On the Hodge to de Rham spectral sequence in positive characteristic, after A. Petrov

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Princeton, February 14, 2023; Chicago, February 27, 2023

CONTENTS

1. The main theorem

1.1. Let k be a field. Let X be a smooth k-scheme. De Rham cohomology of X/k, $H^*_{\mathrm{dR}}(X/k) = H^*(X, \Omega^\bullet_{X/k})$ is the abutment of the *Hodge to de Rham spectral sequence*

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k). \tag{1.1.1}$$

When X/k is proper, its terms are finite dimensional k-vector spaces, and it degenerates at E_1 if and only, for all n,

$$\sum_{i+j=n} h^{i,j} = h^n, \tag{1.1.2}$$

where $h^{i,j} = \dim_k H^j(X, \Omega^i_{X/k})$, $h^n = \dim_k H^n_{\mathrm{dR}}(X/k)$. It follows from Hodge theory that this is the case if k is of characteristic zero.

Suppose now that k is a perfect field of characteristic $p > 0$. It has been known since the early 1960's that there exist projective, smooth k-schemes whose Hodge to de Rham spectral sequences do not degenerate at E_1, already in dimension 2. However, it was shown in 1987 that, in presence of a smooth lift to $W_2(k)$, at least some partial degeneration holds. More precisely:

Theorem 1.2 ([DI]). Let X_0/k be a smooth scheme. The datum of a smooth lift $X_1/W_2(k)$ of X_0 determines a decomposition in $D(X_0^{(1)}) := D(X_0^{(1)}, \mathcal{O}_{X_0^{(1)}})$

$$\oplus_{0 \leqslant i < p} \Omega^i_{X_0^{(1)}/k}[-i] \xrightarrow{\sim} \tau^{<p} F_* \Omega^\bullet_{X_0/k} \tag{1.2.1}$$

(where $X_0^{(1)}$ is the pull-back of X_0 by the Frobenius automorphism of $\mathrm{Spec}(k)$ and $F : X_0 \to X_0^{(1)}$ is the relative Frobenius), inducing the Cartier isomorphism C^{-1} on H^i for $i < p$.

In particular, the datum of X_1 defines a section

$$s : H^1(F_* \Omega^\bullet_{X_0/k})[-1] \to \tau^{\leqslant 1} F_* \Omega^\bullet_{X_0/k}, \tag{1.2.2}$$

of the canonical projection, and (1.2.1) is deduced from s by the multiplicative structure of both sides. Moreover, the existence of such a section s is equivalent to that of a lifting X_1.

If we assume $\dim(X_0) \leqslant p$, by a duality argument it is shown in [DI] that the datum of X_1 forces the existence of a (non-canonical) decomposition

$$\oplus_{0 \leqslant i \leqslant p} \Omega^i_{X_0^{(1)}/k}[-i] \xrightarrow{\sim} F_* \Omega^\bullet_{X_0/k} \tag{1.2.3}.$$

In particular, if X_0/k is proper, smooth, of dimension $\leqslant p$, and admits a smooth lift X_1, by (1.1.2) the Hodge to de Rham spectral sequence degenerates at E_1. That left open the question whether there might exist a proper, smooth X_0/k, of dimension $p+1$, admitting a smooth lift to $W_2(k)$, for which the Hodge to de Rham would not degenerate at E_1. This question has been recently solved affirmatively by Petrov:

Theorem 1.3 [P]. There exists a projective smooth scheme $X/W(k)$, of relative dimension $p+1$, such that the Hodge to de Rham spectral sequence of $X_0 := X \otimes_{W(k)} k$ does not degenerate at E_1, more precisely, for which

$$h^p(X_0) < \sum_{i+j=p} h^{i,j}(X_0).$$

The goal of this talk is to explain the main ideas and steps in the proof of Th. 1.3.

2. An overview of the strategy

2.1. *The obstructions.*

The first (and main conceptual step) is the following: given X_0 smooth over k, admitting a (smooth) lift X_1 over $W_2(k)$, exhibit the obstructions to:

(a) decomposability of $\tau^{\leqslant p} F_* \Omega^\bullet_{X_0/k}$ in $D(X_0^{(1)})$,

(b) for X_0/k assumed moreover to be proper, degeneration at E_1 of the Hodge to de Rham spectral sequence.

Petrov gives an explicit formula for the obstruction $e_{X_1,p}$ to (a) in terms of certain characteristic classes. The relation between this obstruction and that to (b) is rather indirect as it involves the *conjugate spectral sequence*. The obstruction $e_{X_1,p}$ determines in this spectral sequence a higher differential $d_{p+1}^{0,p}$, with target $H^{p+1}(X_0^{(1)}, \mathcal{O})$, whose non-vanishing ensures its non-degeneration at E_2, hence the non-degeneration at E_1 of (1.1.1) for X_0.

The above discussion works more generally for smooth *Artin stacks*. This is crucial, as the construction of $X/W(k)$ in Th. 1.3 is by approximation from a smooth Artin stack over $W(k)$ of the form BG, for a certain finite flat group scheme $G/W(k)$.

2.2. *Petrov's group scheme G.*

Given k (perfect, of characteristic $p > 0$), Petrov chooses an elliptic curve $E/W(k)$ whose reduction E_0 on k is supersingular. (If $k = \mathbb{F}_p$, one can choose a supersingular E_0/k thanks to Honda-Tate, and one lifts it to $\mathbb{Z}_p$.) Let $q = p^2$. On the finite flat commutative $W(k)$-group scheme

$$H = E[p] \otimes_{\mathbf{F}_p} \mathbb{F}_q^p$$

(a product of $2p$ copies of $E[p]$) the group $\mathrm{SL}_p(\mathbb{F}_q)$ acts. Petrov defines the (non-commutative) finite, flat $W(k)$-group scheme G as the semi-direct product

$$G := \mathrm{SL}_p(\mathbb{F}_q) \ltimes H.$$

Petrov's main result is:

Theorem 1.3.a. The differential $d_{p+1}^{0,p}$ for BG_0 is non-zero.

The proof is indirect, using the replacement of BG by a quotient stack $[A/\mathrm{SL}_p(\mathcal{O}_F)]$, for a certain abelian scheme $A/W(k)$ acted on by $\mathrm{SL}_p(\mathcal{O}_F)$ for a certain real quadratic extension F of $\mathbb{Q}$. This proof is the longest and most technical part of the whole paper. It combines inputs of homotopical algebra related to Steenrod operations with delicate results on algebraic group cohomology, in the style of Cline-Parshall-Scott-van der Kallen [CPSvdK].

2.3. *The approximation X.*

By an adaptation of the method of Godeaux-Serre-Raynaud, combined with a vanishing result of Antieau-Bhatt-Mathew [ABM], Petrov constructs a projective, smooth scheme $X/W(k)$, of relative dimension $p+1$, which is equipped with an fppf G-torsor Z, such that the map $X \to BG$ defined by Z induces on the special fibers an *injection* $H^{p+1}(BG_0, \mathcal{O}) \to H^{p+1}(X_0, \mathcal{O})$ on the targets of the differential $d_{p+1}^{0,p}$. By functoriality of the conjugate spectral sequence, Th. 1.3 then follows from Th. 1.3.a.

3. The obstructions

3.1. Let X_0 be a smooth k-scheme, admitting a smooth lift X_1 to $W_2(k)$. For brevity, write $\mathrm{dR}_{X_0/k}$ for $F_*\Omega^\bullet_{X_0/k}$. Then X_1 produces the decomposition (1.2.2) of $\tau^{<p}\mathrm{dR}_{X_0/k}$, but $\tau^{\leqslant p}\mathrm{dR}_{X_0/k}$ may not be decomposable: the map d of degree 1 of the triangle

$$\tau^{<p}\mathrm{dR}_{X_0/k} \to \tau^{\leqslant p}\mathrm{dR}_{X_0/k} \to \Omega^p_{X_0^{(1)}}[-p] \xrightarrow{d}$$

is the obstruction $e_{X_1,p}$ to its decomposability (in $D(X_0^{(1)})$). By (1.2.2) this obstruction is written as a map

$$e_{X_1,p} : \Omega^p_{X_0^{(1)}/k} \to \oplus_{0\leqslant i<p}\Omega^i_{X_0^{(1)}/k}[-i][p+1].$$

Let π_i denote the projection of the right hand side to the ith summand. By a (recent) result of Drinfeld and Bhatt-Lurie [BL], the datum of X_1 decomposes $\tau^{[1,p]}\mathrm{dR}_{X_0/k}$, hence

$$\pi_i(e_{X_1,p}) = 0$$

for $i > 0$. Therefore

$$e_{X_1,p} \in \mathrm{Hom}(\Omega^p_{X_0^{(1)}/k}, \mathcal{O}_{X_0^{(1)}}[p+1]). \tag{3.1.1}$$

3.2. *Relation with the conjugate spectral sequence.* Let X_0 be a smooth k-scheme. The canonical filtration of $\mathrm{dR}_{X_0/k}$ determines a spectral sequence, called the *conjugate spectral sequence*, which, after a renumbering and use of the Cartier isomorphism, reads:

$$E_2^{i,j} = H^i(X_0^{(1)}, \Omega^j_{X_0^{(1)}/k}) \Rightarrow H^{i+j}(\mathrm{dR}_{X_0/k}). \tag{3.2.1}$$

Assume moreover X_0/k proper, and let $h^{i,j} = \dim_k H^j(X_0, \Omega^i_{X_0/k})$, $h^n = \dim_k H^n_{\mathrm{dR}}(X_0/k)$ as in (3.1.2). Consider the Hodge to de Rham spectral sequence (1.1.1) for X_0 and the conjugate spectral sequence (3.2.1). Then we have the equivalences

(1.1.1) degenerates at $E_1 \Leftrightarrow \sum_{i+j=n} h^{i,j} = h^n \Leftrightarrow$ (3.2.1) degenerates at E_2.

Assume now that X_0/k is proper and smooth, and admits a smooth lift X_1. Then (3.2.1) has the following partial degeneration properties:

(i) Since $\tau^{[0,p-1]}\mathrm{dR}_{X_0/k}$ is decomposable (Th. 1.2), we have $E_2^{i,j} = E_\infty^{i,j}$ for $i+j=p$ and $i>0$. For the same reason, we have $E_2^{p+1,0} = E_{p+1}^{p+1,0}$.

(ii) Since $\tau^{[1,p]}\mathrm{dR}_{X_0/k}$ is decomposable (by the result of Drinfeld and Bhatt-Lurie mentioned above), we have $E_2^{0,p} = E_{p+1}^{0,p}$.

Finally:

(iii) The differential

$$d_{p+1}^{0,p} : E_{p+1}^{0,p} = E_2^{0,p} = H^0(X_0^{(1)}, \Omega^p_{X_0^{(1)}/k}) \to H^{p+1}(X_0^{(1)}, \mathcal{O}) = E_2^{p+1,0} = E_{p+1}^{p+1,0}$$

is obtained by applying $e_{X_1,p}$ to $H^0(X_0^{(1)}, -)$.

Therefore we get the following criterion:

(iv) $d_{p+1}^{0,p} \neq 0 \Leftrightarrow h^p < \sum_{i+j=p} h^{i,j}$

In particular, if $d_{p+1}^{0,p} \neq 0$, (3.2.1) does not degenerate at E_2 (hence (1.1.1) does not degenerate at E_1).

A central result in [P] is the following formula:

Theorem 3.3 ([P], Cor. 7.5). The morphism

$$e_{X_1,p} \in H^{p+1}(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k})$$

(3.1.1) (where $T_{X_0^{(1)}/k}$ is the dual of $\Omega^1_{X_0^{(1)}/k}$) is given by

$$e_{X_1,p} = \mathrm{Bock}_{X_1^{(1)}}(\mathrm{ob}_{F,X_1} \circ \alpha(\Omega^1_{X_0^{(1)}/k})),$$

where the morphisms

$$\alpha(\Omega^1_{X_0^{(1)}/k}) : \Omega^p_{X_0^{(1)}/k} \to F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}/k}[p-1],$$

$$\mathrm{ob}_{F,X_1} : F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}} \to \mathcal{O}_{X_0^{(1)}}[1],$$

$$\mathrm{Bock}_{X_1^{(1)}} = \mathrm{Bock}^p_{X_1^{(1)}} : H^p(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k}) \to H^{p+1}(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k})$$

are defined in 3.4. Here $F_{X_0^{(1)}}$ denotes the (p-linear) *absolute Frobenius endomorphism* of $X_0^{(1)}$.

3.4. (a) *The Bockstein.* For $M \in D(X_1^{(1)}, \mathcal{O})$, the map $\mathrm{Bock}^m_{X_1^{(1)}}$ is the boundary map

$$H^m(X_0^{(1)}, i^*M) \to H^{m+1}(X_0^{(1)}, i^*M)$$

deduced from the exact sequence (of $\mathcal{O}_{X_1^{(1)}}$-modules)

$$0 \to i_*\mathcal{O}_{X_0^{(1)}} \xrightarrow{p} \mathcal{O}_{X_1^{(1)}} \to i_*\mathcal{O}_{X_0^{(1)}} \to 0,$$

(where $X_1^{(1)}$ is the pull-back of X_1 by the Frobenius automorphism of $W_2(k)$, and $i : X_0^{(1)} \to X_1^{(1)}$ the inclusion).

(b) *The obstruction to lifting Frobenius.* The map ob_{F,X_1} is the obstruction to lifting the absolute Frobenius of $X_0^{(1)}$ to a (σ-linear) endomorphism of $X_1^{(1)}$ over $W_2(k)$.

(c) *The class* α. In contrast with the classes (a) and (b), the class α does not depend on the lift X_1, and is a particular case of a characteristic class defined in a more general framework.

Let R be an $\mathbb{F}_p$-algebra. Denote by $F = F_R$ the Frobenius endomorphism of R. Let M be an R-module, and let $M^{(1)} : F^*M$. The map $M \to M^{(1)}$ given by $x \mapsto 1 \otimes_F x$ is a homogeneous polynomial map of degree p (in the sense of [R], cf. [SGA 4 XVII, 5.5.2.4], where it's called a p^{ic} map), hence defines a map

$$v : \Gamma^p M \to M^{(1)}.$$

For M flat, $\Gamma^p M = (M^{\otimes p})^{S_p}$, and v is surjective, characterized by sending $x^{\otimes p}$ to $1 \otimes x$, and vanishing on the image of the symmetrization map

$$N : S^p(M) = (M^{\otimes p})_{S_p} \to (M^{\otimes p})^{S_p} = \Gamma^p(M),$$

and the sequence

$$S^p(M) \xrightarrow{N} \Gamma^p(M) \xrightarrow{v} M^{(1)} \to 0 \qquad (3.4.0)$$

is exact. The notation v is a wink to the Verschiebung. Indeed, when M is the R-algebra of a flat, affine, commutative group scheme G/R, the Verschiebung morphism $V : G^{(1)} \to G$ corresponds to the composite $M \to \Gamma^p M \xrightarrow{v} M^{(1)}$, where the first map is induced by the comultiplication $M \to M^{\otimes p}$, which factors through $\Gamma^p M$ (cf. [SGA 3, VII_A, 4.2, 4.3]). For this reason, v could be called a *half-Verschiebung*.

By left Kan extension v extends to a map

$$v : \Gamma^p(M) \to M^{(1)}$$

for $M \in D(R)$, and even for $M \in D(S)$, if S is a scheme (or stack, or even, prestack over $\mathbb{F}_p$). If $M = E[-1]$, with E a flat $\mathcal{O}_S$-module, by Quillen's décalage formula

$$\Gamma^p(E[-1]) \xrightarrow{\sim} \Lambda^p(E)[-p],$$

v can be re-written as a map

$$v : \Lambda^p(E) \to E^{(1)}[p-1].$$

Petrov defines

$$\alpha(E) := v. \tag{3.4.1}$$

The following alternative definition of $\alpha(E)$ is useful in the proof of Th. 3.3. Let R be an $\mathbb{F}_p$-algebra as above, and let M be an R-module. The map $M \to S^p(M), x \mapsto x^p$ factors through an R-linear map

$$f : M^{(1)} \to S^p(M).$$

Here f could be thought of as a *half-Frobenius*, as, when M is a commutative algebra, the composition $M^{(1)} \xrightarrow{f} S^p(M) \to M$, where the second map is defined by the algebra structure of M, is the (relative) Frobenius map.

Assume now that M is flat. Then the sequence (3.4.0) can be extended to an exact sequence

$$0 \to M^{(1)} \xrightarrow{f} S^pM \xrightarrow{N} \Gamma^pM \xrightarrow{v} M^{(1)} \to 0. \tag{3.4.2}$$

It is convenient to introduce, for $M \in D(S)$ as above, *Kaledin's complex*

$$T(M) := \mathrm{Cone}(S^pM \xrightarrow{N} \Gamma^pM) \in D(S). \tag{3.4.3}$$

It fits in triangles

$$T(M)[-1] \to S^pM \to \Gamma^pM \xrightarrow{d} T(M),$$

$$M^{(1)}[1] \to T(M) \to M^{(1)} \to,$$

and the composition $\Gamma^pM \xrightarrow{d} T(M) \to M^{(1)}$ is just $v : \Gamma^pM \to M^{(1)}$. In particular, for $M = E[-1]$ with E flat, pushing out the triangle

$$T(E[-1])[-1] \to S^p(E[-1]) \to \Lambda^pE[-p] \to$$

by $T(E[-1])[-1] \to E^{(1)}[-2]$ we get a triangle

$$E^{(1)}[-2] \to \tau^{\geqslant 2}S^p(E[-1]) \to \Lambda^pE[-p] \xrightarrow{d} E^{(1)}[-1] \tag{3.4.4}$$

where

$$(3.4.5). \qquad d = v = \alpha(E) : \Lambda^p E \to E^{(1)}[p-1].$$

When $p = 2$, Petrov shows that the triangle (3.4.4) boils down to the short exact sequence

$$(3.4.6) \qquad 0 \to E^{(1)} \xrightarrow{f} S^2 E \xrightarrow{m_1 m_2 \mapsto m_1 \wedge m_2} \Lambda^2 E \to 0,$$

whose boundary map is $\alpha(E) \in \mathrm{Ext}^1(\Lambda^2 E, E^{(1)})$.

4. Proof of the central result

The proof of Th. 3.3 exploits Bhatt-Lurie's theory of *diffracted Hodge complexes* and *Sen operators*. In particular, it relies on a formula for the action of the Sen operator Θ_{X_1} on the weight zero part of $\tau^{\leqslant p}\mathrm{dR}_{X_0/k}$ (Th. 4.2). When X_0 admits a formal smooth lift X to $W(k)$, this formula is not needed, and, instead, Th. 3.3 follows from a general theorem on derived commutative algebras (Th. 4.3), applied to the diffracted Hodge algebra $\Omega^{\not D}_{X^{(1)}/W(k)}$. This general theorem, that Petrov says was inspired to him by Steenrod operations deriving from homology classes of symmetric groups, plays a crucial role, later in the paper, in establishing the non-vanishing of the differential $d_{p+1}^{0,p}$.

4.1. *Diffracted Hodge complexes and Sen operators.*

(a) *Over* $W(k)$.

Let X be a smooth formal scheme over $W(k)$. In [BL22a, 4.7.12] there is defined an object

$$(4.1.1) \qquad \Omega^{\not D}_{X/W(k)} \in D(X) = D(X, \mathcal{O}_X),$$

(abbreviated $\Omega^{\not D}_X$ if no confusion can arise), called the *diffracted Hodge complex* of X, with the following structure and properties.

(i) It is a perfect complex (of perfect amplitude in $[0, \dim(X/W(k))]$), and it is equipped with the structure of a *derived commutative algebra* in $D(X)$[1]

[1]In the sense of Mathew: a derived commutative algebra structure on $A \in D(X)$ is given by maps $S^n A \to A$ ($n \in \mathbb{N}$) with compatibility data imposed by the monad structure of $S^\bullet = \oplus S^n$. In particular, a cosimplicial commutative algebra has such a structure, but in general an E_∞-algebra in $D(X)$ has not. The calculation of prismatic cohomology by certain Čech-Alexander complexes furnishes a cosimplicial commutative algebra representing $\Omega^{\not D}_X$.

(ii) There is a natural isomorphism of (graded commutative) algebras

$$H^\bullet(\Omega_X^{\not D}) \xrightarrow{\sim} \Omega^\bullet_{X/W(k)}.$$

(iii) The diffracted Hodge complex is equipped with an operator

$$\Theta_X \in \mathrm{End}(\Omega_X^{\not D}),$$

called the *Sen operator*, which acts as a derivation (i.e., satisfies Leibniz rule) and induces the multiplication by $-i$ on $H^i(\Omega_X^{\not D})$.

(iv) A multiplicative isomorphism

$$\varepsilon : (\Omega_X^{\not D} \otimes^L_{W(k)} k)^{(1)} \xrightarrow{\sim} \mathrm{dR}_{X_0/k} (= F_*\Omega^\bullet_{X_0/k})$$

inducing the Cartier isomorphism C^{-1} on H^i via (ii).

Denote again by Θ_X the endomorphism of $\mathrm{dR}_{X_0/k}$ induced by Θ_X via ε. By (iii) $\prod_{i\in\mathbb{Z}}(\Theta_X + i)$ is nilpotent, hence induces a decomposition into *generalized eigenspaces*

$$\mathrm{dR}_{X_0/k} \xrightarrow{\sim} \oplus_{0\leqslant i\leqslant p-1}(\mathrm{dR}_{X_0/k})_i, \tag{4.1.2}$$

where the summand $(\mathrm{dR}_{X_0/k})_i$ has cohomology H^j concentrated in degree $j \equiv i \bmod p$, and

$$\Theta_X | (\mathrm{dR}_{X_0/k})_i = -i\mathrm{Id} + (\Theta_X)_i,$$

with $(\Theta_X)_i$ nilpotent.

The decomposition (4.1.2) induces, in particular, a decomposition of the partial truncations $\tau^{[a,a+p-1]}\mathrm{dR}_{X_0/k}$, generalizing those of [DI] (1.2.1) and Achinger-Suh. In particular, it provides a section

$$s_{X_0} : H^1(F_*\Omega^\bullet_{X_0/k})[-1] \to \tau^{\leqslant 1}F_*\Omega^\bullet_{X_0/k}, \tag{4.1.3}$$

of the projection, which, by Li-Mondal [8], coincides with that of (1.2.3).

This section s_{X_0} is in fact the reduction mod p of a section

$$s_X : H^1(\Omega^{\not D})[-1] \to \tau^{\leqslant 1}\Omega^{\not D}_{X/W(k)} \tag{4.1.4}$$

of the projection. The reason is that the operator $\prod_{0\leqslant i\leqslant p-1}(\Theta_X + i)$ on $\tau^{\leqslant p-1}\Omega^{\not D}_{X/W(k)}$ is nilpotent, and the decomposition of $W(k)[T]/\prod_{0\leqslant i\leqslant p-1}(T + i)$ produces a decomposition

$$\oplus_{0\leqslant i\leqslant p-1}\Omega^i_{X/W(k)}[-i] \xrightarrow{\sim} \tau^{\leqslant p-1}\Omega^{\not D}_{X/W(k)}. \tag{4.1.5}$$

The diffracted Hodge complex is defined as

$$\Omega^{\not{D}}_{X/W(k)} := R\pi^{\not{D}}_{*}\mathcal{O}_{X^{\not{D}}}, \tag{4.1.6}$$

where $X^{\not{D}}$ denotes the *diffracted Hodge Tate stack*, pull-back by the $V(1)$-section $\eta : \mathrm{Spf}(W(k)) \to \mathrm{WCart}^{\mathrm{HT}}_{W(k)} = (B\mathbb{G}^{\sharp}_{m})_{W(k)}$ of the Hodge-Tate stack $\mathrm{WCart}^{\mathrm{HT}}_{X}$, and

$$\pi^{\not{D}} : X^{\not{D}} \to X$$

denotes the composition

$$X^{\not{D}} \to \mathrm{WCart}^{\mathrm{HT}}_{X} \xrightarrow{\pi^{\mathrm{HT}}} X$$

([BL22b], 3.8).

(b) *Over* $W_n(k)$, $n \geqslant 2$.

For a *flat* scheme $X_{n-1}/W_n(k)$ one defines the *diffracted Hodge complex* $\Omega^{\not{D}}_{X_{n-1}/W_n(k)}$ in a way similar to (4.1.3). For this, the following refinement of ([BL22b], Example 5.15), due to Devalapurkar-Petrov, is needed: there is a canonical isomorphism

$$\mathrm{WCart}^{\mathrm{HT}}_{W_n(k)} \xrightarrow{\sim} \mathbb{G}^{\sharp}_{a,W_n(k)}/\mathbb{G}^{\sharp}_{m,W_n(k)}, \tag{4.1.4}$$

with the action of $\mathbb{G}^{\sharp}_{m}$ on $\mathbb{G}^{\sharp}_{a}$ given by scaling. (Over $W(k)$, $\mathbb{G}^{\sharp}_{a} = \mathrm{W}^{F=0}$, and $\mathbb{G}^{\sharp}_{m} = (\mathrm{W}^{\times})^{F=1}$, where W is the Witt group scheme.).

One then defines the *diffracted Hodge-Tate stack* $X^{\not{D}}_{n-1}$ by the cartesian square

$$\begin{tikzcd}
(X_{n-1}/W_n(k))^{\not{D}} \arrow[r] \arrow[d] & \mathrm{WCar}^{\mathrm{HT}}_{X_{n-1}} \arrow[d] \\
\mathrm{Spec}(W_n(k) \arrow[r, "\eta"] & \mathrm{WCart}^{\mathrm{HT}}_{W_n(k)},
\end{tikzcd}$$

where η is the composition $\mathrm{Spec}(W_n(k) \xrightarrow{0} \mathbb{G}^{\sharp}_{a,W_n(k)} \to \mathbb{G}^{\sharp}_{a,W_n(k)}/\mathbb{G}^{\sharp}_{m,W_n(k)}$. In particular, it has a natural action of $\mathbb{G}^{\sharp}_{m,W_n(k)}$. One defines the *diffracted Hodge complex* $\Omega^{\not{D}}_{X_{n-1}/W_n(k)}$ by

$$\Omega^{\not{D}}_{X_{n-1}/W_n(k)} := R\pi^{\not{D}}_{*}\mathcal{O}_{(X_{n-1}/W_n(k))^{\not{D}}},$$

where $\pi^{\not{D}}$ is the composition

$$\pi^{\not{D}} : (X_{n-1}/W_n(k))^{\not{D}} \to \mathrm{WCart}^{\mathrm{HT}}_{X_{n-1}} \xrightarrow{\pi^{\mathrm{HT}}} X_{n-1}$$

(which is $\mathbb{G}_m^\sharp$-equivariant, with trivial action on the target). This complex has a derived commutative algebra structure and an action of $\mathbb{G}_m^\sharp$, enabling to define a *Sen operator*

$$\Theta_{X_{n-1}} \in \mathrm{End}(\Omega^{\not{D}}_{X_{n-1}/W_n(k)}),$$

with properties analogous to those of Θ_X.

In particular, for $X_1/W_2(k)$ smooth, $X_0 := X_1 \otimes_{W_2(k)} k$, we have an isomorphism of $\mathcal{O}$-algebras $H^\bullet(\Omega^{\not{D}}_{X_1}) \xrightarrow{\sim} \Omega^\bullet_{X_1/W_2(k)}$, with Θ_1 inducing $-i\mathrm{Id}$ on H^i, an isomorphism

$$\varepsilon : (\Omega^{\not{D}}_{X_1} \otimes^L k)^{(1)} \xrightarrow{\sim} \mathrm{dR}_{X_0/k}$$

through which Θ_{X_1} produces a decomposition (4.1.2). When X_0 is lifted to a formal smooth $X/W(k)$, $\Omega^{\not{D}}_{X_1} = \Omega^{\not{D}}_X \otimes^L W_2(k)$, and Θ_X induces Θ_{X_1}, so that the decomposition (4.1.2) depends only on X_1.

The truncation $\tau^{\leqslant p}(\mathrm{dR}_{X_0/k})_0$ has cohomology sheaves H^i concentrated in degrees 0 and p. As Θ_{X_1} vanishes on them, Θ_{X_1} comes from a morphism

$$c_{X_1,p} : \Omega^p_{X_0^{(1)}/k} \to \mathcal{O}_{X_0^{(1)}}[p],$$

which one checks is unique. This morphism, which can be viewed as a class $c_{X_1,p} \in H^p(X_0^{(1)}, T_{X_0^{(1)}/k})$ could be named the *Sen class* in degree $\leqslant p$. A crucial ingredient in the proof of Th. 3.3 is the following formula for $c_{X_1,p}$:

Theorem 4.2 ([P], Th. 7.1). Under the assumptions of Th. 3.3, we have

$$c_{X_1,p} = \mathrm{ob}_{F,X_1} \circ \alpha(\Omega^1_{X_0^{(1)}/k}). \tag{4.2.0}$$

By Th. 3.3 we thus have

$$e_{X_1,p} = \mathrm{Bock}_{X_1^{(1)}}(c_{X_1,p}). \tag{4.2.1}$$

In particular, if $\tau^{\leqslant p}\mathrm{dR}_{X_0/k}$ is not decomposable, the Sen class $c_{X_1,p}$ doesn't vanish, hence the Sen operator Θ_{X_1} is not semi-simple.

Actually, Petrov proves Th. 4.2 in a slightly more general framework, with $X_1/W_2(k)$ assumed only *quasi-syntomic* instead of smooth. This extension enables him to prove 4.2 (even in the smooth case) by flat descent from quasi-regular semi-perfectoid algebras, where filtrations in the derived category are replaced by filtrations on the nose.

As explained at the beginning of the section, when $X_0 = X \otimes_{W(k)} k$ for a smooth formal scheme $X/W(k)$, Th. 4.2 is not needed, in fact Th. 3.3 follows from the following general result, applied to the diffracted Hodge complex $\Omega^{\not D}_{X^{(1)}/W(k)}$ (where $X^{(1)}$ is the pull-back of X by the Frobenius automorphism of $W(k)$):

Theorem 4.3 ([P], Theorem 4.1). Let X be a formal scheme (or an algebraic stack) flat over $W(k)$, let $X_0 = X \otimes_{W(k)} k$ be its special fiber. Let $A \in D^{\geqslant 0}(X, \mathcal{O}_X)$ be a derived commutative algebra satisfying the following conditions:

(i) $H^0(A) = \mathcal{O}_X$,

(ii) $H^1(A)$ is a locally free $\mathcal{O}_X$-module of finite type, and multiplication induces an isomorphism $\Lambda H^1(A) \xrightarrow{\sim} H^*(A)$.

Assume further that there exists a section

$$s : H^1(A)[-1] \to \tau^{\leqslant 1}(A)$$

of the canonical projection. Then:

(1) s induces a decomposition

$$\oplus_{0 \leqslant i \leqslant p-1} H^i(A)[-i] \xrightarrow{\sim} \tau^{\leqslant p-1} A$$

in $D(X)$, 3.4 (c)).

(2) The extension class

$$e_{A,p} : H^p(A) \to (\tau^{\leqslant p-1} A)[p+1]$$

defined by $\tau^{\leqslant p-1} \to \tau^{\leqslant p} \to H^p[-p]$ is equal (up to a natural homotopy) to the composition

$$H^p(A) = \Lambda^p H^1(A) \overset{\alpha(H^1(A))}{\to} H^1(A/p)^{(1)}[p-1] \xrightarrow{s} \tau^{\leqslant 1}(A/p)^{(1)}[p]$$

$$\text{(4.3.1)} \qquad \xrightarrow{\varphi_{A/p}} (\tau^{\leqslant 1}(A/p))[p] \xrightarrow{\mathrm{Bock}_A} (\tau^{\leqslant 1} A)[p+1] \to (\tau^{\leqslant p-1} A)[p+1].$$

Here $\alpha(H^1(A))$ denotes the composition of the reduction mod p map $H^1(A) \to H^1(A/p)$ and the map α relative to $H^1(A/p) = H^1(A)/p$ (3.4 (c)), and for $M \in D(A)$, the Bockstein map $\mathrm{Bock}_M : M/p \to M[1]$ is defined by the triangle $M \xrightarrow{p} M \to M/p$.e

(3) When A is an augmented algebra, by $\varepsilon : A \to \mathcal{O}_X \to A$ inducing an isomorphism on H^0, and s is the associated section, the composition of the last four maps in (4.3.1) can be replaced by the composition

(4.3.2)

$$H^1(A/p)^{(1)}[p-1] \xrightarrow{\varphi_{A/p}} H^1(A/p)[p-1] \xrightarrow{\mathrm{Bock}_{H^1(A)}} H^1(A)[p] \hookrightarrow (\tau^{\leqslant p-1} A)[p+1].$$

For X formal smooth over $W(k)$, applying Th. 4.2 to the diffracted Hodge complex $\Omega^{\not D}_{X/W(k)}$ (with the section s_X (4.1.4), and using a formula for the obstruction $\mathrm{ob}_{\mathrm{F},\mathrm{X}_1}$ to lifting the Frobenius of X_0 to X_1 based on the relation between the cotangent complex of $X_0^{(1)}$ over $W(k)$ and the truncation $\tau^{\leqslant 1}\mathrm{dR}_{X_0/k}$ [7], one obtains a formula for the obstruction $e_{X,p} : \Omega^p_{X/W} \to \tau^{\leqslant p-1}\Omega^{\not D}_{X/W(k)}[p+1]$ to split the canonical filtration of $\tau^{\leqslant p}\Omega^{\not D}_{X/W(k)}$, namely:

Corollary 4.4 ([P], Th. 5.8). The class $e_{X,p}$ is the composition of

$$\Omega^p_{X/W(k)} \overset{\alpha(\Omega^1_{X_0/k})}{\to} \Omega^1_{X_0^{(1)}/k}[p-1] \overset{\mathrm{ob}_{F,X_1}}{\to} \mathcal{O}_{X_0}[p] \overset{\mathrm{Bock}_{\mathcal{O}_X}}{\to} \mathcal{O}_X[p+1]$$

and the canonical map $\mathcal{O}_X[p+1] \to (\tau^{\leqslant p-1}\Omega^{\not D}_{X/W(k)})[p+1]$.

Th. 3.3 then follows from Cor. 4.4 by reduction mod p. In this proof, formula (4.2.0) for the Sen operator $c_{X_1,p}$ is not used.

Proof of Th. 4.3. This is more or less formal from the definition of the class α. The multiplicative structure of A (in the strong sense described above) is crucial. When A can be represented by a component-wise flat cosimplicial commutative algebra, then 4.3 follows from the alternate definition of α given in (3.4.2) - (3.4.5) and the commutative diagram (of cosimplicial sheaves on X:

$$\begin{tikzcd}
S^pA \arrow[r, "N"] \arrow[d, "m"'] & \Gamma^pA \arrow[r] \arrow[d, "m"'] & (A/p)^{(1)} \arrow[d, "\varphi_{A/p}"'] \\
A \arrow[r, "p!"] & A \arrow[r] & A/p.
\end{tikzcd}$$

Remark 4.5. Th. 4.3, which is not used in the proof of Th. 3.3 in the general case (of a lifting X_1), turns out to be crucial in the proof of Th. 1.3.a.

5. Petrov's auxiliary abelian scheme A

5.1. *Construction of real quadratic fields.* Petrov constructs, for any prime number p, a real quadratic field F (depending on p) having the following properties:

(i) p is inert in F, hence $\mathcal{O}_F/p = \mathbb{F}_{p^2}$, and the completion of $\mathcal{O}_F$ at p is $W(\mathbb{F}_{p^2})$;

(ii) The map $\mathcal{O}_F \to \mathcal{O}_F/p = \mathbb{F}_{p^2}$ induces a surjection $\mathcal{O}_F^\times \twoheadrightarrow \{x \in \mathbb{F}_{p^2} | N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x) = \pm 1\}$;

(iii) There exists a unit $u \in \mathcal{O}_F^\times$ whose image $u_1 \in W_2(\mathbb{F}_{p^2})$ satisfies $\mathrm{Fr}_p(u_1) \neq u_1^p$.

This is easy: for $p = 2$, take $F = \mathbb{Q}(\sqrt{5})$, for $p > 2$, take $F = \mathbb{Q}(\sqrt{d^2+1})$ where d is a suitable lift in $\mathbb{Z}$ of $\frac{1}{2}\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(u_0)$, where $u_0 \in \mathbb{F}_{p^2}^\times$ is a generator of the cyclic group $\{x \in \mathbb{F}_{p^2} | N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x) = \pm 1\}$.

5.2. Recall the definition of Petrov's finite flat group scheme $G/W(k)$ (2.2):

$$G := \mathrm{SL}_p(\mathbb{F}_q) \ltimes H.$$

where $E/W(k)$ is an elliptic curve with supersingular reduction E_0 on k, and

$$H = E[p] \otimes_{\mathbf{F}_p} \mathbb{F}_q^p,$$

where $q = p^2$.

Choose F as in 5.1. Define the abelian scheme $A/W(k)$ by

$$A := E \otimes_{\mathbb{Z}} \mathcal{O}_F^{\oplus p}, \tag{5.2.1}$$

where the tensor product is taken in the sense of Serre. It is isomorphic to a sum of $2p$ copies of E and it is equipped with a natural action of $\mathrm{GL}_p(\mathcal{O}_F)$. The action of $\mathrm{GL}_p(\mathcal{O}_F)$ on $A[p]$ factors through $\mathrm{GL}_p(\mathcal{O}_F/p) = \mathrm{GL}_p(\mathbb{F}_q)$ and defines a $\mathrm{GL}_p(\mathbb{F}_q)$-equivariant isomorphism

$$A[p] \xrightarrow{\sim} H.$$

The classifying map $A \to BA[p]$ defined by the $A[p]$-torsor $p : A \to A$ is thus identified to a $\mathrm{GL}_p(\mathcal{O}_F)$-equivariant morphism $A \to BH$, which induces a map

$$f : [A/\mathrm{SL}_p(\mathcal{O}_F) \to [BH/\mathrm{SL}_p(\mathbb{F}_q), \tag{5.2.2}$$

and

$$[BH/\mathrm{SL}_p(\mathbf{F}_q)] \xrightarrow{\sim} BG$$

as $BH \to BG$ is a $G/H = \mathrm{SL}_p(\mathbb{F}_q)$-torsor.

It is easy to see ([P], Lemma 10.2) that f induces an isomorphism

$$f_0^* : H^0(BG_0, L\Omega^p_{BG_0^{(1)}}) \xrightarrow{\sim} H^0([A_0^{(1)}/\mathrm{SL}_p(\mathcal{O}_F], L\Omega^p_{[A_0^{(1)}/\mathrm{SL}_p(\mathcal{O}_F]}), \tag{5.2.3}$$

where subscript zero means reduction on $\mathrm{Spec}(k)$. Therefore, by functoriality of the differential $d_{p+1}^{0,p}$ in the conjugate spectral sequence, in order to prove Th. 1.3.a (hence Th. 1.3), it suffices to prove:

Theorem 1.3.b ([P], Prop. 10.3). The differential $d_{p+1}^{0,p}$ in the conjugate spectral sequence for $[A_0/\mathrm{SL}_p(\mathcal{O}_F)]$ is non-zero.

Remark 5.2.4. As shown in [DI], $\mathrm{dR}_{A_0/k}$ is decomposable, hence the conjugate spectral sequence of A_0/k degenerates at E_2. However, Th. 1.3.b shows that there is no $\mathrm{SL}_p(\mathcal{O}_F)$-equivariant decomposition of $\mathrm{dR}_{A_0/k}$.

5.3. The proof of Th.1.3.b consists of two steps.

(1) The first (and, by far, the most difficult) one consists in showing that the truncated *global dR cohomology* complex $\tau^{\leqslant p}R\Gamma(A_0^{(1)}, \mathrm{dR}_{A_0/k})$ is *not* $\mathrm{SL}_p(\mathcal{O}_F)$*-equivariantly decomposable*, more precisely, the canonical map

$$H^p_{\mathrm{dR}}([A_0/\mathrm{SL}_p(\mathcal{O}_F)]/k) \to H^p_{\mathrm{dR}}(A_0/k)^{\mathrm{SL}_p(\mathcal{O}_F)}$$

is *not surjective*.

This relies on delicate results on the cohomology of SL_p (in both algebraic and discrete settings), and especially, refinements of basic results of Cline-Parshall-Scott-van der Kallen [CPSvdK] (= [4]).

(2) The second one consists in showing that, in contrast, the truncated *global Hodge cohomology complex* $\tau^{\leqslant p}R\Gamma(A_0^{(1)}, \oplus\Omega^i_{A_0^{(1)}}[-i])$ *is* $\mathrm{SL}_p(\mathcal{O}_F)$*-equivariantly decomposable*. This (much easier) step uses the supersingularity of E_0 in a crucial way. Then a showdown between Hodge and de Rham implies the non-vanishing of $d^{0,p}_{p+1}$.

6. Cohomology of SL_p

Let V be a p-dimensional k-vector space, and, as usual, let $V^{(1)}$ denote its pull-back by the absolute Frobenius of k. There are three key results:

(a) The k-group scheme $\mathrm{GL}(V)$, and *a fortiori* its subgroup $\mathrm{SL}(V)$, acts naturally on V, so that V can be viewed as a rank p vector bundle on $B\mathrm{SL}(V)$. One can consider its class

$$\alpha(V) : \Lambda^p V \to V^{(1)}[p-1]$$

in $D(B\mathrm{SL}(V), \mathcal{O})$ (3.4.1). As the determinant trivializes $\Lambda^p V$, $a(V)$ can be rewritten as a class

$$\alpha(V) \in \mathrm{Ext}^{p-1}(\mathcal{O}, V^{(1)}) = H^{p-1}(B\mathrm{SL}(V), V^{(1)}) \qquad (6(a))$$

where the right hand side is often more classically written $H^{p-1}(\mathrm{SL}(V), V^{(1)})$.

Theorem 6.1 ([P] Prop. 12.1, Lemma 12.4). The class $\alpha(V)$ (6(a)) is non-zero. Moreover, if $p > 2$, we have

$$H^i(\mathrm{SL}(V), V^{(1)}) = 0$$

for all $i \neq p-1$, and

$$H^{p-1}(\mathrm{SL}(V), V^{(1)}) = k.$$

(b) Assume now that $k = \mathbb{F}_q$, with $q > p$. Then the inclusion of $\mathrm{SL}(V)(\mathbb{F}_q) = \mathrm{SL}_p(\mathbb{F}_q)$ into $\mathrm{SL}(V)$ as a discrete subgroup induces a map $B\mathrm{SL}_p(\mathbb{F}_q) \to B\mathrm{SL}(V)$.

Theorem 6.2 ([P] Prop. 13.1). The restriction map

$$H^{p-1}(\mathrm{SL}(V), V^{(1)}) \to H^{p-1}(\mathrm{SL}_p(\mathbb{F}_q), V^{(1)})$$

is injective.

This is a refinement, in this particular situation, of a general result of Cline-Parshall-Scott-van der Kallen [4] asserting that for a split semi-simple group $G/\mathbb{F}_q$, with $q = p^r$, and a finite dimensional G-module E, then, for any given n, the restriction map $H^n(G, E^{(e)}) \to H^n(G(\mathbb{F}_q), E^{(e)})$ is injective for sufficiently large e and r (depending on n) (where $E^{(e)}$ is the pull-back of E by the eth power of the absolute Frobenius).

(c) Finally, assume that $k = \mathbb{F}_q$, with $q = p^2$. Let $\widetilde{V}$ be a free $W_2(k)$-module lifting V, and denote again by $\widetilde{V}^{(1)}$ its pull-back by the Frobenius of $W_2(k)$. Let $F/\mathbb{Q}$ be a real quadratic extension constructed as in 5.1. We have natural maps $\mathcal{O}_F \to W(k) \to W_2(k) \to k$, which, in particular, define homomorphisms $\mathrm{SL}_p(\mathcal{O}_F) \to \mathrm{SL}_p(W_2(k)) \to \mathrm{SL}_p(k)$, hence actions of $\mathrm{SL}_p(\mathcal{O}_F)$ on V, $\widetilde{V}$, $\widetilde{V}^{(1)}$. The $\mathrm{SL}_p(\mathcal{O}_F)$-equivariant exact sequence

$$0 \to V^{(1)} \xrightarrow{p} \widetilde{V}^{(1)} \to V^{(1)} \to 0$$

defines *Bockstein homomorphisms*

$$\mathrm{Bock}^i : H^i(\mathrm{SL}_p(\mathcal{O}_F), V^{(1)}) \to H^{i+1}(\mathrm{SL}_p(\mathcal{O}_F), V^{(1)}).$$

Theorem 6.3 ([P], Prop. 14.1). Let

$$\alpha(V) \in H^{p-1}(\mathrm{SL}_p(\mathcal{O}_F), V^{(1)})$$

denote the pull-back of the class (6(a)) by the composite homomorphism $\mathrm{SL}_p(\mathcal{O}_F) \to \mathrm{SL}_p(k) \to \mathrm{SL}(V)$. Then

$$\mathrm{Bock}^{p-1}(\alpha(V)) \in H^p(\mathrm{SL}_p(\mathcal{O}_F), V^{(1)})$$

is non-zero.

6.4. *Glimpses on the proofs of Th. 6.1, Th. 6.2, Th. 6.3.*

Proof of Th. 6.1. For $p = 2$, it is elementary to see that $(S^2V)^{\mathrm{SL}_2} = 0$, hence the short exact sequence (3.4.6) admits no $\mathrm{SL}_2(V)$-equivariant splitting.

For $p > 2$, the proof is much more difficult. A crucial preliminary step is to analyze $S^p(V[-1])$ in terms of derived coinvariants. For X an $\mathbb{F}_p$-prestack, and $M \in D(X)$ denote by $M \mapsto M^{\otimes p} \in D(X, \mathcal{O}_X[S_p])$ the functor defined by left Kan extension from pth power tensor product on finite type projective modules over an $\mathbb{F}_p$-algebra, and $M \mapsto (M^{\otimes p})_{hS_p}$ the derived coinvariant functor, defined by

$$(M^{\otimes p})_{hS_p} := M^{\otimes p} \otimes^L_{\mathbb{F}_p[S_p]} \mathbb{F}_p.$$

Kaledin's complex (3.4.3), together with Quillen's décalage formula, shows that $S^p(V[-1])$ is cohomologically concentrated in degrees 1, 2, and p, with $H^1 = H^2 = V^{(1)}$, and $H^p = \Lambda^p V$. The S_p-equivariant map $V[-1]^{\otimes p} \to S^p(V[-1])$ (with trivial action on the target) thus induces a map

$$\tau^{\geqslant 1}(V[-1]^{\otimes p})_{hS_p} \to S^p(V[-1]). \tag{6.4.1}$$

The crucial result is:

Lemma 6.4.2 ([P] Lemma 3.8). The map (6.4.1) is an isomorphism.

The proof relies on Priddy's calculation of the cohomology of the left hand side, which gives the same values as for the right hand side, the known relation between $H^j S^\bullet \mathbb{F}_p[-i]$ and natural transformations from H^i to H^j of $\mathbb{F}_p$-cosimplicial commutative algebras, and finally, the fact that the Bockstein $H^1(B\mathbb{F}_p, \mathbb{F}_p) \to H^2(B\mathbb{F}_p, \mathbb{F}_p)$ is an isomorphism.

From Lemma 6.4.2, the proof of Th. 6.1 uses classical results on the (algebraic) cohomology of reductive groups (Kempf's vanishing theorem, Jantzen's good filtrations) and for the assertions relative to $H^i(\mathrm{SL}(V), V^{(1)})$, another interpretation of $S^p(V[-1])$ in terms of a certain truncated de Rham complex, inspired by Friedlander-Suslin [FS] ([6]).

Proof of Th. 6.2. The strategy is the same as in [CPSvdK]. Let $B \subset \mathrm{SL}(V)$ be a Borel subgroup. By Kempf's vanishing theorem, $H^i(\mathrm{SL}(V)/B, \mathcal{O}) = 0$ for $i > 0$, hence the restriction map $R\Gamma(\mathrm{SL}(V), E) \to R\Gamma(B, E)$ is an isomorphism for any $\mathrm{SL}(V)$-module E. Therefore to show injectivity for the restriction map for $\mathrm{SL}(V)$ it suffices to do it for B. Writing $B = T \ltimes U$, with U the unipotent radical, and using that $H^i(T, M) = H^i(T(\mathbb{F}_q), M) = 0$ for $i > 0$ and p-torsion coefficients M, one has

$$H^i(B, E) = H^i(U, E)^T, \ H^i(B(\mathbb{F}_q), E) = H^i(U(\mathbb{F}_q), E)^{T(\mathbb{F}_q)}.$$

So one is reduced to showing injectivity of the restriction map

$$H^{p-1}(U, V^{(1)})^T \to H^{p-1}(U(\mathbb{F}_q), V^{(1)})^{T(\mathbb{F}_q)}$$

The starting point is the known structure of $H^\bullet(\mathbb{G}_a, k)$ ([CPSvdK, Th. 4.1]) and $H^\bullet(\mathbb{G}_a(\mathbb{F}_q), k)$ as a k-algebra: a symmetric algebra on H^1 for $p = 2$, and, for $p > 2$, the tensor product of an exterior algebra on H^1 and a symmetric algebra on H^2, with an isomorphim $H^1 \xrightarrow{\sim} H^2$ given by the Bockstein. See ([P], Lemma 13.4). The proof then goes differently for $p = 2$ and $p > 2$. For $p > 2$, choosing a basis (e_j) of V and identifying B the the group of upper triangular matrices, the proof relies on a delicate combinatorial analysis of the T-invariants of the cohomology groups $H^i(U, \chi_j^p)$, where χ_j is the character of $T = B/U$ through which it acts on e_j.

Proof of Th. 6.3. This is the final bouquet. Two important ingredients:

(a) the subgroup $A_p \xrightarrow{\sim} \mathbb{G}_a^{p-1} \subset U$ of matrices acting trivially on $V/\langle e_1\rangle$ detects cohomology classes of $V^{(1)}$ in degree $\leqslant p-1$;

(b) as the group of units $\mathcal{O}_F^\times$ is infinite, if $\chi : \mathcal{O}_F^\times \to k^\times$ is a non-trivial character, then $R\Gamma(T(\mathcal{O}_F), \chi) = 0$.

6.5. *Application to the proof of 5.3 (1).* The defect of surjectivity of the map

$$H^p_{\mathrm{dR}}([A_0/\mathrm{SL}_p(\mathcal{O}_F)]/k) \to H^p_{\mathrm{dR}}(A_0/k)^{\mathrm{SL}_p(\mathcal{O}_F)}$$

is controlled by the exact sequence

$$H^p_{\mathrm{dR}}([A_0/\mathrm{SL}_p(\mathcal{O}_F)]/k) \to H^p_{\mathrm{dR}}(A_0/k)^{\mathrm{SL}_p(\mathcal{O}_F)} \xrightarrow{\delta} H^{p+1}(\mathrm{SL}_p(\mathcal{O}_F, \tau^{\leqslant p-1}R\Gamma_{\mathrm{dR}}(A_0/k)).$$

One has to show that $\delta \neq 0$.

A general result ([P], Prop. 9.3) on de Rham cohomology of abelian varieties A_0/k acted on by a discrete group, itself a consequence of generalization of Th. 4.3 (3) to equivariant augmented derived commutative algebras, applied to the augmented algebra $R\Gamma_{\mathrm{dR}}(A_0/k)$, implies that:

(i) there is an $\mathrm{SL}_p(\mathcal{O}_F)$-equivariant isomorphism

$$\tau^{\leqslant p-1}R\Gamma_{\mathrm{dR}}(A_0/k) \xrightarrow{\sim} \oplus_{i\leqslant p-1} H^i_{\mathrm{dR}}(A_0/k)[-i];$$

(ii) δ lands in the direct summand $H^p(\mathrm{SL}(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0/k))$ and is given by the composition of

$$H^p_{\mathrm{dR}}(A_0/k)^{\mathrm{SL}_p(\mathcal{O}_F)} = \Lambda^p H^1_{\mathrm{dR}}(A_0/k)^{\mathrm{SL}_p(\mathcal{O}_F)} \xrightarrow{c} H^p(\mathrm{SL}(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0/k)^{(1)}),$$

where

$$c := \mathrm{Bock}^{p-1}(\alpha(H^1_{\mathrm{dR}}(A_0/k)))$$

(Bockstein being relative to the lift $H^1_{crys}(A_0/W_2(k))$ of $H^1_{\mathrm{dR}}(A_0/k)$), and

$$F^*_{A_0} : H^p(\mathrm{SL}(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0/k)^{(1)}) \to H^p(\mathrm{SL}(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0/k)).$$

is the Frobenius morphism. For $V = \mathbb{F}_q^p$ as in Th. 6.3, choosing $\xi \in H^1_{\mathrm{dR}}(E_0/k)$ such that $F^*_{E_0}(\xi) \neq 0$ and looking at the corresponding embedding of $k \otimes_{\mathbb{F}_p} V$ into $H^1_{\mathrm{dR}}(E_0) \otimes_k (k \otimes V) \hookrightarrow H^1_{\mathrm{dR}}(A_0/k)$, one finds that Th. 6.3 implies the non-vanishing of δ.

6.6. *Proof of 5.3 (2).* It relies on the following general result:

Proposition 6.6.1 ([P], Prop. 9.1). Let A be an abelian scheme over $W(k)$ endowed with an action of a discrete group Z.

(i) There is a Z-equivariant isomorphism

$$\tau^{\leqslant p-1} R\Gamma(A, \mathcal{O}) \xrightarrow{\sim} \oplus_{0 \leqslant i \leqslant p-1} H^i(A, \mathcal{O})[-i].$$

(ii) The extension class $e : H^p(A, \mathcal{O}) \to \tau^{\leqslant p-1} R\Gamma(A, \mathcal{O})[p+1]$ in $D_U(W(k))$ defined by $\tau^{\leqslant p} R\Gamma(A, \mathcal{O})$ is the composition of the maps

$$\Lambda^p H^1(A, \mathcal{O}) \xrightarrow{\alpha} H^1(A_0, \mathcal{O})^{(1)})[p-1] \xrightarrow{F^*_{A_0}} H^1(A_0, \mathcal{O})[p-1] \xrightarrow{\text{Bock}} H^1(A, \mathcal{O})[p]$$

and the inclusion of the direct summand $H^1(A, \mathcal{O})[p]$ in $\tau^{\leqslant p-1} R\Gamma(A, \mathcal{O})[p+1]$.

In particular, if $F^*_{A_0} : H^1(A_0, \mathcal{O}) \to H^1(A_0, \mathcal{O})$ is zero, then there is a Z-equivariant isomorphism

$$\tau^{\leqslant p} R\Gamma(A, \mathcal{O}) \xrightarrow{\sim} \oplus_{0 \leqslant i \leqslant p-1} H^i(A, \mathcal{O})[-i],$$

and, more generally, for all $0 \leqslant j \leqslant p$, a Z-equivariant isomorphism

$$\tau^{\leqslant p} R\Gamma(A, \Omega^j[-j]) \xrightarrow{\sim} \oplus_{0 \leqslant i \leqslant p} (H^i(A, \Omega^j[-j]))[-i],$$

This follows from Th. 4.3 (3) applied to the derived commutative algebra $R\Gamma(A, \mathcal{O})$ in $D_Z(W(k))$, augmented by the map $e^* : R\Gamma(A, \mathcal{O}) \to W(k)$ deduced from the identity section, as $F^*_{A_0}$ can be identified with the Frobenius morphism of the derived commutative algebra $R\Gamma(A_0, \mathcal{O})$. The last assertion comes from the fact that $\Omega^1_{A/W(k)} = \pi^* e^* \Omega^1_{A/W(k)}$, where $\pi : A \to \mathrm{Spec}(W(k))$.

Then 5.3 (2) follows as A_0 is a sum of $2p$ copies of the supersingular elliptic curve E_0, hence $F^*_{A_0}$ is zero on $H^1(A_0, \mathcal{O})$.

7. The showdown Hodge vs dR: end of proof of the main theorem

The rough story is that in a lifted situation acted on by a discrete group, where the $\tau^{\leqslant p}$-truncation of Hodge cohomology is equivariantly decomposable but that of de Rham cohomology is not, then Hodge and de Rham shoot at each other, and the result is the non-vanishing of $d^{0,p}_{p+1}$. More precisely:

Proposition 7.1 ([P], Lemma 10.6). Let X_1 be a smooth scheme over $W_2(k)$ acted on by a discrete group Z, and let $X_0 = X_1 \otimes k$. Assume that:

(i) for all $0 \leqslant j \leqslant p$, there is a Z-equivariant isomorphism

$$\tau^{\leqslant p} R\Gamma(X_0, \Omega^j[-j]) \xrightarrow{\sim} \oplus_{0 \leqslant i \leqslant p}(H^i(X_0, \Omega^j[-j])[-i];$$

(ii) The map

$$H^p_{\mathrm{dR}}([X_0/Z]/k) \to H^p_{\mathrm{dR}}(X_0/k)^Z$$

is not surjective.

Then, in the conjugate spectral sequence for the stack $[X_0/Z]$, the differential

$$d^{0,p}_{p+1} : H^0([X_0/Z]^{(1)}, L\Omega^p_{[X_0/Z]^{(1)}/k}) \to H^{p+1}([X_0/Z]^{(1)}, \mathcal{O})$$

(see 3.2) does not vanish.

The conclusion is equivalent to saying that the map

$$H^p_{\mathrm{dR}}([X_0/Z]^{(1)}/k) \to H^0(X_0^{(1)}, \Omega^p)^Z$$

induced by $\mathrm{Fil}^{\mathrm{conj}}_p \to \mathrm{gr}^{\mathrm{conj}}_{\mathrm{p}}$ is not surjective. The proof is formal.

7.2. *End of proof of main theorem (Th. 1.3.a).* As explained in 5.2, it is enough to prove Th. 1.3.b, i.e., that the differential $d^{0,p}_{p+1}$ for $[A_0/\mathrm{SL}_p(\mathcal{O}_F)]$ is non-zero. We check conditions (i) and (ii) for $X = A$ and $Z = \mathrm{SL}_p(\mathcal{O}_F)$. Condition (i) is satisfied by 5.3 (2), proved in 6.6. Condition (ii) is satisfied by 5.3 (1), proved in 6.5. That finishes the proof of Th. 1.3.a, hence of Th. 1.3, modulo the approximation results in 2.3, which we will omit.

8. Complement 1: Sen operators and Kodaira-Spencer classes

8.1. As observed after (4.2.1), for X_0/k smooth and liftable to $X_1/W_2(k)$, if $\tau^{\leqslant p}\mathrm{dR}_{X_0/k}$ is not decomposable, then the Sen class $c_{X_1,p}$ is not zero. This is the case for the smooth projective schemes X_0/k (of dimension $p+1$) constructed by Petrov (2.3). For X_0/k of dimension $< p$ (smooth and liftable to $W_2(k)$), $\mathrm{dR}_{X_0/k}$ is decomposable, and we have (trivially) $c_{X_1,p} = 0$, as $\Omega^p_{X_0/k} = 0$. For X_0/k of dimension p, it is still true that $\mathrm{dR}_{X_0/k}$ is decomposable (see (1.2.3)), but *a priori* it might happen that $c_{X_1,p}$ does not vanish. This is indeed the case, as Petrov shows. In fact, the examples he constructs are, in contrast to the bizarre looking BG, quite familiar varieties. They arise from a beautiful relation that Petrov discovered, for reductions mod p of smooth pencils of relative dimension $p-1$ over $W_2(k)$, between the Sen class $c_{X_1,p}$, the Kodaira-Spencer class of the pencil, and the obstruction to lifting the Frobenius of the base curve.

8.2. Let

$$f : X_1 \to Y_1$$

be a smooth morphism between smooth $W_2(k)$-schemes, with $\dim(Y_1) = 1$, $\dim(X_1/Y_1) = p - 1$. As usual, denote by the subscript $(-)_0$ the reduction mod p. Let

$$\mathrm{KS}_{f_0} \in \mathrm{Hom}(\Omega^1_{X_0/Y_0}, f_0^*\Omega^1_{Y_0/k}[1]) = \mathrm{Hom}(T_{Y_0/k}, Rf_{0*}T_{X_0/Y_0})[1])$$

denote the extension given by the short exact sequence

$$0 \to f_0^*\Omega^1_{Y_0/k} \to \Omega^1_{X_0} \to \Omega^1_{X_0/Y_0} \to 0,$$

and let

$$\mathrm{ks}_{f_0} : T_{Y_0/k} \to R^1 f_{0*} T_{X_0/Y_0}$$

the morphism deduced by applying H^0. Applying $(-)^{\otimes p-1}$ to it and composing with the cup-product produces a map

$$\mathrm{ks}_{f_0}^{p-1} : \mathcal{O}_{Y_0} \to T_{Y_0/k}^{\otimes 1-p} \otimes R^{p-1} f_{0*} \Lambda^{p-1} T_{X_0/Y_0}.$$

On the other hand, the obstruction ob_{F,Y_1} to lifting the absolute Frobenius of Y_0 is a map

$$\mathrm{ob}_{F,Y_1} : \mathcal{O}_{Y_0} \to F_0^* T_{Y_0/k}[1] = T_{Y_0/k}^{\otimes p}[1].$$

Finally, the (pull-back by $(F_0^*)^{-1}$ of) the Sen class of Th. 4.2 is a class

$$c_{X_1,p} \in H^p(X_0, \Lambda^p T_{X_0/k}) = H^1(Y_0, T_{Y_0/k} \otimes R^{p-1} f_{0*} \Lambda^{p-1} T_{X_0/Y_0}).$$

Petrov's relation is:

Theorem 8.3 ([P], Th. 8.1). *Up to multiplication by an element of $\mathbb{F}_p^*$, we have*

$$c_{X_1,p} = \mathrm{ob}_{F,X_1} \cdot \mathrm{ks}_{f_0}^{p-1}. \tag{8.3.0}$$

Proof. Formula (4.2.0) gives

$$c_{X_1,p} = \mathrm{ob}_{F,X_1} \cdot \alpha(\Omega^1_{X_0/k}), \tag{8.3.1}$$

where

$$\alpha(\Omega^1_{X_0/k}) \in H^{p-1}(X_0, (F_{X_0}^* T_{X_0/k})^\vee \otimes \Lambda^p T_{X_0/k})$$

is the class (3.4.5) for $\Omega^1_{X_0/k}$. To identify (up to a unit) the right hand sides of (8.3.0) and (8.3.1), Petrov uses a general formula ([P], Lemma 8.2) for the class $\alpha(E)$ when E is given as an extension of vector bundles $0 \to L \to E \to$

$E' \to 0$, with L a line bundle. For $p = 2$ its proof is elementary, but for $p > 2$, Petrov deduces it from Th. 6.1 and additional technical results used in the proof of Th. 6.2.

Corollary 8.4 ([P] Prop. 8.4). Assume $k = \overline{\mathbb{F}}_p$. There exists a projective smooth scheme $X/W(k)$, of relative dimension p, such that the Sen class $c_{X,p} \in H^p(X_0, \Lambda^p T_{X_0/k})$ is non-zero.

Proof. Such a scheme is defined as

$$X := (S/Y)^{p-1} (= S \times_Y \cdots \times_Y S) \ (p-1 \text{ factors})$$

for suitable choices of:

- $Y/W(k)$: a geometrically connected smooth projective curve
- $h : S \to Y$: a smooth projective morphism of relative dimension 1,

such that the Kodaira-Spencer map ks_{h_0} is an injection of vector bundles, whose cokernel is a sum of line bundles of degree $< (1/p-1)deg(\omega_{Y_0})$ (in particular, Y_0 and the fibers of h have genus $\geqslant 2$). The scheme Y is constructed as a complete intersection of ample divisors in an appropriate compactification of M_g for $g \geqslant \max(4, \frac{p}{3}+1)$.

9. Complement 2: de Rham decomposability of quasi-F-split varieties

Theorem 9.1 (A. Petrov, Oct. 22). Let Y/k be a smooth scheme. Denote by F the (p-linear) *absolute* Frobenius endomorphism of Y. Consider the ring homomorphism

$$\sigma_Y : \mathcal{O}_Y \to F_* W(\mathcal{O}_Y)/p$$

sending a to $[a]^p$, where $[a]$ denotes the Teichmüller representative. Then

$$F_* W(\mathcal{O}_Y)/p \otimes_{\mathcal{O}_{Y,\sigma}} F_* \Omega^\bullet_{Y/k}$$

is (non-canonically) decomposable, i.e., isomorphic to the direct sum of its $H^i[-i]$.

Remark 9.2. As $F : \mathcal{O}_Y \to \mathcal{O}_Y$ factors through σ_Y, $F^* F_* \Omega^\bullet_{Y/k}$ is decomposable as well. This corollary had been observed independently by B. Bhatt and V. Vologodsky.

Proof of Th. 9.1. Consider the *relative Hodge-Tate stack*

$$\pi^{\mathrm{HT}} : \mathrm{WCart}^{\mathrm{HT}}_Y \to Y$$

(relative to the crystalline prism $(W(k),(p))$). For R a *reduced* k-algebra, $\mathrm{WCart}_Y^{\mathrm{HT}}(R) = Y(W(R)/p)$ (where $W(R)/p$ is the discrete groupoid given by the natural action of $W(R)^*$). By ([BL22b], Prop. 5.12), it is a gerbe banded by $T^\sharp_{Y/k}$, and by ([BL22b], 6.4), we have

$$\mathrm{dR}_{Y/k} = R\pi_*^{HT}\mathcal{O}.$$

As the (formal) scheme $W(Y)_k$ lifts to the δ-scheme $W(Y)$, the proof of ([BL22b], Prop. 5.12) shows that the pull-back of π^{HT} by the map

$$\sigma_Y : W(Y)_k \to Y$$

corresponding to the ring homomorphism σ_Y above admits a section. Therefore the above gerbe is trivial:

$$\mathrm{WCart}_Y^{\mathrm{HT}} \xrightarrow{\sim} BT^\sharp_{Y/k}.$$

Viewing it as the affine stack $T^\sharp_{Y/k}[1]$, we find, by Quillen's décalage formula,

$$R\pi_*^{HT}\mathcal{O} \xrightarrow{\sim} \Gamma^\bullet(T^\vee_{Y/k}[-1]) = \Lambda^\bullet(\Omega^1_{Y/k})[-\bullet] = \oplus_i \Omega^i_{Y/k}[-i].$$

Recall the following definition, due to Yobuko:

Definition 9.3 ([Y],4.1). Let m be a positive integer. A projective, smooth scheme Y/k is called m *quasi-F-split* if the map induced by σ_Y

$$\sigma_{Y,m} : \mathcal{O}_Y \to F_*W_m(\mathcal{O}_Y)/p$$

splits as $\mathcal{O}_Y$-modules.

For $m = 1$, Y is called *F-split*, this is an earlier notion, due to Mehta-Ramanathan.

Corollary (A. Petrov). If Y is m-quasi-F-split for some m, then $F_*\Omega^\bullet_{Y/k}$ is decomposable.

Proof. By Th. 9.1, The pull-back of $\mathrm{dR}_{Y/k}$ by $\sigma_{Y,m}$ is decomposable. The conclusion follows from the projection formula $(\sigma_{Y,m})_*(\sigma_{Y,m})^*M = (\sigma_{Y,m})_*\mathcal{O} \otimes M$, using an $\mathcal{O}_Y$-linear retraction of $\sigma_{Y,m}$.

This is the case, for example, if Y is a Calabi-Yau variety of finite height. Indeed, by Yobuko ([Y], 4.5), if $h \in \mathbf{N} \cup \infty$ is the height of the Artin-Mazur formal group Φ^n ($n = \dim(Y)$), then $h = h^s$, where h^s is the minimum $m > 0$ for which Y is m quasi-F-split, or ∞ it no such m exists.

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