6. DE RHAM-WITT COMPLEX AND LOG CRYSTALLINE COHOMOLOGY

$\alpha : L \to k$: a fine log str. on $s = \mathrm{Spec}(k)$

$W_n(L)$: Teichmüller lifting of L to $W_n(s)$:

$W_n(L) = L \oplus \mathrm{Ker}(W_n(k)^* \to k^*)$

$L \to W_n(k) : a \mapsto [\alpha(a)]$

Example : $L = (\mathbb{N} \to k, 1 \mapsto 0)^a$ (standard log point)

$W_n(L) = (\mathbb{N} \to W_n(k), 1 \to 0)^a$

$(X, M)/(s, L)$: fine log scheme,

log smooth and Cartier type $/(s, L)$

Log crystalline site

$\text{Crys}((X, M)/(W_n(s), W_n(L))$

• objects : log DP thickenings (U, T, N, δ),

$U \to X$ étale, $i : (U, M) \to (T, N)$ exact ($\Leftrightarrow$ strict)

closed immersion, $T/W_n(s)$;

with DP δ on $\text{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$

compatible with can. DP on $pW_n(k)$

- morphisms : obvious
- covering families : $(U_i, T_i)_{i\in I} \to (U, T)$:

$(T_i \to T)_{i\in I}$ étale cover, $U_i = U \times_T T_i$

Log crystalline topos

$((X, M)/(W_n(s), W_n(L))_{\mathrm{crys}}$ (or $(X/W_n)_{\mathrm{logcrys}}$)

structural sheaf $\mathcal{O}_{X/W_n} : (U, T) \to \mathcal{O}_T$

canonical morphism

$u = u_{(X,M)/(W_n(s)W_n(L))} : (X/W_n)_{\mathrm{logcrys}} \to X_{\text{ét}}$

Calculation of $Ru_*\mathcal{O}_{X/W_n}$

If $i : X \to Z$ closed immersion,

$Z/(W_n, W_n(L)$ log smooth

DP-envelope $j = X \to D_n$, $D_n \to Z$

exact closed immersion,

s. t. $i =$ composite $X \to D_n \to Z$,

Ker$j =$ DP-ideal,

universal for these properties

(uses Kato's local exactification :

(local) factorization of $i : X \to Y \to Z$,

$X \to Y$ exact closed immersion, $Y \to Z$ log étale.)

Then :

$$Ru_* \mathcal{O}_{X/W_n} = \mathcal{O}_{\mathcal{D}} \otimes \Omega^{\cdot}_{Z/(W_n, W_n(L))}.$$

NB. $i : X \to Z$ exists only locally

(even if X/k projective),

general case : cohomological descent

Remark

Log crystalline site, topos,

calculation of $Ru_*\mathcal{O}$ generalize to

$(X, M) \to (S, L, I, \gamma)$ map of fine log schemes,

p nilpotent on S, γ DP on I extendable to X

back to previous case :

Log crystalline (Hyodo-Kato) cohomology :

$H^m((X,M)/(W_n,W_n(L)) :=$

$H^m(((X,M)/(W_n,W_n(L))_{\text{logcrys}}, \mathcal{O}_{X/W_n})$

$= H^m(X, Ru_*\mathcal{O}_{X/W_n})$

$(= H^m(X, \mathcal{O}_D \otimes \Omega^{\cdot}_{Z/(W_n,W_n(L))})$

for $i : X \to Z$)

comes equipped with

- Frobenius operator

$$\varphi : H^m((X,M)/(W_n, W_n(L)) \to H^m((X,M)/(W_n, W_n(L))$$

defined by (Frobenius on schemes, p on monoids)

Cartier type $\Rightarrow$

$\varphi = \sigma$-linear isogeny (Berthelot-Ogus, Hyodo-Kato)

$\exists\ \sigma^{-1}$-linear ψ, $\varphi\psi = \psi\varphi = p^r$

$(r = \dim(X) = \operatorname{rk} \Omega^1_{(X,M)/(s,L)})$

• (for (s, L) = standard log point)

monodromy operator

$N : H^m((X,M)/(W_n, W_n(L)) \to H^m((X,M)/(W_n, W_n(L))$

N = residue at $t = 0$ of Gauss-Manin connection on $H^m((X,M)/(W_n < t >, \text{can}))$ rel. to W_n (with trivial log str.)

basic relation :

$$N\varphi = p\varphi N$$

for X/k proper, get (φ, N)-module structure on

$K_0 \otimes H^m((X, M)/(W, W(L)))$,

where

$H^m((X, M)/(W, W(L))) = \text{proj.lim.}\, H^m((X, M)/(W_n, W_n(L))$

a f. g. W-module,

N is nilpotent

Log de Rham-Witt complex

imitate Katz-I-Raynaud's re-construction :

change notations : $Y = (Y, M)$ log smooth, Cartier type $/(k, L)$

$W_n\omega^i_Y = \sigma^n_* R^i u_{(Y,M)/(W_n, W_n(L)^*}\mathcal{O})$,

d : Bockstein

F : given by restriction from W_{n+1} to W_n

V : given by multiplication by p

R : uses Cartier type, variant of Mazur-Ogus gauges th.

get pro-complex

$$W_{\cdot}\omega_{Y}^{\cdot},$$

with operators F, V satisfying standard formulas ($FV = VF = p$, $FdV = d$, etc.)

new feature : $\exists$ can. homomorphism

$$\text{dlog} : M \to W_n\omega^1_Y,$$

$(a \mapsto \text{dlog}\,\tilde{a} \in \mathcal{H}^1(\mathcal{O}_D \otimes \Omega^\cdot_{(Z,M_Z)/(W_n,W_n(L))})$, $\tilde{a} \in M_Z \mapsto a \in M$,

$Y \subset (Z, M_Z)/(W_n, W_n(L))$ log smooth embedding,

$D =$ pd-envelope)

satisfying

$F\,\mathrm{dlog} = \mathrm{dlog}$, $[\alpha(a)]\,\mathrm{dlog}\,a = d[\alpha(a)]$

$W_n\omega^0_Y = W_n\mathcal{O}_Y$, and

$W_n\omega^*_Y$ generated as $W_n(\mathcal{O}_Y)$-algebra by
$dW_n\mathcal{O}_Y$, $\mathrm{dlog}\,M^{gp}$

comparison th. generalizes :

$$Ru_{(Y,M)/(W_n,W_n(L)*}\mathcal{O} \xrightarrow{\sim} W_n\omega^{\cdot}_Y$$

$$R\Gamma((Y,M)/(W_n,W_n(L)) \xrightarrow{\sim} R\Gamma(Y,W_n\omega^{\cdot}_Y)$$

slope spectral sequence, higher Cartier isom., etc. generalize, too

7. THE HYODO-KATO ISOMORPHISM

$S = \mathrm{Spec}\, A$, A complete dvr, char. $(0, p)$,

$K = \mathrm{Frac}(A)$, $k = A/\pi A$ residue field, perfect

$K_0 = \mathrm{Frac}(W(k))$

X/S semi-stable reduction

$Y = X \otimes k$ the special fiber

goal : for X/S proper, define (K_0, φ, N)-structure on $H^m_{dR}(X_K/K)$
using log crystalline cohomology of Y

- case X/S smooth : Berthelot-Ogus isomorphism

$$K \otimes_W H^m(Y/W) \xrightarrow{\sim} H^m_{dR}(X_K/K)$$

$\Rightarrow$ (K_0, φ)-structure ($N = 0$)

- general case : use log str.

M_X : can. log str. on X, induced by special fiber

$(M_X = \mathcal{O}_X \cap j_* \mathcal{O}^*_{X_K}$, $j : X_K \subset X)$

M_Y : induced log str. on Y

L : $(\mathbb{N} \to k, 1 \mapsto 0)^a$: log str. on $s = \operatorname{Spec} k$,

induced by standard log str. on S, $(\mathbb{N} \to A, 1 \to \pi)^a$

Then $(s, L) =$ standard log point, and

$$(Y, M_Y) \to (s, L)$$

of semi-stable type, in particular,

log smooth and Cartier type

Put $A_n = A \otimes \mathbb{Z}/p^n\mathbb{Z} = A \otimes_W W_n$, $S_n = \mathrm{Spec}\, A_n$

$X_n = X \otimes \mathbb{Z}/p^n\mathbb{Z} = X \otimes_W W_n$, with induced log str.

M_{S_n} :

$$\begin{array}{ccccccc} Y & \longrightarrow & X_1 & \longrightarrow & X_n & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S_1 & \longrightarrow & S_n & \longrightarrow & S \end{array}$$

Note : M_{S_n} associated with $\mathbf{N} \to A_n$, $1 \mapsto \pi$, while $W_n(L)$ associated with $\mathbf{N} \to W_n$, $1 \mapsto 0$.

Consider projective systems (a) and (b) of $D^+(X_n, A_n) = D^+(Y, A_n)$:

(a) = de Rham-Witt

$$A_n \otimes^L_{W_n} Ru_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O}$$
$$\xrightarrow{\sim} A_n \otimes^L_{W_n} W_n\omega^{\cdot}_{Y/W_n}$$

(b) = de Rham

$$\omega^{\cdot}_{X_n/A_n} := \Omega^{\cdot}_{(X_n,M_{X_n})/(S_n,M_{S_n})}$$

THEOREM 7.1 (Hyodo-Kato)

There exists a canonical isomorphism

$$\rho_\pi : \mathbb{Q} \otimes (A. \otimes^L_{W.} W.\omega^{\cdot}_{Y/W.}) \xrightarrow{\sim} \mathbb{Q} \otimes \omega^{\cdot}_{X./S.}$$

in $\mathbb{Q} \otimes$ proj.sys. $D^+(Y, A_n)$

(for additive cat. C, $\mathrm{Hom}_{\mathbb{Q}\otimes C} = \mathbb{Q} \otimes \mathrm{Hom}_{\mathcal{C}}$,
$\mathbb{Q} \otimes K$: image of K in $\mathbb{Q} \otimes \mathcal{C}$)

COROLLARY 7.2

For X/S proper (and semistable), ρ_π induces an isomorphism :

$$\rho_\pi : K \otimes_W H^m((Y, M_Y)/(W, W(L)) \xrightarrow{\sim} H^m_{dR}(X_K/K)$$

(gives (K_0, φ, N) structure on $H^m_{dR}(X_K/K)$)

Remarks

(1) 7.1 valid for X/S log smooth, Cartier type

(2) ρ_π depends on π :

$\rho_{\pi u} = \rho_\pi \exp(\log(u)N)$, $u \in A^*$

(3) if X/S smooth, then :

$$H^m(Y/W) \xrightarrow{\sim} H^m((Y, M_Y)/(W, W(L)),$$

ρ_π = Berthelot-Ogus isomorphism, independent of π.

Highlights of proof

(Rough) idea : (a), (b) come from

F-crystal on $W < t >$ with log pole at $t = 0$

use Frobenius ($t \mapsto t^p$), an isogeny, contracting the disc, to connect :

(a) = log crys side = fiber at 0,

(b) = dR side = general fiber

Main ingredients :

- Berthelot-Ogus's method, to reduce to rigidity th. $/W_n < t >$ mod bounded p-torsion

- de Rham-Witt, lifting of higher Cartier isomorphisms to construct rigidification

- Reduction to rigidity th.

Embed $\operatorname{Spec} A$ into $\operatorname{Spec} W[t]$, $t \to \pi$,
with log str. $\mathbb{N} \to W[t], 1 \to t$

$\operatorname{Spec} R_n =$ dp-envelope of $\operatorname{Spec} A_n$ in $\operatorname{Spec} W_n[t]$
$=$ dp-envelope of $S_1 = \operatorname{Spec} A_1$ in $\operatorname{Spec} W_n[t]$:
$A_1 \leftarrow A_n \leftarrow R_n \leftarrow W_n[t]$

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_n & & & & \\ \downarrow & & \downarrow & & & & \\ S_1 & \longrightarrow & \mathrm{Spec} A_n & \longrightarrow & \mathrm{Spec} R_n & \longrightarrow & \mathrm{Spec} W_n[t] \end{array}$$

Base change in crystalline cohomology $\Rightarrow$

$$\omega^{\cdot}_{X_n/A_n} (= Ru_{X_1/A_n *}\mathcal{O}) = A_n \otimes^{L}_{R_n} Ru_{X_1/R_n *}\mathcal{O}$$

$$\omega^{\cdot}_{X_n/A_n}(= Ru_{X_1/A_n *}\mathcal{O}) = A_n \otimes^L_{R_n} Ru_{X_1/R_n *}\mathcal{O}$$

recall goal : relate $\omega^{\cdot}_{X_n/A_n}$ to $A_n \otimes_{W_n} W_n\omega^{\cdot}_{Y/k}$

done in 2 steps

- Step 1 : relate $Ru_{X_1/R_n *}\mathcal{O}$ to $Ru_{Y/W_n<t>*}\mathcal{O}$

using Frobenius isogeny (Berthelot-Ogus's method)

- Step 2 : relate $Ru_{Y/W_n<t>*}\mathcal{O}$ to $Ru_{Y/(W_n, W_n(L))*}\mathcal{O}$

$(= W_n\omega^{\cdot}_{Y/k})$ using Hyodo-Kato rigidity theorem

Step 1 : relate $Ru_{X_1/R_n *}\mathcal{O}$ to $Ru_{Y/W_n<t>*}\mathcal{O}$

recall : $W_n < t >$ = dp-envelope of W_n in $W_n[t]$

= dp-envelope of k in $W_n[t]$,

R_n = dp-envelope of A_1 in $W_n[t]$

$\Rightarrow$ gets diagram (of log schemes),

with action of Frobenius ($t \mapsto t^p$ on $W_n[t]$)

$$\begin{array}{ccccc} Y & \hookrightarrow & X_1 & & \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}\,k & \hookrightarrow & \mathrm{Spec}\,A_1 & & \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}\,W_n < t > & \hookrightarrow & \mathrm{Spec}\,R_n & \longrightarrow & \mathrm{Spec}\,W_n[t] \end{array}$$

in particular, gets :

$$\varphi : Ru_{X_1/R_n *}\mathcal{O} \to Ru_{X_1/R_n *}\mathcal{O},$$

which is an isogeny ($\Leftarrow$Cartier type) :

$\exists\psi$, $\psi\varphi = \varphi\psi = p^r$, $r = \dim Y$

For $N \geq \log_p e$, $e = [K : K_0]$, $(\pi A)^{p^N}) \subset (pA) \Rightarrow$
$F^N : X_1/S_1 \to X_1/S_1$ factors through Y/k :

$$\begin{array}{ccccc}
X_1 & \longrightarrow & Y & \hookrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec} A_1 & \longrightarrow & \mathrm{Spec} k & \hookrightarrow & \mathrm{Spec} A_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec} R_n & \longrightarrow & \mathrm{Spec} W_n < t > & \hookrightarrow & \mathrm{Spec} R_n
\end{array}$$

where composite horizontal maps = F^N

φ^N = isogeny $\Rightarrow$

$$\varphi^N : \mathbf{Q} \otimes ((R., F^N) \otimes^L_{W.} Ru_{Y/W.<t>*}\mathcal{O}) \xrightarrow{\sim} \mathbf{Q} \otimes Ru_{X_1/R.*}\mathcal{O}$$

completes step 1 : relate $Ru_{X_1/R_n*}\mathcal{O}$ to $Ru_{Y/W_n<t>*}\mathcal{O}$

Step 2 : relate $Ru_{Y/W_n<t>*}\mathcal{O}$ to $Ru_{Y/(W_n,W_n(L))*}\mathcal{O}$

$$\begin{array}{ccccc} (Y, M_Y) & & & & \\ \downarrow & & & & \\ (\mathrm{Spec}\,k, L) & \hookrightarrow & (\mathrm{Spec}\,W_n, W_n(L)) & \hookrightarrow & (\mathrm{Spec}\,W_n < t >, \mathrm{can}) \end{array}$$

gives canonical Frobenius equivariant map

$$Ru_{Y/(W.<t>,can)*}\mathcal{O} \to Ru_{Y/(W.,W.(L))*}\mathcal{O}$$

hence

$$(*) \qquad \mathbf{Q}\otimes Ru_{Y/(W.<t>,can)*}\mathcal{O} \to \mathbf{Q}\otimes Ru_{Y/(W.,W.(L))*}\mathcal{O}$$

THEOREM 7.3 (Hyodo-Kato rigidity theorem)

(i) (*) admits a unique Frobenius equivariant section

$$s : \mathbf{Q}\otimes Ru_{Y/(W.,W.(L))*}\mathcal{O} \to \mathbf{Q}\otimes Ru_{Y/(W.<t>,can)*}\mathcal{O}$$

(ii) The map

$$h : \mathbb{Q}\otimes(W. < t > \otimes_{W.} Ru_{(Y/(W.,W.(L))*}\mathcal{O}) \to \mathbb{Q}\otimes Ru_{(Y/W.<t>)*}\mathcal{O}$$

defined by s is an isomorphism.

7.3 $\Rightarrow$ 7.1

Recall : φ^N = isogeny $\Rightarrow$

$$\varphi^N : \mathbf{Q} \otimes ((R_\cdot, F^N) \otimes^L_{W_\cdot} Ru_{Y/W_\cdot<t>*}\mathcal{O}) \xrightarrow{\sim} \mathbf{Q} \otimes Ru_{X_1/R_\cdot *}\mathcal{O}$$

Similarly :

$$\varphi^{-N} : \mathbf{Q} \otimes (R_\cdot \otimes^L_{W_\cdot} Ru_{(Y/(W_\cdot, W_\cdot(L))*}\mathcal{O})$$

$$\xrightarrow{\sim} \mathbf{Q} \otimes ((R_\cdot, F^N) \otimes^L_{W_\cdot} Ru_{(Y/(W_\cdot, W_\cdot(L))*}\mathcal{O})$$

Then :

$$h_\pi = \varphi^N h \varphi^{-N} :$$

$$\mathbb{Q} \otimes (R. \otimes^L_{W.} Ru_{(Y/(W.,W.(L)))*}\mathcal{O}) \xrightarrow{\sim} \mathbb{Q} \otimes Ru_{X_1/R.*}\mathcal{O}$$

As

$$A_n \otimes^L_{R_n} Ru_{X_1/R_n*}\mathcal{O} = \omega^\cdot_{X_n/A_n},$$

$$\rho_\pi = \mathbb{Q} \otimes (A. \otimes^L_{R.} h_\pi) : \mathbf{Q} \otimes W.\omega^\cdot_Y \xrightarrow{\sim} \mathbf{Q} \otimes \omega^\cdot_{X./A.}$$

(notations h_π, ρ_π because π used in $W[t] \to A$, $t \mapsto \pi$)

Proof of Hyodo-Kato rigidity th.

Proof of (i) :

base change in crystalline cohomology $\Rightarrow$:

$$Ru_{Y/(W_n,W_n(L))*}\mathcal{O} = W_n \otimes^L_{W_n<t>} Ru_{Y/(W_n<t>,can)*}\mathcal{O}$$

$\Rightarrow$ φ-equivariant triangle ($I_n = \mathrm{Ker}\, W_n < t > \to W_n$)

$$I_n \otimes^L_{W_n<t>} Ru_{Y/(W_n<t>,can)*}\mathcal{O} \to$$

$$Ru_{Y/W_n<t>,can))*}\mathcal{O} \to Ru_{Y/(W_n,W_n(L))*}\mathcal{O}) \to$$

Idea :

powers of φ close to zero on left vertex,

isogeny on right vertex $\Rightarrow$

exists unique φ-equivariant section of

$$\mathbf{Q} \otimes Ru_{Y/W.<t>,can))*}\mathcal{O} \to \mathbf{Q} \otimes Ru_{Y/(W.,W.(L))*}\mathcal{O})$$

More precisely :

- $\varphi\psi = \psi\varphi = p^r$ on right vertex ($\Leftarrow$ Cartier type)
- $\varphi^i(I_n) \subset (p^i)!I_n$ (use DP on I_n)

• Put : $B := Ru_{Y/W.<t>,can))*}\mathcal{O}$, $C := Ru_{Y/(W.,W.(L))*}\mathcal{O})$

$\varepsilon : B \to C$

Key point (Hyodo-Kato) :

p^{2cr+d} kills Ker and Coker of

$\mathrm{Hom}_{\varphi}(C, \varepsilon) : \mathrm{Hom}_{\varphi}(C, B) \to \mathrm{Hom}_{\varphi}(C, C)$

where $c \in \mathbf{N}$, $d \in \mathbf{N}$ such that

$p^{rc+1} | p^c!$,

$p^{r(i+1)} | (p^i)! p^d \ \forall i \geq 0$.

(Note : $v_p(p^i!) = (p^i - 1)/(p - 1)$)

Proof of (ii) :

relies on existence of

lifted higher Cartier isomorphism

Recall :

if $(Y, M_Y)/(k, L) \subset (Z., M.)/(W.[t], 1 \mapsto t)$

= compatible system of log smooth embeddings,

get :

$$R^q u_{Y/W_n<t>*}\mathcal{O} = \mathcal{H}^q(\mathcal{O}_{D_n} \otimes \omega^{\cdot}_{Z_n/W_n<t>})$$

(D_n = dp-envelope of Y in Z_n)

THEOREM 7.4 (Hyodo-Kato)

(a) $\exists$ unique homomorphism of graded algebras

$$c_n : W_n\omega_Y^* \longrightarrow R^* u_{Y/W_n<t>*}\mathcal{O},$$

$(a_0, \cdots, a_{n-1}) \mapsto \Sigma_{0\le i\le n-1}\, p^i \tilde{a}_i^{p^{n-i}}$, in degree 0

$d(a_0, \cdots, a_{n-1}) \mapsto \Sigma_{0\le i\le n-1}\, \tilde{a}_i^{p^{n-i}-1} d\tilde{a}_i$, $\mathrm{dlog}\, b \mapsto \mathrm{dlog}\, \tilde{b}$, in degree 1

($\tilde{a}_i \in \mathcal{O}_{D_n}$ (resp. $\tilde{b} \in M_{Z_n}$) lifts a_i (resp. b)).

(b) The composition of c_n with the inverse higher Cartier isom. C^n

$$R^q u_{Y/(W_n,W_n(L)^*}\mathcal{O} \xrightarrow{C^n} W_n\omega^q_Y \xrightarrow{c_n} R^q u_{Y/W_n<t>^*}\mathcal{O}$$

is a φ-equivariant section of the can. projection, and induces a $W_n < t >$-linear isomorphism

$$h^q_n : W_n < t > \otimes_{W_n} R^q u_{Y/(W_n,W_n(L)^*}\mathcal{O} \xrightarrow{\sim} R^q u_{Y/W_n<t>^*}\mathcal{O}.$$

Remark :

Olsson (Astérisque 316, 4.6.9, 9.4.1é) has shown that

$$c_n : W_n\omega_Y^* \to R^* u_{Y/W_n<t>*}\mathcal{O},$$

is a map of dga (for a Bockstein d on rhs), and recovered $h_n^q =$ iso.

Uses :

- Langer-Zink de Rham-Witt complex for alg. stacks
- dictionary (discovered by Lafforgue) :

(log scheme) = (scheme over a certain alg. stack)

End of proof of (ii) :

Recall :

$$h : \mathbb{Q}\otimes(W. < t > \otimes_{W.} Ru_{(Y/(W.,W.(L))^*}\mathcal{O}) \to \mathbb{Q}\otimes Ru_{(Y/W.<t>)^*}\mathcal{O}$$

$$h_n^q : W_n < t > \otimes_{W_n} R^q u_{Y/(W_n,W_n(L)^*}\mathcal{O} \xrightarrow{\sim} R^q u_{Y/W_n<t>^*}\mathcal{O}.$$

One shows :

$$\mathcal{H}^q(h) = \mathbb{Q} \otimes h_.^q,$$

(uniqueness of φ-equivariant sections)

Proof of 7.4 :

uses explicit presentations of $W_n\omega^*_Y$ by generators and relations, e. g. as a quotient of $\omega^\cdot_{W_n(Y)/(W_n,W_n(L))}$ and structure of Ker$W_{n+1}\omega^q_Y \to W_n\omega^q_Y$

Remarks

Ogus (1995) : alternate proof, variants and generalizations of 7.2 for log F-crystals

But : 7.1, 7.3 crucial for crystalline interpretation of

$$B_{\mathrm{st}}^{+} \otimes_W H^m((Y, M_Y)/(W, W(L))$$

via Künneth formulas at finite levels :

$$H^0(\overline{S}_1/R_n) \otimes_{R_n} H^m(X_1/R_n) \xrightarrow{\sim} H^m(\overline{X}_1/R_n),$$

where

lhs $= H^0(\overline{S}_1/R_n) \otimes_{W_n} H^m(Y/(W_n, W_n(L))$

mod bounded p-torsion by Hyodo-Kato 7.1

($\overline{X}_1 = X_1 \otimes \mathcal{O}_{\overline{K}}$, etc.,

can. log. str. on $\overline{S}_1$, $\overline{X}_1$, Y, R_n)

and construction of comparison map

$$B_{\mathsf{st}} \otimes H^m(X_{\overline{K}}, \mathbb{Q}_{p)}) \to B_{\mathsf{st}} \otimes H^m((Y, M_Y)/(W, W(L))$$