Grothendieck at Pisa : crystals and Barsotti-Tate groups

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1. Grothendieck at Pisa

Grothendieck visited Pisa twice, in 1966, and in 1969. It is on these occasions that he conceived his theory of crystalline cohomology and wrote foundations for the theory of deformations of p-divisible groups, which he called Barsotti-Tate groups. He did this in two letters, one to Tate, dated May 1966, and one to me, dated Dec. 2-4, 1969. Moreover, discussions with Barsotti that he had during his first visit led him to results and conjectures on specialization of Newton polygons, which he wrote in a letter to Barsotti, dated May 11, 1970.

May 1966 coincides with the end of the SGA 5 seminar [77]. Grothendieck was usually quite ahead of his seminars, thinking of questions which he might consider for future seminars, two or three years later. In this respect his correspondence with Serre [18] is fascinating. His local monodromy theorem, his theorems on good and semistable reduction of abelian varieties, his theory of vanishing cycles all appear in letters to Serre from 1964. This was to be the topic for SGA 7 [79], in 1967-68. The contents of SGA 6 [78] were for him basically old stuff (from before 1960), and I think that the year 1966-67 (the year of SGA 6) was a vacation of sorts for him, during which he let Berthelot and me quietly run (from the notes he had given to us and to the other contributors) a seminar which he must have considered as little more than an exercise.

In 1960 Dwork's proof [24] of the rationality of the zeta function of varieties over finite fields came as a surprise and drew attention to the power of p-adic analysis. In the early sixties, however, it was not p-adic analysis but étale cohomology which was in the limelight, due to its amazing development by Grothendieck and his collaborators in SGA 4 [76] and SGA 5. Étale cohomology provided a cohomological interpretation of the zeta function, and paved the way to a proof of the Weil conjectures. Moreover, it furnished interesting ℓ-adic Galois representations. For example, if, say, X is proper and smooth over a number field k, with absolute Galois group $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$, then for each prime number ℓ, the cohomology groups $H^i(X \otimes \overline{k}, \mathbf{Q}_\ell)$ are continuous, finite dimensional $\mathbf{Q}_\ell$-representations of Γ_k (of dimension b_i, the i-th Betti number of $X \otimes \mathbf{C}$, for any embedding $k \hookrightarrow \mathbf{C}$). These representations have local counterparts : for each finite place v of k and choice of an embedding of $\overline{k}$ in $\overline{k_v}$, the groups $H^i(X \otimes \overline{k_v}, \mathbf{Q}_\ell)$ are naturally identified to $H^i(X_{\overline{k}}, \mathbf{Q}_\ell)$, and the (continuous) action of the decomposition group

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$\Gamma_{k_v} = \mathrm{Gal}(\overline{k}_v/k_v) \subset \Gamma_k$ on them corresponds to the restriction of the action of Γ_k. For ℓ not dividing v, the structure of these local representations had been well known since 1964 : by Grothendieck's local monodromy theorem, an open subgroup of the inertia group $I_v \subset \Gamma_{k_v}$ acts by unipotent automorphisms. For ℓ dividing v, the situation was much more complicated and it's only with the work of Fontaine in the 70's and 80's and the development of the so-called *p-adic Hodge theory* that a full understanding was reached. However, the first breakthroughs were made around 1965, with the pioneering work of Tate on p-divisible groups. Together with Dieudonné theory, this was one of the main sources of inspiration for Grothendieck's letters.

2. From formal groups to Barsotti-Tate groups

2.1. *The Tate module of an abelian variety*

As Serre explains in his Bourbaki talk [69], numerous properties of abelian varieties can be read from their group of division points. More precisely, if A is an abelian variety over a field k of characteristic p, $\overline{k}$ an algebraic closure of k, and ℓ a prime number, one can consider the *Tate module* of A,

$$T_\ell(A) := \varprojlim_n A(\overline{k})[\ell^n],$$

(where, for a positive integer m, $[m]$ denotes the kernel of the multiplication by m), which is a free module of rank r over $\mathbf{Z}_\ell$, equipped with a continuous action of $\mathrm{Gal}(\overline{k}/k)$. For $\ell \neq p$, one has $r = 2g$, where $g = \dim A$, and it has been known since Weil that when k is finite, this representation determines the zeta function of A. For k of characteristic $p > 0$, and $\ell = p$, one has $r \leq g$, and it was observed in the 50's that in this case it was better to consider not just the kernels of the multiplications by p^n on the $\overline{k}$-points of A, but the *finite algebraic group schemes* $A[p^n]$, and especially their identity components $A[p^n]^0$, whose union is the *formal group* of A at the origin, a smooth commutative formal group of dimension g. For example, when $g = 1$ (A an elliptic curve), this group has dimension 1 and height 1 or 2 according to whether $r = 1$ (ordinary case) or $r = 0$ (supersingular case).

In the late 50's and early 60's formal groups were studied by Cartier, Dieudonné, Lazard, and Manin, mostly over perfect fields or sometimes over complete local noetherian rings with perfect residue fields. The notion of p-divisible group, which was first introduced by Barsotti [2] under the name "equidimensional hyperdomain", was formalized and studied by Serre and Tate (around 1963-66) before Grothendieck got interested in the topic. Let me briefly recall a few salient points of what was known at that time.

2.2. *Dieudonné theory, p-divisible groups*

Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ the ring of Witt vectors on k, σ the automorphism of W defined by the absolute Frobenius of k, i. e. $a = (a_0, a_1, \cdots) \mapsto a^\sigma = (a_0^p, a_1^p, \cdots)$. Dieudonné theory associates with a finite commutative algebraic p-group G over k its *Dieudonné module*,

$$M(G) = \mathrm{Hom}(G, CW),$$

where CW is the fppf sheaf of Witt covectors on $\mathrm{Spec}\, k$. This $M(G)$ is a W-module of finite length, equipped with a σ-linear operator F and a σ^{-1}-operator V satisfying the relation $FV = VF = p$, defined by the Frobenius F and the Verschiebung V on G. The above definition is due to Fontaine [31]. Classically (cf. [62], [23]) one first defined $M(G)$ for G *unipotent* as $\mathrm{Hom}(G, \mathrm{CW}_u)$, where $\mathrm{CW}_u = \varinjlim W_n \subset \mathrm{CW}$ is the sheaf of unipotent covectors, and treated the multiplicative case by Cartier duality.

In general, by a *Dieudonné module*, one means a W-module, with operators F and V as above. The Dieudonné module of G is a contravariant functor of G, and this functor defines an anti-equivalence from the category of finite commutative algebraic p-groups over k to that of Dieudonné modules of finite length over W. The functor $G \mapsto M(G)$ is extended to formal groups, viewed as direct limits of connected finite commutative p-groups, and gives an embedding of the category of formal groups into a suitable category of Dieudonné modules.

A central result in the theory is the *Dieudonné-Manin classification theorem*, which describes the category of finitely generated Dieudonné modules up to isogeny. More precisely, let K denote the fraction field of W. Define an *F-space*[2] as a finite dimensional K-vector space equipped with a σ-linear automorphism F. The Dieudonné-Manin theorem says that, if k is algebraically closed, the category of F-spaces is semi-simple, and for each pair of integers (r, s), with $r = 0$ and $s = 1$ or $r \neq 0$ and $s > 0$ coprime, there is a unique isomorphism class of simple objects, represented by $E_{r,s} = K_\sigma[T]/(T^s - p^r)$, where $K_\sigma[T]$ is the non-commutative polynomial ring with the rule $Ta = a^\sigma T$, and F acts on $E_{r,s}$ by multiplication by T. Grothendieck was to revisit this theorem in his 1970 letter to Barsotti ([39], Appendix). We will discuss this in §5.

The Dieudonné module of a formal group is not necessarily finitely generated over W. For example, for the formal group $G = \widehat{\mathbf{G}_a}$, one has $M(G) = k_\sigma[[F]]$, with $V = 0$. Such phenomena do not occur, however, for p-divisible groups. Recall (cf. [69]) that given a base scheme S and an integer $h \geq 0$, a *p-divisible group* (*Barsotti-Tate group* in Grothendieck's terminology) *of height* h over S is a sequence of finite locally free commutative group schemes G_n of

[2] F-isocrystal, in today's terminology

rank p^{nh} over S and homomorphisms $i_n : G_n \to G_{n+1}$, for $n \geq 1$, such that the sequences

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1}$$

are exact. For $n \geq 0$, $m \geq 0$, one then gets short exact sequences of group schemes over S

$$0 \to G_n \to G_{n+m} \xrightarrow{p^n} G_m \to 0.$$

The abelian sheaf $G := \varinjlim_n G_n$ on the fppf site S_{fppf} of S is then p-divisible, p-torsion, and $\operatorname{Ker} p.\mathrm{Id}_G$ is G_1, which in particular is finite locally free (of rank p^h). The sequence (G_n, i_n) is reconstructed from G by $G_n := \operatorname{Ker} p^n \mathrm{Id}_G$. It was to avoid confusion with the more general notion of p-divisible abelian sheaf - and also to pay tribute to Barsotti and Tate - that Grothendieck preferred the terminology *Barsotti-Tate group* to denote an abelian sheaf G on S_{fppf} which is p-divisible, p-torsion, and such that $\operatorname{Ker} p\mathrm{Id}_G$ is finite locally free.

The Cartier duals $G_n^\vee = \mathcal{H}om(G_n, \mathbf{G}_m)$, with the inclusions dual to $p : G_{n+1} \to G_n$, form a p-divisible group of height h, called the *dual* of G, denoted $G^\vee$. The basic examples of p-divisible groups are : $(\mathbf{Q}_p/\mathbf{Z}_p)_S = ((\mathbf{Z}/p^n\mathbf{Z})_S)$, its dual $(\mathbf{Q}_p/\mathbf{Z}_p)(1)_S = (\mu_{p^n,S})$, and the p-divisible group of an abelian scheme A over S

$$A[p^\infty] = (A[p^n])_{n\geq 1}.$$

When $S = \operatorname{Spec} K$, for K a field of characteristic $\neq p$, with an algebraic closure $\overline{K}$, a p-divisible group G of height h over S is determined by its *Tate module*

$$T_p(G) := \varprojlim G_n(\overline{K}),$$

a free $\mathbf{Z}_p$-module of rank h, equipped with a continuous action of $\mathrm{Gal}(\overline{K}/K)$.

If $S = \operatorname{Spec} k$, with k as above, let G be a p-divisible group of height h over S. Then, G is determined by its *Dieudonné module*

$$M(G) := \varprojlim M(G_n),$$

a free W-module of rank h. And $M(G^\vee)$ is the dual $M(G)^\vee$, with F and V interchanged. The functor $G \mapsto M(G)$ is an (anti)-equivalence from the category of p-divisible groups over k to the full subcategory of Dieudonné modules consisting of modules which are free of finite rank over W.

Suppose now that $S = \operatorname{Spec} R$, where R is a complete discrete valuation ring, with perfect residue field k of characteristic $p > 0$ and fraction field K of characteristic zero, and let $\overline{K}$ be an algebraic closure of K. Let G be a p-divisible group of height h over S. Then two objects of quite a different nature are associated with G :

• the Tate module of G_K, $T_p(G_K)$, a free $\mathbf{Z}_p$-module of rank h on which $\mathrm{Gal}(\overline{K}/K)$ acts continuously

• the Dieudonné module of G_k, a free W-module of rank h, equipped with semi-linear operators F and V satisfying $FV = VF = p$.

Understanding the relations between these two objects, as well as their relations with the differential invariants associated with an abelian scheme A over S when $G = A[p^\infty]$ (such as $\mathrm{Lie}(A)$, its dual, and the de Rham cohomology group $H^1_{\mathrm{dR}}(A/S)$), was the starting point of p-adic Hodge theory.

2.3. *The theorems of Tate and Serre-Tate*

Let me briefly recall the main results, see [69] and [70] for details. Let $S = \operatorname{Spec} R$, k, K, $\overline{K}$ as before.

Theorem 2.3.1 (Tate) ([70], Th. 4). The functor $G \mapsto G_K$ from the category of p-divisible groups over S to that of p-divisible groups over K is fully faithful, i. e., for p-divisible groups G, H over S, the map

$$\mathrm{Hom}(G, H) \to \mathrm{Hom}(G_K, H_K)(\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)})(T_p(G_K), T_p(H_K))$$

is bijective.

(Actually, Tate shows that 2.3.1 holds more generally for R local, complete, integral, normal, with perfect residue field k (of characteristic $p > 0$) and fraction field K of characteristic zero, but the proof is by reduction to the complete discrete valuation ring case.)

The equal characteristic analogue of 2.3.1 was to be established only many years later, by de Jong in 1998 [21].

Theorem 2.3.2 (Tate) ([70], Th. 3, Cor. 2). Let $C := \widehat{\overline{K}}$, be the completion of $\overline{K}$, with its continuous action of $\mathrm{Gal}(\overline{K}/K)$. Let G be a p-divisible group over S. Then there is a natural decomposition, equivariant under $\mathrm{Gal}(\overline{K}/K)$,

$$T_p(G_K) \otimes C \xrightarrow{\sim} (\mathrm{t}_G \otimes C(1)) \oplus ((\mathrm{t}_{G^\vee})^\vee \otimes C),$$

where, for a p-divisible group H over S, t_H denotes the Lie algebra of its identity component, a formal group over S.

Note that, in particular, if $d = \dim(\mathrm{t}_G)$, $d^\vee = \dim(\mathrm{t}_{G^\vee})$, one has $d + d^\vee = h$, where h is the height of G, a relation which can already be read on the Dieudonné modules of G_k and $G_k^\vee$.

When $G = A[p^\infty]$, for an abelian scheme A over S, the decomposition of 1.3.2 gives, by passing to the duals, a $\mathrm{Gal}(\overline{K}/K)$-equivariant decomposition of the form

$$H^1(A_{\overline{K}}, \mathbf{Z}_p) \otimes C \xrightarrow{\sim} (H^0(A_K, \Omega^1_{A_K/K}) \otimes C(-1)) \oplus (H^1(A_K, \mathcal{O}_{A_K}) \otimes C).$$

In his seminar at the Collège de France in 1966-67, Tate conjectured a generalization of this decomposition in higher dimension, the so-called *Hodge-Tate decomposition*, which was fully proven only in 1998, by Tsuji and de Jong as a corollary of the proof of Fontaine-Jannsen's conjecture C_{st} ([73], [20], [6]), after partial results by many authors (Raynaud, Fontaine, Bloch-Kato, Fontaine-Messing, Hyodo, Kato) (different proofs of C_{st} as well as of the related conjectures C_{cris}, C_{pst}, C_{dR} - by Faltings [27], Niziol [61], Beilinson [3], [4] - have been given since then). A report on this is beyond the scope of these notes. In ([70], p. 180) Tate also asked for a similar decomposition for suitable rigid-analytic spaces over K. This question was recently solved by Scholze [67].

Theorem 2.3.3 (Serre-Tate). Let R be a local artinian ring with perfect residue field k of characteristic $p > 0$, and let A_0 be an abelian variety over k. Then the functor associating with a lifting A of A_0 over R the corresponding lifting $A[p^\infty]$ of the p-divisible group $A_0[p^\infty]$ is an equivalence from the category of liftings of A_0 to that of lifting of $A_0[p^\infty]$.

Serre and Tate did not write up their proof, sketched in notes of the Woods Hole summer school of 1964 [54]. The first written proof appeared in Messing's thesis ([58], V 2.3). A different proof was found by Drinfeld, see [48]. Another proof, based on Grothendieck's theory of deformations for Barsotti-Tate groups, is given in [43] (see 4.2 (ii)).

3. Grothendieck's letter to Tate : crystals and crystalline cohomology

That was roughly the state of the art when Grothendieck came on the scene. In the form of a riddle, a natural question was : what do the following objects have in common :

- a Dieudonné module
- a p-adic representation of the Galois group of a local field K as above
- a de Rham cohomology group ?

At first sight, nothing. However, a p-adic Galois representation is, in a loose sense, some kind of analogue of a local system on a variety over $\mathbf{C}$. Local systems arising from the cohomology of proper smooth families can be interpreted in terms of relative de Rham cohomology groups, with their Gauss-Manin connection. In characteristic zero at least, integrable connections correspond to compatible systems of isomorphisms between stalks at infinitesimally near points. On the other hand, by Oda's thesis [62] the reduction mod p of the Dieudonné module of the p-divisible group $A[p^\infty]$ of an abelian variety A over k is isomorphic to $H^1_{\mathrm{dR}}(A/k)$ (see the end of this section). Recall also that in his letter to Atiyah (Oct. 14, 1963) [34] Grothendieck had asked for an algebraic interpretation of the Gauss-Manin connection in the proper smooth case and discussed the H^1_{dR} of abelian schemes. These were probably some of the ideas that Grothendieck had in mind when he wrote his famous letter to Tate, of May 1966. Here is the beginning of this letter :

"Cher John,

J'ai réfléchi aux groupes formels et à la cohomologie de de Rham, et suis arrivé à un projet de théorie, ou plutôt de début de théorie, que j'ai envie de t'exposer, pour me clarifier les idées.

Chapitre 1 La notion de cristal.

Commentaire terminologique : Un cristal possède deux propriétés caractéristiques : la *rigidité*, et la faculté de *croître* dans un voisinage approprié. Il y a des cristaux de toute espèce de substance : des cristaux de soude, de soufre, de modules, d'anneaux, de schémas relatifs etc."

Grothendieck refined and expanded his letter in a seminar he gave at the IHÉS in December, 1966, whose notes were written up by Coates and Jussila [35]. The contents are roughly the following :

3.1. *Crystals.* The word is as beautiful as the mathematical objects themselves. Starting with a scheme S over a base T, Grothendieck considers the category $\mathcal{C}$ of T-thickenings of open subschemes of S, i. e. T-morphisms $i : U \hookrightarrow V$ where U is an open subscheme of S and i is a locally nilpotent closed immersion, with maps from $U \hookrightarrow V$ to $U' \hookrightarrow V'$ given by the obvious commutative diagrams. He calls *crystal in modules* on S (relative to T) a cartesian section over $\mathcal{C}$ of the fibered category of quasi-coherent modules over the category $\mathrm{Sch}_{/T}$ of T-schemes. More generally, given a fibered category $\mathcal{F}$ over $\mathrm{Sch}_{/T}$, he calls crystal in objects of $\mathcal{F}$ a cartesian section of $\mathcal{F}$ over $\mathcal{C}$. He gives a few examples (especially for T of characteristic $p > 0$, showing that crystals in modules with additional Frobenius and Verschiebung structures can be viewed as a family of Dieudonné modules parametrized by the points of S) and, for S smooth over T, he gives a description of a crystal in modules in terms of a quasi-coherent module E on S equipped with what we now call a *stratification*, namely an isomorphism $\chi : p_1^*E \xrightarrow{\sim} p_2^*E$, where p_1, p_2 are the projections to X from the formal completion of the diagonal $S \hookrightarrow S \times_T S$, such χ satisfying a natural cocycle descent condition on the formal completion of the diagonal in $S \times_T S \times_T S$.

He also introduces the first avatar of what was to become the *crystalline site*, which he calls the "crystallogenic site" ("site cristallogène"), consisting of thickenings $U \hookrightarrow V$ as above, with covering families those families $(U_i \hookrightarrow V_i)_{i \in I} \to (U \hookrightarrow V$ such that $(V_i \to V)$ is Zariski covering and $U_i = U \cap V_i$. He notes that crystals in modules can be re-interpreted in terms of certain sheaves on this site, and that for $f : X \to Y$ (in $\mathrm{Sch}_{/T}$), the functoriality will not be for the sites, but for the corresponding topoi (a point which will later be crucial in Berthelot's theory [5]). He adds that he expects that for f proper and smooth, then the $R^i f_*$ of the structural sheaf of the crystallogenic topos of X should give the relative de Rham cohomology sheaves $R^i f_* \Omega^{\cdot}_{X/Y}$ endowed with their Gauss-Manin connection.

However, in the course of his letter Grothendieck realizes that these definitions will have to be modified to take into account characteristic $p > 0$ phenomena. He adds a handwritten note in the margin : "fait mouche en car. 0, et pas en car. $p > 0$". We will discuss this in the next section.

3.2. *De Rham cohomology as a crystal.* This is of course the most striking observation. In his letter, Grothendieck, carefully enough, writes : "Chapitre 2 : la cohomologie de de Rham est un cristal. L'affirmation du titre n'est pour l'instant qu'une hypothèse ou un vœu pieux, mais je suis convaincu qu'elle est essentiellement correcte." He gives two pieces of evidence for his claim.

(a) He mentions Monsky-Wahnitzer's work on the independence of de Rham cohomology of ("weakly complete") liftings to $W(k)$ of smooth affine k-schemes. He criticizes the authors for not being able to globalize their construction to proper schemes (except for curves) and having to work $\otimes\mathbf{Q}$. A couple of years later, Berthelot's thesis solved the globalization problem, however a full understanding of Monsky-Washnitzer cohomology was reached by Berthelot again, but only in the 80's, with his theory of *rigid cohomology* (where $\otimes\mathbf{Q}$ is essential).

(b) He says that he has found an algebraic construction of the Gauss-Manin connection on $R^i f_* \Omega^{\bullet}_{X/S}$ for a smooth morphism X/S (and S over some base T), or rather on the object $Rf_* \Omega^{\bullet}_{X/S}$ of the derived category $D(S, \mathcal{O}_S)$, adding that, however, he has not yet checked the integrability condition. He also asks for a crystalline interpretation (i. e. in terms of cohomology of a suitable crystalline site) of this connection, and of the corresponding Leray spectral sequence for $X \to S \to T$. In his lectures at the IHÉS [35], he gave the details of his construction and explained the link between (a) and (b). His construction, close in spirit to Manin's, based on local liftings of derivations, was used later by Katz in [47]. As for the integrability, Katz and Oda found a simple, direct proof in [46], based on the analysis of the Koszul filtration of the absolute de Rham complex $\Omega^{\bullet}_{X/T}$. However, the crystalline interpretation requested by Grothendieck, which was to be given by Berthelot in his thesis [5], and the (dual) approach, in char. zero, via $\mathcal{D}$-modules was to give a deeper insight into this structure.

As for the link between (a) and (b), Grothendieck's observation was the following. Suppose X is proper and smooth over $S = \mathrm{Spec} W[[t]]$ ($t = (t_1, \cdots, t_n)$), and $\mathcal{H}^i_{dR}(X/S)$ is free of finite type for all i. Let $u : \mathrm{Spec} W \to S$, $v : \mathrm{Spec} W \to S$ be sections of S such that $u \equiv v \bmod p$. We then get two schemes over W, $X_u := u^*X$, $X_v := v^*X$ such that $X_u \otimes k = X_v \otimes k = Y$, and two de Rham cohomology groups, $H^i_{dR}(X_u/W) = u^*\mathcal{H}^i(X/S)$, $H^i_{dR}(X_v/W) = v^*\mathcal{H}^i(X/S)$. By the Gauss-Manin connection

$$\nabla : \mathcal{H}^i_{dR}(X/S) \to \Omega^1_{S/W} \otimes \mathcal{H}^i_{dR}(X/S),$$

we get an *isomorphism*

$$\chi(u, v) : H^i_{dR}(X_u/W) \tilde{\to} H^i_{dR}(X_v/W),$$

defined by

$$u^*(a) \mapsto \sum_{m \geq 0} (1/m!)(u^*(t) - v^*(t))^m v^*(\nabla(D)^m a)$$

for $a \in \mathcal{H}^i_{dR}(X/S)$, with the usual contracted notations, where

$$D = (D_1, \cdots, D_n),\ D_i = \partial/\partial t_i$$

(note that $(1/m!)(u^*(t) - v^*(t))^m \in W$ and that the series converges p-adically : this is easy for $p > 2$, was proved by Berthelot in general [5]). These isomorphisms satisfy $\chi(u, u) = \mathrm{Id}$ and $\chi(v, w)\chi(u, v) = \chi(u, w)$, for $w \equiv u \bmod p$.

This suggested to Grothendieck that, for Y/k proper, smooth, given two proper smooth liftings X_1, X_2 of Y over W, one could hope for an isomorphism (generalizing $\chi(u, v)$)

$$\chi_{12} : H^i_{dR}(X_1/W) \tilde{\to} H^i_{dR}(X_2/W)$$

with $\chi_{23}\chi_{12} = \chi_{13}$. Monsky-Washnitzer's theory provided such an isomorphism (after tensoring with $\mathbf{Q}$) in the affine case, for good liftings X_i. This hope was to be realized by the construction of *crystalline cohomology* groups $H^i(Y/W)$ (depending only on Y, with no assumption of existence of lifting), providing a canonical isomorphism :

$$\chi : H^i(Y/W) \tilde{\to} H^i_{dR}(X/W)$$

for any proper smooth lifting X/W of Y, such that for X_1, X_2 as above, $\chi_2 = \chi_{12}\chi_1$. Grothendieck sketched the construction in [35] (which worked for $p > 2$), the general case was done and treated in detail by Berthelot [5].

But let us come back to Grothendieck's letter. For $f : X \to S$ proper and smooth, of relative dimension d, S being over some base T, Grothendieck (boldly) conjectures :

(*) $Rf_*\Omega^\bullet_{X/S}$ should be a perfect complex on S, underlying a structure of crystal relative to T, commuting with base change, and that for each prime number p, on the corresponding object H_p for the reduction mod p of f, the Frobenius operator $F : H_p^{(p)} \to H_p$ should be an *isogeny*, with an operator V in the other direction, satisfying $FV = VF = p^d$.

He analyzes the case where X/S is an abelian scheme, and makes two critical observations.

(3.2.1) For $S = \operatorname{Spec} W$ and X/S an abelian scheme, $H^1_{\mathrm{dR}}(X/S)$, which should be the value on S of the sought after *crystal* defined by $X \otimes k$, equipped with the operators F and V defined by Frobenius and Verschiebung, should be the *Dieudonné module* of the p-divisible group associated with $X \otimes k$.

(3.2.2) He realizes that in char. $p > 0$ his assertion that de Rham cohomology is a crystal in the sense defined at the beginning of his letter is wrong. In fact, for S smooth over T, a (quasi-coherent) crystal on S/T would correspond (cf. 3.1) to a *stratified module* M relative to T. And, it would not be possible to put such a stratification on $\mathcal{H}^1_{\mathrm{dR}}(A/S)$ relative to $T = \operatorname{Spec} k$ for an *elliptic curve* A over S, in a way which would be functorial in A and compatible with base change, for S of finite type, regular, and of dimension ≤ 1 over k. He gives the example of an elliptic curve A/S, S a smooth curve over T, with a rational point s where A_s has Hasse invariant zero. In this case, the (absolute) Frobenius map $F : H^1(A_s, \mathcal{O})^{(p)} \to H^1(A_s, \mathcal{O})$ is zero, so $F^2 : H^1_{\mathrm{dR}}(A_s/s)^{(p^2)} \to H^1_{\mathrm{dR}}(A_s/s)$ is zero, hence, because of the stratification, F^2 would be zero on the completion of S at s, hence in a neighborhood of s, which is not the case when S is modular. This observation led him, in [35], to call the previously defined site (the "crystallogenic site") the *infinitesimal site*, and define a new site (and, accordingly, a new notion of crystal), putting *divided powers* on the ideals of the thickenings. Technical problems arose for $p = 2$, as the natural divided powers of the ideal pW are not p-adically nilpotent, but these were later solved by Berthelot in his thesis, dropping the restriction of nilpotence on the divided powers introduced in [35], on schemes where p is locally nilpotent.

Why add divided powers ? In [35] Grothendieck explains that the introduction of divided powers was "practically imposed by the need to define the first Chern class $c_1(L) \in H^2(X_{\mathrm{cris}}, \mathcal{J})$ of an invertible sheaf L on X", as the obstruction to lifting L to X_{cris}, using the logarithm

$$\log : 1 + \mathcal{J} \to \mathcal{J},\ \log(1+x) = \sum_{n \geq 1} (-1)^{n-1}(n-1)!(x^n/n!),$$

where $\mathcal{J}$ is the kernel of the natural surjective map from $\mathcal{O}_{X_{\rm cris}}$ to $\mathcal{O}_{X_{\rm zar}}$. While this was certainly a motivation, it seems to me that the primary motivation was to make de Rham cohomology a crystal.

For S smooth over T, and E a quasi-coherent module on S, a stratification on E relative to T is given by an action of the ring $\mathrm{Diff}_{S/T} = \cup \mathrm{Diff}^n_{S/T}$ of differential operators of S over T. In general, an integrable connection ∇ on E relative to T does not extend to an action of $\mathrm{Diff}_{S/T}$. But it does extend to an action of the ring of *PD-differential operators* $\text{PD-Diff}_{S/T} = \cup\ \text{PD-Diff}^n_{S/T}$ (PD for "puissances divisées"), where $\text{PD-Diff}^n_{S/T}$ is the dual (with values in $\mathcal{O}_S$) of the *divided power envelope* $D^n_{S/T}$ of the ideal of the n-th infinitesimal neighborhood of S in $S \times_T S$. In terms of local coordinates $(x_i)_{1 \le i \le r}$ on S, the associated graded (for the filtration by the order) of Diff is a divided power polynomial algebra on generators δ_i corresponding to $\partial/\partial x_i$, while that of PD-Diff is a usual polynomial algebra on the δ_i's. And crystals, for a suitable site defined by thickenings with divided powers, were to correspond exactly (for S/T smooth and schemes T where p is locally nilpotent) to modules with an integrable connection (satisfying an additional condition of *p-nilpotency*), the datum of ∇ being equivalent to that of the action of $\text{PD-Diff}_{S/T}$, i.e. to a PD-analogue of a stratification relative to T.

A good definition of a *crystalline site* was worked out by Berthelot in [5], and the first part of Grothendieck's conjecture (*) above proven in ([5], V 3.6). The existence of V satisfying $FV = VF = p^d$ was shown by Berthelot-Ogus in ([11], 1.6).

As for Grothendieck's expectation (3.2.1) above, it was proved by Oda ([62], 5.11) that $H^1_{\rm dR}(A/k)(\xrightarrow{\sim} H^1_{\rm dR}(X/S) \otimes k)$ (where $A := X \otimes k$) is isomorphic to the Dieudonné module of $A[p]$, i. e. to M/pM, where M is the Dieudonné module of the p-divisible group G associated with A. With Berthelot's definition of crystalline cohomology, $H^1_{\rm dR}(X/S)$ is isomorphic to the crystalline cohomology group $H^1(A/W)$. The isomorphism between $H^1(A/W)$ (with the operators F and V) and M was first proved by Mazur-Messing in [56]. Different proofs were given later ([8], ([42], II 3.11.2)).

For a survey of crystalline cohomology (up to 1990), see [44].

4. Grothendieck's letter to Illusie : deformations of Barsotti-Tate groups

This letter has two parts. In the first part, Grothendieck describes a (conjectural) theory of first order deformations for flat commutative group schemes. This theory was developed in the second volume of my thesis [41]. In the second part, he applies it to Barsotti-Tate groups, stating theorems of existence and classification of deformations for Barsotti-Tate groups and truncated ones. He gave the proofs in his course at the Collège de France in

1970-71. These proofs were written up in [43].

4.1. *Deformations of flat commutative group schemes.*

"Marina, les 2-4 déc. 1969. Cher Illusie, Le travail avance, mais avec une lenteur ridicule. J'en suis encore aux préliminaires sur les groupes de Barsotti-Tate sur une base quelconque - il n'est pas encore question de mettre des puissances divisées dans le coup ! La raison de cette lenteur réside en partie dans le manque de fondements divers. (...) De plus, à certains moments, je suis obligé d'utiliser une théorie de déformations pour des schémas en groupes plats mais non lisses, qui doit certainement être correcte, et qui devrait sans doute figurer dans ta thèse, mais que tu n'as pas dû écrire encore, sans doute. Je vais donc commencer par te soumettre ce que tu devrais bien prouver. (...)"

Grothendieck then proposes a theory of (first-order) deformations for group schemes G/S which are locally of finite presentation and flat. He says he is mainly interested in the commutative case, but that the non commutative case should also be considered (which I did in [41]). In both cases, the invariant which controls the deformations is the *co-Lie complex* of G/S,

$$\ell_G := Le^* L_{G/S},$$

where $e : S \to G$ is the unit section, $L_{G/S}$ the cotangent complex of G/S, and $Le^* : D^-(G, \mathcal{O}_G) \to D^-(S, \mathcal{O}_S)$ the derived functor of e^*. This complex appeared for the first time in the work of Mazur-Roberts [57]. As G/S is locally a complete intersection, this is a perfect complex, of perfect amplitude in $[-1, 0]$ (and $L_{G/S}$ is recovered from it by $L_{G/S} = \pi^* \ell_G$, where $\pi : G \to S$ is the projection). When G is *commutative, finite and locally free*, ℓ_G is related to the Cartier dual G^* of G by the following beautiful formula (proposed by Grothendieck in his letter, and proven by him in his course at Collège de France, see ([56], 14.1)) : if M is a quasi-coherent $\mathcal{O}_S$-module, there is a natural isomorphism (in $D(S, \mathcal{O}_S)$),

$$R\mathcal{H}om_{\mathcal{O}_S}(\ell_G, M) \xrightarrow{\sim} \tau_{\leq 1} R\mathcal{H}om(G^*, M).$$

After having stated a (conjectural) theory of obstructions for deformations of G in the commutative case, Grothendieck realizes that he needs more. In fact, he sees that he will need to apply this theory to the truncations $G(n) = \operatorname{Ker} p^n \mathrm{Id}_G$ of Barsotti-Tate groups G. Such truncations are $\mathbf{Z}/p^n$-modules, and deformations should preserve this structure. But then, in order to get a common theory for commutative group schemes and commutative group schemes annihilated by p^n, it is natural to introduce a ring A of *complex multiplication* acting on G (hence on ℓ_G); obstruction groups (and related ones) should involve this A-linear structure. He adds that whether A should be a constant ring or a more or less arbitrary sheaf of rings, he has not yet tried to think about it. In [41] I treat the case where A is a (non necessarily commutative) ring scheme satisfying a mild hypothesis with respect to the given infinitesimal thickening. For applications to deformations of abelian schemes and Barsotti-Tate groups, the case where A is the scheme defined by a constant commutative ring (in fact, $\mathbf{Z}$ or $\mathbf{Z}/n$) suffices. One of the main results is the following.

Theorem 4.1.1 ([41], VII 4.2.1). Let $i : S \hookrightarrow S'$ be a closed immersion defined by an ideal I of square zero. Let A be a (constant) ring, and G a scheme in A-modules over S, flat and locally of finite presentation over S. Let us work with the fppf topology on S. Consider the differential graded ring $A \otimes^L_{\mathbf{Z}} \mathcal{O}_S$ [3] and let

$$\ell^\vee_G \in D^{[0,1]}(A \otimes^L_{\mathbf{Z}} \mathcal{O}_S)$$

be the *Lie complex* of G/S, defined in ([41], VII (4.1.5.4)), whose image in $D(\mathcal{O}_S)$ is the dual $R\mathcal{H}om(\ell_G, \mathcal{O}_S)$ of the co-Lie complex of G. Then :

(i) There is an obstruction

$$o(G, i) \in \mathrm{Ext}^2_A(G, \ell^\vee_G \otimes^L_{\mathcal{O}_S} I)$$

whose vanishing is necessary and sufficient for the existence of a deformation G' of G over S' as a scheme in A-modules, flat over S'.

(ii) This obstruction depends functorially on G in the following sense : if $u : F \to G$ is a homomorphism of (flat and locally of finite presentation) schemes in A-modules over S, then

$$u^*(o(G, i)) = \ell^\vee_u(o(F, i)),$$

where u^* and $\ell^\vee_u$ are the natural functoriality maps.

(iii) When $o(G, i) = 0$, the set of isomorphism classes of deformations G' of G over S' is a torsor under $\mathrm{Ext}^1_A(G, \ell^\vee_G \otimes^L_{\mathcal{O}_S} I)$, and the group of automorphisms of a given deformation G' is $\mathrm{Ext}^0_A(G, \ell^\vee_G \otimes^L_{\mathcal{O}_S} I)$.

The proof is long and technical, involving complicated diagrams and a delicate homotopical stabilization process. Those diagrams were suggested by Grothendieck's attempts to calculate $\mathrm{Ext}^i_{\mathbf{Z}}(G, -)$ by *canonical resolutions* of G by finite sums of $\mathbf{Z}[G^r]$ ($r \geq 1$), which he had described, for $i \leq 2$, in ([37], VII 3.5), and that he recalls in his letter to me (such resolutions,

[3] When $A = \mathbf{Z}/n$, this is simply the differential graded ring $[\mathcal{O}_S \xrightarrow{n} \mathcal{O}_S]$, in degrees 0 and -1.

called *Moore complexes*, were constructed by Deligne [22]). Variants, called *MacLane resolutions*, involving sums of $\mathbf{Z}[G^r \times \mathbf{Z}^s]$, and taking into account the multiplicative structures, are given in ([41] VI 11.4.4)) (see also [13]). The method, however, is flexible, and can be applied to many other kinds of deformation problems (such as morphisms, with source and target, or only source or target extended). The functoriality statement (iii) (and similar properties for homomorphisms of rings $A \to B$) are of course crucial in the applications, where the obstruction group Ext^2 may be nonzero, but the obstruction is zero, because of functoriality constraints.

At the end of ([41], VII) I write that Deligne's theory of Picard stacks might yield simpler proofs of the above results. But over forty years have elapsed, and no such simpler proof has yet appeared.

4.2. *Deformations of BT's and BT_n's.*

The second part of his letter starts with what Grothendieck calls "fascicule de résultats sur les groupes de BT (= Barsotti-Tate) et les groupes de BT tronqués ("part soritale")". As he had explained in the first part, while the goal was to show the existence of infinitesimal liftings of BT's and classify them, the key objects of study were in fact n-truncated BT's.

Given an integer $n \geq 1$, an *n-truncated BT* (or *BT_n*) G over a base scheme S is defined as an abelian sheaf on S (for the fppf topology), which is annihilated by p^n, flat over $\mathbf{Z}/p^n$, and such that $G(1) := \mathrm{Ker}\, p\mathrm{Id}_G$ is finite locally free over S (its rank is then of the form p^h, where h is called the height of G). When $n = 1$, one imposes an additional condition, namely that on $S_0 = V(p) \subset X$, one has $\mathrm{Ker}\, V = \mathrm{Im}\, F$, where $V : G_0^{(p)} \to G_0$ and $F : G_0 \to G_0^{(p)}$ are the Verschiebung and the Frobenius morphisms respectively, with $G_0 = G \times_S S_0$.

If G is a BT over S, one shows that for all $n \geq 1$, $G(n) := \mathrm{Ker}\, p^n\mathrm{Id}_G$ is a BT_n, which raises the question whether any BT_n is of the form $G(n)$ for a BT G. Grothendieck tackles this question in his letter, simultaneously with that of existence and classification of deformations of BT's and BT_n's. In order to apply the general obstruction theory, one needs precise information on co-Lie complexes of truncated BT's. If S is a scheme where $p^N\mathcal{O}_S = 0$, for an integer $N \geq 1$, and G is a BT_n, with $n \geq N$, then the co-Lie complex ℓ_G enjoys nice properties. In particular $\omega_G := \mathcal{H}^0(\ell_G)$ and $n_G := \mathcal{H}^{-1}(\ell_G)$ are locally free of the same rank, which, when $G = \mathcal{G}(n)$ for a BT $\mathcal{G}$ on S, is the dimension of the formal Lie group associated to $\mathcal{G}$. After a subtle analysis of the relations between these invariants and the behavior of $\mathrm{Ext}^*(-, M)$, M a quasi-coherent $\mathcal{O}_S$-module, under exact sequences $0 \to G(n) \to G(n+m) \to G(m) \to 0$, Grothendieck obtains the following theorem (first stated in ([36], 6.3)), see ([39], 4.1), ([43], 4.4)) :

Theorem 4.2.1. Let $n \geq 1$. Let $i : S \to S'$ be a nilimmersion, with S' affine.

(1) Let G be a BT_n on S. There exists a BT_n G' on S' extending G.

(2) Let H be a BT on S. There exists a BT H' on S' extending H.

(3) If $E(H, S')$ (resp. $E(H(n), S')$) denotes the set of isomorphism classes of BT's (resp. BT_n's) on S' extending H (resp. $H(n)$)), then the natural map

$$E(H, S') \to E(H(n), S')$$

is surjective, and bijective if i is nilpotent of level k and there exists $N \geq 1$ such that $p^N \mathcal{O}_S = 0$ and $n \geq kN$.

(4) For $k = 1$ and $n \geq N$ as in (3), $E(G, S')$ (resp. $E(H, S')$) is a torsor under $t_{G^\vee} \otimes t_G \otimes I$ (resp. $t_{H^\vee} \otimes t_H \otimes I$), where I is the ideal of i, and the automorphism group of a deformation of G (resp. H) on S' is $t_{G^\vee} \otimes t_G \otimes I$ (resp. 0). Here t_G denotes the dual of ω_G, and $G^\vee$ the Cartier dual of G.

(5) If S is the spectrum of a complete noetherian local ring with perfect residue field, for any BT_n G on S there exists a BT H on S such that $G = H(n)$.

(For *formal* BT's H, (2) and (4) had been proven by Cartier and Lazard [17], [53].)

Just to give an idea on how 4.2.1 is derived from 4.1.1, let me observe that assertions (4) for a BT H follow from the following facts :

• deforming H on S' is equivalent to deforming the projective system $H(\cdot) = H(n)_{n \geq 1}$ on S',

• the obstruction $o(H, i)$ to deforming $H(.)$ lies in

$$\mathrm{Ext}^2_{\mathbf{Z}/p^\cdot}(H(\cdot), \ell^\vee_{H(\cdot)} \otimes^L_{\mathcal{O}_S} I),$$

and this group is zero,

• the isomorphism classes of deformations of H on S' form a torsor under

$$\mathrm{Ext}^1_{\mathbf{Z}/p^\cdot}(H(\cdot), \ell^\vee_{H(\cdot)} \otimes^L_{\mathcal{O}_S} I),$$

and this group is canonically isomorphic to $t_{H^\vee} \otimes t_H \otimes I$,

• the automorphism group of a deformation H' on S' is

$$\mathrm{Ext}^0_{\mathbf{Z}/p^\cdot}(H(\cdot), \ell^\vee_{H(\cdot)} \otimes^L_{\mathcal{O}_S} I),$$

which is zero.

This theorem had immediate applications, and cast long shadows.

Among the immediate applications, we have (i) and (ii) below, already mentioned by Grothendieck in his letter :

(i) The *pro-representability of the deformation functor* of a BT H over a perfect field k of characteristic $p > 0$, namely the fact that the functor of deformations of H over the category of artinian local $W(k)$-algebras of residue field k is pro-represented by a smooth formal scheme

$$S = \mathrm{Spf}\,(W(k)[[t_{ij}]]_{1 \leq i \leq d, 1 \leq j \leq d^\vee}$$

where d ($= \mathrm{rk}\,\omega_H$) is the dimension of H, and $d^\vee$ that of its Cartier dual $H^\vee$ (as after 2.3.2 , one has $d + d^\vee = h$, where h is the height of H).

(ii) A *short proof of the existence of infinitesimal liftings of abelian schemes and of Serre-Tate's theorem 1.3.3*, see ([43], Appendice). Concerning Serre-Tate's theorem, Grothendieck made an interesting comment. At the beginning of [48], Katz recalls that this theorem, in the case of a g-dimensional *ordinary* abelian variety over k (assumed to be algebraically closed), implies the existence of "a remarkable and unexpected structure of *group* on the corresponding formal moduli space". At the end of his letter (6.7), Grothendieck explains why, in fact, this structure was expected. His explanation relies on a theory of deformations of *extensions* in the general context of flat group schemes in A-modules (as in 4.1.1), which he applies to BT's or BT_n's. Unfortunately, this (beautiful) part of his letter was not discussed in [41] nor [43].

As for the long shadows :

(iii) Property 4.2.1 (5) (which did not appear in [36] nor [39], but was an easy consequence of the theory) implies a *formula for the different $d_{G/S}$* of a BT_n G over the spectrum S of a complete discrete valuation ring R with perfect residue field k of characteristic $p > 0$, with dimension d, namely

$$d_{G/S} = p^{nd}\mathcal{O}_S,$$

As a corollary, when k is algebraically closed and the fraction field K of R is of characteristic zero, this implies a *formula for the determinant of the Tate module* of G_K,

$$\Lambda^h_{\mathbf{Z}/p^n} G_K \xrightarrow{\sim} (\mathbf{Z}/p^n)(d),$$

where h is the height of G, see ([43], (4.9.2), (4.10)). These results were used by Raynaud in [65] to effectively bound the modular height in an isogeny class of abelian varieties, an improvement of Faltings' theorem.

(iv) The existence of infinitesimal liftings of BT's and BT_n's was the starting point for the study of their *Dieudonné theory* from a *crystalline* view point. In his letter to Tate [33] (2.6), Grothendieck makes the following observation. For an abelian scheme $f : A \to S$ over a base S, with dual abelian scheme $A^\vee$, consider the *universal extension* $G(A)$ of A by a vector bundle (a construction due to Serre),

$$0 \to (\mathrm{t}_{\mathrm{A}^\vee})^\vee \to G(A) \to A \to 0,$$

where

$$\mathrm{t}_{\mathrm{A}^\vee} \xrightarrow{\sim} R^1 f_* \mathcal{O}_A$$

is the Lie algebra of $A^\vee$. The Lie algebra of $G(A)$ is the *dual* $\mathcal{H}^1_{\mathrm{dR}}(A/S)^\vee$ of $\mathcal{H}^1_{\mathrm{dR}}(A/S)$, with its natural filtration :

$$0 \to (\mathrm{t}_{\mathrm{A}^\vee})^\vee \to \mathcal{H}^1_{\mathrm{dR}}(A/S)^\vee \to (f_*\Omega^1_{A/S})^\vee \to 0,$$

dual to the Hodge filtration of $\mathcal{H}^1_{\mathrm{dR}}(A/S)$,

$$0 \to f_*\Omega^1_{A/S} \to \mathcal{H}^1_{\mathrm{dR}}(A/S) \to R^1 f_* \mathcal{O}_A \to 0.$$

The crystalline nature of $\mathcal{H}^1_{\mathrm{dR}}$ led Grothendieck to conjecture that the universal extension $G(A)$ itself should be crystalline. Of course, as explained above, the definition of "crystalline" in [33] had to be modified, but with this modified definition, Grothendieck statement was indeed correct. A variant of this extension for BT's (also proposed by Grothendieck in [33], chap. 3) together with the local liftability statement (4.2.1 (1)) enabled Messing [58] to construct the *Dieudonné crystals* associated to BT's. The theory was developed in several directions afterwards (Mazur-Messing [56], Berthelot-Breen-Messing [7], [8], [9]). For a description of the state of the art on this subject in 1998, see de Jong's survey [21]. New breakthroughs were made quite recently by Scholze, using his theory of perfectoid spaces [68], giving in particular a classification of BT's over the ring of integers of an algebraically closed complete extension of $\mathbf{Q}_p$.

Other types of "Dieudonné theories" have been considered. The oldest one is *Cartier's theory* of p-typical curves ([14], [16], see also [53], [75]), which works well for formal groups (even in mixed characteristic). This theory has had a wide range of applications (including K-theory and homotopy theory). For those pertaining to the theory of the *de Rham-Witt complex*, see [44] for a brief survey. More recently, we have the theories of Breuil-Kisin (for finite flat commutative group schemes) ([15], [51]), and Zink's theory of *displays* (see Messing's Bourbaki report [59]), which plays an important role in the study of Rapoport-Zink spaces (see (vi) below).

(v) *The mysterious functor, Fontaine's rings and p-adic Hodge theory.* In his talk at the Nice ICM [38] Grothendieck explains that, given a base S where the prime number p is locally nilpotent, and a BT G on S, if $\mathbf{D}(G)$

denotes its Dieudonné crystal (constructed in [58]), the "value" $\mathbf{D}(G)_S$ of $\mathbf{D}(G)$ on S, a locally free $\mathcal{O}_S$-module of rank equal to the height of G, comes equipped with a canonical filtration by a locally direct summand $\mathrm{Fil}(\mathbf{D}(G)_S)$ (namely ω_G), and that if S' is a thickening of S equipped with nilpotent divided powers, then, up to isomorphisms, liftings of G to S' correspond bijectively to liftings of $\mathrm{Fil}(\mathbf{D}(G)_S)$ to a locally direct summand of $\mathbf{D}(G)_{S'}$.

He gives the following corollary. Let R be a complete discrete valuation ring of perfect reside field k of characteristic p and fraction field K of characteristic zero. Let $K_0 := \mathrm{Frac}(W)$, $W = W(k)$. Then the functor associating to a BT G on R *up to isogeny* the pair (M, Fil) consisting of the F-space (see footnote 1) $M = \mathbf{D}(G_k) \otimes_W K_0$ (a K_0-vector space of dimension equal to the height of G, equipped with a σ-linear automorphism F) and the *K-submodule* $\mathrm{Fil} = \mathrm{Fil}\,\mathbf{D}(G_k)_R \otimes_R K \subset M$, is *fully faithful*.

Grothendieck then observes that, on the other hand, in view of Tate's theorem (2.3.1), G is "known" (up to a unique isomorphism) when its Tate module $T_p(G_K)$ is known, therefore raising the question : is there a "more or less algebraic" way of reconstructing (M, Fil) from the datum of the Galois module $T_p(G_K)$? He also proposes to investigate analogues of this question for cohomology in higher degrees, with F-crystals (coming from crystalline cohomology of varieties over k) equipped with longer filtrations (coming from liftings to R). This is the so-called problem of the *mysterious functor*, that he discussed in his talks at the Collège de France (but did not mention explicitly in [38]). As for the original problem (for BT's), in ([31], V 1.4) Fontaine explains how to obtain $T_p(G_K)$ from (M, Fil), but does not define the functor in the other direction. This problem, together with its expected generalizations in higher dimension and the desire to understand its relation with Hodge-Tate's decompositions (2.3.2)), was the starting point of Fontaine's construction of his "Barsotti-Tate rings" (B_{cris}, B_{dR}, B_{st}), and the true beginning of *p-adic Hodge theory*.

(vi) *Rapoport-Zink moduli spaces.* The formal moduli space S in (i) pro-represents the deformation functor of "naked" BT's. In the past fifteen years, variants of this moduli space for BT's endowed with additional structures of isogeny type or complex multiplication type have been constructed by Rapoport-Zink and intensely studied by many other authors. These spaces play the role of local analogues of Shimura varieties arising from moduli of abelian varieties with similar additional structures. They have been used by Harris and Taylor [40] to establish the local Langlands correspondence for GL_n over p-adic fields. A new, simpler proof was recently given by Scholze [66]. See also [64], [30], [68].

(vii) *Traverso's conjectures.* If G is a BT over an algebraically closed field k of characteristic $p > 0$, there exists a positive integer n such that G is determined up to isomorphism (resp. isogeny) by $G(n)$ (cf. ([71], Th. 3)). The least such n is denoted by n_G (resp. b_G, which is called the *isogeny cutoff* of G). A conjecture of Traverso [72] predicted that the isogeny cutoff of G satisfies the inequality $b_G \leq \lceil dd^\vee/(d+d^\vee) \rceil$ for G of dimension d and codimension $d^\vee$ $(= \dim G^\vee)$, with $dd^\vee > 0$. This conjecture was proved by Nicole and Vasiu [60]. On the other hand, Traverso conjectured that $n_G \leq \min(d, d^\vee)$, but recently, Lau, Nicole and Vasiu [52] disproved this conjecture, giving the correct (sharp) bound $\lfloor 2dd^\vee/(d+d^\vee) \rfloor$. This result makes a critical use of 4.2.1. Let me also mention related work of Vasiu [74] and Gabber and Vasiu [32] presenting progress on the search for invariants and classification of truncated BT's.

5. Grothendieck's letter to Barsotti : Newton and Hodge polygons

In 1966-67, during the SGA 6 seminar, Berthelot, Grothendieck and I would often take a walk after lunch in the woods of the IHÉS. It is in the course of one of these walks that Grothendieck told us that he had had a look at Manin's paper [55] and thought about his classification theorem (cf. 2.2). What he explained to us that day, he was to write it up years later, in his letter to Barsotti of May 11, 1970 ([39], Appendice).

Grothendieck observes first that, instead of indexing the simple objects $E_{r,s} = K_\sigma[T]/(T^s - p^r)$ of the category of F-isocrystals on k by pairs of integers (r, s) in lowest terms, it is better to index them by rational numbers, i. e. write $E_{r,s} = E_{r/s}$. He calls $\lambda = r/s$ the *slope* of $E_{r,s}$, a terminology which he attributes to Barsotti. In this way, Manin's theorem implies that any F-isocrystal M admits a canonical (finite) decomposition

$$(*) \qquad M = \oplus_{\lambda \in \mathbf{Q}} M_\lambda,$$

where M_λ is *isoclinic* of slope λ, i. e. a direct sum of copies of E_λ. This decomposition is compatible with tensor products, and, when k is the algebraic closure of a perfect field k_0 descends to k_0. Ordering the slopes of M in nondecreasing order $\lambda_1 \leq \cdots \leq \lambda_n$ $(n = \mathrm{rk}(M))$, one defines the *Newton polygon* $\mathrm{Nwt}(M)$ of M as the graph of the piecewise linear function $0 \mapsto 0$, $i \in [1, n] \mapsto \lambda_1 + \cdots + \lambda_i$. If m_λ the multiplicity of λ in M (i. e. the number of times that λ appears in the preceding sequence), λm_λ is an integer, in particular, the breakpoints lie in $\mathbf{Z}^2$. When k is the algebraic closure of $\mathbf{F}_q$, $q = p^a$, then (by a result of Manin), this Newton polygon is the Newton polygon of the polynomial $\det(1 - F^a t, M)$. As for the relation with BT's, Grothendieck notes that the (F-iso)crystals corresponding to BT's are those with slopes in the interval $[0, 1]$ (slope 0 (resp. 1) for the ind-étale (resp.

multiplicative) ones), and that the decomposition (*) can be refined into a decomposition

$$M = \oplus_{i \in \mathbf{Z}} M_i(-i), \tag{**}$$

where M_i has slopes in $[0, 1)$ and $(-i)$ is the Tate twist, consisting in replacing F by $p^i F$.

Now, the main points in Grothendieck's letter are :

• the sketch of proof of a specialization theorem for F-crystals

• a conjecture on the specialization of BT's

• comments on a conjecture of Katz.

I will briefly discuss these points, each of them has had a long posterity.

5.1. *The specialization theorem.* Roughly speaking, it says that, if M is an absolute F-crystal on a scheme S of characteristic $p > 0$, which one can think of a family of F-crystals M_s parametrized by the points s of S, then the Newton polygon of M_s (i. e. of $M_{\overline{s}}$ for a perfect over field $k(\overline{s})$ of $k(s)$) *rises under specialization* of s (and the endpoints don't change). For a more precise statement and a full proof, see ([49], 2.3.1).

Such F-crystals arise for example from relative crystalline cohomology groups of proper smooth schemes X/S (which, in view of (**), as Grothendieck puts it, produces "a whole avalanche of BT's over k (up to isogeny)"). In this case a variant (and a refinement) of this specialization theorem - which is not a formal consequence of it - was given by Crew [19].

5.2. *Specialization of BT's.* Grothendieck explains that, in the case of a BT $\mathcal{G}$ over S, of height h and dimension d, with S as in 5.1, the specialization theorem says that, if $G' = \mathcal{G}_s$ is a specialization of $G = \mathcal{G}_t$ ($s \in \overline{\{t\}}$), and (λ_i) (resp. (λ'_i)) ($1 \leq i \leq h$) is the sequence of slopes of G (resp. G'), then we have

$$\sum \lambda_i = \sum \lambda'_i \tag{1}$$

(both sums being equal to d), and

$$\lambda_1 \leq \lambda'_1,\ \lambda_1 + \lambda_2 \leq \lambda'_1 + \lambda'_2,\ \cdots,\ \sum_{1 \leq i \leq j} \lambda_i \leq \sum_{1 \leq i \leq j} \lambda'_i, \cdots \tag{2}$$

((1) expressing that both polygons have the same endpoints $(0, 0)$, (h, d)). He conjectures that, conversely, given a BT $G_0 = G'$ over a perfect field k of characteristic $p > 0$, and denoting by S the formal modular variety of G_0 (4.2 (i)), with reduction $S_0 = S \otimes_{W(k)} k$, with universal BT $\mathcal{G}$ over S_0, given any nondecreasing sequence of rational numbers λ_i ($1 \leq i \leq h$) between 0 and 1, the conditions (1) and (2) are sufficient for the existence of a fiber of $\mathcal{G}$ at some point of S_0 having this sequence as sequence of slopes. This conjecture was eventually proven by Oort in 2000 [63].

5.3. *Katz's conjecture.* At the end of his letter, Grothendieck says that his specialization theorem was suggested to him by "a beautiful conjecture of Katz", which he recalls and formulates in a greater generality. This is the following statement :

Conjecture 5.3.1. Let k be a perfect field of characteristic $p > 0$, $W = W(k)$, $K_0 = \mathrm{Frac}(W)$, X/k a proper and smooth scheme, $i \in \mathbf{Z}$, $H^i(X/W)$ the i-th crystalline cohomology group of X/k, with its σ-linear endomorphism F. Let $\mathrm{Nwt}_i(X)$ be the Newton polygon of the F-isocrystal $(H^i(X/W) \otimes K_0, F)$. Let $\mathrm{Hdg}_i(X)$ be the *Hodge polygon* of $H^i_{\mathrm{Hdg}}(X/k) = \oplus H^{i-j}(X, \Omega^j_{X/k})$, starting at 0 and having slope j with multiplicity $h^{j,i-j} = \dim_k H^{i-j}(X, \Omega^j_{X/k})$. Then $\mathrm{Nwt}_i(X)$ lies on or above $\mathrm{Hdg}_i(X)$.

As recalled in ([50], p. 343), such an inequality was proved for the first time by Dwork, for the middle dimensional primitive cohomology of a projective smooth hypersurface of degree prime to p ([25], §6). Conjecture 5.3.1 was established first by Mazur for X liftable to W, and then by Ogus in general ([10], §8), with a refinement when X/k has nice cohomological properties, namely $H^*(X/W)$ is torsion-free, and the Hodge to de Rham spectral sequence of X/k degenerates at E_1. See [44] for a survey.

Grothendieck adds that in the case where X/k "lifts to characteristic zero", one should have a stronger inequality, involving the Hodge numbers of the lifted variety. Namely, if X'/R is a proper and smooth scheme over R, R a complete discrete valuation ring with residue field k and fraction field K of characteristic 0, such that $X' \otimes k = X$, then one can consider the Hodge numbers $h'^{j,i-j} = \dim_k H^{i-j}(X'_K, \Omega^j_{X'_K/K})$, which satisfy

$$h'^{0,i} \leq h^{0,i}, \cdots, h'^{j,i-j} \leq h^{j,i-j}, \cdots,$$

so that the Hodge polygon $\mathrm{Hdg}_i(X'_K)$, constructed similarly to $\mathrm{Hdg}_i(X)$ but with the numbers $h'^{j,i-j}$, lies on or above $\mathrm{Hdg}_i(X)$. Then he proposes :

Conjecture 5.3.2. With the above notations, $\mathrm{Nwt}_i(X/k)$ lies on or above $\mathrm{Hdg}_i(X'_K)$.

Grothendieck says that he has some idea on how to attack 5.3.1, but not 5.3.2 "for the time being". Actually, 5.3.2 was to follow from the proof of Fontaine's conjecture C_{cris}, which implies that the filtered φ-module

$$((H^i(X_k/W), F), \mathrm{Fil}^j H^i_{\mathrm{dR}}(X'_K/K)),$$

where Fil^j denotes the Hodge filtration, is *weakly admissible*.

Inequalities 5.3.1 and 5.3.2 have applications to Chevalley-Warning type congruences on numbers of rational points of varieties over finite fields (or over discrete valuation rings R as above with k finite). See [44] and [45] for 5.3.1. As an example of application of 5.3.2, quite recently Berthelot, Esnault and Rülling used a variant of 5.3.2 (for proper flat schemes having *semistable reduction* over k), following from the proof of Fontaine-Jannsen's C_{st}-conjecture, together with several other cohomological techniques (Berthelot's *rigid cohomology*, Witt vectors cohomology) to prove the following theorem :

Theorem 5.3.3. ([12]) *Let X/R with R as above and $k = \mathbf{F}_q$. Assume :*
(i) X regular, and proper and flat over R ;
(ii) X_K geometrically connected ;
(iii) $H^i(X_K, \mathcal{O}_{X_K}) = 0$ for all $i > 0$.
Then $|X_k(\mathbf{F}_{q^n})| \equiv 1$ mod q^n for all $n \geq 1$.

See the introduction of [12] for a discussion of the analogy of this result with that of Esnault [26] based on ℓ-adic techniques.

5.4. *New viewpoints on slopes.* The analogy between the notions of slopes and Newton polygons for F-crystals and those of slopes and Harder-Narasimhan filtrations for vector bundles on curves is not fortuitous. There is a common framework for the two notions, which was recently discovered by André [1]. Fargues [28] exploited this to construct a Harder-Narasimhan filtration on finite flat commutative group schemes over valuation rings of mixed characteristics, and similar filtrations play an important role in Fargues-Fontaine's work [29] on p-adic Galois representations.

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References

[1] Y. André, *Slope filtrations*, Confluentes Mathematici **1** (2009), 1-85.

[2] I. Barsotti, *Analytical methods for abelian varieties in positive characteristic*, Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962), Librairie Universitaire, Louvain, 77–85.

[3] A. Beilinson, *p-adic periods and de Rham cohomology*, J. of the AMS **25** (2012), 715-738.

[4] A. Beilinson, *On the crystalline period map*, preprint (2013), http://arxiv.org/abs/1111.3316v4.

[5] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math. 407, Springer-Verlag (1974).

[6] P. Berthelot, *Altérations de variétés algébriques*, Séminaire N. Bourbaki, 1995-1996, n. 815, 273-311, Astérisque 241, 1997.

[7] P. Berthelot and W. Messing, *Théorie de Dieudonné cristalline I*, in Journées de Géométrie Algébrique de Rennes I, Astérisque 63, 1979, 17-37.

[8] P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné Cristalline II*, Lecture Notes in Math. 930, Springer-Verlag, 1982.

[9] P. Berthelot and W. Messing, *Théorie de Dieudonné Cristallin III: Théorèmes d'Équivalence et de Pleine Fidélité*, in The Grothendieck Festschrift Vol. I (Eds. P. Cartier, L. Illusie, N.M. Katz, G. Laumon, Y. Manin, K.A. Ribet), 173-247, Progress in Math. 86, Birkhäuser, 1990.

[10] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Math. Notes, vol. 21, Princeton Univ. Press, Princeton, NJ, 1978.

[11] P. Berthelot and A. Ogus, *F-Isocrystals and De Rham Cohomology. I*, Invent. math. **72** (1983), 159-199.

[12] P. Berthelot, H. Esnault, K. Rülling, *Rational points over finite fields for regular models of algebraic varieties of Hodge type* ≥ 1, Ann. of Math. **176**, 1 (2012), 413-508, arXiv 1009.0178.

[13] L. Breen, *Extensions du groupe additif*, Pub. math. I.H.É.S **48** (1978), 39-125.

[14] P. Cartier, *Groupes formels associés aux anneaux de Witt généralisés.*, C. R. Acad. Sci. Paris Sér. A-B **265** (1967), A49-A52.

[15] C. Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.

[16] P. Cartier, *Modules associés à un groupe formel commutatif. Courbes typiques.*, C. R. Acad. Sci. Paris Sér. A-B **265** (1967), A129-A132.

[17] P. Cartier, *Relèvements des groupes formels commutatifs*, Séminaire Bourbaki, 1968-69, 359, 217-230, Lecture Notes in Math. 179, Springer-Verlag, 1972.

[18] P. Colmez et J.-P. Serre, éditeurs, *Correspondance Grothendieck-Serre*, Documents Mathématiques 2, SMF, 2001.

[19] R. Crew, *Specialization of crystalline cohomology*, Duke Math. J. **53** (1986), 749-757.

[20] A. J. de Jong, *Smoothness, semi-stability and alterations*, Pub. math. IHÉS **83** (1996), 51-93.

[21] J. de Jong, *Barsotti-Tate groups and crystals*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Documenta Mathematica II: 259–265.

[22] P. Deligne, Letter to L. Breen, Aug. 4, 1970 (unpublished).

[23] M. Demazure, *Lectures on p-divisible groups*, Lecture Notes in Mathematics 302, Springer-Verlag, 1972.

[24] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631-648.

[25] B. Dwork, *A deformation theory for the zeta function of a hypersurface*, Proc. ICM Stockholm (1962), 247-259.

[26] H. Esnault, *Deligne's integrality theorem in unequal characteristic and rational points over finite fields*, Annals of Math. **164** (2006), 715–730.

[27] G. Faltings, *Almost étale extensions*, Cohomologies p-adiques et applications arithmétiques (II), Astérisque 279, SMF (2002), 185-270.

[28] L. Fargues, *La filtration de Harder-Narasimhan des schémas en groupes finis et plats*, J. für die reine und angewandte Mathematik (Crelle J.) **645** (2010), 1-39.

[29] L. Fargues and J.-. Fontaine, *Vector bundles and p-adic Galois representations*, AMS/IP Studies in Advanced Math. **51**, 5th International Congress of Chinese Mathematicians, AMS 2012.

[30] L. Fargues, E. Mantovan, *Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales* Astérisque 291, 2004.

[31] J.-M. Fontaine, *Groupes p-divisibles sur les corps locaux*, Astérisque 47-48, SMF, 1977.

[32] O. Gabber and A. Vasiu, *Dimensions of group schemes of automorphisms of truncated Barsotti-Tate groups*, to appear in IMRN.

[33] A. Grothendieck, Letter to J. Tate, May 1966.

[34] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Pub. math. I.H.É.S. **29** (1966), 95-103.

[35] A. Grothendieck, *Crystals and the de Rham cohomology of schemes* (IHES, Dec. 1966), notes by J. Coates and O. Jussila, in *Dix exposés sur la cohomologie des schémas*, Advanced Studies in Pure Math. 3, North-Holland, Masson, 1968.

[36] A. Grothendieck, Letter to L. Illusie, 2-4 Dec. 1969.

[37] A. Grothendieck, *Groupes de Monodromie en Géométrie Algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1967-69 (SGA 7 I), Lecture Notes in Math. 288, Springer-Verlag, 1972.

[38] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux*, Actes du Congrès International des Mathématiciens, Nice (1970, 431-436, Gauthier-Villars, 1971.

[39] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Sém. Math. Sup. 45, Presses de l'Université de Montréal, 1974.

[40] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, Princeton University Press, 2001.

[41] L. Illusie, *Complexe Cotangent et Déformations II*, Lecture Notes in Math. 283, Springer-Verlag, 1972.

[42] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. ÉNS, 4e série, t. 12 (1979), 501-661.

[43] L. Illusie, *Déformations de groupes de Barsotti-Tate, d'après A. Grothendieck*, in Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell, Lucien Szpiro, Astérisque **127**, SMF, 1985, 151-198.

[44] L. Illusie, *Crystalline cohomology*, in Motives (eds. U. Jannsen, S. Kleiman, J-P. Serre), Proceedings of Symposia in Pure Math. 55, 1, 43-70.

[45] L. Illusie, *Miscellany on traces in ℓ-adic cohomology : a survey*, Japanese Journal of Math. **1-1** (2006), 107-136.

[46] N. Katz and T. Oda, *On the differentiation of De Rham cohomology classes with respect to parameters*, J. Math. Kyoto Univ. Volume 8, Number 2 (1968), 199-213.

[47] N. Katz, *Nilpotent connections and the monodromy theorem : applications of a result of Turrittin.* Pub. math. I.H.É.S. **39** (1970), p. 175-232.

[48] N. Katz, *Serre-Tate local moduli*, Surfaces Algébriques, Séminaire de Géométrie Algébrique d'Orsay 1976-78, 138-202, Lecture Notes in Math. 868, Springer-Verlag, 1981.

[49] N. Katz, *Slope filtration of F-crystals*, in Journées de Géométrie Algébrique de Rennes I, Astérisque 63, 1979, 113-163.

[50] N. Katz and J. Tate, *Bernard Dwork (1923-1998)*, Notices of the AMS **46**, n.3 (1999), 338-343.

[51] M. Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, 459–496, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.

[52] E. Lau, M.-H. Nicole, and A. Vasiu, *Stratification of Newton polygon strata and Traverso's conjectures for p-divisible groups*, to appear in Ann. of Math.

[53] M. Lazard, *Commutative Formal Groups*, Lecture Notes in Math. 443, Springer-Verlag, 1975.

[54] J. Lubin, J-P. Serre, and J. Tate, *Elliptic curves and formal groups*, Woods Hole Summer Institute, mimeographed notes, 1964.

[55] Y. Manin, *The theory of commutative formal groups over fields of finite characteristic*, English trans., Russian Math. Surv. **18**, 1963.

[56] B. Mazur and W. Messing, *Universal Extensions and One Dimensional Crystalline Cohomology*, Lecture Notes in Math. 370, Springer-Verlag, 1974.

[57] B. Mazur and L. Roberts, *Local Euler characteristics*, Invent. math. **2** (1970), 201-234.

[58] W. Messing, *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes*, Lecture Notes in Math. 264, Springer-Verlag, 1972.

[59] W. Messing, *Travaux de Zink*, Séminaire Bourbaki, 2005-2006, n. 964, n. 964, 341-364.

[60] M.-H. Nicole and A. Vasiu, *Traverso's Isogeny Conjecture for p-Divisible Groups*, Rend. Sem. Univ. Padova, **118** (2007), 73-83.

[61] W. Niziol, *Semistable conjecture via K-theory*, Duke Math. J. **141** (2008), 151-178.

[62] T. Oda, *The First de Rham Cohomology Group and Dieudonné Modules*, Ann. Scient. ÉNS, 3e série, t. 2 (1969), 63-135.

[63] F. Oort, *Newton polygons and formal groups : conjectures by Manin and Grothendieck*, Ann. of Math. **152** (2000), 183-206.

[64] M. Rapoport and T. Zink, *Period Spaces for p-divisible Groups*, Ann. Math. Stud. 141, Princeton University Press, Princeton 1996.

[65] M. Raynaud, *Hauteurs et isogénies*, in Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell, Lucien Szpiro, Astérisque **127**, SMF, 1985, 199-234.

[66] P. Scholze, *The Local Langlands Correspondence for* GL_n *over p-adic fields*, preprint, Bonn, 2010, to appear in Invent. Math.

[67] P. Scholze, *p-adic Hodge theory for rigid-analytic spaces*, preprint, 2013.

[68] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, preprint, 2013.

[69] J-P. Serre, *Groupes p-divisibles (d'après J. Tate)*, Séminaire Bourbaki, 1966/67, n. 318, in Réédition du Séminaire Bourbaki 1948-1968, volume 10, Années 1966-67, 1967-68, SMF.

[70] J. Tate, *p-Divisible Groups*, Proc. of a Conference on Local Fields, Driebergen 1966 (Ed. T. A. Springer), 158-183, Springer-Verlag, 1987.

[71] C. Traverso, *Sulla classificazione dei gruppi analitici di caratteristica positiva*, Ann. Scuola Norm. Sup. Pisa, **23**, 3 (1969), 481-507.

[72] C. Traverso, *Specialization of Barsotti-Tate groups*, Symposia Mathematica, Vol. XXIV (Sympos., INDAM, Rome, 1979), 1-21, Academic Press, 1981.

[73] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Inv. Math. **137** (1999), 233-411.

[74] A. Vasiu, *Reconstructing p-divisible groups from their truncations of small level*, Comm. Math. Helv. **85** (2010), 165–202.

[75] T. Zink. *Cartiertheorie kommutativer former Gruppen*, Teubner, 1984.

[76] [SGA 4] *Théorie des topos et cohomologie étale des schémas*, Séminaire de géométrie algébrique du Bois-Marie 1963-64, dirigé par M. Artin, A. Grothendieck, J.-L. Verdier, Lecture Notes in Math. 269, 270, 305, Springer-Verlag, 1972, 1973.

[77] [SGA 5] *Cohomologie ℓ-adique et fonctions L*, Séminaire de Géométrie Algébrique du Bois-Mari.e 1965/66, dirigé par A. Grothendieck, Lecture Notes in Math. 589, Springer-Verlag, 1977.

[78] [SGA 6] *Théorie des Intersections et Théorème de Riemann-Roch*, Séminaire de géométrie algébrique du Bois-Marie 1966-67, dirigé par P. Berthelot, A. Grothendieck, L. Illusie, Lecture Notes in Math. 225, Springer-Verlag, 1971.

[79] [SGA7] *Groupes de monodromie en géométrie algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1967-1969, I dirigé par A. Grothendieck, II par P. Deligne et N. Katz, Lecture Notes in Math. 288, 340, Springer-Verlag 1972, 1973.