G_0, \mathrm{Lie}_{G_0} \otimes^L J)) \overset{u_0^*}{\to} \mathrm{Ext}^2_A(F_0, \mathrm{Lie}_{G_0} \otimes^L J)) \overset{\mathrm{Lie}_{u_0}}{\leftarrow} \mathrm{Ext}^2_A(F_0, \mathrm{Lie}_{F_0} \otimes^L J).$$

(b) When $o(G) = 0$, the set of isomorphism classes of extensions G' is an affine space under $\mathrm{Ext}^1_A(G_0, \mathrm{Lie}_{G_0} \otimes^L J)$, and the automorphism group of a given extension G' is $\mathrm{Hom}_A(G_0, \mathrm{Lie}_{G_0} \otimes^L J)$.

We have similar results for deformations of morphisms. Let F' and G' be schemes in A-modules, flat and locally of finite presentation over S', let $F := F'_S$, $G := G'_S$, $F_0 := F'_{S_0}$, $G_0 := G'_{S_0}$, and let $f : F \to G$ be an S-morphism of schemes in A-modules. Let $f_0 := f_{S_0} : F_0 \to G_0$.

(a_1) There exists an obstruction

$$o(f) \in \mathrm{Ext}^1_A(F_0, \mathrm{Lie}_{G_0} \otimes^L J)$$

whose vanishing is necessary and sufficient for the existence of a morphism $f' : F' \to G'$ extending f. This obstruction is functorial in F' and G' in the obvious way.

(b_1) When $o(f) = 0$, the set of extensions f' of f is an affine space under $\mathrm{Hom}_A(F_0, \mathrm{Lie}_{G_0} \otimes^L J)$.

1.10. *Proof of 1.5* ([7], 4.5).

Proof of 1.5 (i). Observe first that the flatness criterion along the fibers implies that if G' is a commutative, finite, locally free group scheme over S' annihilated by p^n that extends G, G' is automatically a BT_n. Therefore, by 1.5 (a) the obstruction to finding a BT_n on S' extending G is an element

$$o(G) \in \mathrm{Ext}^2_{\mathbb{Z}/p^n}(G_0, \mathrm{Lie}_{G_0} \otimes^L J).$$

Using Lemma 1.8, one can observe that $o(G)$ actually lies in a *smaller* group. Indeed, consider the commutative square

$$\begin{array}{ccc} \mathrm{Ext}^2_{\mathbb{Z}/p^n}(G_0, \mathrm{Lie}_{G_0} \otimes^L J) & \xrightarrow{(1)} & \mathrm{Ext}^2_{\mathbb{Z}}(G_0, \mathrm{Lie}_{G_0} \otimes^L J) \\ \uparrow{\scriptstyle (3)} & & {\scriptstyle (4)}\uparrow \\ \mathrm{Ext}^2_{\mathbb{Z}/p^n}(G_0, \mathrm{t}_{G_0} \otimes^L J) & \xrightarrow{(2)} & \mathrm{Ext}^2_{\mathbb{Z}}(G_0, \mathrm{t}_{G_0} \otimes^L J), \end{array} \tag{1.10.1}$$

where (1) and (2) are the forgetful maps, and ((3) and (4) are induced by $\mathrm{t}_{G_0} \to \mathrm{Lie}_{G_0}$. By Lemma 1.8 (3), (2) is an isomorphism. By Lemma 1.8 (4), (3) is an isomorphism. Finally, as S is affine and ν_{G_0} is locally free of finite type (Lemma 1.8 (1)), Lie_{G_0} decomposes into $\mathrm{t}_{G_0} \oplus \nu_{G_0}[-1]$, so that (4) is the injection of a direct summand. So, denoting again by $o(G)$ its image by the isomorphism $(2) \circ (3)^{-1}$, we have

$$o(G) \in \mathrm{Ext}^2_{\mathbf{Z}}(G_0, \mathrm{t}_{G_0} \otimes^L J) \subset \mathrm{Ext}^2_{\mathbf{Z}}(G_0, \mathrm{Lie}_{G_0} \otimes^L J)$$

where the inclusion is that of a direct summand. In other words, the existence of a BT_n on S' extending G is equivalent to the existence of a commutative, finite locally free commutative group scheme on S' extending G, and the obstruction lies in the direct summand $\mathrm{Ext}^2_{\mathbf{Z}}(G_0, \mathrm{t}_{G_0} \otimes^L J)$. We have a similar picture with $H(n+1)$. Therefore, by functoriality of the obstruction (1.9 (a) applied to $A = \mathbf{Z}$ and u the inclusion $G = H(n) \hookrightarrow H(n+1)$, we have

$$u_0^* o(H(n+1)) = \mathrm{t}_{u_0} o(G),$$

where

$$\mathrm{Ext}^2_{\mathbb{Z}}(H_0(n+1), \mathrm{t}_{H_0(n+1)} \otimes J) \overset{u_0^*}{\to} \mathrm{Ext}^2_{\mathbb{Z}}(G_0, \mathrm{t}_{H_0(n+1)} \otimes J) \overset{\mathrm{t}_{u_0}}{\leftarrow} \mathrm{Ext}^2_{\mathbb{Z}}(G_0, \mathrm{t}_{G_0} \otimes J)$$

are the functoriality maps. By Lemma 1.8 (2), t_{u_0} is an isomorphism, while $u_0^* = 0$ by Lemma 1.8 (5). Hence $o(G) = 0$.

Proof of 1.5 (ii). Let $F := H(n+1)$. If F', F'' are deformations of F over S', denoting by $[-]$ the isomorphism class of a deformation, by 1.9 (b), we have

$$[F'] - [F''] \in \mathrm{Ext}^1_{\mathbb{Z}/p^{n+1}}(F_0, \mathrm{Lie}_{F_0} \otimes^L J) \overset{1.8(4)}{\overset{\sim}{\to}} \mathrm{t}_{F_0^*} \otimes \nu_{F_0} \otimes J,$$

and this element is the obstruction $o(\mathrm{Id}_F, F', F'')$ to extending Id_F to an isomorphism $F' \overset{\sim}{\to} F''$ ([7], 3.3 (a)). Showing that

$$\mathrm{Def}(H(n+1), i) \to \mathrm{Def}(H(n), i)$$

is bijective is equivalent to showing that

$$(*) \qquad \mathrm{t}_{F_0^*} \otimes \nu_{F_0} \otimes J \to \mathrm{t}_{G_0^*} \otimes \nu_{G_0} \otimes J, [F'] - [F''] \mapsto [F'(n)] - [F''(n)]$$

is bijective. By functoriality of this obstruction (1.9 (a_1)) for the morphism $p : F = H(n+1) \to G = H(n)$, we have

$$p^* o(\mathrm{Id}_G, G', G'') = o(p : F \to G, F', G'') = p_* o(\mathrm{Id}_F, F', F'')$$

where

$$\mathrm{t}_{G_0^*} \otimes \nu_{G_0} \otimes J \xrightarrow{p^*} \mathrm{t}_{F_0^*} \otimes \nu_{G_0} \otimes J \xleftarrow{p_*} \mathrm{t}_{F_0^*} \otimes \nu_{F_0} \otimes J$$

are the functoriality maps. By Lemma 1.8 (2), these maps are isomorphisms, and the image of $[F'] - [F'']$ by (*) is $(p^*)^{-1}p_*([F'] - [F''])$, which finishes the proof.

2. Complements

2.1. Grothendieck's theorem ([7], Th. 4.4) contains several refinements and corollaries to both Th. 1.2 and Prop. 1.5. We list some of them here.

(i) In the situation of Th. 1.2, if $\mathrm{Def}(G, i)$ denotes the set of isomorphism classes of deformations G' of G over S', for $n' \leqslant n$, the map

$$\mathrm{Def}(G, i) \to \mathrm{Def}(G(n'), i),\ G' \mapsto G'(n')$$

is surjective.

(ii) Assumptions as in Prop. 1.5, but no BT H is given. Let G be a BT_n on S. The set of isomorphism classes of deformations G' of G over S' is an affine space under $\mathrm{t}_{G_0^*} \otimes \mathrm{t}_{G_0} \otimes J$ and the group of automorphisms of a deformation G' (inducing the identity on G) is isomorphic to $\mathrm{t}_{G_0^*} \otimes \mathrm{t}_{G_0} \otimes J$. For $n > n'$, the corresponding homomorphism

$$\mathrm{Aut}(G') \to \mathrm{Aut}(G'(n'))$$

vanishes.

(iii) Assumptions as in Prop. 1.5. For any $n \geqslant 1$, the map

$$\mathrm{Def}(H) \to \mathrm{Def}(H(n)),\ H' \mapsto H'(n)$$

is bijective. Moreover, the group of automorphisms of a deformation H' of H (inducing the identity on H) is reduced to zero.

(iv) If S is noetherian, local, complete with perfect residue field of characteristic p, then, for any BT_n G on S there exists a BT H on S such that $G = H(n)$.

2.2. *An apparent contradiction.*[4] Let k be an algebraically closed field of characteristic $p > 0$, $R = k[[t]]$. Consider the situation of Prop. 1.5, with $S' = \mathrm{Spec}(R/(t^{p+1}))$, $S = \mathrm{Spec}(R/(t^p))$, $S_0 : \mathrm{Spec}(k)$, $J = t^p(R/(t^{p+1}))$, $K = t(R/(t^{p+1}))$. Take H to be the BT group $\mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)^*$ on S (where $(\mathbb{Q}_p/\mathbb{Z}_p)^*$ means the dual of $\mathbb{Q}_p/\mathbb{Z}_p$ (often denoted $\mathbb{Q}_p/\mathbb{Z}_p(1)$, where

[4](after F. Oort and O. Gabber, private discussion, March 2006)

(1) means Tate twist, a notation we will avoid here, for risk of confusion with the kernel of p). If we believe 2.1 (iii), $H' \mapsto H'(1)$ is a bijection from the set of isomorphism classes of deformations of H over S' to the set of isomorphism classes of deformations of $E := H(1) = \mathbb{Z}/p \times \mu_p$ over S'. However, there seems to be two non-isomorphic deformations H', H'' of H over S' with $H'(1) \xrightarrow{\sim} H''(1) \xrightarrow{\sim} \mathbb{Z}/p \times \mu_p$ on S', namely H' the trivial deformation $\mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)^*$ of H, and H'' the (non-trivial) extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $(\mathbb{Q}_p/\mathbb{Z}_p)^*$ given by the unit $1+t^p \in R/(t^{p+1})$. Indeed, $H''(1)$ is the μ_p-torsor

$$H''(1) = S'[X]/(X^p - (1+t^p))$$

on S'. However, this torsor has the section $s : X \mapsto 1+t$, hence is trivial.

What went wrong? In fact, nothing. The isomorphism $\sigma : H'(1) \xrightarrow{\sim} H''(1)$ given by s is not an isomorphism of deformations of E, because it reduces mod t^p to a non-trivial automorphism of E, given by $1+t \in \mu_p(R[t]/(t^p))$, and $1+t$ doesn't lift to a p-th root of unity in $R[t]/(t^{p+1})$.

2.3. Let k be a perfect field of characteristic $p > 0$. For $n \geqslant 1$, $h \geqslant 1$ fixed, let B denote the restriction to $\mathrm{Spec}(W(k))$ of the stack $BT_{n,h}$ considered after Th. 1.2. Let G_0 an object of $B(k)$ (i.e., a BT_n of height h on $\mathrm{Spec}(k)$). Let $T_{G_0}(B)$ be the tangent Picard stack of B at G_0. Then 2.1 (ii) implies

$$H^{-1}(T_{G_0}(B)) = H^0(T_{G_0}(B)) = \mathrm{t}_{G_0^*} \otimes \mathrm{t}_{G_0},$$

where G_0^* is the Cartier dual of G_0. In particular,

$$\dim(B/W(k)) = \mathrm{rk}(T_{G_0}(B)) = 0.$$

The smoothness of $B/W(k)$ implies that B has a smooth cover by a smooth $W(k)$-scheme X. In particular, the triangle associated with $X \to B \to \mathrm{Spec}(W(k))$ implies that $L\Omega^1_{B/W(k)}$, as an object of $D(B)$ ($:= \lim_{\mathrm{Spec(R)} \to \mathrm{B}} D(R)$) is perfect, with perfect amplitude in $[0,1]$, and rank zero.

Not much seems to be known on the global structure of $L\Omega^1_B$.

2.4. *The Deligne-Grothendieck formula.*

Let $i : S_0 \hookrightarrow S$ be a closed immersion defined by an ideal J of square zero. Let G be a commutative group scheme over S, which is flat and locally of finite presentation, and let $G_0 = G_{S_0}$. Let us work with the fppf topology on Sch/S. Then there is a canonical isomorphism in $D(S, \mathcal{O}_S)$

$$\mathrm{Fib}(G \to i_* G_0) \xrightarrow{\sim} i_*(\mathrm{Lie}_{G_0} \otimes^L J).$$

This isomorphism was conjectured by Grothendieck ([3], (4.1.9)) after reading [8] and was proved by Deligne shortly afterwards (unpublished). A recent proof for G affine is given in ([2], Th. 5.1.13), using Clausen-Scholze's theory of animated rings.

Grothendieck (*loc. cit.*, (4.1.12)) observes that $R^1 i_* G_0 = 0$ (for the fppf topology, and already for the syntomic topology) and asks whether $R^q i_* G_0 = 0$ for $q \geqslant 2$. The answer in general seems to be unknown.[5] Assuming it is positive (at least for $q = 2$), Grothendieck deduces an obstruction theory for deformations of extensions $0 \to H \to E \to G \to 0$ over S, for given G and H, reducing on S_0 to a given extension $0 \to H_0 \to E_0 \to G_0 \to 0$ (obstruction in $\mathrm{Ext}^2(G, \mathrm{Lie}_{H_0} \otimes^L J$, set of isomorphism classes of deformations a torsor under $\mathrm{Ext}^1(G, \mathrm{Lie}_{H_0} \otimes^L J)$, and automorphism group of a fixed deformation being $\mathrm{Ext}^0(G, \mathrm{Lie}_{H_0} \otimes^L J$. From this he deduces, by arguments similar to those used in the proof of Th. 1.2 that, if G and H are BT's over S, S being assumed affine, any extension E_0 of G_0 by H_0 [6] can be deformed to an extension E of G by H, and the obvious map $E(E_0, i) \to E(E_0(n), i)$ is surjective (with additional refinements as in 2.1). It would be desirable to complete this program, especially as the obstruction theory for deformations of extensions is not addressed in ([6] VII 4.2).

Extensions of G_0 by H_0 can be viewed as homotopy fibers of maps $G_0 \to H_0[1]$ (this is the starting point of Grothendieck's reasoning above). Similarly, *biextensions* of G_0 by (P_0, Q_0) can be viewed as maps $P_0 \otimes^L_Z Q_0 \to G_0[1]$ ([4], Cor. 3.6.5). This seems to cry for new foundational material on commutative group objects in higher, derived Artin stacks. One can hope that such foundations could also lead to a better understanding (and simplification) of the obstruction theory of [6].

2.5. *A question.*

(Raynaud's example of *drooping* (or *sagging*). Let K be a complete discrete valuation field of mixed characteristic, with ring of integers $\mathcal{O}_K$ and perfect residue field $k = \mathcal{O}_K/\mathfrak{m}$. Assume that $\mathbb{Q}_p(\zeta_p) \subset K$, where ζ_p is a primitive p-th root of 1, so that in particular $e \geqslant p - 1$. Let $S = \mathrm{Spec}(\mathcal{O}_K)$. Let

$$u : (\mathbb{Z}/p\mathbb{Z})_S \to (\mu_p)_S$$

be the morphism of group schemes sending 1 to ζ_p. Then $u_K : (\mathbb{Z}/p)_K \to (\mu_p)_K$ is an isomorphism while $u_k : (\mathbb{Z}/p) \to (\mu_p)_k$ is the trivial morphism. Pulling back by u the extension

$$0 \to (\mu_p)_S \to (\mu_{p^2})_S \to (\mu_p)_S \to 0$$

[5] If G is finite locally free, then the answer is yes, because G has a 2-term resolution $0 \to G \to H^0 \to H^1 \to 0$ with H^i smooth, affine, commutative (the *Bégueri resolution* ([1],2.2)), see ([2], Th. 5.2.7).

[6] Such an extension is automatically a BT ([5], III 5.2).

gives an extension

$$0 \to (\mu_p)_S \to E \to (\mathbb{Z}/p)_S \to 0,$$

where E is a commutative, locally free group scheme over S, whose generic fiber E_K is isomorphic to μ_{p^2}, and special fiber E_k isomorphic to $\mu_p \oplus \mathbb{Z}/p$.

Similarly (Raynaud, p-torsion du schéma de Picard, 4.2), pushing out by u the extension

$$0 \to (\mathbb{Z}/p)_S \to H_S \to (\mathbb{Z}/p)_S^2 \to 0,$$

where H_S is the constant Heisenberg group of rank 3 on S, produces an extension

$$0 \to (\mu_p)_S \to G \to (\mathbb{Z}/p)_S^2 \to 0,$$

where G is a finite locally free group scheme over S whose generic fiber is isomorphic to H, in particular, is non-commutative, and special fiber is isomorphic to the sum $\mu_p \oplus (\mathbb{Z}/p)^2$.

Let $S_n = \mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^{n+1})$. Let G_0 be a commutative, finite locally free group scheme on $S_0 = \mathrm{Spec}(k)$. For a deformation G_n of G over S_n (as a commutative group scheme, or as a $\mathbb{Z}/p^r$-module scheme if G_0 is annihilated by p^r) the theory of deformation $\mathcal{T}_n$ of G_n to S_{n+1} (obstruction, isomorphism classes, automorphisms of an extension) depends only on G_0 (as $\pi^n/\pi^{n+1} \xrightarrow{\sim} k$), and is the same as if S was $\mathrm{Spec}(W(k))$. However, it seems that the above examples can't occur over $W(k)$ (they seem to require $e \geqslant p-1$). Where does the discrepancy lie? Is it in the projective system of $\mathcal{T}_n$'s? [7]

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[7] Gabber (private communication) observes that using the pth root of unity $1+p \in \mathbb{Z}/p^2$ one can construct similarly to Raynaud's second example a non-commutative finite flat group scheme G_1 over $\mathbb{Z}/p^2$ such that $G_0 = G_1 \times_{\mathrm{Spec}(\mathbb{Z}/p^2)} \mathrm{Spec}(\mathbb{F}_p)$ is commutative. However, for p odd, G_1 has no lift to $\mathbb{Z}/p^3$. Gabber has shown more generally that if R is a discrete valuation ring with residue characteristic $p > 0$, uniformizer π, fraction field K, and valuation $v : K^* \to \mathbb{Z}$, then, given $m > 0$, there exists a finite flat group scheme G over R such that G_K is non-commutative and G is commutative mod π^m if and only if $v(p) \geqslant (p-1)m$ (with the convention $v(0) = \infty$) (letter to Shizhang Li, May 5, 2022).

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