Elementary abelian ℓ-groups
and mod ℓ equivariant étale cohomology algebras

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This is a report on joint work with W. Zheng [8]. It grew out of questions that Serre asked me about traces for finite group actions. These questions were the subject of the previous joint papers ([6], [7]). They led us to consider more generally actions of algebraic groups and revisit, in the context of mod ℓ étale cohomology, a theory of equivariant cohomology developed in the early 70's by Quillen for actions of compact Lie groups on topological spaces ([14], [15]).

1. Finite ℓ-group actions, fixed point sets and localizations

Let k be an algebraically closed field of characteristic p and ℓ a prime number $\neq p$. Let X be a separated k-scheme of finite type, acted on by a finite ℓ-group G. Serre ([18], 7.2) observed that we have the following identity

$$\chi(X) \equiv \chi(X^G) \bmod \ell. \tag{1.1}$$

Here X^G is the fixed point set of G, and $\chi = \chi(-, \mathbf{Q}_\ell) = \sum (-1)^i \dim H^i(-, \mathbf{Q}_\ell)$ denotes an Euler-Poincaré ℓ-adic characteristic. It has been known since the early sixties that this integer does not depend on ℓ, as follows from Grothendieck's cohomological formula for the zeta function of a variety over a finite field. Recall also that, by a theorem of Laumon [11], $\chi = \chi_c := \sum (-1)^i \dim H^i_c(-, \mathbf{Q}_\ell)$.

The proof of (1.1) is immediate : by dévissage one reduces to the case where $G = \mathbf{Z}/\ell\mathbf{Z}$; in this case, as $|G| = \ell \neq p$ and G acts freely on $X - X^G$, by a theorem of Deligne (cf. ([6], 4.3)) we have $\chi_c(X - X^G) = \ell\chi_c((X - X^G)/G)$, hence $\chi_c(X) = \chi_c(X^G) + \chi_c(X - X^G)$, and (1.1) follows from Laumon's result. When $G = \mathbf{Z}/\ell\mathbf{Z}$, for $g \in G$ we have a more precise result :

$$\mathrm{Tr}(g, H^*_c(X, \mathbf{Q}_\ell)) = \chi(X^G) + \chi(X - X^G)/G)\mathrm{Reg}_G(g), \tag{1.2}$$

where $\mathrm{Tr}(g, H^*_c) := \sum (-1)^i \mathrm{Tr}(g, H^i_c)$ and Reg_G denotes the character of the regular representation of G. In fact ((([6], (2.3)) $\mathrm{Tr}(g, H^*_c) = \mathrm{Tr}(g, H^*)$ (an equivariant form of Laumon's theorem).

In particular, if ℓ does not divide $\chi(X)$, then X^G is not empty. This is the case, for example, if X is the standard affine space $\mathbf{A}^n_k$ of dimension n over k, as (1.1) implies $\chi(X^G) \equiv 1 \bmod \ell$. Serre ([18], 1.2) remarks that in this case one can show $X^G \neq \emptyset$ in a much more elementary way : reduce to the case where k is the algebraic closure of a finite field $k_0 = \mathbf{F}_q$ and the action of G on $X = \mathbf{A}^n_k$ comes from an action of G on $X_0 = \mathbf{A}^n_{k_0}$. Then we have the stronger property $X_0(k_0)^G \neq \emptyset$, as $|X_0(k_0)| = q^n$ and ℓ divides the cardinality of any non trivial orbit. Given a field K and an action of a finite ℓ-group G on $\mathbf{A}^n_K$, Serre ([18], *loc. cit.*) asks whether $\mathbf{A}^n_K(K)^G$ is not empty. This is the case for $n \leq 2$ (elementary for $n = 1$, by Esnault-Nicaise ([5], 5.12) for $n = 2$). The answer is unknown for $n = 3$, $K = \mathbf{Q}$, $|G| = 2$. In the positive direction, in addition to the case where K is finite, Esnault-Nicaise ([5], 5.17) prove that the answer is yes if K is a henselian discrete valuation field of characteristic zero whose residue field is of characteristic $\neq \ell$, and which is either algebraically closed or finite of cardinality q with $\ell | q - 1$. In the case $K = k$, Smith's theory gives more than the existence of a fixed point. Indeed we have :

Theorem 1.3 ([18], 7.9) ([6], 7.3, 7.8). *Let X be an algebraic space separated and of finite type over k endowed with an action of a finite ℓ-group G. Then, if X is mod ℓ acyclic, so is X^G.*

Here, we say that Y/k is *mod ℓ acyclic* if $H^*(Y, \mathbf{F}_\ell) = H^0(Y, \mathbf{F}_\ell) = \mathbf{F}_\ell$. It is shown in *loc. cit.* that the conclusion of 1.3 still holds if the assumption $\ell \neq p$ made at the beginning of this section is dropped.

Sketch of proof of 1.3. As in the proof of (1.1) we may assume by dévissage that $G = \mathbf{Z}/\ell\mathbf{Z}$. In this case, Serre's proof exploits the action of the algebra $\mathbf{F}_\ell[G]$ on $\pi_*(\mathbf{Z}/\ell\mathbf{Z})$, where $\pi : X \to X/G$ is the projection. The proof given in [6], which uses equivariant cohomology, is close in spirit to that of Borel [2] in the topological case. Let us first give a general definition.

For an algebraic space Y separated and of finite type over k endowed with an action of a finite group G, $R\Gamma(Y, \mathbf{F}_\ell)$ is an object of $D^+(\mathbf{F}_\ell[G])$. The equivariant cohomology complex of Y is defined as

$$R\Gamma_G(Y, \mathbf{F}_\ell) := R\Gamma(G, R\Gamma(Y, \mathbf{F}_\ell)), \tag{1.3.1}$$

which we will abbreviate here to $R\Gamma_G(Y)$. It has a natural multiplicative structure, and $H^*_G(Y) = H^*R\Gamma_G(Y)$ is a graded algebra over the graded $\mathbf{F}_\ell$-algebra $H^*_G = H^*(G, \mathbf{F}_\ell)$. For $G = \mathbf{Z}/\ell\mathbf{Z}$, we have

$$H^*_{\mathbf{Z}/\ell\mathbf{Z}} = \begin{cases} \mathbf{F}_\ell[x] & \text{if } \ell = 2 \\ \mathbf{F}_\ell[x]/(x^2) \otimes \mathbf{F}_\ell[y] & \text{if } \ell > 2, \end{cases} \tag{1.3.2}$$

where x is the tautological generator of $H^1_{\mathbf{Z}/\ell\mathbf{Z}}$, and, for $\ell > 2$, $y = \beta x$, where $\beta : H^1_{\mathbf{Z}/\ell\mathbf{Z}} \xrightarrow{\sim} H^2_{\mathbf{Z}/\ell\mathbf{Z}}$ is the Bockstein operator (associated with the exact sequence $0 \to \mathbf{F}_\ell \to \mathbf{Z}/\ell^2\mathbf{Z} \to \mathbf{F}_\ell \to 0$).

Coming back to the proof of 1.3, the key point is that (for $G = \mathbf{Z}/\ell\mathbf{Z}$) the restriction map

$$H^*_G(X) \to H^*_G(X^G) = H^*_G \otimes H^*(X^G), \tag{1.3.3}$$

which is a map of graded H_G^*-modules, becomes an isomorphism after inverting $\beta x \in H^2_{\mathbf{Z}/\ell\mathbf{Z}}$. Indeed, the assumption that X is mod ℓ acyclic implies that $H_G^*(X) = H_G^*$, hence $H^*(X^G)$ has to be of rank one over $\mathbf{F}_\ell$. The assertion about (1.3.3) follows from the fact that $H_G^*(X, j_!\mathbf{F}_\ell)$ is of bounded degree, where $j : X - X^G \hookrightarrow X$ is the inclusion, as X/G is of finite ℓ-cohomological dimension.

The above key point is similar to various *localization formulas* considered by Quillen, Atiyah-Segal, Goresky-Kottwitz-MacPherson. For actions of elementary abelian ℓ-groups[1] we have the following result, which is an analogue of Quillen's theorem ([14], 4.2) :

Theorem 1.4 ([6], 8.3). *Let X be an algebraic space separated and of finite type over k endowed with an action of an elementary abelian ℓ-group G of rank r, and let*

$$e := \prod_{\xi \in H_G^1 - \{0\}} \beta\xi \in H_G^{2(\ell^r - 1)},$$

where $\beta : H_G^1 \to H_G^2$ is the Bockstein operator. Then the restriction map

$$H_G^*(X)[e^{-1}] \to H_G^*(X^G)[e^{-1}]$$

is an isomorphism.

The proof in ([6], 8.3) is by dévissage on G. In ([14], 4.2) it is deduced from general structure theorems for $H_G^*(X, \mathbf{F}_\ell)$ for actions of compact Lie groups G on certain topological spaces X. This led us to investigate algebraic analogues of these results.

2. Quotient stacks and equivariant cohomology algebras : finiteness theorems

2.1. If G is a compact Lie group, we have a classifying space BG, which is the base of a universal G-torsor PG, whose total space is contractible. If X is a G-space[2], i. e. a topological space endowed with a continuous action of G, the projection $w : PG \times X \to BG$ induced by $PG \to BG$ factors through the quotient

$$PG \wedge^G X := (PG \times X)/G,$$

[1] An elementary abelian ℓ-group is a group G isomorphic to the direct product of a finite number r of cyclic groups of order ℓ. The integer r is called the rank of G.

[2] We let groups act on spaces on the right.

where G acts by $(p, x)g = (pg, xg)$, giving a commutative diagram

$$\text{(2.1.1)} \qquad \begin{array}{ccc} PG \times X & \xrightarrow{u} & PG \wedge^G X \; . \\ \downarrow{\scriptstyle w} & \swarrow{\scriptstyle v} & \\ BG & & \end{array}$$

Here u makes $PG \times X$ into a G-torsor over $PG \wedge^G X$, and v is a locally trivial fibration of fiber X. The torsor u is universal in the sense that, up to homotopy, maps from a compact space T to $PG \wedge^G X$ correspond to pairs of a G-torsor P on T and an equivariant map from P to $PG \times X$.

Let ℓ be a prime number. The equivariant mod ℓ cohomology complex of X, $R\Gamma_G(X, \mathbf{F}_\ell)$, is defined, à la Borel, by

$$\text{(2.1.2)} \qquad R\Gamma_G(X, \mathbf{F}_\ell) := R\Gamma(PG \wedge^G X, \mathbf{F}_\ell).$$

Its cohomology,

$$H^*_G(X, \mathbf{F}_\ell) := H^* R\Gamma_G(X, \mathbf{F}_\ell),$$

is a graded $\mathbf{F}_\ell$-algebra over $H^*(BG, \mathbf{F}_\ell) = H^*_G(pt, \mathbf{F}_\ell)$. Using v one can rewrite (2.1.2) as

$$\text{(2.1.3)} \qquad R\Gamma_G(X, \mathbf{F}_\ell) = R\Gamma(BG, R\Gamma(X, \mathbf{F}_\ell)),$$

where by abuse $R\Gamma(X, \mathbf{F}_\ell) \in D^+(BG, \mathbf{F}_\ell)$ denotes $Rv_*\mathbf{F}_\ell$ (a locally constant complex of value $R\Gamma(X, \mathbf{F}_\ell)$). This equivariant cohomology is studied in Quillen's papers [14], [15]. One of the main results is that $H^*(BG, \mathbf{F}_\ell)$ is a finitely generated $\mathbf{F}_\ell$-algebra, and, if $H^*(X, \mathbf{F}_\ell)$ is finite dimensional, then $H^*_G(X, \mathbf{F}_\ell)$ is finite over $H^*(BG, \mathbf{F}_\ell)$ ([14], 2.1, 2.2, 2.3).

2.2. Similar results are available in the setting of mod ℓ étale cohomology and actions of algebraic groups. From now on we denote by k an algebraically closed field of characteristic $p \geq 0$ and ℓ a prime number $\neq p$. Let G be an algebraic group over k, and let X be an algebraic space of finite type over k [3], endowed with an action of G. Consider the quotient stack $[X/G]$ ([12], 3.4.2). This is an Artin stack[4] of finite type over k, which comes equipped with a surjection $u : X \to [X/G]$ making X into a universal G-torsor over

[3] By an algebraic group over k we mean a k-group scheme of finite type. By an algebraic space X over k we mean the quotient of a k-scheme by an étale equivalence relation ; we do not assume X to be quasi-separated.

[4] By an Artin stack over k we mean a stack in groupoids $\mathcal{X}$ over the big fppf site of $\operatorname{Spec} k$ such that the diagonal $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is representable (by algebraic spaces) and there exists a smooth surjective k-morphism $X \to \mathcal{X}$ with X a k-algebraic space.

$[X/G]$, in the sense that the groupoid of points of $[X/G]$ with values in a k-algebraic space T consists of pairs of a G_T-torsor P on T and an equivariant map $P \to X$. The quotient stack $[\operatorname{Spec} k/G]$ is called the *classifying stack* of G and is denoted BG. We have a 2-commutative diagram similar to (2.1.1) :

$$\begin{array}{ccc} X & \xrightarrow{u} & [X/G] \, , \\ \downarrow{\scriptstyle w} & \swarrow{\scriptstyle v} & \\ BG & & \end{array} \tag{2.2.1}$$

where $v : [X/G] \to [\operatorname{Spec} k/G] = BG$ is an fppf locally trivial fibration of fiber X. The equivariant mod ℓ cohomology complex of X, $R\Gamma_G(X, \mathbf{F}_\ell)$, is defined, similarly to (2.1.2), by

$$R\Gamma_G(X, \mathbf{F}_\ell) := R\Gamma([X/G], \mathbf{F}_\ell). \tag{2.2.2}$$

The cohomology on the right hand side is that of the smooth-étale site of $[X/G]$. As above, using v, one can rewrite it as

$$R\Gamma_G(X, \mathbf{F}_\ell) = R\Gamma(BG, R\Gamma(X, \mathbf{F}_\ell)), \tag{2.2.3}$$

where by abuse $R\Gamma(X, \mathbf{F}_\ell) \in D^+(BG, \mathbf{F}_\ell)$ denotes $Rv_*\mathbf{F}_\ell$. Thus, if G is a finite (discrete) group, the definition given in (2.2.2) agrees with that given in (1.3.1). The cohomology

$$H^*_G(X, \mathbf{F}_\ell) := H^* R\Gamma_G(X, \mathbf{F}_\ell)$$

is a graded $\mathbf{F}_\ell$-algebra over the graded algebra $H^*(BG, \mathbf{F}_\ell)$.

By definition, $R\Gamma_G(X, \mathbf{F}_\ell)$ depends only on the stack $[X/G]$, which can have various descriptions as a quotient stack. For example, if G is a subgroup of an algebraic group G' over k, we have a natural equivalence

$$[X/G] \xrightarrow{\sim} [X \wedge^G G'/G'], \tag{2.2.4}$$

called *induction formula*, and hence an isomorphism

$$H^*_G(X, \mathbf{F}_\ell) \xrightarrow{\sim} H^*_{G'}(X \wedge^G G', \mathbf{F}_\ell). \tag{2.2.5}$$

Here $X \wedge^G G'$ is the quotient of $X \times G'$ by the diagonal action of G, an fppf locally trivial fibration of fiber X over the homogeneous space $G'/G = \operatorname{Spec} k \wedge^G G'$.

The following theorem is similar to the results of Quillen mentioned at the end of 2.1 :

Theorem 2.3 ([8], 4.6). *With the notations of 2.2, assume X of finite presentation over k. Then $H^*(BG, \mathbf{F}_\ell)$ is a finitely generated $\mathbf{F}_\ell$-algebra, and $H^*_G(X, \mathbf{F}_\ell)$ is a finite $H^*(BG, \mathbf{F}_\ell)$-module.*

In an earlier version of [8] this result was first proved by Illusie-Zheng in the case that G is an affine group, or a semi-abelian variety. A few more cases were suggested to us by Brion. The day after my talk at the conference, Deligne kindly provided me with a proof of the general case [3].

Remark 2.4. The result established in ([8], 4.6) is slightly more general. One can replace $\mathbf{F}_\ell$ by a noetherian $\mathbf{Z}/n\mathbf{Z}$-algebra Λ, with n invertible in k : $H^*(BG, \Lambda)$ is a finitely generated Λ-algebra. One can also replace $\mathbf{F}_\ell$ by an object K of $D^b_c([X/G], \Lambda)$, i. e. the full subcategory of $D^b([X/G], \Lambda)$ consisting of complexes (over the lisse-étale site) with bounded, cartesian, constructible cohomology (the datum of a cartesian, constructible sheaf of Λ-modules on $[X/G]$ is equivalent to the datum of a constructible sheaf of Λ-modules F on X together with an action of G on F compatible with the action of G on X ; if G acts trivially on X, this action factors through the finite discrete group $\pi_0(G)$). Then $H^*_G(X, K) := H^*([X/G], K)$ is a finite $H^*(BG, \Lambda)$-module.

Examples 2.5. Let Λ be as in 2.4.

(a) Let r be an integer ≥ 1. For $1 \leq i \leq r$, let $c_i \in H^{2i}(BGL_{r,k}, \Lambda(i))$ be the i-th Chern class of the tautological bundle $\mathcal{O}^r$ over $BGL_{r,k}$. Let $\Lambda[x_1, \cdots, x_r]$ be the polynomial algebra over Λ on generators x_i of degree $2i$ for $1 \leq i \leq r$. Then $H^q(BGL_{r,k}, \Lambda) = 0$ for q odd, and the homomorphism of Λ-algebras

$$\Lambda[x_1, \cdots, x_r] \to H^{2*}(BGL_{r,k}, \Lambda(*)) := \oplus_{i \geq 0} H^{2i}(BGL_{r,k}, \Lambda(i))$$

sending x_i to c_i is an isomorphism.

This result has been known since the 60's. A proof of the analogous result for de Rham cohomology, consisting in approximating $BGL_{r,k}$ by grassmannians, was communicated to me by Deligne in 1967. The argument is sketched by Behrend in ([1], 2.3.2).

(b) Let G be a semi-abelian variety over k, extension of an abelian variety A of dimension g by a torus T of dimension r. Then we have a short exact sequence of free Λ-modules

$$0 \to H^1(A, \Lambda) \to H^1(G, \Lambda) \to H^1(T, \Lambda) \to 0,$$

of successive ranks $2g$, $2g + r$, r, and isomorphisms of algebras

$$H^*(G, \Lambda) \simeq \Lambda^*_\Lambda(H^1(G, \Lambda)),$$

$$H^{2*}(BG, \Lambda) \simeq S^*_\Lambda(H^1(G, \Lambda)),$$

where $H^1(G, \Lambda)$ is placed in degree 2, and $H^q(BG, \Lambda) = 0$ for q odd.

2.6. *Sketch of proof of 2.3.* When G is affine, one can embed G into $GL_{r,k}$ for some r, and by the induction formula (2.2.5) one is reduced to the case $G = GL_{r,k}$. One can then imitate Quillen's proof, using 2.5 (a). The general case is reduced to this one, using the general structure of algebraic groups over k. We may assume G reduced. Then G has a filtration $1 \subset G_2 \subset G_1 \subset G_0 = G$, with G_{i+1} normal in G_i, and successive quotients : G_0/G_1 a finite discrete group, G_1/G_2 an abelian variety, G_2 a connected affine group. Very roughly, the idea is that the abelian variety layer A in this dévissage can be replaced by the inductive system of its division points

$$A[m\ell^\infty] = \text{“}\varinjlim_{n \geq 1}\text{”} A[m\ell^n]$$

for a suitable integer $m \geq 1$ (e. g. the order of G_0/G_1), where $A[d]$ denotes the kernel of the multiplication by d, using the fibration $BA[m\ell^n] \to BA$ with fiber $A/A[m\ell^n] \xrightarrow{\sim} A$, and the fact that the transition map

$$H^*(A/A[m\ell^{n+1}], \mathbf{F}_\ell) \to H^*(A/A[m\ell^n], \mathbf{F}_\ell)$$

vanishes in positive degree.

3. The amalgamation and stratification theorems

3.1. If G is a compact Lie group and X a G-space which is either compact of paracompact and of finite ℓ-cohomological dimension, and such that $H^*(X, \mathbf{F}_\ell)$ is finite dimensional, Quillen relates the size of the (finitely generated) graded algebra $H^*_G(X) := H^*_G(X, \mathbf{F}_\ell)$, i.e. the dimension of the spectrum of the (commutative) reduced algebra $H^*_G(X)_{\text{red}}$, to elementary abelian ℓ-subgroups of G. He shows that this dimension is equal to the maximum rank of an elementary abelian ℓ-subgroup of G fixing a point in X. He deduces this from a more precise theorem describing the spectrum, up to a homeomorphism, as an amalgamated sum of spectra of reduced algebras $H^*(BA, \mathbf{F}_\ell)_{\text{red}}$, for A varying among elementary abelian ℓ-subgroup of G with non empty fixed point sets ([15], 8.10). Again, analogous results are available in the algebraic setting, which we will now discuss.

3.2. First, recall the structure of $H^*_A := H^*(BA, \mathbf{F}_\ell)$ for an elementary abelian ℓ-group A of rank r ([14], §4) ([6], §8). Let $\check{A} := \mathrm{Hom}(A, \mathbf{F}_\ell)$. Then

$$H^1_A = \check{A},$$

the Bockstein map

$$\beta : H^1_A \to H^2_A$$

defined by the exact sequence $0 \to \mathbf{F}_\ell \to \mathbf{Z}/\ell^2\mathbf{Z} \to \mathbf{F}_\ell \to 0$ is injective, and we have an isomorphism of graded $\mathbf{F}_\ell$-algebras

$$H_A^* = \begin{cases} S(\check{A}) & \text{if } \ell = 2 \\ \Lambda\check{A} \otimes S(\beta\check{A}) & \text{if } \ell > 2. \end{cases}$$

If $\{x_1, \cdots, x_r\}$ is a basis of $\check{A}$ over $\mathbf{F}_\ell$, then

$$H_A^* = \begin{cases} \mathbf{F}_\ell[x_1, \cdots, x_r] & \text{if } \ell = 2 \\ \Lambda(x_1, \cdots, x_r) \otimes \mathbf{F}_\ell[y_1, \cdots, y_r] & \text{if } \ell > 2. \end{cases}$$

In particular, $\mathrm{Spec}\,(H_A^*)_{\mathrm{red}}$ is the affine space $\mathbf{A}^r_{\mathbf{F}_\ell}$.

3.3. Let k, ℓ, G and X be as in 2.2. By analogy with ([15], 8.1) we define the following category

$$\mathcal{A}_{(G,X,\ell)}. \tag{3.3.1}$$

Objects of $\mathcal{A}_{(G,X,\ell)}$ are pairs (A, C), where A is an elementary abelian ℓ-subgroup of G, and C is a connected component of the fixed point space X^A (in particular, is not empty). For objects (A, C) and (A', C') of $\mathcal{A}_{(G,X,\ell)}$, maps from (A, C) to (A', C') are defined by

$$\mathrm{Hom}_{\mathcal{A}_{(G,X,\ell)}}((A, C), (A', C')) = \mathrm{Trans}_G((A, C), (A', C'))(k),$$

where $\mathrm{Trans}_G((A, C), (A', C'))$, the *transporter* of (A, C) into (A', C'), is the closed subscheme of G representing the functor on k-schemes

$$S \mapsto \{g \in G(S) | g^{-1}A_S g \subset A'_S, C_S g \supset C'_S\}$$

(see ([8], 6.4)). Composition is defined by composition of transporters. When no confusion can arise, we will abbreviate $\mathcal{A}_{(G,X,\ell)}$ into $\mathcal{A}_{(G,X)}$ and write $\mathcal{A}_G$ for $\mathcal{A}_{(G,\mathrm{Spec}\,k)}$.

3.4. In the rest of this section we will consider projective systems indexed by a smaller category $\mathcal{A}^\flat_{(G,X)}$. As in ([14], (8.2)), the map

$$\mathrm{Trans}_G((A, C), (A', C'))(k) \to \mathrm{Hom}(A, A')$$

sending g to the homomorphism $a \mapsto g^{-1}ag$ induces an injection

$$\mathrm{Cent}_G(A, C)(k)\backslash\mathrm{Trans}_G((A, C), (A', C'))(k) \hookrightarrow \mathrm{Hom}(A, A'), \tag{3.4.1}$$

where

$$\mathrm{Cent}_G(A, C)(k) = \{g \in G(k) | Cg = C \text{ and } g^{-1}ag = a \text{ for all } a \in A\}.$$

The injection (3.4.1) is compatible with composition. The category

$$\mathcal{A}^{\flat}_{(G,X,\ell)} \tag{3.4.2}$$

having the same objects as $\mathcal{A}_{(G,X,\ell)}$, but with maps defined by the left hand side of (3.4.1) is the analogue of the category defined by Quillen in [14]. As in *loc. cit.*,

$$W(A,C) := \mathrm{Cent}_G(A,C)(k)\backslash \mathrm{Trans}_G((A,C),(A,C))(k) \subset \mathrm{End}(A) \tag{3.4.3}$$

is a (finite) group, called the *Weyl group* of (A,C). For $X = \mathrm{Spec}\, k$ we write $W(A, \mathrm{Spec}\, k) = W(A)$. For G connected, reductive, with Weyl group $W = N_T(G)/T$, where T is a maximal torus of G, if $T[\ell]) := \mathrm{Ker}\, \ell : T \to T$, $W(T[\ell])$ is a quotient of W and $W(T[\ell]) = W$ if $\ell > 2$; for $G = \mathrm{GL}_n$, $W(T[\ell]) = W$. See ([8], 6.7).

Lemma 3.5. *The category $\mathcal{A}^{\flat}_{(G,X,\ell)}$ is equivalent to a finite category, more precisely :*

(a) For any objects (A,C) and (A',C') of $\mathcal{A}_{(G,X)}$, the set of homomorhisms in $\mathcal{A}^{\flat}_{(G,X)}$ from (A,C) to $(A',C'))$ is finite.

(b) The set of isomorphism classes of objects of $\mathcal{A}^{\flat}_{(G,X)}$ is finite.

Assertion (a) is trivial. For (b), the main point is the following fact, which was communicated to us by Serre : the set of conjugacy classes of elementary abelian ℓ-subgroups of G is finite. This follows from the boundedness of the ranks of such subgroups, and the fact that if H is a finite group of order prime to p, the orbits of G acting on $\mathrm{Hom}_{\mathrm{gp}}(H,G)$ by conjugation are open.

3.6. For $(A,C) \in \mathcal{A} := \mathcal{A}_{(G,X)}$, we have a restriction homomorphism

$$H^*_G(X) \to H^*_A(C) = H^*_A \otimes H^*(C) \to H^*_A \otimes H^0(C) = H^*_A, \tag{3.6.1}$$

where $H^*_G(X) := H^*_G(X, \mathbf{F}_\ell)$, etc. For $g \in G(k)$, the map

$$\theta_g : H^*_G(X) \to H^*_G(X)$$

induced by $(h,x) \mapsto (g^{-1}hg, xg)$ is the identity.

For $g \in \mathrm{Hom}_{\mathcal{A}(G,X)}((A,C),(A',C'))$, we thus get a commutative triangle

$$\begin{array}{ccc} H^*_G(X) & & \\ \downarrow & \searrow & \\ H^*_{A'} & \xrightarrow{\theta_g} & H^*_A \end{array} ,$$

and hence a canonical map

$$a_{(G,X)} : H^*_G(X) \to \varprojlim_{(A,C)\in \mathcal{A}_{(G,X)}} H^*_A. \tag{3.6.2}$$

Note that $\theta_g : H^*_{A'} \to H^*_A$ depends only on the image of g in $\mathcal{A}^\flat_{(G,X)}$, so that in the above projective limit, we can replace the index category by $\mathcal{A}^\flat_{(G,X)}$ (which is equivalent to finite category).

The following result (*amalgamation theorem*) is an analogue of ([14], 6.2), ([15], 8.5).

Theorem 3.7 ([8], 6.11). *Assume X separated. Then the homomorphism $a_{(G,X)}$ is a uniform F-isomorphism.*

That $a_{(G,X)}$ is a uniform F-isomorphism means that its kernel and cokernel are annihilated by a power of $F : a \mapsto a^\ell$, i. e., there exists an integer $N \geq 1$ such that for any a in the kernel (resp. target) of $a_{(G,X)}$, $F^N a = 0$ (resp. $F^N a \in \operatorname{Im} a_{(G,X)}$).

Remark. When G is an elementary abelian ℓ-group, the localization theorem 1.4 for $X^G = \emptyset$, namely that $H^*_G(X)[e^{-1}] = 0$, is an easy corollary of 3.7. However, it is not clear how to transpose to the algebraic setting the arguments of Quillen in ([14], 4.2) to reduce to this case. The proof in ([6], 8.3) uses an independent method.

Theorem 3.7 has the following geometric consequence, which justifies the terminology "amalgamation theorem". Define

$$\underline{(G,X)} := \operatorname{Spec} H^{\varepsilon *}_G(X)_{\mathrm{red}}, \tag{3.7.1}$$

where $\varepsilon = 1$ if $\ell = 2$, and $\varepsilon = 2$ otherwise. In particular, for an elementary abelian ℓ-group A,

$$\underline{A} := \underline{(A, \operatorname{Spec} k)} = \operatorname{Spec} (H^{\varepsilon *}_A)_{\mathrm{red}},$$

a standard affine space of dimension equal to the rank of A (3.2). The map (3.6.1) induces a morphism or schemes

$$(A,C)_* : \underline{A} \to \underline{(G,X)}, \tag{3.7.2}$$

hence $a_{(G,X)}$ (3.6.2) induces a morphism of schemes

$$\varinjlim_{(A,C)\in \mathcal{A}^\flat_{(G,X)}} \underline{A} \to \underline{(G,X)}. \tag{3.7.3}$$

It follows from 3.7 that (3.7.3) is a universal homeomorphism. If $(A_i, C_i)_{i\in I}$ is a finite set of representatives of isomorphism classes of objects of $\mathcal{A}^\flat_{(G,X)}$, by a corollary to the finiteness theorem 2.3 (see ([8], 4.8)) $\coprod_{i\in I} \underline{A_i}$ is finite over $\underline{(G,X)}$ and the limit on the left hand side is the quotient of $\coprod_{i\in I} \underline{A_i}$ by a (finite) equivalence relation over $\underline{(G,X)}$, see ([8], 11.1). Therefore we get the following corollary, similar to ([14], 7.7) :

Corollary 3.8. *The dimension of $\underline{(G,X)}$ is the maximal rank of an elementary abelian ℓ-subgroup of G fixing a point in X.*

Example : The dimension of $\operatorname{Spec} H^*(B\mathrm{GL}_{n,k})$ is n (2.5 (a)), which is also the rank of $\operatorname{Ker} \ell : T \to T$, where T is a maximal torus of $\mathrm{GL}_{n,k}$.

The structure of $\underline{(G,X)}$ in relation with (3.7.2) can be described more precisely. We have the following *stratification theorem*, similar to ([15], 10.2, 12.1) :

Theorem 3.9 ([8], 11.2). *Denote by $V_{(A,C)}$ the reduced subscheme which is the image of the (finite) map $\underline{A} \to \underline{(G,X)}$ (3.7.2). Let*

$$\underline{A}^+ := \underline{A} - \cup_{A'<A}\underline{A}',$$

$$V^+_{(A,C)} := V_{(A,C)} - \cup_{A'<A} V_{(A',C|A')}$$

(where $A' < A$ means $A' \subset A$ and $A' \neq A$ and $C|A'$ the component of $X^{A'}$ containing C). Then :

(a) The Weyl group $W(A,C)$ (3.5.3) acts freely on $\underline{A}^+$ and the map $\underline{A}^+ \to V^+_{(A,C)}$ given by (3.7.2) induces a homeomorphism

$$\underline{A}^+/W(A,C) \to V^+_{(A,C)}.$$

(b) The subschemes $V_{(A,C)}$ of $\underline{(G,X)}$ are the integral closed sub-cones of $\underline{(G,X)}$ that are stable under the Steenrod operations on $H^{\varepsilon *}_G(X)$ (see 3.12).

(c) Let $(A_i, C_i)_{i\in I}$ be a finite set of representatives of isomorphism classes of objects of $\mathcal{A}^\flat_{(G,X)}$. Then the $V_{(A_i,C_i)}$'s form a finite stratification of $\underline{(G,X)}$, namely $\underline{(G,X)}$ is the disjoint union of the $V^+_{(A_i,C_i)}$, and $V_{(A_i,C_i)}$ is the closure of $V^+_{(A_i,C_i)}$.

The proof is based on 2.3 and 3.7, and is entirely analogous to that of ([15], 10.2, 12.1).

Examples 3.10. In the following, we set

$$R^{\varepsilon *}(G,X) := \varprojlim_{(A,C)\in\mathcal{A}^\flat_{(G,X)}} H^{\varepsilon *}_A,$$

and

$$(G,X)_{\text{proj}} := \operatorname{Proj} H_G^{\varepsilon *}(X)_{\text{red}},$$

so that $\underline{(G,X)} - \{0\}$ is a $\mathbf{G}_m$-bundle over $(G,X)_{\text{proj}}$. We also omit $\mathbf{F}_\ell$ from the notations of cohomology groups.

(a) For $X = \mathbf{P}^1_k$, with the natural action of $G = \mathbf{G}_{m,k}$, the category $\mathcal{A}^\flat_{(G,X)}$ has three objects : $(\{0\}, X)$, $(\mu_\ell, 0)$, (μ_ℓ, ∞), and $\underline{(G,X)}$ is a cone on $\{0\} \cup \{\infty\}$, a union of two lines, $V_{(\mu_\ell,0)}$ and $V_{(\mu_\ell,\infty)}$, their intersection being the point $V_{(\{0\},X)}$. In other words, $\operatorname{Proj} R^{\varepsilon *}(G,X)_{\text{red}} = \{0, \infty\}$.

(b) More generally, let X be the projective space $\mathbf{P}^n_k$, with its natural action of the torus $T = \mathbf{G}_m^{n+1}/\mathbf{G}_m$ of rank n by $(x_0 : \cdots : x_n)g = (x_0 g : \cdots : x_n g)$. Then $\operatorname{Proj} R^{\varepsilon *}(T,X)_{\text{red}} = X - T$ (hence $(T,X)_{\text{proj}}$ is homeomorphic to $X - T$). The stratification of 3.9 (c) is deduced from the standard stratification of $X - T$. Similar results hold for toric varieties ([8], 6.15). For example, if X is the toric variety $\operatorname{Spec} k[P]$ for a fine and saturated monoid P such that $P^* = 0$, with the action of the torus $T = \operatorname{Spec} k[P^{\text{gp}}]$, then $\underline{(T,X)}$, with its stratification (3.9 (c)), is homeomorphic to $\mathbf{V}(P^{\text{gp}} \otimes \mathbf{F}_\ell)$, with the stratification given by the faces of P.

(c) Let n be an integer ≥ 1 and let T be the standard maximal torus in $G = \mathrm{GL}_{n,k}$. Let $W (= N_T(G)/T = S_n)$ be the Weyl group. A complement to 2.5 (a) is that the restriction map $H^*(BG) \to H^*(BT)$ induces an isomorphism

$$H^*(BG) \xrightarrow{\sim} H^*(BT)^W.$$

The map $a_{(G,X)}$ (3.6.2) (for $X = \operatorname{Spec} k$) is the composition with the restriction $H^*(BT)^W \to H^*(BT[\ell])^W$, where $T[\ell] = \operatorname{Ker} \ell : T \to T$ (recall that $W(T[\ell]) = W$ (3.4.3)). It is injective and its cokernel is annihilated by F. The space $\underline{(G,X)}$ is the affine space $\mathbf{A}^n$ over $\mathbf{F}_\ell$.

Question 3.11. In the situation of 3.7, let $d_\ell(G,X)$ be the dimension of $\operatorname{Spec} H_G^{\varepsilon *}(X, \mathbf{F}_\ell)$, and let us write $d_\ell(G)$ for $d_\ell(G, \operatorname{Spec} k)$. We have $d_\ell(G,X) = d_\ell(G_{\text{red}}, X_{\text{red}})$. Is there an integer N such that, for all $\ell > N$, $d_\ell(G,X)$ is independent of ℓ ? For $d_\ell(G)$, does it suffice to take for N the supremum of the orders of G_1/G_0 and the Weyl group W of G_2/R_u in the dévissage mentioned in 2.6 (with G reduced), where R_u denotes the unipotent radical of G_2 ?

If $[X/G]$ is a Deligne-Mumford stack, it follows from 3.7 that $d_\ell(G,X) = 0$ as soon as ℓ does not divide the orders of the (finite) inertia groups. On the other hand, it is easy to see that $d_\ell(G,X)$ is bounded by an integer independent of ℓ. Indeed, we have $d_\ell(G,X) \leq d_\ell(G)$ (by 2.3 or 3.8), $d_\ell(G) \leq d_\ell(G_0/G_1) + d_\ell(G_1/G_2) + d_\ell(G_2)$, and, for $\ell > N$, $d_\ell(G_0/G_1) = 0$ (as $\ell > |G_0/G_1|$), $d_\ell(G_1/G_2) = 2\dim(G_1/G_2)$ (2.5 (b)), $d_\ell(G_2) = d_\ell(G_2/R_u)$, and as

$\ell > |W|$ all elementary abelian ℓ-subgroups of G_2/R_u are toral ([17], 1.2.2), whence $d_\ell(G_2) = \mathrm{rk}(G_2/R_u)$.

3.12. *Steenrod operations.* The graded algebra $H_G^*(X) := H_G^*(X, \mathbf{F}_\ell)$ admits Steenrod operations, preserving $H_G^{\varepsilon *}(X)$. These are homomorphisms

$$P^i : H_G^*(X) \to \begin{cases} H_G^{*+i}(X) & \text{if } \ell = 2 \\ H_G^{*+2(\ell-1)i}(X) & \text{if } \ell > 2. \end{cases} \tag{3.12.1}$$

For $\ell = 2$, P^i is sometimes denoted Sq^i. Their construction is a particular case of Steenrod operations on $H^*(T, \mathbf{F}_\ell)$ for a topos T, see [4], [16], ([8], 11.6). They satisfy the following properties, where we write H^* for $H_G^*(X)$: $P^i = 0$ for $i < 0$, $P^0 = \mathrm{Id}$; for $x \in H^q$, $P^i x = 0$ for $q < i$ if $\ell = 2$, $P^i x = 0$ for $q < 2i$ if $\ell > 2$; for $x \in H^i$ (resp. $x \in H^{2i}$), $P^i x = x^\ell$ if $\ell = 2$ (resp. $\ell > 2$) ; if one defines

$$P_t : H^* \to H^*[t]$$

by $P_t(x) = \sum_{i \geq 0} P^i(x) t^i$, so that $P_t(x) = x + x^\ell t$ for $x \in H^1$, $\ell = 2$ (resp. $x \in H^2$, $\ell > 2$), then P_t is a ring homomorphism (Cartan's formula).

4. A stack-theoretic reformulation of the amalgamation theorem, and a generalization.

The source of the homomorphism $a_{(G,X)}$ (3.6.2) depends only on the stack $\mathcal{X} = [X/G]$, but the target involves fixed points of the action of subgroups of G on X. However, one can rewrite this target as a limit over a certain category $\mathcal{C}$ of points of $\mathcal{X}$. This reformulation makes sense on any Artin stack, and can also be extended to include constructible coefficients. Such a reformulation is in fact needed to prove 3.7. Indeed a crucial continuity property ([14], 5.6) used by Quillen in his proof of the analogous topological result has to be replaced by an analysis of specialization of points in $\mathcal{C}$.

4.1. Let k and ℓ be as in 2.2, and let $\mathcal{X}$ be an Artin stack over k (see footnote 3).

(a) We define a *geometric point* of $\mathcal{X}$ to be a representable morphism $x : S \to \mathcal{X}$, where S is a strictly local scheme. A morphism from $x : S \to \mathcal{X}$ to $y : T \to \mathcal{X}$ is a morphism $f : S \to T$ together with a 2-morphism $u : x \to yf$. By inverting morphisms (f, u) such that f sends the closed point of S to the closed point of T, we get a category

$$\mathcal{P}_{\mathcal{X}} \tag{4.1.1}$$

called the category of geometric points of $\mathcal{X}$. In the case $\mathcal{X}$ is a scheme, this category is equivalent to the usual category of geometric points of $\mathcal{X}$. If Λ

is a noetherian ring, and F is a constructible sheaf of Λ-modules on $\mathcal{X}$, the natural map

$$\Gamma(\mathcal{X}, F) \to \varprojlim_{x \in \mathcal{P}_{\mathcal{X}}} F_x \tag{4.1.2}$$

is an isomorphism ([8], 7.12). Here, for $(x : S \to \mathcal{X}) \in \mathcal{P}_{\mathcal{X}}$, $F_x = \Gamma(S, F) = F_s$ is the stalk of F at the closed point s of S.

(b) We will need a bigger category of geometric points, depending on ℓ. We define an *ℓ-elementary point* of $\mathcal{X}$ to be a representable morphism $x : \mathcal{S} \to \mathcal{X}$, where $\mathcal{S}$ is isomorphic to a quotient stack $[S/A]$, where S is a strictly local scheme endowed with an action of an elementary abelian ℓ-group A acting trivially on the closed point of S. Note that the representability condition imposes that $x : [S/A] \to \mathcal{X}$ induces an injection $A \hookrightarrow \mathrm{Aut}_{\mathcal{X}}(s \to \mathcal{X})$. A morphism from $x : [S/A] \to \mathcal{X}$ to $y : [T/B] \to \mathcal{X}$ is an isomorphism class of pairs (φ, α), where $\varphi : [S/A] \to [T/B]$ is an $\mathcal{X}$-morphism, and $\alpha : x \to y\varphi$ is a 2-morphism; an isomorphism between pairs (φ, α) and (ψ, β) is a 2-morphism $c : \varphi \to \psi$ such that $\beta = c\alpha$. Such a pair (φ, α) is represented by a morphism of $\mathcal{X}$-schemes $f : S \to T$ and a group homomorphism $u : A \to B$ (and if (f_1, u_1), (f_2, u_2) are two such pairs, then $u_1 = u_2$ and there exists a unique $r \in B$ such that $f_1 r = f_2$). By inverting morphisms (φ, α) such that f sends s to t and the (unique) homomorphism $u : A \to B$ is an isomorphism, we get a category

$$\mathcal{C}_{\mathcal{X},\ell}, \tag{4.1.3}$$

called the category of *ℓ-elementary points* of $\mathcal{X}$, abbreviated to $\mathcal{C}_{\mathcal{X}}$ if no confusion can arise. It follows readily from the definitions that the obvious functor

$$\mathcal{P}_{\mathcal{X}} \to \mathcal{C}_{\mathcal{X}} \tag{4.1.4}$$

is fully faithful, and one can show that if $\mathcal{X}$ is of finite type over k and F a constructible sheaf of Λ-modules as in (a), the natural map

$$\Gamma(\mathcal{X}, F) \to \varprojlim_{x \in \mathcal{C}_{\mathcal{X},\ell}} F_s^A \tag{4.1.5}$$

is again an isomorphism (compatible with (4.1.2)), where, for $(x : [S/A] \to \mathcal{X}) \in \mathcal{C}_{\mathcal{X}}$, $F_s^A := \Gamma([S/A], x^*F) = \Gamma(BA, F_s)$ (s the closed point of S).

4.2. Replacing F_x by $H^q([S/A], F)$ in (4.1.5) leads to the announced reformulation and generalization of 3.7. Let $D_c^+(\mathcal{X}, \mathbf{F}_\ell)$ denote the full subcategory of $D^+(\mathcal{X}, \mathbf{F}_\ell)$ consisting of complexes of sheaves of $\mathbf{F}_\ell$-modules over the lisse-étale site of $\mathcal{X}$ with bounded below, cartesian, constructible cohomology. For $K \in D_c^+(\mathcal{X}, \mathbf{F}_\ell)$, and $q \in \mathbf{Z}$,

$$(x : \mathcal{S} \to \mathcal{X} \in \mathcal{C}_{\mathcal{X}}) \mapsto H^q(\mathcal{S}, K|\mathcal{S})$$

is a projective system (of $\mathbf{F}_\ell$-vector spaces) indexed by $\mathcal{C}_{\mathcal{X}}$, and the restriction maps $H^q(\mathcal{X}, K) \to H^q(\mathcal{S}, K|\mathcal{S})$ are compatible with the transition morphisms, hence yield a homomorphism

$$a_{\mathcal{X},K} : H^q(\mathcal{X}, K) \to \varprojlim_{(\mathcal{S}\to\mathcal{X})\in\mathcal{C}_{\mathcal{X}}} H^q(\mathcal{S}, K). \tag{4.2.1}$$

For $\mathcal{X}$ a quotient stack of the form $[X/G]$ as in 2.2, with X separated, and K the constant sheaf $\mathbf{F}_\ell$, the right hand side of (4.2.1) is naturally identified with the right hand side of (3.5.2) and $a_{\mathcal{X},K}$ with $a_{(G,X)}$ ([8], 8.6).

More generally, the right hand side of (4.2.1) can be described in terms of a certain inverse limit involving fixed point sets X^A for elementary abelian ℓ-subgroups A of G.

For a pair (A, C) in $\mathcal{A}_{(G,X)}$, we have a restriction map $H^q_G(X, K) \to H^q_A(C, K|C)$, and an edge homomorphism $H^q_A(C, K|C) = H^q(BA, K|C) \to H^0(C, \mathcal{H}^q_A(K))$, where $\mathcal{H}^q_A(K)$ denotes the cohomology sheaf $\mathcal{H}^q$ of the complex $R\Gamma(BA, K|C)$ on C, hence a composition

$$H^q_G(X, K) \to H^0(C, \mathcal{H}^q_A(K|C)). \tag{4.2.2}$$

For a map $(\theta_g : A \hookrightarrow A', Cg \supset C')$ in $\mathcal{A}_{(G,X)}(k)$, we don't have a map $H^0(C', \mathcal{H}^q_{A'}(K|C')) \to H^0(C, \mathcal{H}^q_A(K|C))$, but instead a commutative square

$$\begin{array}{ccc} H^q_G(X, K) & \longrightarrow & H^0(C, \mathcal{H}^q_A(K)) \\ \downarrow & & \downarrow \\ H^0(C', \mathcal{H}^q_{A'}(K)) & \longrightarrow & H^0(C'g^{-1}, \mathcal{H}^q_A(K)) \end{array}, \tag{4.2.3}$$

where the right vertical map is the restriction and the lower horizontal map is given by the isomorphism $H^0(C', \mathcal{H}^q_{A'}(K)) \xrightarrow{\sim} H^0(C'g^{-1}, \mathcal{H}^q_{gA'g^{-1}}(K))$ followed by the restriction to A. Let $R^q_G(X, K)$ be the set of families

$$(x_{(A,C)} \in H^0(C, \mathcal{H}^q_A(K)))_{(A,C)\in\mathcal{A}_{(G,X)}}$$

such that for any map $g : (A, C) \to (A', C')$ in $\mathcal{A}_{(G,X)}$ the images of $x_{(A,C)}$ and $x_{(A',C')}$ in $H^0(C'g^{-1}, \mathcal{H}^q_A(K))$ coincide. We therefore get a map

$$a_{(G,X;K)} : H^q_G(X, K) \to R^q_G(X, K). \tag{4.2.4}$$

For $K = \mathbf{F}_\ell$, the right hand side of (4.2.4) coincides with that of (3.6.2).

Here is an alternate description of $R^q_G(X,K)$ ([8], 6.18). Let $\mathcal{A}^\natural_G$ denote the following category. Objects of $\mathcal{A}^\natural_G$ are triples (A, A', g), where A, A' are elementary abelian ℓ-subgroups of G and g is an element of $G(k)$ such that the conjugation $c_g : s \mapsto g^{-1}sg$ maps A into A'. Morphisms in $\mathcal{A}^\natural_G$ from (A, A', g) to (Z, Z', h) are pairs $(a, b) \in G(k) \times G(k)$ such that $g = ahb$, $c_a : A \to Z$, $c_b : Z' \to A'$. For $(A, A', g) \in \mathcal{A}^\natural_G$, we have an equivariant map $(1, c_g) : (X^{A'}, A) \to (X, G)$, where A acts trivially on $X^{A'}$ via $c_g : A \to A'$, hence a morphism

$$[1/c_g] : [X^{A'}/A] \to [X/G].$$

On the other hand, we have the second projection

$$\pi : [X^{A'}/A] = BA \times X^{A'} \to X^{A'}.$$

Consider the sheaf $R^q\pi_*[1/c_g]^*K$ $(= \mathcal{H}^q_A([1/c_g]^*K))$ on $X^{A'}$. A map $(a, b) : (A, A', g) \to (Z, Z', h)$ in $\mathcal{A}^\natural_G$ induces a morphism

$$(b, a)^* : H^0(X^{Z'}, R^q\pi_*[1/c_h]^*K) \to H^0(X^{A'}, R^q\pi_*[1/c_g]^*K),$$

and

$$R^q_G(X,K) = \varprojlim_{\mathcal{A}^\natural_G} H^0(X^{A'}, R^q\pi_*[1/c_g]^*K). \tag{4.2.5}$$

Now, we have the key compatibility, whose proof is not formal :

Lemma 4.3 ([8], 8.6). *For $\mathcal{X} = [X/G]$ as in 2.2, with X separated, $K \in D^+_c(\mathcal{X}, \mathbf{F}_\ell)$, and each integer q, there is a natural isomorphism*

$$\varepsilon : \varprojlim_{(\mathcal{S}\to\mathcal{X})\in\mathcal{C}_\mathcal{X}} H^q(\mathcal{S}, K) \xrightarrow{\sim} R^q_G(X,K)$$

making the following diagram commute :

$$\begin{array}{ccc} H^q([X/G],K) & \longrightarrow & \varprojlim_{(\mathcal{S}\to\mathcal{X})\in\mathcal{C}_\mathcal{X}} H^q(\mathcal{S},K) \, , \\ & \searrow & \downarrow \varepsilon \\ & & R^q_G(X,K) \end{array}$$

where the horizontal and oblique arrows are given respectively by (4.2.1) and (4.2.4), with the identification (4.2.5).

Thanks to 4.3, the following result generalizes 3.7 :

Theorem 4.4 ([8], 8.3). *Let $\mathcal{X}$ be an Artin stack of finite presentation over k which is either a Deligne-Mumford stack with finite inertia or a quotient stack of the form $[X/G]$ for a separated algebraic space of finite type X over k and G an algebraic group over k, and let K be an object of $D^+_c(\mathcal{X}, \mathbf{F}_\ell)$ having a multiplicative structure. Let*

$$R^*(\mathcal{X}, K) := \oplus_q \varprojlim_{(\mathcal{S}\to\mathcal{X})\in\mathcal{C}_\mathcal{X}} H^q(\mathcal{S}, K).$$

Then

$$a_{\mathcal{X},K} : H^*(\mathcal{X}, K) \to R^*(\mathcal{X}, K)$$

is a uniform F-isomorphism.

By a *multiplicative structure on K* we mean a multiplication map $m : K \otimes K \to K$ and a unit map $e : \mathbf{F}_\ell \to K$ satisfying the usual associativity and commutativity conditions with respect to the constraints of the symmetric monoidal category $D^+_c(\mathcal{X}, \mathbf{F}_\ell)$ ([8], §3). Such a structure makes $H^*(\mathcal{X}, K)$ into a graded $\mathbf{F}_\ell$-algebra.

A common generalization of the two cases of 4.4 would be the case where $\mathcal{X}$ has a *stratification by global quotients*, i. e. $\mathcal{X}_{\mathrm{red}}$ has a stratification by locally closed substacks such that each stratum is isomorphic to a quotient stack $[X/G]$ as in 4.4. Indeed, by a theorem of Kresch [10], if for any geometric point $x \to \mathcal{X}$ the fiber at x of the inertia $I_\mathcal{X}$ is affine, in particular if $\mathcal{X}$ is a Deligne-Mumford stack with finite inertia, then $\mathcal{X}$ has a stratification by global quotients of the form $[X/G]$ with G affine. When X has a stratification by global quotients, one can still show that the kernel of $a_{\mathcal{X},K}$ is annihilated by a power of $F : a \mapsto a^\ell$ ([8], *loc. cit.*).

On the other hand, recall that in the case $\mathcal{X} = [X/G]$ and K is in $D^b_c(\mathcal{X}, K)$, the source, $H^*(\mathcal{X}, K)$, of $a_{\mathcal{X},K}$ is a finitely generated $\mathbf{F}_\ell$-algebra (2.4). The target, $R^*(\mathcal{X}, K)$, is finitely generated, too ([8], 6.17). In fact ([8], 8.3 (a)), $R^*(\mathcal{X}, K)$ is finitely generated when $\mathcal{X}$ is an Artin stack of finite presentation over k admitting a stratification by global quotients. One can therefore ask :

Questions 4.5. Let $\mathcal{X}$ be an Artin stack of finite presentation over k admitting a stratification by global quotients, and let K be an object of $D^b_c(\mathcal{X}, \mathbf{F}_\ell)$ endowed with a multiplicative structure.

(a) Is $H^*(\mathcal{X}, \mathbf{F}_\ell)$ a finitely generated $\mathbf{F}_\ell$-algebra ?

(b) Is $a_{\mathcal{X},K}$ a uniform F-isomorphism ?

5. Outline of proof of 4.4.

We roughly follow the pattern of Quillen's proof for the analogous results ([14], 6.2), ([15], 8.5). In *loc. cit.* the starting point is to analyze $H^*_G(X, \mathbf{F}_\ell)$

via the Leray spectral sequence of the quotient map

$$\pi : PG \wedge^G X \to X/G,$$

which is a proper map whose fiber at a point y of X/G is $PG \wedge^G Gx$ for a point x in X above y, so that $(R^q\pi_*\mathbf{F}_\ell)_y = H^q_G(Gx, \mathbf{F}_\ell) \tilde{\to} H^q(BG_x, \mathbf{F}_\ell)$, where $G_x \subset G$ is the stabilizer of x. In the situation of 4.4, with $\mathcal{X} = [X/G]$, the quotient X/G doesn't make sense in general, but with some additional hypotheses on $\mathcal{X}$, a coarse moduli space will be a satisfactory substitute.

5.1. In particular, suppose that $\mathcal{X}$ in 4.4 is a Deligne-Mumford stack with finite inertia. Then, by the Keel-Mori theorem [9] there exists a coarse moduli space morphism

$$f : \mathcal{X} \to Y. \tag{5.1.1}$$

Recall that this means that Y is an algebraic space over k, f is initial among maps from $\mathcal{X}$ to a k-algebraic space, and for any algebraically closed field K over k, f induces an isomorphism from the set of isomorphism classes of objects of $\mathcal{X}(K)$ to $Y(K)$. In addition, f is proper, and is a universal homeomorphism. For an algebraically closed geometric point y of Y, the reduced fiber[5] $f^{-1}(y)_{\mathrm{red}}$ consists of a single isomorphism class of objects of $\mathcal{X}$ over y, namely that of the quotient stack $[Gx/G]$ ($\tilde{\to}$ $[\operatorname{Spec} k(y) \wedge^{G_x} G/G] \tilde{\to} BG_x$ (2.2.4)) for a geometric point x of X above y.

A key fact, similar to ([14], 3.2), is that the edge homomorphism

$$e : H^*(\mathcal{X}, K) \to H^0(Y, R^*f_*K) \tag{5.1.2}$$

of the Leray spectral sequence of f is a uniform F-isomorphism. As in *loc. cit.* it is a simple consequence of the multiplicative structure of the spectral sequence.

As f is proper, by (an easy case of) ([13], 9.14) the sheaves R^qf_*K are constructible and for any geometric point y of Y, we have

$$(R^qf_*K)_y \tilde{\to} H^q(BG_x, K|BG_x) \tag{5.1.3}$$

(for a geometric point x of $\mathcal{X}$ above y). Therefore, by (4.1.2) applied to Y and R^qf_*K, we have

$$H^0(Y, R^qf_*K) \tilde{\to} \varprojlim_{y \in \mathcal{P}_Y} H^q(BG_x, K|BG_x). \tag{5.1.4}$$

[5]Fibers of f are not necessarily reduced, as the example of a Kummer cover of the affine line already shows.

Assume now :

(*) *All inertia groups G_x of $\mathcal{X}$ are elementary abelian ℓ-groups.*

Then, by (4.1.5), (5.1.4) induces an isomorphism

$$H^0(Y, R^q f_* K) \xrightarrow{\sim} \varprojlim_{[S/A] \in \mathcal{C}_{\mathcal{X}}} H^q([S/A], K). \tag{5.1.5}$$

The composite of (5.1.2) and (5.1.5) is the canonical map

$$a_{\mathcal{X},K} : H^*(\mathcal{X}, K) \to R^*(\mathcal{X}, K)$$

of 4.4, and it is a uniform F-isomorphism.

5.2. Suppose now that $\mathcal{X}$ in 4.4 is a quotient stack $[X/G]$. Imitating Quillen's method in [14], one reduces to the situation of 5.1 with the additional assumption (*). By the dévissage used in the proof of 2.3 (cf. 2.6), one first reduces to the case where G is affine. One then embeds G into some general linear group $L := \mathrm{GL}_{n,k}$ and let G act diagonally on $X_0 := X \times (T[\ell]\backslash L)$, where $T \subset L$ is a maximal torus. The map $G \times X_0 \to X_0 \times X_0$, $(g, x) \times (x, xg)$ is then finite and unramified, and all the inertia groups G_x are elementary abelian ℓ-subgroups of G. Therefore $\mathcal{X}_0 := [X_0/G]$ satisfies the assumptions of 5.1, plus (*). So $a_{\mathcal{X}_0,K}$ is a uniform F-isomorphism. One repeats the operation with $\mathcal{X}_1 := [X_1/G]$, where $X_1 = X \times (T[\ell]\backslash L) \times (T[\ell]\backslash L)$, and gets that $a_{\mathcal{X}_1,K}$ is a uniform F-isomorphism. A descent argument yields the conclusion for $\mathcal{X}$.

5.3. When $\mathcal{X}$ in 4.4 is a Deligne-Mumford stack with finite inertia, but does not necessarily satisfy (*), one applies 4.4 to the stacks BG_x (and $K|BG_x$) appearing in the right hand side of (5.1.4). A closer analysis shows that one can find a bound N for the power of F annihilating the kernel and the cokernel of $a_{BG_x,K}$ which is independent of the order of G_x. One then concludes using (5.1.4) and (4.1.5).

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