Recent developments on zeta values over finite fields,
after S. Mondal

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Introduction

Let p be a prime number. Let X be a proper and smooth scheme over the finite field $k = \mathbb{F}_q$ with $q = p^a$ elements. Let $n \in \mathbb{Z}$. Since Artin–Tate's pioneering work in the early 1960s [62], the local behavior of the zeta function $\zeta(X, s)$ near n has been extensively studied up to now. Recall that $\zeta(X, s) = Z(X, q^{-s})$, with $Z(X, t)$ a rational function of t (1.1). Let ρ_n be the order of the pole of $\zeta(X, s)$ at $s = n$, and define $Z^*(X, q^{-n})$ by

$$Z(X, t) = (1 - q^n t)^{-\rho_n} Z^*(X, t)$$

with $Z^*(X, q^{-n}) \neq 0$. Various cohomological formulas for the *special value* $Z^*(X, q^{-n})$ have been given, all depending on the semisimplicity conjecture for the eigenvalue q^n of Frobenius acting on the ℓ-adic cohomology group $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell)$ ($\ell \neq p$, and $\overline{k}$ an algebraic closure of k). In [61] Tate proved that, if X is an abelian variety, Frobenius acts semisimply on $H^*(X_{\overline{k}}, \mathbb{Q}_\ell)$, but little progress has been made on the semisimplicity conjecture since then.

In [51], using a new construction in homological algebra, that he called *Bockstein stabilization*, Mondal proved an *unconditional* cohomological formula for the p-adic absolute value of $Z^*(X, q^{-n})$ ([51], Theorem 1.1). The purpose of these notes is to provide an introduction to his work.

To put things into perspective, we start by recalling classical material on the zeta function and its realizations in various cohomology theories. We then discuss the starting point, namely Tate's and Artin–Tate's conjectures, especially in the (crucial, and seminal) case of surfaces. We then formulate three basic questions arising from it, and sketch their posterity. In section 2 we describe Mondal's method of Bockstein stabilization. The main result is Proposition 1 in 2.3. In the next three sections we discuss applications of his method to formulas for the ℓ-adic and p-adic absolute values of the special value $Z^*(X, q^{-n})$. The main results are Theorem 3 in 3.1, Theorem 4 in 4.1, with its generalization Theorem 5 in 4.5, and Theorem 6 in 5.5. Theorem 5 gives an unconditional formula using Milne–Ramachandran's approach via Ekedahl's category $D^b_c(\mathcal{R}_k)$ of coherent complexes over the Raynaud ring $\mathcal{R}_k$, while Theorem 6 is Mondal's work proper, based on the formalism of dualizable prismatic F-gauges of Bhatt–Lurie [9]. In the last section we sketch how Ekedahl's results in ([22], II) should in fact show that the two methods are equivalent.

When working with derived categories, we tacitly assume that they are taken in the ∞-categorical sense, but often the classical ones, i.e., the homotopy categories of the ∞-ones, suffice.

Summary

1. A brief historical review

1.1. The zeta function over finite fields

Let p be a prime number, $q = p^a$, $a \geqslant 1$. Let $X/\mathbb{F}_q$ be a scheme of finite type. Its zeta function is

$$\zeta(X, s) = \prod_{x \in |X|} (1 - (\sharp k(x))^{-s})^{-1}$$

where $|X|$ denotes the set of closed points of X and, for $x \in |X|$, $\sharp k(x)$ denotes the cardinality of the residue field $k(x)$. As $\sharp k(x) = q^{\deg(x)}$, where $\deg(x) = [k(x) : \mathbb{F}_q]$, we have

$$\zeta(X, s) = Z(X, q^{-s}), \tag{1.1.1}$$

where

$$Z(X, t) := \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}.$$

A simple calculation, using that

$$\sharp X(\mathbb{F}_{q^n}) = \sum_{x \in |X|, \deg(x)|n} \deg(x),$$

shows that

$$Z(X,t) = \exp(\sum_{n \geqslant 1} \sharp X(\mathbb{F}_{q^n}) \frac{t^n}{n}).$$

It was conjectured by Weil, and proved by Dwork [20] (by p-adic methods), then by Grothendieck [26] (using étale cohomology) that $Z(X,t)$ is a rational function of t. When X is proper and smooth, of dimension d, more is known. By the Weil conjectures ([14], [16]),

$$Z(X,t) = \frac{P_1(X,t) \cdots P_{2d-1}(X,t)}{P_0(X,t) \cdots P_{2d}(X,t)}, \tag{1.1.2}$$

where $P_i(X,t)$ is a polynomial with coefficients in $\mathbb{Z}$, of the form

$$P_i(X,t) = \prod_{1 \leqslant j \leqslant b_i} (1 - \alpha_{ij} t),$$

where the α_{ij}'s are algebraic integers, all of whose complex conjugates are of absolute value $q^{i/2}$, in other words, are *q-Weil integers of weight i*. We will write $P_i(t)$ for $P_i(X,t)$ when no confusion can arise.

If X is geometrically connected, we have

$$P_0(t) = 1 - t, \;\; P_{2d}(t) = 1 - q^d t.$$

The polynomials P_i's have various cohomological interpretations.

(a) *ℓ-adic cohomology*

Let $\overline{k}$ be an algebraic closure of $k = \mathbb{F}_q$. Let Fr denote the *relative* Frobenius endomorphism of $X_{\overline{k}}/\overline{k}$, i.e. $Fr_X \otimes 1_{\overline{k}}$, where Fr_X is the identity on the underlying space and raises sections of $\mathcal{O}_X$ to the qth power. Let ℓ be a prime number different from p. Then the étale cohomology groups $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces, zero for $i \notin [0, 2d]$. For each i, Fr induces an *automorphism* Fr^* of $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$. This automorphism coincides with the automorphism induced by transportation of structure by the *inverse* σ^{-a} of the generator σ^a of $\mathrm{Gal}(\overline{k}/k)$ where $\sigma(x) = x^p$. It follows from Grothendieck's trace formula ([26], ([29], XIV), ([15], [Rapport])) that

$$Z(X,t) = \prod_{0 \leqslant i \leqslant 2d} \det(1 - Fr^* t, H^i(X_{\overline{k}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}, \tag{1.1.3}$$

and from ([14], [16]) that, for all i, $\det(1 - Fr^* t, H^i(X_{\overline{k}}, \mathbb{Q}_\ell))$ has integer coefficients, independent of ℓ, and

$$P_i(t) = \det(1 - Fr^* t, H^i(X_{\overline{k}}, \mathbb{Q}_\ell)). \tag{1.1.4}$$

In (1.1.2) the integer b_i is the dimension of $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$, the ith Betti number of X, and the α_{ij}'s, i.e., the *reciprocal roots* of P_i, are the eigenvalues of Fr^* on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$. In particular, they are *ℓ-adic units* for all ℓ. It follows from Poincaré duality that $\alpha \mapsto q^d/\alpha$ induces a bijection from the reciprocal roots of P_i to those of P_{2d-i}.

(b) *Crystalline cohomology*

Let $W(k)$ be the ring of Witt vectors on k, $K = K(k)$ its fraction field, and denote again by σ the automorphism of K induced by the Frobenius $\sigma : x \mapsto x^p$ of k. The crystalline cohomology groups $H^i_{\mathrm{crys}}(X/W(k))$ (usually denoted simply $H^i(X/W(k))$) are finitely generated $W(k)$-modules, zero for $i \notin [0, 2d]$. Let F be the *absolute* Frobenius endomorphism of X, i.e., F is the identity on the underlying space and raises sections of $\mathcal{O}_X$ to the p-th power. This endomorphism induces a σ-linear endomorphism F^* of $H^i(X/W(k))$. This isomorphism is an *isogeny*, i.e., induces a (σ-linear) automorphism F^* of $H^i(X/W(k)) \otimes K$. Berthelot ([4], VII, 3.2.3) proved that the K-linear automorphism F^{*a} (recall that $q = p^a$) gives rise to a decomposition similar to (1.1.3)

$$(1.1.5) \qquad Z(X,t) = \prod_{0 \leqslant i \leqslant 2d} \det(1 - F^{*a}t, H^i(X/W(k)) \otimes K)^{(-1)^{i+1}}.$$

Katz and Messing [35], using [16], proved[1] that

$$(1.1.6) \qquad P_i(t) = \det(1 - F^{*a}t, H^i(X/W(k)) \otimes K).$$

In particular, b_i is the dimension over K of $H^i(X/W(k)) \otimes K$, and the α_{ij}'s are the eigenvalues of F^{*a} on $H^i(X/W(k)) \otimes K$.

The α_{ij}'s are not, in general, p-adic units. With its σ-linear automorphism F^*, $H^i(X/W(k)) \otimes K$ is an *F-isocrystal*, and as such, by a theorem of Dieudonné–Manin ([18], [40], see also [17], [5]), has a canonical decomposition, indexed by rational numbers $\lambda \in \mathbb{Q}$, called the *slope decomposition*

$$(1.1.7) \qquad H^i(X/W(k)) \otimes K = \oplus_{\lambda \in \mathbb{Q}} (H^i(X/W(k)) \otimes K)_\lambda,$$

where

$$(H^i(X/W(k)) \otimes K)_\lambda \otimes_K K(\overline{k}) \xrightarrow{\sim} E_\lambda^{\oplus n_\lambda}$$

for some integer $n_\lambda \geqslant 0$, where E_λ is the F-isocrystal over $K(\overline{k})$ defined by

$$E_\lambda := K(\overline{k})_\sigma[T]/(T^s - p^r)$$

[1] Assuming X/k projective. The general proper smooth case was deduced from the projective one using de Jong's alterations (([55], Remark 2.2 (4)), [59], [12]).

for $\lambda = r/s$, $r \in \mathbb{Z}$, $s > 0$, $(r,s) = 1$, F acting semi-linearly by left multiplication by T. The numbers λ in (1.1.7) are called the *slopes* of $H^i(X/W(k))$. The *multiplicity* m_λ of a slope λ is $n_\lambda \dim(E_\lambda)$. Because the lattice $H^i(X/W(k))/\text{torsion}$ is F^*-stable, the slopes for which the corresponding summand is not zero are nonnegative.

Another result of Dieudonné–Manin is that the slopes of $(H^i(X/W(k)) \otimes K, F^*)$ are the p-adic valuations, normalized by $v(q) = 1$, of the eigenvalues of its K-linear automorphism F^{*a}, and the multiplicity of a slope λ is the number of eigenvalues of F^{*a} of valuation λ. It follows from Poincaré duality in crystalline cohomology ([4], VII, Th. 2.1.3) that the slopes of $H^i(X/W(k)) \otimes K$ lie in $[0,d]$ (and the weak Lefschetz theorem [3] implies that, if X is projective, they lie in $[0,i]$ if $0 \leqslant i \leqslant d$ and $[i-d,d]$ for $i \geqslant d$ [5]).

Further insight into the structure of $H^i(X/W(k))$ (and, in particular, on its slopes), are provided by its interpretation as the cohomology of the *de Rham–Witt complex* $W\Omega^\bullet_X$. The components $W\Omega^i_X$ are equipped with a σ-linear (resp. σ^{-1}-linear-) endomorphism F (resp. V) satisfying $FV = VF = p$, $FdV = d$, and there is a canonical isomorphism in $D(W(k))$ ([31], II Th. 1.4),

$$R\Gamma(X/W(k)) \xrightarrow{\sim} R\Gamma(X, W\Omega^\bullet_X) \tag{1.1.8}$$

(where the cohomology on the right hand side is calculated for the Zariski topology). By (1.1.8) the σ-linear endomorphism $\varphi = F^*$ of $R\Gamma(X/W(k))$ is induced by the endomorphism of the complex $W\Omega^\bullet_X$ given by p^iF in degree i. See [56] for a proof of a refinement of a conjecture of Katz–Mazur–Ogus on the relation of the slopes of Frobenius with the Hodge numbers $h^{i,j} = \dim_k H^j(X, \Omega^i_X)$.

In fact, the isomorphism (1.1.8) provides $R\Gamma(X/W(k))$, a perfect complex of $W(k)$-modules endowed with the σ-linear endomorpism φ, with a richer structure, namely that of an object of $D^b_c(\mathcal{R}_k)$, where $\mathcal{R}_k$ is the *Raynaud ring* on k, a graded ring generated by the Dieudonné ring $W(k)_\sigma[F,V]/(FV = VF = p)$ in degree zero and an element d in degree 1, of square zero, satisfying $FdV = d$ ([32], I (1.1)). The category $D^b_c(\mathcal{R}_k)$ is a certain full subcategory of the derived category $D(\mathcal{R}_k)$ of *graded* $\mathcal{R}_k$-modules.[2]

(c) *Prismatic and syntomic cohomologies*

[2] A graded $\mathcal{R}_k$-module is a complex $(M^\bullet, d)$ of $W(k)$-modules, where each M^i is equipped with σ- (resp. σ^{-1}-) linear operators F (resp. V), satisfying $FV = VF = p$, $FdV = d$. See ([33], Definition 2.4.6) for the definition of $D^b_c(\mathcal{R}_k)$.

New perspectives on crystalline cohomology were brought by the the prismatic cohomology theory of Bhatt-Scholze [6], and the associated stacks and their variants constructed by Drinfeld [19] and Bhatt–Lurie ([7], [8], [9]).

Let $f : X \to \mathrm{Spec}(k)$ be the projection. It induces a morphism $f^{\Delta} : X^{\Delta} \to \mathrm{Spec}(k)^{\Delta}$ between the corresponding prismatic stacks. Let $\mathcal{O}$ denote the structural sheaf of rings of X^{Δ}. Consider $Rf^{\Delta}_{*}\mathcal{O} \in D_{\mathrm{qc}}(\mathrm{Spec}(k)^{\Delta})$, where $D_{\mathrm{qc}}(\mathrm{Spec}(k)^{\Delta})$ is equivalent, by σ^*, to $D(W(k))$. There is a canonical identification

$$\sigma^* Rf^{\Delta}_{*}\mathcal{O} \xrightarrow{\sim} R\Gamma(X/W(k)),$$

by which $\sigma^*\varphi$, where φ is the σ-linear endomorphism of $Rf^{\Delta}_{*}\mathcal{O}$ (deduced from the map $\varphi_{X^{\Delta}} : X^{\Delta} \to X^{\Delta}$ ([9], 3.1.2), corresponds to the σ-linear endomorphism $\varphi = F^*$ of the right hand side considered in (b). In particular,

$$P_i(t) = \det(1 - \varphi^a t, R^i f^{\Delta}_{*}\mathcal{O} \otimes K). \tag{1.1.9}$$

This formula doesn't bring any extra information to (1.1.6). However, $Rf^{\Delta}_{*}\mathcal{O}$ underlies a finer object, namely we have

$$Rf^{\Delta}_{*}\mathcal{O} = j^{*}_{\Delta} Rf^{\mathrm{syn}}_{*}\mathcal{O},$$

where $f^{\mathrm{syn}} : X^{\mathrm{syn}} \to \mathrm{Spec}(k)^{\mathrm{syn}}$ is the morphism induced on the *syntomic stacks*, $\mathcal{O}$ the structural sheaf of X^{syn}, and $j_{\Delta} : \mathrm{Spec}(k)^{\Delta} \to \mathrm{Spec}(k)^{\mathrm{syn}}$ the canonical map ([9], 4.1.1) (an étale map). This finer object encodes the *Nygaard filtration* $\mathrm{Fil}^{\bullet}$ on $R\Gamma(X/W(k))$ together with its *F-gauge structure*. As we will see in section 6, this extra structure is essentially equivalent to that given to $R\Gamma(X/W(k))$ by the structure of an object of $D^b_c(\mathcal{R}_k)$, and the extra information they provide on the $H^i(X/W(k))$ is crucial for the analysis, in [51], of the *special values* of $\zeta(X,s)$ at integers.

1.2. Poles at integers

It follows from (1.1.2) that $x \in \mathbb{C}$ is a zero of $\zeta(X,s)$ if and only if $q^{-x} - t$ divides one of the polynomials $P_{2i+1}(t)$ for $0 \leqslant i \leqslant d-1$, i.e., if and only if q^x is an eigenvalue of Fr^* on $H^{2i+1}(X_{\overline{k}}, \mathbb{Q}_\ell)$. Since those eigenvalues are Weil numbers of weight $2i+1$, x must lie on the line $\mathrm{Re}(s) = i + \frac{1}{2}$. In particular, no integer $n \in \mathbb{Z}$ can be a zero of $\zeta(X,s)$. The subject of zeroes of $\zeta(X,s)$ doesn't seem to have attracted much attention. The case of $s = \frac{1}{2}$ was studied in [57]. Quite recently, this case was revisited by Mondal [53], using the main results of [51].

Let $n \in \mathbb{Z}$. Then $\zeta(X,s)$ has a pole at $s = n$ if and only if $q^{-n} - t$ divides one of the polynomials $P_{2i}(t)$ for $0 \leqslant i \leqslant d$, i.e., if and only if q^n is an eigenvalue of Fr^* on $H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell)$, and as these are Weil numbers of weight $2i$, this can happen only if $n = i$. In particular, $\zeta(X, s)$ has no pole at $s = n$ if $n < 0$ or $n > d$. For $0 \leqslant n \leqslant d$, $\zeta(X, s)$ may or may not have a pole at $s = n$.

Let $\rho_n = \rho_n(X)$ be the *order of the pole* of $\zeta(X, s)$ at $s = n$, i.e. the nonnegative integer ρ_n such that $(1 - q^n t)^{\rho_n} Z(X, t)$ has no pole at $t = q^{-n}$ and the *special value*

$$Z^*(X, q^{-n}) := \lim_{t \to q^{-n}} (1 - q^n t)^{\rho_n} Z(X, t) \tag{1.2.1}$$

is a nonzero rational number. As we observed above, for $i \neq 2n$, we have $P_i(X, q^{-n}) \neq 0$. It is convenient to define

$$P_i^*(X, q^{-n}) := P_i(X, q^{-n})$$

if $i \neq 2n$, and $P_{2n}^*(X, q^{-n})$ by

$$P_{2n}(X, t) = (1 - q^n t)^{\rho_n} P_{2n}^*(X, t)$$

with $P_{2n}^*(X, t)$ in $\mathbb{Q}[t]$ (actually, in $\mathbb{Z}[t]$, as its coefficients are algebraic integers), and $P_{2n}^*(X, q^{-n}) \neq 0$. Thus

$$Z^*(X, q^{-n}) = \prod_i P_i^*(X, q^{-n})^{(-i)^{i+1}}.$$

The number ρ_n has a well-known, deep conjectural geometric significance. By definition, it is the multiplicity of q^n as an eigenvalue of Fr^* on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell)$, or equivalently of 1 as an eigenvalue of Fr^* on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$. If this eigenvalue 1 has multiplicity 1 in the *minimal polynomial* of Fr^* on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$, i.e., if Fr^* has no nontrivial Jordan block with 1 on the diagonal (*a fortiori* if Fr^* is *semisimple*),[3] then

$$\rho_n = \dim_{\mathbb{Q}_\ell} H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))^{\mathrm{Gal}(\overline{k}/k)}.$$

The Tate conjecture ([60], Conjecture 1, and (12) p. 101) (assuming semisimplicity of Fr^* on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$) asserts that, if X is projective,

$$H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))^{\mathrm{Gal}(\overline{k}/k)} = \mathrm{Im}(\mathrm{cl} : \mathrm{CH}^n(X) \to H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n)), \tag{1.2.2}$$

where $\mathrm{CH}^n(X)$ is the Chow group of codimension n cycles, and cl is the cycle class map.

[3]It is generally conjectured, after Weil, that if X is projective, Fr^* acts semisimply on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ for all i. Tate proved it for X an abelian variety ([61], p. 138).

The conjecture is trivial for $n = 0$. For $n = 1$, the conjecture and the semisimplicity of the action of Fr^* were proven by Tate for products of curves and abelian varieties [61] (see also [46], [63]). They are known for K3 surfaces if $p > 2$ [39], and for very few other examples (see [63]). If X is a surface (i.e., $d = 2$), it was proven by Artin–Tate ([62], Theorem 5.2) that (1.2.2), for $n = 1$, is equivalent to the finiteness of $\mathrm{Br}(X)(\ell)$,[4] see ([64], Proposition 5.1.2)[5] for an analog for the p-part. I will come back to this in 1.3. Essentially nothing is known for $n > 1$, even for abelian varieties (see [46]).

As we recalled in the introduction, the special values (1.2.1) have been the focus of extensive studies for the past sixty years, starting with Tate's seminal Bourbaki exposé [62]. Recently, Mondal [51] gave an *unconditional* cohomological formula for the p-adic absolute values of $Z^*(X, q^{-n})$, $n \in [0, d]$, depending on new invariants attached to the crystalline cohomology of X, that he called the *stable Bockstein characteristics*. Slightly afterwards, a K-theoretic attempt was made by Hyslop [30]. However, Mondal[6] found a counterexample to ([30], Theorem 2.13), which seems to bar hope for the method in [30] to work.

1.3. Example: the case of a surface

The interplay between the semisimplicity conjecture, the Tate conjecture, and the calculation of the special value $Z^*(X, q^{-n})$ is best seen in the first nontrivial example, that of a smooth, geometrically connected, projective surface X/k ($k = \mathbb{F}_q$), i.e., $d = 2$, $n = 1$. In this case, (1.1.2) reads

$$Z(X,t) = \frac{P_1(t)P_3(t)}{(1-t)P_2(t)(1-q^2t)}. \tag{1.3.1}$$

Write $P_1(t) = \prod_{1 \leqslant j \leqslant b_1}(1 - \alpha_{1,j}t)$. As, by Poincaré duality, $b_1 = b_3$ and $\alpha \mapsto q^2/\alpha$ is a bijection of the set of reciprocal roots of P_1 to the set of reciprocal roots of P_3, we have $P_3(t) = \prod_{1 \leqslant j \leqslant b_1}(1 - \frac{q^2}{\alpha_{1,j}}t)$. As in any complex embedding, $\alpha_{1,j}\overline{\alpha_{1,j}} = q$, and $\alpha \mapsto \overline{\alpha}$ is a permutation of the reciprocal roots, since P_1 has integer coefficients, we deduce that $P_3(t) = P_1(qt)$ and (1.3.1) can be rewritten

$$Z(X,t) = \frac{P_1(t)P_1(qt)}{(1-t)P_2(t)(1-q^2t)}. \tag{1.3.2}$$

[4] In the notation of [62], $\mathrm{Br}(X)(\ell) := \cup_{r \geqslant 1}\mathrm{Ker}(\ell^r : \mathrm{Br}(X) \to \mathrm{Br}(X))$ is the ℓ^∞-torsion of $\mathrm{Br}(X)$.

[5] The fact that $T_p\mathrm{Br}(X_{\overline{k}})$ is finitely generated and torsion-free over $\mathbb{Z}_p$, used in the construction of the exact sequence ([64] (5.1.1)), relies on Milne's finiteness theorem ([41], Theorem 2.1), where the restriction $p \neq 2$ can be lifted, see [49], ([31], II Prop. 5.9) and ([32], IV Cor. 3.4).

[6] Private communication, 4/11/26.

As above, denote by $\rho_1 = \rho_1(X) \geqslant 0$ the order of the pole of $Z(X,t)$ at $t = q^{-1}$ (equivalently, of $\zeta(X,s)$ at $s=1$). With the notation of (1.2.1), we thus have

$$P_2(t) = (1-qt)^{\rho_1} P_2^*(t) \tag{1.3.3}$$

with $P_2^*(t) \in \mathbb{Z}[t]$ and $P_2^*(q^{-1}) \neq 0$. Equivalently,

$$P_2(q^{-s}) \sim (s-1)^{\rho_1}(\log q)^{\rho_1} P_2^*(q^{-1})$$

when $s \to 1$.

If we are given a proper, flat k-morphism $f\colon X \to Y$ to a smooth, proper, geometrically connected curve Y/k, with geometricallly connected, smooth generic fiber X_η over the generic point η of Y, then the behavior of $\zeta(X,s)$ near $s=1$ is closely related to the behavior, near $s=1$, of the L-function associated with the Jacobian $J = \mathrm{Pic}^0_{X_\eta/\eta}$ of X_η. In 1966, Artin and Tate [62] investigated this relation, and formulated a "geometric analog" of the Birch and Swinnerton-Dyer conjecture, involving the Néron-Severi group $\mathrm{NS}(X)$ and the Brauer group $\mathrm{Br}(X)$.

By definition, $\mathrm{Br}(X) = H^2(X, \mathbf{G}_m)$. This is a torsion group. The Néron-Severi group scheme $\mathrm{NS}_{X/k}$ is by definition $\mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k}$, and $\mathrm{NS}(X)$ is defined as $\mathrm{NS}_{X/k}(k)$. Let $G = \mathrm{Gal}(\overline{k}/k)$. We have a G-equivariant exact sequence

$$0 \to \mathrm{Pic}^0_{X/k}(\overline{k}) \to \mathrm{Pic}_{X/k}(\overline{k}) \to \mathrm{NS}_{X/k}(\overline{k}) \to 0,$$

where $\mathrm{NS}_{X/k}(\overline{k})$, denoted $\mathrm{NS}(X_{\overline{k}})$ in the sequel, is a *finitely generated* abelian group ([2], XIII, 5.1). By Lang's theorem, $H^1(G, \mathrm{Pic}^0_{X/k}(\overline{k})) = 0$, hence we get

$$\mathrm{NS}(X) = \mathrm{NS}(X_{\overline{k}})^G,$$

and an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0.$$

In particular, $\mathrm{NS}(X)$ is finitely generated. Artin and Tate made the following conjectures 1 and 2.

Conjecture 1. *Artin–Tate conjecture (AT)* ([62], (C), p. 426). The Brauer group $\mathrm{Br}(X)$ is finite, $\rho_1(X)$ is equal to the rank of $\mathrm{NS}(X)$,[7] and

$$P_2^*(X, q^{-1}) = \frac{[\mathrm{Br}(X)]|\det(D_i.D_j)|}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tors}}]^2}, \tag{1.3.4}$$

[7] Usually denoted $\rho(X)$ and called the *Picard number* of X.

where: $[E]$ denotes the cardinality of a set E, $(D_i)_{1 \leqslant i \leqslant \rho(X)}$ is a base of $\mathrm{NS}(X)$ modulo torsion[8], $(D_i.D_j)$ is the intersection number, and

$$\alpha(X) := \chi(X, \mathcal{O}_X) - 1 + \dim(\mathrm{Pic}^{0,\mathrm{red}}_{X/k}).$$

Here $\mathrm{Pic}^{0,\mathrm{red}}_{X/k}$, sometimes denoted $\mathrm{Pic\,Var}(X)$, is dual to an Albanese variety of X, and has dimension $\frac{b_1}{2}$, where

$$b_1 = \dim H^1(X_{\overline{k}}, \mathbb{Q}_\ell) = \dim H^1(X/W(k)) \otimes K.$$

Remark. The special value $Z^*(X, q^{-1})$ (1.2.1) is easily calculated from $P_2^*(X, q^{-1})$. By (1.3.2),

$$Z^*(X, q^{-1}) = \frac{P_1(q^{-1})P_1(1)}{(1-q^{-1})(1-q)P_2^*(q^{-1})}.$$

The quantities $P_1(q^{-1})$, $P_1(1)$ have nice geometric interpretations (see ([51], Lemma 7.4), whose proof is elementary): if $A := (\mathrm{Pic}^{0,\mathrm{red}}_{X/k})^\vee$ is the Albanese variety of X, then:

$$(1.3.4a) \qquad P_1(1) = [A(k)], \;\; P_1(q^{-1}) = \frac{[A(k)]}{q^{\dim(A)}}.$$

Thus, (1.3.4) is equivalent to:

$$(1.3.4b) \qquad Z^*(X, q^{-1}) = \frac{[A(k)]^2[\mathrm{NS}(X)_{\mathrm{tors}}]^2 q^{\chi(X,\mathcal{O}_X)}}{q(1-q)(1-q^{-1})[\mathrm{Br}(X)]|\det(D_i.D_j)|}.$$

Conjecture 2. *The equivalence conjecture*: the Artin–Tate conjecture for X is equivalent to the Birch and Swinnerton-Dyer conjecture for J.

The Birch and Swinnerton-Dyer conjecture asserts, in particular, that the Tate–Shafarevich group $Ш(J/K)$ is finite and that the order of the zero of the L-function of J at $s = 1$ is equal to the rank of the Mordell–Weil group $J(K)$.

Conjecture 2 is now a theorem. It has a long history. Artin–Grothendieck ([27], (4.7)) proved that Br(X) is finite if and only if $Ш(J/K)$ is finite. But it's only quite recently that a (correct) proof of Conjecture 2 was given, see [38].

[8] More precisely, given $V := \sum_{1 \leqslant i \leqslant \rho(X)} \mathbb{Z}D_i$, a torsion-free submodule of $\mathrm{NS}(X)$ sent isomorphically to $\mathrm{NS}(X)/\mathrm{tors}$, $\det(D_i.D_j)$ does not depend on the choices of V and the D_i's, and as $\mathrm{NS}(X)/V \xrightarrow{\sim} \mathrm{NS}(X)_{\mathrm{tors}}$, the quotient $\frac{\det(D_i.D_j)}{[\mathrm{NS}(X)_{\mathrm{tors}}]^2}$ is the *discriminant* $\Delta(\mathrm{NS}(X))$ of the intersection form in the sense of ([38], 1.4).

Conjecture 1 is still wide open. Artin and Tate proved that if $\rho_1(X)$ is equal to the rank of $\mathrm{NS}(X)$, then, for any $\ell \neq p$, the two sides of (1.3.4) have the same ℓ-adic absolute value $|-|_\ell$, where $|x|_\ell = \ell^{-v_\ell(x)}$ for $x \in \mathbb{Q}$.

Actually, Artin and Tate in *loc. cit.* proved more, namely they proved:

Theorem 1 ([62], Theorem 5.2)
(a) The following (i) – (iv) are equivalent.

(i) $\mathrm{Br}(X)(\ell)$ is finite.[9]
(ii) The canonical map

$$\mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G,$$

where $G = \mathrm{Gal}(\overline{k}/k)$, is an isomorphism.
(iii) $\mathrm{rk}(\mathrm{NS}(X)) = \mathrm{rk}_{\mathbb{Z}_\ell} H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G$.
(iv) $\rho_1(X) = \mathrm{rk}(\mathrm{NS}(X))$.

(b) If (i) – (iv) are satisfied, then the two sides of (1.3.4) have the same ℓ-adic absolute value, i.e.,

$$(1.3.5) \qquad |P_2^*(X, q^{-1})|_\ell = \left| \frac{[\mathrm{Br}(X)] \det(D_i.D_j)}{q^{\alpha(X)} [\mathrm{NS}(X)_{\mathrm{tors}}]^2} \right|_\ell .$$

The equivalence of (i) – (iii) in (a) and, as $\mathrm{rk}_{\mathbb{Z}_\ell} H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G \leqslant \rho_1(X)$, the fact that (iv) implies (iii) follow from the exact sequence

$$(1.3.6) \qquad 0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G \to T_\ell(\mathrm{Br}(X)) \to 0$$

(cf. ([64], (5.1.1))).

The fact that (i) implies (iv) and assertion (b) are delicate. Let me recall the main points. In fact, (iv) is the conjunction of the following (iv1), (iv2):

(iv1) (*partial semisimplicity conjecture*) q is a simple root of the minimal polynomial of Fr^* on $H^2(X_{\overline{k}}, \mathbb{Q}_\ell)$; equivalently, 1 is a simple root of the minimal polynomial of Fr^* on $H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))$, i.e.,

$$\rho_1(X) = \dim H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^G$$

$$(= \dim \mathrm{Ker}(Fr^* - 1 : H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)) \to H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)))).$$

(iv2) (*Tate conjecture*) The canonical map

$$\mathrm{NS}(X) \otimes \mathbb{Q}_\ell \to H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^G,$$

[9] For a torsion abelian group A, we denote by $A(\ell)$ its ℓ-primary summand, i.e., $\varinjlim_n A[\ell^n]$.

is an isomorphism.

By the exact sequence (1.3.6), (i) implies (iv2). The problem is to show that (i) implies (iv1) and to prove (1.3.5).

By an elementary, but nontrivial, linear algebra lemma ([62], Lemma z4), (iv1) is equivalent to the fact that the canonical map

$$f : H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))_G,$$

induced by the *identity map* of $H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))$ is an ℓ-quasi-isomorphism.

In [62], a map $f : A \to B$ of $\mathbb{Z}_\ell$-modules is called a *quasi-isomorphism* if $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are finite, and in this case the integer $z(f)$ is defined as

$$z(f) = \ell^{\lg(\operatorname{Ker}(f)) - \lg(\operatorname{Coker}(f))} = [\operatorname{Ker}(f)]/[\operatorname{Coker}(f)],$$

where lg denotes the length of a finite length $\mathbb{Z}_\ell$-module, and $[S]$ denotes the cardinality of a finite group S. The terminology "ℓ-quasi-isomorphism" is that of [48]. It avoids confusion with the usual notion of quasi-isomorphism of complexes, and remembers the prime ℓ.

Moreover, by the same lemma, if (iv1) is satisfied, then

$$z(f) = |P_2^*(X, q^{-1})|_\ell. \tag{1.3.7}$$

The strategy is to use the compatibility of the cycle class map $\operatorname{NS}(X) \otimes \mathbb{Z}_\ell \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))$ with intersection pairing and Poincaré duality to realize f, up to an ℓ-quasi-isomorphism, as the map

$$e : \operatorname{NS}(X) \otimes \mathbb{Z}_\ell \to \operatorname{Hom}(\operatorname{NS}(X), \mathbb{Z}_\ell)$$

defined by the intersection pairing, exploiting the isomorphism

$$h : \operatorname{NS}(X) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G$$

provided by (1.3.6) as $T_\ell(\operatorname{Br}(X)) = 0$ by (i). The non-degeneracy of this pairing implies that e is an ℓ-quasi-isomorphism, whose Artin–Tate's z-invariant is

$$z(e) = \frac{|\det(D_i.D_j)|_\ell}{|\operatorname{NS}(X)_{\ell-\mathrm{tors}}|_\ell}.$$

However, it turns out that f is not isomorphic to e, but that there is a canonical commutative diagram

$$\begin{array}{ccc} \operatorname{NS}(X) \otimes \mathbb{Z}_\ell & \xrightarrow{e} & \operatorname{Hom}(\operatorname{NS}(X), \mathbb{Z}_\ell) \\ {\scriptstyle h\,\wr}\downarrow & & \uparrow{\scriptstyle g^*} \\ H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G & \xrightarrow{f} & H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))_G, \end{array}$$

where g^* is an ℓ-quasi-isomorphism with z-invariant

$$z(g^*) = \left| \frac{[\mathrm{NS}(X)_{\ell-\mathrm{tors}}]}{[\mathrm{Br}(X)(\ell)]} \right|_{\ell}^{-1}.$$

See ([41], section 5) for an alternative (and more detailed) proof of this fact. It follows that f is an ℓ-quasi-isomorphism, with invariant $z(f) = z(e)z(g^*)^{-1}$, so that, by (1.3.7), we get the desired equality (1.3.5)

$$|P_2^*(X, q^{-1})|_\ell = \left| \frac{[\mathrm{Br}(X)]|\det(D_i.D_j)|}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tors}}]^2} \right|_\ell .$$

1.4. p-adic variants and higher dimensional problems

The fact that (1.3.5) holds for all $\ell \neq p$ implies that (1.3.4) holds up to a factor $\pm p^\nu$. Thus Theorem 1 (and its proof) immediately raised the following questions.

Question 1. Assuming that conditions (i) – (iv) in Theorem 1 hold, is it true that $\mathrm{Br}(X)(p)$ is finite (equivalently, that $\mathrm{Br}(X)$ is finite), and do we have

$$|P_2^*(X, q^{-1})|_p = \left| \frac{[\mathrm{Br}(X)]\det(D_i.D_j)}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tors}}]^2} \right|_p, \tag{1.4.1}$$

where $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{Q}$?

Question 2. Assuming only the partial semisimplicity conjecture (iv1), can one give a *cohomological expression* for $P_2^*(X, q^{-1})$, up to sign, in terms of ℓ-adic and p-adic cohomologies of X?

Question 3. Can one unconditionally (i.e., independently of any conjecture) give a *cohomological expression* for $P_2^*(X, q^{-1})$, up to sign, in terms of ℓ-adic and p-adic cohomologies of X?

Let me briefly discuss the posterity of these questions.

Posterity of question 1. Tate raised question 1 in [62], adding that it would be "a good test for any p-adic cohomology theory." Tate's exposé was given in February 1966. In December 1966 Grothendieck [28] introduced *crystalline cohomology*, that was supposed to fill in the gap at p in the family of ℓ-adic cohomologies. The theory was developed by Berthelot in his thesis [4], giving in particular the cohomological expression (1.1.5) for the zeta function. However, $\mathrm{Br}(X)(p)$ involved flat cohomology, $H^*_{\mathrm{fppf}}(X, \mu_{p^n})$, which *a priori* had no clear relation with crystalline cohomology, nor with $Z(X, t)$.

The situation drastically changed in 1974 with the construction, by Bloch, of his *complex of sheaves of typical curves* $TC\mathcal{K}_{\bullet+1}$ on X [10], calculating $R\Gamma(X/W(k))$ as $R\Gamma(X, TC\mathcal{K}_{\bullet+1})$ for $\dim(X) < p$ and $p > 2$, a construction that was generalized in [31] to arbitrary dimension and characteristic, under the name of the *de Rham–Witt complex* $W\Omega^\bullet_X$. Bloch's construction was reported on by Berthelot in [5], and opened the way to an understanding of the so far mysterious relations between crystalline cohomology, flat cohomology and Serre's Witt vectors cohomology. Using this new tool, Milne solved question 1 in ([41], Theorem 4.1). In *loc. cit.* he assumed $p \neq 2$, but as explained in [49], this hypothesis can be removed, by replacing the reference [10] by [31]. His proof closely follows the pattern of the proof of Theorem 1. His duality theorem [42] plays an important role. The formula for the invariant $\alpha(X)$ of (1.3.4) given in ([41], p. 529) turned out to be later understood as an avatar of *Crew–Milne's formula*[10] ([13], p. 84, Remark) ([22], IV Theorem 3.2).

Posterity of question 2. Again, the answer is yes, and Milne [43] gave a solution in a much more general framework, for proper smooth schemes X/k and special zeta values at arbitrary integers n. The key idea was that, instead on concentrating on the polynomial $P_{2n}(X, t)$ (and the special value $P^*_{2n}(X, q^{-n})$), one should instead consider all the cohomology groups $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$, with their Frobenius automorphisms, and the analogous ones for crystalline cohomology, which contribute to the zeta function and the special value $Z^*(X, q^{-n})$. His main result is the following.

Theorem 2 ([43], Theorem 0.1). Let X/k be proper and smooth, and let $n \in \mathbb{Z}$. Assume that, for all $\ell \neq p$ the minimal polynomial of Fr^* acting on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$ does not have 1 as a multiple root, and that the minimal polynomial of $\gamma := (\sigma^a)^*$ acting on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_p(n))$ does not have 1 as a multiple root, where the notation is the following:

$$R\Gamma(X, \mathbb{Z}_p(n)) := R\varprojlim_m R\Gamma(X, W_m\Omega^n_{\log}[-n]),$$

$W_m\Omega^i_{\log} = W_m\Omega^i_{X,\log}$ is the ith logarithmic Hodge–Witt sheaf ([32], IV 3.1) on X, and the cohomology on the right and side is taken for the étale topology, so that

$$R\Gamma(X, \mathbb{Z}_p(n)) = \mathrm{Fib}(1 - \gamma : R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)) \to R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)).$$

Consider the rational number $Z^*(X, q^{-n})$ which is the special value defined in (1.2.1). Its ℓ-adic ($\ell \neq p$) and p-adic absolute values are given by the

[10] See ([22], p. 82).

following formulas:

(a) $$|Z^*(X, q^{-n})|_\ell^{-1} = \chi(X, \mathbb{Z}_\ell(n))^\times,$$

(b) $$|Z^*(X, q^{-n})|_p^{-1} = \chi(X, \mathbb{Z}_p(n))^\times \cdot q^{\chi(X, \mathcal{O}_X, n)}.$$

In (a) and (b) the notation is as follows:

The rational number $\chi(X, \mathbb{Z}_\ell(n))^\times$ (resp. $\chi(X, \mathbb{Z}_p(n))^\times$) is the so-called ℓ-adic (resp. p-adic) *multiplicative Bockstein characteristic* defined in (1.4.7) (resp. (1.4.9)).

The number $\chi(X, \mathcal{O}_X, n)$ is the integer defined by

$$\chi(X, \mathcal{O}_X, n) := \sum_{j, 0 \leqslant i \leqslant n} (-1)^{i+j} (n-i) h^{ij},$$

where $h^{ij} := \dim_k H^j(X, \Omega^i_{X/k})$.

Main ideas in the proof of Theorem 2. Identify $G = \mathrm{Gal}(\overline{k}/k)$ ($k = \mathbb{F}_q$, $q = p^a$) with $\widehat{\mathbb{Z}}$ by the canonical generator $\gamma = \sigma^a$, $\sigma : x \mapsto x^p$. For X/k proper and smooth of any dimension, and any $n \in \mathbb{Z}$, consider the exact triangle (where γ acts as $(Fr^*)^{-1}$)

(1.4.2) $$R\Gamma(X, \mathbb{Z}_\ell(n)) \to R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n)) \overset{1-\gamma}{\to} R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n)) \to .$$

It gives rise to short exact sequences (of the degenerate Hochschild-Serre spectral sequence)

(1.4.3) $$0 \to H^{i-1}(X_{\overline{k}}, \mathbb{Z}_\ell(n))_G \to H^i(X, \mathbb{Z}_\ell(n)) \to H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n))^G \to 0$$

The canonical basis $\theta \in H^1(G, \mathbb{Z}_\ell)$ (given by the projection $\widehat{\mathbb{Z}} \to \mathbb{Z}_\ell$) gives rise, by functoriality to a map

$$\theta : R\Gamma(X, \mathbb{Z}_\ell(n)) \to R\Gamma(X, \mathbb{Z}_\ell(n))[1].$$

It was observed by Rapoport and Zink ([58], Satz 1.3) (see also ([34], pp. 37–39)) that, after having chosen a suitable complex of $\mathbb{Z}_\ell[G]$-modules K representing $R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n))$, θ can be identified with the map of simple complexes associated with the map of double complexes

$$\begin{array}{ccccc} K & \xrightarrow{1-\gamma} & K & & \\ & & \uparrow{\scriptstyle \mathrm{Id}} & & \\ & & K & \xrightarrow{1-\gamma} & K \end{array}$$

(here, it is the quotient $\mathbb{Z}_\ell$ of $G = \widehat{\mathbb{Z}}$ that appears, instead of $\mathbb{Z}_\ell(1)$ in ([34], pp. 37–39), and the vertical map $1 \otimes T$ in the diagram below (3.6.3) in ([34], *loc. cit*) is the identity map, T being chosen as $1 \in \mathbb{Z}_\ell$). See (([47], p. 10), ([48], p. 822)) for a similar exposition. It was shown in [58] that $\theta \circ \theta = 0$, and one can check that (in today's language) we thus get an infinite *complex* (of objects of a suitable derived stable ∞-category)

$$(1.4.4) \qquad B_\ell(X,n) := (\cdots \to R\Gamma(X, \mathbb{Z}_\ell(n))[-1] \xrightarrow{\theta} R\Gamma(X, \mathbb{Z}_\ell(n)) \xrightarrow{\theta} \cdots)$$

(see (2.1.3), (2.1.3') for a generalization, cf. ([51], 4.1)). This complex gives rise to a (bounded) complex of finitely generated $\mathbb{Z}_\ell$-modules, called a *Bockstein complex*) (see (2.1.4) for a generalization),

$$(1.4.5) \qquad \mathrm{Bock}_\ell(X,n) := (\cdots \to H^i(X, \mathbb{Z}_\ell(n)) \xrightarrow{d^i} H^{i+1}(X, \mathbb{Z}_\ell(n)) \to \cdots)$$

which fits into a commutative diagram ([43], 6.5)),

$$(1.4.5') \qquad \begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \uparrow & & \downarrow \\
H^{i-2}(X_{\overline{k}}, \mathbb{Z}_\ell(n))_G & & H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n))^G & \xrightarrow{f_i} & H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n))_G \\
\delta\downarrow & & \uparrow & & \delta\downarrow \\
H^{i-1}(X, \mathbb{Z}_\ell(n)) & \xrightarrow{d^{i-1}} & H^i(X, \mathbb{Z}_\ell(n)) & \xrightarrow{d^i} & H^{i+1}(X, \mathbb{Z}_\ell(n)) \\
\downarrow & & \delta\uparrow & & \downarrow \\
H^{i-1}(X_{\overline{k}}, \mathbb{Z}_\ell(n))^G & \xrightarrow{f_{i-1}} & H^{i-1}(X_{\overline{k}}, \mathbb{Z}_\ell(n))_G & & H^{i+1}(X_{\overline{k}}, \mathbb{Z}_\ell(n))^G \\
\downarrow & & \uparrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where the (vertical) short exact sequences are (1.4.3), and the f_i's are analogs of the map f above, before in (1.3.7), i.e., are the compositions of the canonical inclusions and projections

$$H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n))^G \hookrightarrow H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n)) \twoheadrightarrow H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n))_G.$$

The key point (see ([48], Lemmas 5.1 and 5.2)), which is elementary, is the following:

(i) for each i, f_i is an ℓ-quasi-isomorphism if and only if 1 is a simple root of the minimal polynomial of Fr^* on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))$ (i.e., if $\dim(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))^G)$ is the multiplicity ρ_i of 1 as an eigenvalue or Fr^*), and in this case, with $z(f_i) := [\mathrm{Ker}(f_i)]/[\mathrm{Coker}(f_i)]$, for

$$f_i : H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))^G \to H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))_G$$

induced by the identity of $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))$, we have

$$z(f_i) = |\prod_{j,\lambda_j \neq 1} (1-\lambda_{ij})|_\ell = |Q_i^*(X,1)|_\ell,$$

where the $\lambda_{ij}'s$ are the eigenvalues of Fr^* on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))$, and

$$Q_i(X,t) = \det(1 - Fr^* t, H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))) = (1-t)^{\rho_i} Q_i^*(X,t)$$

with $Q_i^*(X,1) \neq 0$; for $i \neq 2n$, 1 is not an eigenvalue of Fr^* on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))$, $\rho_i = 0$, and f_i is an ℓ-quasi-isomorphism.

(ii) all the f_i's are ℓ-quasi-isomorphisms (equivalently, f_n is an ℓ-quasi-isomorphism) if and only if all the cohomology groups $H^i(\mathrm{Bock}_\ell(X,n))$ are of finite length, and when this condition is satisfied, then

$$\prod_i [H^i(\mathrm{Bock}_\ell(X,n)]^{(-1)^i} = \prod_i z(f_i)^{(-1)^i}$$

It follows from (i) and (ii) that, if for all $\ell \neq p$ the minimal polynomial of Fr^* acting on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$ does not have 1 as a multiple root, we have

(1.4.6) $$|Z^*(X, q^{-n})|_\ell^{-1} = \chi(X, \mathbb{Z}_\ell(n))^\times$$

where

(1.4.7) $$\chi(X, \mathbb{Z}_\ell(n))^\times := \prod_i [H^j(\mathrm{Bock}_\ell(X,n))]^{(-1)^i}$$

(sometimes denoted $\chi(X, \mathbb{Z}_\ell(n), e)$) is the *multiplicative Bockstein characteristic* of the complex $\mathrm{Bock}_\ell(X,n)$ (1.4.5), which has bounded, finite length cohomology.

From the semisimplicity assumption at 1, one deduces that $H^i(X, \mathbb{Z}_\ell(n))_{\mathrm{tor}}$[11] is finite for all i, and that

(1.4.7′) $$\chi(X, \mathbb{Z}_\ell(n))^\times = \prod_i [H^i(X, \mathbb{Z}_\ell(n))_{\mathrm{tor}}]^{(-1)^i} |\det(d^{2n})|_\ell,$$

where $d^{2n} : H^{2n}(X, \mathbb{Z}_\ell(n)) \to H^{2n+1}(X, \mathbb{Z}_\ell(n))$ is the map d_ℓ^{2n} of (1.4.5), which is shown to be an isomorphism modulo torsion and $\det(d^{2n})_\ell$ is defined as the determinant of a matrix of d_ℓ^{2n} mod torsion.

[11]One denotes by M_{tor} the torsion subgroup of an abelian group.

Finally, using Gabber's theorem [24] to the effect that $H^i(X_{\overline{k}}, \mathbb{Z}_\ell)$ is torsion-free for almost all ℓ, one can rewrite

$$(1.4.7'') \qquad \chi(X, \widehat{\mathbb{Z}}(n))^\times = \frac{\prod_i [H^i(X, \widehat{\mathbb{Z}}(n))_{\mathrm{tor}}]^{(-1)^i}}{\det(d^{2n})},$$

with $\det(d^{2n}) = \prod_\ell \det(d^{2n})_\ell \cdot \det(d^{2n})_p$, as formulated by Milne ([43], p. 298).

The ℓ part, $\ell \neq p$, is, somehow, the easy part. The difficult part is the p-part (see Remark 4.1.2).

First of all, one has a Bockstein complex $\mathrm{Bock}_p(X,n)$ similar to $\mathrm{Bock}_\ell(X,n)$:

$$(1.4.8) \qquad \mathrm{Bock}_p(X,n) = (\cdots \to H^i(X, \mathbb{Z}_p(n)) \xrightarrow{d^i} H^{i+1}(X, \mathbb{Z}_p(n)) \to \cdots).$$

The eigenvalues of $\gamma = \sigma^a$ on $H^i(X_{\overline{k}}, \mathbb{Q}_p(n))$ are the quotients q^n/λ_{ij}, where the λ_{ij}'s are those eigenvalues of $(F^a)^*$ on $H^i(X/W(k)) \otimes K$ for which $v_q(\lambda_{ij}) = n$ ([48], 5.4)). It is again true (and formal, by ([48], Lemmas 5.1 and 5.2)) that if the minimal polynomial of $(\sigma^a)^*$ acting on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_p(n))$ does not have 1 as a multiple root, then $\mathrm{Bock}_p(X,n)$ has finite length cohomology groups, and its *multiplicative Bockstein characteristic* is defined as

$$(1.4.9) \qquad \chi(X, \mathbb{Z}_p(n))^\times := \prod_i [H^i(\mathrm{Bock}_p(X,n))]^{(-1)^i} = \prod_i z(f_i)^{(-1)^i},$$

and sometimes denoted $\chi(X, \mathbb{Z}_p(n), e)$, where

$$f_i : H^i(X_{\overline{k}}, \mathbb{Z}_p(n))^G \to H^i(X_{\overline{k}}, \mathbb{Z}_p(n))_G$$

is defined in the same way as in the ℓ-adic case. However, $z(f_i)$ is not given by $|P_i^*(X, q^{-n})|_p$ (as in (i) above) but involves a factor which is a power of q, namely

$$|z(f_i)|_p = |P_i^*(q^{-n})|_p \cdot q^{(\sum_{j, v_q(\alpha_{i,j})<n} -(n - v_q(\alpha_{i,j})) + T^{n-1,i-n}},$$

where $\alpha_{i,j}$ is the set of eigenvalues of F^{*a} on $H^i(X/W(k)) \otimes K$, and $T^{n-1,i-n}$ is the dimension of the connected unipotent part U^i of the perfect group scheme over k that $H^i(X_{\overline{k}}, \mathbb{Q}_p(n))$ underlies ([32], IV 3.3). The integers $T^{r,s}$ are invariants associated to $R\Gamma(X/W(k))$ viewed as an object of $D^b_c(\mathcal{R}_k)$ (([22], (6.1) p. 14), ([33], (3.1.3), ([48], p. 810)). The formula

$$(1.4.10) \qquad |Z^*(X, q^{-n})|_p^{-1} = \chi(X, \mathbb{Z}_p(n))^\times \cdot q^{\chi(X, \mathcal{O}_X, n)}$$

then follows from the (formal) identity

$$\chi(X, \mathbb{Z}_p(n))^\times = \prod_i z(f_i)^{(-1)^i},$$

and the crucial ingredient provided by Crew–Milne's formula ([22], IV Theorem 3.2), already implicitly used in the case of a surface, as we have seen above. The factor $q^{\chi(X,\mathcal{O}_X,n)}$ is sometimes called *Milne's correcting factor*.

The main tool in the proof of (1.4.9) was that $R\Gamma(X/W(k))$ belongs to $D^b_c(\mathcal{R}_k)$ (as recalled at the end of 1.1 (b)). It was therefore tempting to generalize the p-part of Theorem 2 to objects of $D^b_c(\mathcal{R}_k)$. This is carried out in [48], which not only generalizes [43] but offers a better understanding of the proof.

Several reformulations of Theorem 2 with motivic variants of the characteristics $\chi(X, \widehat{\mathbb{Z}}(n))^\times$ defined using Bloch's complexes $\mathbb{Z}(n)$ and Lichtenbaum's Weil étale cohomology [37], with $\mathbb{Z}_\ell$- or $\mathbb{Z}_p$-cohomology replaced by $\mathbb{Z}$-cohomology have been proposed and studied by several authors, starting with Milne himself [43] (see e.g. ([25], Theorem 1.3), [44], [47], [50]).

Posterity of Question 3. The first attempt to address this question was made only very recently, by Mondal [51]. Without any semisimplicity assumption at 1 for the action of Fr^* on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$ ($\ell \neq p$, and $H^{2n}(X_{\overline{k}}, \mathbb{Q}_p(n))$, Mondal defined a *stabilized* multiplicative Bockstein characteristic $\chi_s(X, \widehat{\mathbb{Z}}(n))^\times = \prod_{\ell \neq p} \chi_s(X, \mathbb{Z}_\ell(n))^\times \cdot \chi_s(X, \mathbb{Z}_p(n))^\times$, giving rise to a formula similar to that of Theorem 2 (Corollary 4.1.3)

$$Z^*(X, q^{-n}) = \pm \chi_s(X, \widehat{\mathbb{Z}}(n))^\times q^{\chi(X,\mathcal{O}_X,n)}.$$

The problem with removing the semisimplicity assumption is that the nilpotent part of $1-\gamma$ on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$ may have nontrivial Jordan blocks, which makes the dimension of the kernel of $1-\gamma$ strictly smaller than the number of eigenvalues 1 of γ, hence the invariant $z(f_{2n})$, for $f_{2n} : H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))^G \to H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))_G$ is no longer defined. Mondal's ingenious idea is to kill the Jordan blocks by taking powers of $1-\gamma$ and making a suitable rescaling. Actually, Mondal works in the p-adic context, but his technique applies to the ℓ-adic one as well, as he suggested ([51], 7.7). For the p-part, Mondal works more generally with perfect objects of $D_{\mathrm{qc}}(\mathrm{Spec}(k)^{\mathrm{syn}})$, of which $Rf^{\mathrm{syn}}_*\mathcal{O}$ for $f : X \to \mathrm{Spec}(k)$ proper and smooth is an example, with $R\Gamma(X/W(k))$ the underlying object of $D(W(k))$ (cf. (1.8)). Using Ekedahl's equivalence between $D^b_c(\mathcal{R}_k)$ and $D_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$, that will be reviewed in section 6, Mondal can recover Theorem 0.1 of [48], in the sense that under the semisimplicity assumption of *loc. cit.* Mondal's stabilized Bockstein characteristic χ_s coincides with χ of *loc. cit.* but the method is totally different.

2. Bockstein stabilization

2.1. Let $\mathcal{A}$ be an abelian category, $M \in D(\mathcal{A})$, $u \in \mathrm{End}(M)$, $L = L(M,u) := \mathrm{Fib}(u)$, so that we have a distinguished triangle

$$L \xrightarrow{a} M \xrightarrow{u} M \xrightarrow{b} L[1]. \tag{2.1.1}$$

Let

$$d_u := b \circ a : L \to L[1].$$

We have

$$(d_u)[1] \circ d_u = b[1] \circ a[1] \circ b \circ a$$

hence

$$(d_u)[1] \circ d_u = 0 \tag{2.1.2}$$

as $a[1] \circ b = 0$ in the triangle $M \xrightarrow{b} L[1] \xrightarrow{a[1]} M[1] \xrightarrow{u[1]} M[1]$ rotated from (2.1.1). So we get a complex of objects (of the additive category $D(\mathcal{A})$):

$$B(M,u) = (\cdots \to L \xrightarrow{d_u} L[1] \xrightarrow{d_u[1]} L[2] \to \cdots), \tag{2.1.3}$$

generalizing the construction in (1.4.5). It produces a complex of objects of $\mathcal{A}$

$$\mathrm{Bock}(M,u) = (\cdots \to H^i(L) \xrightarrow{H^i(d_u)} H^{i+1}(L) \to \cdots), \tag{2.1.4}$$

called the *Bockstein complex* of u (denoted $\mathrm{Bock}^\bullet(M,u)$ in [51]).

Here we implicitly worked with the *naive* derived category. As explained in ([51], 4.6) the above constructions can be upgraded to the context of stable derived ∞-categories. Replacing M by $\widehat{M} := \lim_n \mathrm{Cofib}(u^n)$, and viewing $(\cdots \to \widehat{M} \xrightarrow{u} \widehat{M} \xrightarrow{u} \cdots)$ as a decreasing filtered object $(\widehat{M}, F^*)$ of the derived ∞-category $D(\mathcal{A})$, d_u underlies the map $\delta : \mathrm{gr}^i \to \mathrm{gr}^{i+1}[1]$ arising from the fiber sequence $\mathrm{gr}^{i+1} \to F^i/F^{i+2} \to \mathrm{gr}^i$. The vanishing (2.1.2) can be upgraded to a null-homotopy from $\delta \circ \delta$ to 0 (see ([1], Lemma 3.18)). In this language, the Bockstein complex (2.1.4) is the E_1-page of the spectral sequence of $(\widehat{M}, F^*)$ (cf. ([1], Theorem 3.31)). The complex $B(M,u)$ (2.1.3) can be re-written

$$B(M,u) = (\cdots \to \mathrm{gr}^{-1}[-1] \xrightarrow{d_u} \mathrm{gr}^0 \xrightarrow{d_u} \mathrm{gr}^1[1] \to \cdots) \tag{2.1.3$'$}$$

(with all the gr^i identified with $L[1]$), and the Bockstein complex (2.1.4), shifted by 1, can be identified with the Beilinson (perverse) cohomology object ${}^B H^0(\widehat{M}, F^*)$ for the Beilinson t-structure on the filtered derived category (cf. ([1], 4.7)).

Let p be a prime number, and take for $\mathcal{A}$ the category of $\mathbb{Z}_p$-modules. Let $M \in D(\mathbb{Z}_p)$. We will say that $M \in D(\mathbb{Z}_p)$ is *of finite length* if M is in D^b, and $H^i(M)$ is of finite length for all i. If M is of finite length, Mondal defines the *finite length Euler characteristic* of M as

$$\chi^l(M) := \sum_i (-1)^i \mathrm{lg}_{\mathbb{Z}_p} H^i(M). \tag{2.1.5}$$

One defines the *multiplicative Euler characteristic* of M as

$$\chi(M)^\times := p^{\chi^l(M)} = \prod_i [H^i(M)]^{(-1)^i}. \tag{2.1.6}$$

Finally, for $(M \in D^b(\mathbb{Z}_p), u \in \mathrm{End}(M))$, *if* $\mathrm{Bock}(M,u)$ *(2.1.4) is of finite length*, Mondal calls the finite length Euler characteristic

$$\chi^l(\mathrm{Bock}(M,u)) = \sum_i (-1)^i \mathrm{lg}_{\mathbb{Z}_p} H^i(\mathrm{Bock}(M,u))$$

the *Bockstein characteristic* of (M,u). Associated to it is the *multiplicative Bockstein characteristic*

$$\chi^l(\mathrm{Bock}(M,u))^\times := p^{\chi^l(\mathrm{Bock}(M,u))} = \prod_i [H^i(\mathrm{Bock}(M,u))]^{(-1)^i}$$

If M is of finite length, so is $\mathrm{Bock}(M,u)$, and

$$\chi^l(\mathrm{Bock}(M,u)) = \chi^l(M).$$

But it may happen that M is not of finite length but $\mathrm{Bock}(M,u)$ is, as the example of $(M = \mathbb{Z}_p, u = 0)$ shows (see example (b) below).

Examples 2.2. (a) Let $\beta_p \in \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Z}/p, \mathbb{Z}/p)$ be the class of the extension

$$0 \to \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

It generates the usual *Bockstein operation* $H^i(X, \mathbb{Z}/p) \to H^{i+1}(X, \mathbb{Z}/p)$ in the cohomology of a space (or topos) X. The morphism of distinguished triangles, where $\mathbb{Z}/p[-1] = L(\mathbb{Z}_p, p)$,

$$\begin{array}{ccccccc}
\mathbb{Z}/p[-1] & \xrightarrow{a} & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{b} & \mathbb{Z}/p \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle b} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\
\mathbb{Z}/p[-1] & \xrightarrow{\beta} & \mathbb{Z}/p & \xrightarrow{p} & \mathbb{Z}/p^2 & \longrightarrow & \mathbb{Z}/p
\end{array}$$

shows that

$$\beta = d_p : \mathbb{Z}/p[-1] \to \mathbb{Z}/p.$$

(b) For a $\mathbb{Z}_p$-module M placed in degree zero and an endomorphism u of M, set $M^u := \operatorname{Ker}(u)$, $M_u := \operatorname{Coker}(u)$. Then $L(u)$ is the complex $M \xrightarrow{u} M$ concentrated in degrees 0 and 1, and $\operatorname{Bock}(M, u)$ is the complex, concentrated in degrees 0 and 1

$$\operatorname{Bock}(M, u) = (M^u \xrightarrow{d_u} M_u)$$

where d_u is the composition of the inclusion $M^u \hookrightarrow M$ and the projection $M \twoheadrightarrow M_u$. For $M = H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n)), u = 1 - \gamma$ (and ℓ instead of p), this map d_u is the map f_i in the diagram (1.4.5'). *When* $\operatorname{Bock}(M, u)$ *is of finite length*, we have

$$\chi^l(\operatorname{Bock}(M, u))^\times = z(d_u) = [\operatorname{Ker}(d_u)]/[\operatorname{Coker}(d_u)],$$

with the notation of [62] (see (i) after (1.4.5)).

(c) For $M \in D^b(\mathbb{Z}_p), u \in \operatorname{End}(M))$, the Bockstein complex $\operatorname{Bock}(M, u)$ fits in the middle row of a commutative diagram similar to (1.4.5'), where $L(M, u) := \operatorname{Fib}(u)$:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \uparrow & & \downarrow \\
H^{i-2}(M)_u & & H^i(M)^u & \xrightarrow{f_i} & H^i(M)_u \\
\delta\downarrow & & \uparrow & & \delta\downarrow \\
H^{i-1}(L(M,u)) & \xrightarrow{d^{i-1}} & H^i(L(M,u)) & \xrightarrow{d^i} & H^{i+1}(L(M,u)) \\
\downarrow & & \delta\uparrow & & \downarrow \\
H^{i-1}(M)^u & \xrightarrow{f_{i-1}} & H^{i-1}(M)_u & & H^{i+1}(M)^u \\
\downarrow & & \uparrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

The restriction of the upper middle vertical map to $\operatorname{Ker}(d^i)$ sends it surjectively to $\operatorname{Ker}(f_i)$, and induces an exact sequence

$$0 \to \operatorname{Coker}(f_{i-1}) \to H^i(\operatorname{Bock}(M, u)) \to \operatorname{Ker}(f_i) \to 0.$$

In particular, as observed in (ii) before (1.4.6), $\operatorname{Bock}(M, u)$ is of finite length if and only if the complexes $(f_i : H^i(M)^u \to H^i(M)_u)$ are of finite length, and in this case

$$\chi^l(\operatorname{Bock}(M, u))^\times = \prod_i z(f_i)^{(-1)^i}. \tag{2.2.1}$$

2.3. Let γ be the automorphism of $M = \mathbb{Z}_p^2$ given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then, 1 is not a simple root of the minimal polynomial of $\gamma \otimes \mathbb{Q}_p$, the map $f(\gamma) : M^\gamma \to M_\gamma$ induced by the identity of M is not a p-quasi-isomorphism: if $u := 1 - \gamma$, we have $\mathrm{Bock}(M, u) = (\mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p)$, which is not of finite length. Let $Q(M, \gamma, t) := \det(1 - \gamma t : M \otimes \mathbb{Q}_p \to M \otimes \mathbb{Q}_p)$. We have $Q(M, t, \gamma) = (1-t)^2 = (1-t)^2 Q^*(\gamma, 1)$, with $Q^*(\gamma, 1) = 1$, but no formula for $Q^*(\gamma, 1)$ of the type of those in (i) following diagram (1.4.5') is available. However, if γ is replaced by γ^m for $m \geqslant 2$, then $Q(M, \gamma^m, t) = Q(M, \gamma, t) = (1-t)^2$, but this time γ^m is semisimple, $f(\gamma^m) : M^{\gamma^n} \to M_{\gamma^n}$ is the identity of $\mathbb{Z}_p$, and $Q^*(\gamma^m, 1) = z(f(\gamma^m)) = 1$. This observation is at the origin of Mondal's stabilization process. In general, for an automorphism γ, and questions relative to the eigenvalue 1, it is essentially equivalent, but technically more convenient, to work with powers of $u = 1 - \gamma$ rather than with powers of γ.

The basic homological algebra input for this is provided by the following propositions 1 and 2.

Proposition 1 ([51], Proposition 5.1). Let $M \in D^b(\mathbb{Z}_p)$, $u \in \mathrm{End}(M)$. Assume that the following (i), (ii) hold:

(i) $M \otimes \mathbb{Q}_p$ is a perfect complex of $\mathbb{Q}_p$-vector spaces;

(ii) $L(M, u) := \mathrm{Fib}(u)$ is a perfect complex of $\mathbb{Z}_p$-modules.

Then, there exists an integer $r_0 \geqslant 0$ such that, for all $r \geqslant r_0$,

(1) $\mathrm{Bock}(M, u^r)$ (2.1.4) is of finite length,

(2) $\chi^l(\mathrm{Bock}(M, u^r))/r$ is an integer, independent of $r \geqslant r_0$.

Sketch of proof. Easy dévissages reduce the proof of (1) and (2) to the case where M is a single $\mathbb{Z}_p$-module, placed in degree zero.

By (ii), M^u and M_u are finitely generated. By example (b), $\mathrm{Bock}(M, u) := (M^u \xrightarrow{d_u} M_u)$. So $\mathrm{Bock}(M, u)$ would be of finite length if and only if d_u was a p-quasi-isomorphism, i.e., if and only if $d_u \otimes \mathbb{Q}_p$ was an isomorphism. By (i), $M \otimes \mathbb{Q}_p$ is a finite dimensional $\mathbb{Q}_p$-vector space. So $d_u \otimes \mathbb{Q}_p : (M \otimes \mathbb{Q}_p)^u \to (M \otimes \mathbb{Q}_p)_u$ would be an isomorphism if and only if $u \otimes \mathbb{Q}_p$ had no nontrivial Jordan block for the eigenvalue zero, i.e., zero would be a simple root of the minimal polynomial of $u \otimes \mathbb{Q}_p$, equivalently the multiplicity of the eigenvalue zero would equal the dimension of $(M \otimes \mathbb{Q}_p)^u$. This might not be the case. But let r_0 be the smallest integer $\geqslant 1$ such that $u^{r_0} \otimes \mathbb{Q}_p$ annihilates the nilpotent summand of $M \otimes \mathbb{Q}_p$, or, equivalently, such that, in the increasing chain of $\mathbb{Q}_p$-vector spaces

$$\mathrm{Ker}(u \otimes \mathbb{Q}_p) \subset \mathrm{Ker}((u \otimes \mathbb{Q}_p)^2) \subset \cdots \subset \mathrm{Ker}((u \otimes \mathbb{Q}_p)^r) \subset \cdots,$$

we have

$$\mathrm{Ker}((u \otimes \mathbb{Q}_p)^{r_0}) = \mathrm{Ker}((u \otimes \mathbb{Q}_p)^r)$$

for all $r \geqslant r_0$. For $r \geqslant r_0$, no nontrivial Jordan block for u^r relative to the eigenvalue 0 can survive, hence $\mathrm{Bock}(M, u^r)$ is of finite length, which proves (1).

To prove (2), using Lemma 2.3.1 below to calculate $\chi^l(\mathrm{Bock}(M, u^{r(r+1)}))$ in two ways, one finds that, for $r \geqslant r_0$, we have

$$(r+1)\chi^l(\mathrm{Bock}(M, u^r)) = r\chi^l(\mathrm{Bock}(M, u^{r+1})),$$

which shows that $\chi^l(\mathrm{Bock}(M, u^r))$ is divisible by r, and the quotient $\chi^l(\mathrm{Bock}(M, u^r))/r$ is an integer independent of $r \geqslant r_0$.

Lemma 2.3.1. ([51], Corollary 4.14). Let u, v be endomorphisms of a $\mathbb{Z}_p$-module M. Assume that $\mathrm{Fib}(u)$ and $\mathrm{Fib}(v)$ are perfect complexes, and that $\mathrm{Bock}(M, u)$, $\mathrm{Bock}(M, v)$, and $\mathrm{Bock}(M, uv)$ are of finite length. Then either of the squares

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{Id}} & M \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle uv} \\ M & \xrightarrow{u} & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{v} & M \\ {\scriptstyle uv}\downarrow & & \downarrow{\scriptstyle u} \\ M & \xrightarrow{\mathrm{Id}} & M, \end{array}$$

induces a fiber sequence $\mathrm{Fib}(v) \to \mathrm{Fib}(uv) \to \mathrm{Fib}(u) \to$, and

$$\chi^l(\mathrm{Bock}(M, uv)) = \chi^l(\mathrm{Bock}(M, u)) + \chi^l(\mathrm{Bock}(M, v)).$$

This follows from the additivity of Bockstein characteristics on fiber sequences $(M', u') \to (M, u) \to (M'', u'')$, where the fibers of (M', u') and (M'', u'') are perfect complexes, and $\mathrm{Bock}(M', u')$, $\mathrm{Bock}(M'', u'')$, *and* $\mathrm{Bock}(M, u)$ are of finite length. The difficulty here is that, in general, the formation of the complex $B(M, u)$ does not commute with extensions, and in a fiber sequence as above, it may happen that $\mathrm{Bock}(M', u')$ and $\mathrm{Bock}(M'', u'')$ are of finite length, but $\mathrm{Bock}(M, u)$ is not, as already the example of $M = \mathbb{Z}_p^2$, u given by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ shows.

Definition 2.4. ([51], Definition 5.2)). For $(M \in D^b(\mathbb{Z}_p), u \in \mathrm{End}(M))$ satisfying the hypotheses (i) and (ii) of Proposition 1, Mondal defines the *stable Bockstein characteristic* of (M, u) as the integer

$$\chi_s(\mathrm{Bock}(M, u)) := \frac{\chi^l(\mathrm{Bock}(M, u^r))}{r}$$

for $r \geqslant r_0$, with r_0 as in the conclusion of Proposition 1.

In analogy with 2.1, one defines the *multiplicative stable Bockstein characteristic* of (M, u) by

$$\chi_s(\mathrm{Bock}(M,u))^\times := p^{\chi_s(\mathrm{Bock}(M,u))} = |\chi_s(\mathrm{Bock}(M,u))|_p^{-1}.$$

In the situation of Example 2.2 (b), with M a single $\mathbb{Z}_p$-module, $L(u) = (M \overset{u}{\to} M)$ a perfect complex of $\mathbb{Z}_p$-modules and $M \otimes \mathbb{Q}_p$ a perfect complex of $\mathbb{Q}_p$-vector spaces, it is convenient to denote the multiplicative stable Bockstein characteristic of u by

$$z_s(d_u) := p^{\frac{\chi^l(d_{u^r})}{r}} \tag{2.4.1}$$

for $r \geqslant r_0$, where $(d_{u^r} : M^{u^r} \to M_{u^r}) = \mathrm{Bock}(M, u^r)$.

The stable Bockstein characteristic enjoys better properties than the Bockstein characteristic. It is additive under fiber sequences (and hence on compositions): if (u', u, u'') is an endomorphism of a fiber sequence $M' \to M \to M''$ in $D^b(\mathbb{Z}_p)$ such that $M' \otimes \mathbb{Q}_p$, $M'' \otimes \mathbb{Q}_p$ are perfect complexes (in $D(\mathbb{Q}_p)$), and $\mathrm{Fib}(u')$ and $\mathrm{Fib}(u'')$ are perfect complexes (in $D(\mathbb{Z}_p)$), then ([51], Proposition 5.3)

$$\chi_s(\mathrm{Bock}(M,u)) = \chi_s(\mathrm{Bock}(M',u')) + \chi_s(\mathrm{Bock}(M'',u'')). \tag{2.4.2}$$

2.5. **Proposition 2.** ([51], Proposition 5.5) Let M be a finitely generated $\mathbb{Z}_p$-module, and $u \in \mathrm{End}(M)$. If $(\alpha_i)_{i \in I}$ is the set (possibly empty) of nonzero eigenvalues in $\overline{\mathbb{Q}}_p$ of $u \otimes \mathbb{Q}_p$, then, with the notation (2.4.1),

$$z_s(d_u) = |\prod_{i \in I} \alpha_i|_p. \tag{2.5.1}$$

(In particular, $z_s(d_u) = 1$ if $I = \emptyset$. This is the case, for example, when M is torsion.)

The proof is an easy application of the stability properties of the stable Bockstein characteristic. One can reformulate (2.5.1) in the following way. Write

$$\det(T - u \otimes \mathbb{Q}_p) = T^r R(T)$$

with r the multiplicity of the eigenvalue zero of $u \otimes \mathbb{Q}_p$, and $R(0) \neq 0$. Then

$$z_s(d_u) = |R(0)|_p. \tag{2.5.2}$$

In this way, Proposition 2 appears as a generalization of ([62], Lemma z.4, p. 434) (see also ([48], Lemma 5.1) (cited in point (i) of the discussion of the proof of Theorem 2), as in the case where u is a p-quasi-isomorphism, $z_s(d_u) = z(d_u)$ (in this case, $r = \mathrm{rk}(M^u)$, but in general only $\mathrm{rk}(M^u) \leqslant r$ holds).

Remark 2.6.[12] It is straightforward to extend the above definitions and results to the case $\mathbb{Z}_p$ is replaced by a discrete valuation ring R, p by a uniformizer π, and $\mathbb{Q}_p$ by the field of fraction K of R. For $M \in D^b(R)$ and $u \in \mathrm{End}(M)$, such that $M \otimes K$ is a perfect complex of K-vector spaces and $\mathrm{Fib}(u)$ is a perfect complex of R-modules, the analogue of the conclusion of Proposition 1 holds, and one defines the *stable Bockstein characteristic* of (M, u) by

$$\chi_s(\mathrm{Bock}(M, u)) = \frac{\chi_l(\mathrm{Bock}(M, u^r))}{r}$$

for $r \geqslant r_0$. When the residue field $k = R/(\pi)$ is finite, one defines the *multiplicative stable Bockstein characteristic* of (M, u) by

$$\chi_s(\mathrm{Bock}(M, u))^\times = (\sharp k)^{\chi_s(\mathrm{Bock}(M,u))}.$$

3. Application to zeta values: the ℓ-adic case

3.1. We come back to the situation considered at the beginning of *Posterity of Question 3*. Let X be proper, smooth over $k = \mathbb{F}_q$, $q = p^a$, $\overline{k}$ an algebraic closure of k, $G = \mathrm{Gal}(\overline{k}/k)$, with generator σ^a, where $\sigma x = x^p$. Let ℓ be a prime different from p. Fix $n \in \mathbb{Z}$. We define the special value $Z^*(X, q^{-n})$ of $Z(X, t)$ at $t = q^{-n}$ as in (1.2.1):

$$Z^*(X, q^{-n}) := \lim_{t \to q^{-n}} (1 - q^n t)^{\rho_n} Z(X, t),$$

where ρ_n is the order of the zero of $P_{2n}(X, t)$ at $t = q^{-n}$. Let γ be the automorphism of $R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n))$ given by the relative Frobenius Fr^*, or the inverse of σ^a in the Galois action by transportation of structure (so that ρ_n is the multiplicity of the eigenvalue 1 of γ on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$). We do not assume that 1 is a simple root of the minimal polynomial of γ.

Let u the endomorphism $1 - \gamma$ of $R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n))$. As $R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n))$ is a perfect complex of $\mathbb{Z}_\ell$-modules, the same is true of $\mathrm{Fib}(u)$, and $R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell(n))$ is a perfect complex of $\mathbb{Q}_\ell$-vector spaces. Thus, imitating Mondal's definition ([51], 1.3, 6.4), we define the *stable Bockstein ℓ-adic characteristic* of $(X, \mathbb{Z}_\ell(n))$ by

$$\chi_s(X, \mathbb{Z}_\ell(n)) := \chi_s(\mathrm{Bock}(R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n)), u)) \tag{3.1.1}$$

[12] This observation is due to Mondal (private communication).

and its multiplicative associate

$$\chi_s(X, \mathbb{Z}_\ell(n))^\times = \ell^{\chi_s(X, \mathbb{Z}_\ell(n))}.$$

Theorem 3. With the above hypotheses and notation, we have

$$|Z^*(X, q^{-n})|_\ell^{-1} = \chi_s(X, \mathbb{Z}_\ell(n))^\times \tag{3.1.2}$$

This is an ℓ-adic analog of Mondal's main theorem ([51], Theorem 6.5). It generalizes the ℓ-adic part of Milne's theorem Theorem 2, as 1 is not assumed to be a simple root of the minimal polynomial of Fr^* on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_\ell(n))$, and when 1 is a simple root, then the left hand side is $\chi(X, \mathbb{Z}_\ell(n))^\times$ of Theorem 2.

Proof. Since $H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n))$ is finitely generated over $\mathbb{Z}_\ell$ (hence $\mathrm{Fib}(H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n)), u)$ perfect), by the additivity of stable Bockstein characteristics under fiber sequences, we have

$$\chi_s(X, \mathbb{Z}_\ell(n)) = \sum_i (-1)^i \chi_s(\mathrm{Bock}(H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n)), u)),$$

hence

$$\chi_s(X, \mathbb{Z}_\ell(n))^\times = \prod z_s(H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n)), d_u)^{(-1)^i}.$$

By Proposition 2 in 2.5, we have

$$z_s(H^i(X_{\overline{k}}, \mathbb{Z}_\ell(n)), d_u) = |\prod_j \alpha_{i,j}|_\ell$$

where $\alpha_{i,j}$ runs through the nonzero eigenvalues $\alpha_{i,j}$ of $1-\gamma$ on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(n))$, i.e. the $1 - q^{-n}\lambda_{i,j}$, where the $\lambda_{i,j}$'s are the eigenvalues of Fr^* on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ distinct from q^n. Hence, with the notation of (1.2.1),

$$\prod_j \alpha_{i,j} = P_i^*(X, q^{-n}),$$

yielding (3.1.2), since $Z^*(X, q^{-n}) = \prod_i P_i^*(X, q^{-n})^{(-1)^{i+1}}$.

3.2. Under the assumptions of Theorem 3, there is, however, no visible relation between $\chi_s(X, \mathbb{Z}_\ell(n))^\times$ and torsion in $H^*(X, \mathbb{Z}_\ell(n))$, generalizing (1.4.7'). The definition of the stable Bockstein characteristics involves taking rth powers of $1-\gamma$. These are loosely related to $1-\gamma^r$, but γ^r is a generator of $\mathrm{Gal}(\overline{k}/\mathbb{F}_{q^r})$, so $1-\gamma^r$ is related to poles at n of $Z(X_{\mathbb{F}_{q^r}}, t)$.

4. Application to zeta values: the p-adic case, via coherent complexes over the Raynaud ring

4.1. **Theorem 4** ([51], Theorem 1.1, Remark 1.4). Let X/k be proper and smooth, let $R\Gamma(X, \mathbb{Z}_p(n))$ and $R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n))$ be defined as in 1.4, Theorem 2, i.e.,

$$R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)) := R\Gamma(X_{\overline{k}}, W\Omega^n_{\log}[-n]),$$

$$R\Gamma(X, \mathbb{Z}_p(n)) := R\Gamma(X, W\Omega^n_{\log}[-n]) \xrightarrow{\sim} \mathrm{Fib}(1-\gamma : R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)) \to R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n))),$$

where $\gamma := \sigma^a \in \mathrm{Gal}(\overline{k}/k)$. Let $u := 1 - \gamma$.

(1) (i) $R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p$ is a perfect complex of $\mathbb{Q}_p$-vector spaces.

(ii) $R\Gamma(X, \mathbb{Z}_p(n))(= \mathrm{Fib}(u : R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)) \to R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n))$ is a perfect complex of $\mathbb{Z}_p$-modules.

(2) Let

$$\chi_s(X, \mathbb{Z}_p(n)) := \chi_s(\mathrm{Bock}(R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)), u))$$

be the stable Bockstein characteristic of $(X, \mathbb{Z}_p(n))$, which is defined thanks to (1), and

$$\chi_s(X, \mathbb{Z}_p(n))^\times := p^{\chi_s(X, \mathbb{Z}_p(n))}$$

be its multiplicative associate. Then we have

$$|Z^*(X, q^{-n})|_p^{-1} = \chi_s(X, \mathbb{Z}_p(n))^\times \cdot q^{\chi(X, \mathcal{O}_X, n)}, \tag{4.1.1}$$

where

$$\chi(X, \mathcal{O}_X, n) := \sum_{j, i \leqslant n} (-1)^{i+j}(n-i)h^{ij},$$

with $h^{ij} := \dim H^j(X, \Omega^i_{X/k})$, as in Theorem 2.

Remark 4.1.1. When the minimal polynomial of γ acting on $H^{2n}(X_{\overline{k}}, \mathbb{Q}_p(n))$ does not have 1 as a multiple root, then

$$\chi_s(X, \mathbb{Z}_p(n))^\times = \chi(X, \mathbb{Z}_p(n))^\times,$$

and (4.1.1) reduces to (1.4.10).

Remark 4.1.2. In the proof of Theorem 3, the finite generation over $\mathbb{Z}_\ell$ of the $H^i(X_{\overline{k}}, \mathbb{Z}_\ell)$'s, which is the starting point, makes the calculation of the stable Bockstein characteristic $\chi_s(\mathrm{Bock}(R\Gamma(X_{\overline{k}}, \mathbb{Z}_\ell(n)), u))$ very easy. In the situation of Theorem 4, the analogous cohomology groups $H^i(X_{\overline{k}}, \mathbb{Z}_p(n))$'s are not necessarily finitely generated over $\mathbb{Z}_p$, due to the possible presence of infinite torsion in them. This renders the calculation of the analogous stable Bockstein characteristic much more difficult.

Combining Theorem 3 and Theorem 4, we get:

Corollary 4.1.3. Let X/k be proper and smooth, and let $n \in \mathbb{Z}$. With the notation of (3.1.1) and 4.1 (2), let

$$\chi_s(X, \widehat{\mathbb{Z}}(n))^\times := \prod_{\ell \neq p} \chi_s(X, \mathbb{Z}_\ell(n))^\times \cdot \chi_s(X, \mathbb{Z}_p(n))^\times.$$

Then we have

$$Z^*(X, q^{-n}) = \pm \chi_s(X, \widehat{\mathbb{Z}}(n))^\times \cdot q^{\chi(X, \mathcal{O}_X, n)}.$$

Example 4.1.4. Let X/k be a proper and smooth surface, and let $n = 1$. Recall that, by (1.3.2) and (1.3.4a), we have

$$Z^*(X, q^{-1}) = -\frac{[A(k)]^2}{q^{-1}(1-q)^2 q^{\dim(A)} P_2^*(X, q^{-1})},$$

where A is the Albanese variety of X. As $\chi(X, \mathcal{O}_X, 1) = \chi(X, \mathcal{O}_X)$, by Corollary 4.1.3, this gives

$$P_2^*(X, q^{-1}) = \pm \frac{[A(k)]^2}{(1-q)^2 \chi_s(X, \widehat{\mathbb{Z}}(1))^\times \cdot q^{\alpha(X)}},$$

where $\alpha(X) = \dim(A) + \chi(X, \mathcal{O}_X) - 1$ is the Artin-Tate invariant.[13] Comparing with the conjectural Artin-Tate formula (1.3.4), one finds

$$(4.1.4a) \qquad \pm \frac{[A(k)]^2}{(1-q)^2 \chi_s(X, \widehat{\mathbb{Z}}(1))^\times} = \frac{[\mathrm{Br}(X)] |\det(D_i.D_j)|}{[\mathrm{NS}(X)_{\mathrm{tors}}]^2}.$$

The last remaining cases of (another) conjecture of Artin-Tate on the existence of a symplectic pairing on the (conjecturally finite) Brauer group having been proven by Carmeli-Feng [11], under Conjecture 1 the order of $\mathrm{Br}(X)$ is a square, and thus, by (4.1.4a) the rational number

$$(4.1.4b) \qquad |\chi_s(X, \widehat{\mathbb{Z}}(1)^\times| \cdot |\det(D_i.D_j)|$$

is a square. However, the invariant (4.1.4b) is defined *unconditionally*. Mondal asks whether one could prove unconditionally that (4.1.4b) is a square.

It is known (unconditionally) that the non-divisible quotient $\mathrm{Br}(X)_{\mathrm{nd}}$ of the Brauer group (i.e., the quotient by the subgroup of divisible elements) is finite (see e.g. ([65], Lemma 1.2) for the ℓ-part; the proof for the p-part is analogous). The above mentioned symplectic pairing is actually constructed

[13] This invariant, which is defined unconditionally, was shown by Mumford ([54], Lecture 27, Theorem p. 196) to be non-negative. Little seems to be known about it.

(unconditionally) on $\mathrm{Br}(X)_{\mathrm{nd}}$. It would be interesting to find a relation between the invariants $\chi_s(X, \widehat{\mathbb{Z}}(1))^\times$ and $\mathrm{Br}(X)_{\mathrm{nd}}$.

4.2. We will deduce Theorem 4 from a general result about the zeta function of objects of Ekedahl's category $D^b_c(\mathcal{R}_k)$ (see end of 1.1 (b)), in the same way as in the proof of ([48], Theorem 0.2).

The graded $\mathcal{R}_k$-module structure on $W\Omega^\bullet_X$ makes $R\Gamma(X, W\Omega^\bullet_X)$ an object of $D^b_c(\mathcal{R}_k)$, a complex of graded $\mathcal{R}_k$-modules, satisfying certain finiteness conditions. It can be viewed as a bicomplex $P(X) = P^{\bullet,\bullet}(X) = (\cdots \to P^{\bullet,j}(X) \to P^{\bullet,j+1}(X) \to \cdots)$, where the first degree corresponds to the degree of the components of $W\Omega^\bullet_X$ (cf. ([32], 2.1) and ([48], 1.6). The grading of $R\Gamma(X, W\Omega^\bullet_X)$ is associated with the *naive* filtration of $W\Omega^\bullet_X$ (not the Nygaard filtration):

$$\mathrm{gr}^i R\Gamma(X, W\Omega^\bullet_X) = R\Gamma(X, W\Omega^i_X[-i]).$$

The simple associated complex is canonically isomorphic to $R\Gamma(X/W(k))$:

$$R\Gamma(X/W(k)) \xrightarrow{\sim} sR\Gamma(X, W\Omega^\bullet_X),$$

a perfect complex of $W(k)$-modules, with the σ-linear endomorphism φ of $R\Gamma(X/W(k))$ corresponding to the endomorphism $\underline{F}$ of $W\Omega^\bullet_X$ given by $p^i F$ in degree i. In particular, $sR\Gamma(X, W\Omega^\bullet_X) \otimes K$ (for $K = K(k)$ the fraction field of $W(k)$) is a perfect complex of $K(k)$-vector spaces, and the cohomology groups $(H^i(sR\Gamma(X, W\Omega^\bullet_X) \otimes K, \varphi)$ $(= (H^i(X/W(k)) \otimes K, \varphi))$ are F-isocrystals, as recalled in 1.1 (b).

It is therefore natural to define the zeta function of an object P of $D^b_c(\mathcal{R}_k)$, as in ([48], p. 802), by

$$Z(P,t) := \prod_i \det(1 - F^{*a}t, H^i(sP \otimes K))^{(-1)^{i+1}}. \tag{4.2.1}$$

The polynomials

$$P_i(P,t) := \det(1 - F^a t, H^i(sP \otimes K))$$

lie *a priori* in $K[t]$, but in fact, in $\mathbb{Q}_p[t]$ as the K-linear endomorphism F^a commutes with the (σ-linear one) F. Hence $Z(P,t)$ lies in $\mathbb{Q}_p(t)$. For $P = P(X) := R\Gamma(X, W\Omega^\bullet_X)$ we recover $Z(X,t)$ (1.1.6).

Let $n \in \mathbb{Z}$, and ρ_n be the order of the *pole* of $Z(P,t)$ at $t = q^{-n}$. One defines $Z^*(P, q^{-n})$ by

$$Z(P,t) = (1 - q^n t)^{-\rho_n} Z^*(P,t) \tag{4.2.2}$$

with $Z^*(P, q^{-n}) \neq 0$, in other words,

$$Z^*(P, q^{-n}) = \lim_{t \to q^{-n}} (1 - q^n t)^{\rho_n} Z(P, t).$$

One can recover $R\Gamma(X, \mathbb{Z}_p(n))$ defined in Theorem 2 from $P(X)$, in the following way. The exact sequence of étale pro-sheaves

$$0 \to W_\bullet \Omega^n_{X,\log} \to W_\bullet \Omega^n_X \xrightarrow{1-F} W_\bullet \Omega^n_X \to 0$$

gives a canonical isomorphism

$$R\Gamma(X, W\Omega^n_{X,\log}[-n]) \xrightarrow{\sim} \mathrm{Fib}(1 - F : R\Gamma(X, W\Omega^n_X[-n]).$$

By the main result of ([45] (see also ([48], Example 4.1)), it can be re-written

$$R\Gamma(X, W\Omega^n_{X,\log}[-n]) \xrightarrow{\sim} R\Gamma_{\mathrm{abs}}(X, \mathbb{Z}_p(n))$$

where

$$R\Gamma_{\mathrm{abs}}(X, \mathbb{Z}_p(n)) := R\mathrm{Hom}_{\mathcal{R}_k}(W(k), R\Gamma(X, W\Omega^\bullet_X)(n)),$$

with the Tate twist (r) for an object $P^{\bullet,\bullet}$ of $D^b_c(\mathcal{R}_k)$ is $P^{\bullet,\bullet}(r) = P^{\bullet+r,\bullet-r}$ as in [45]. We will simply write $R\Gamma(X, \mathbb{Z}_p(n))$ for $R\Gamma_{\mathrm{abs}}(X, \mathbb{Z}_p(n))$. In particular, we have a canonical isomorphism

$$R\Gamma(X, \mathbb{Z}_p(n)) \xrightarrow{\sim} \mathrm{Fib}(1 - \gamma : R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n)) \to R\Gamma(X_{\overline{k}}, \mathbb{Z}_p(n))).$$

For $P \in D^b_c(\mathcal{R}_k)$, as in ([48], Theorem 0.2), we define

$$\text{(4.2.3)} \quad R\Gamma(P, \mathbb{Z}_p(n)) := \mathrm{Fib}(1 - F : \mathrm{gr}^n P[-n]) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{R}_k}(W(k), P(n)).$$

Thus, if

$$P(X) := R\Gamma(X, W\Omega^\bullet_X) \in D^b_c(\mathcal{R}_k),$$

we have

$$\text{(4.2.4)} \qquad R\Gamma(P(X), \mathbb{Z}_p(n)) = R\Gamma(X, \mathbb{Z}_p(n)).$$

In order to apply Mondal's Proposition 1 above, we need the following result, which in particular implies (1) of 4.1.

Proposition 4.3. Let $P \in D^b_c(\mathcal{R}_k)$ and $n \in \mathbb{Z}$. Let u be the endomorphism $1 - \gamma$ of $M := R\Gamma(P_{W(\overline{k})}, \mathbb{Z}_p(n))$. Then:
(i) $M \otimes \mathbb{Q}_p$ is a perfect complex of $\mathbb{Q}_p$-vector spaces.
(ii) $\mathrm{Fib}(u)$ is a perfect complex of $\mathbb{Z}_p$-modules.

Remark. By Ekedahl's equivalence $D^b_c(\mathcal{R}_k) \xrightarrow{\sim} D_{\mathrm{perf}}(k^{\mathrm{syn}}, \mathcal{O})$ ([22], Theorem 5.3), Proposition 4.3 should follow from ([9], Proposition 4.5.1) and ([51], Corollary 2.21), see section 6. We give an independent proof below.

Proof. Up to replacing P by $P(-n)$ we may assume $n = 0$. By (4.2.3) we thus have

$$M = \mathrm{Fib}(1 - F : \mathrm{gr}^0 P_{W(\overline{k})} \to \mathrm{gr}^0 P_{W(\overline{k})}).$$

By dévissage we may assume that P is an elementary coherent graded $\mathcal{R}_k$-module (([33], 2.4), ([48], 1.4)), of type I or II as in ([48], 1.3).

Type I. P is a $\mathcal{R}^0_k$-module placed in one degree, which is finitely generated over $W(k)$. We may assume that P is in degree zero, otherwise $\mathrm{gr}^0 P = 0$. We will see that:

(*) in this case, M is perfect over $\mathbb{Z}_p$,

which implies (i) and (ii).

By Ekedahl's finer dévissage ([22], p. 10), we may further assume that P is of one of the following types:

• P is free over $W(k)$, with F bijective. Then (cf. ([31], II Lemme 5.3)) $P_{W(\overline{k})}$ possesses a $W(\overline{k})$-basis consisting of fixed points of F, which reduces to $P_{W(\overline{k})} = W(\overline{k})$, in which case $M = \mathbb{Z}_p$.

• P is free over $W(k)$, with F (and V) topologicaly nilpotent. Then $1-F$ on $P_{W(\overline{k})}$ is bijective, and $\mathrm{Fib}(1 - F) = 0$.

• P is killed by p, with F bijective. Then $P_{\overline{k}}$ possesses a $\overline{k}$-basis consisting of fixed points of F, so we are reduced to $P_{\overline{k}} = \overline{k}$, in which case $M = \mathbb{F}_p$.

• $P = k$, with $F = V = 0$. Then $\mathrm{Fib}(1 - F) = 0$.

In all these cases, (*) holds, which finishes the proof for Type I.

Type II. $P = \mathbf{U}_i[m]$ for some $i \in \mathbb{Z}$ and $m \in \mathbb{Z}$. As $\mathbf{U}_i$ is concentrated in degrees 0 and 1, we may assume that $m = 0$ or $m = 1$.

For $m = 0$, $\mathrm{gr}^0(\mathbf{U}_i)_{\overline{k}} = (\mathbf{U}_i)^0_{\overline{k}} = \prod_{n \geqslant 0} \overline{k} V^n$. We have

$$(1 - F)(\sum_{n \geqslant 0} a_n V^n) = a_0 - (a_0)^p + \sum_{n \geqslant 1} a_n V^n.$$

So $1 - F$ is surjective on $(\mathbf{U}_i)^0_{\overline{k}}$, with kernel $\mathbb{F}_p$. So $M = \mathbb{F}_p$, $u = 1 - \gamma$ is zero and $\mathrm{Fib}(u) = \mathbb{F}_p \oplus \mathbb{F}_p[-1]$.

For $m = 1$, $\mathrm{gr}^0(\mathbf{U}_i[1])_{\overline{k}} = (\mathbf{U}_i)^1_{\overline{k}} = \prod_{n \geqslant i} \overline{k} dV^n$.

We have

$$(1 - F)(\sum_{n \geqslant i} a_n dV^n) = \sum_{n \geqslant i} (a_n - a^p_{n+1}) dV^n.$$

So $1-F$ is surjective on $(\mathbf{U}_i)^1_{\overline{k}}$, with kernel isomorphic to $\overline{k}$, with basis $e_i = \sum_{n\geqslant i} dV^n$ with $\lambda e_i = \sum_{n\geqslant i} \lambda^{p^{-(n-i)}} dV^n$. Thus

$$M = \overline{k} \cdot e_i.$$

So (i) follows trivially: $M \otimes \mathbb{Q}_p = 0$.

Since γ sends λe_i to $\lambda^p e_i$, we have

$$\mathrm{Fib}(u) = \mathbb{F}_p,$$

which finishes the proof.

By 2.3, Proposition 1, we get:

Corollary 4.4. Let $P \in D^b_c(\mathcal{R}_k)$, $n \in \mathbb{Z}$, and

$$M = R\Gamma(P_{W(\overline{k})}, \mathbb{Z}_p(n)) := R\mathrm{Hom}_{\mathcal{R}_{\overline{k}}}(W(\overline{k}), P_{W(\overline{k})}(n))$$

as in 4.3. The multiplicative stable Bockstein characteristic (Definition 2.4)

$$\chi_s(P, \mathbb{Z}_p(n))^\times := \chi_s(\mathrm{Bock}(M, 1-\gamma))^\times \tag{4.4.1}$$

is defined.

Finally, we recall Ekedahl's definition of the *Hodge numbers* of $P \in D^b_c(\mathcal{R}_k)$:

$$h^{i,j}(P) = \dim_k H^j((\mathcal{R}_k)_1 \otimes^L_{\mathcal{R}_k} P)^i. \tag{4.4.2}$$

For $P = P(X)$, $h^{i,j}(P) = \dim_k H^j(X, \Omega^i_{X/k})$.

Theorem 4 will follow from the following theorem;

4.5. **Theorem 5**. Let $P \in D^b_c(\mathcal{R}_k)$, $n \in \mathbb{Z}$. With the notation (4.2.2) and (4.4.1), (4.4.2), we have

$$|Z^*(P, q^{-n})|_p^{-1} = \chi_s(P, \mathbb{Z}_p(n))^\times \cdot q^{\chi(P,n)}, \tag{4.5.1}$$

where

$$\chi(P, n) := \sum_{j, i\leqslant n} (-1)^{i+j}(n-i)h^{i,j}(P),$$

with $h^{i,j}(P)$ as in (4.4.2).

Proof. First note that $\chi(P, n)$ is additive on distinguished triangles $P' \to P \to P'' \to$ in $D^b_c(R_k)$. Indeed, such a triangle gives a triangle $P'(n) \to P(n) \to P''(n) \to$, and then a triangle $\mathcal{R}_1 \otimes^L_R P'(n) \to \mathcal{R}_1 \otimes^L_R P(n) \to \mathcal{R}_1 \otimes^L_R P''(n) \to$ in $D^b_c(k[d])$, and in particular, for each $i \in \mathbb{Z}$, a triangle in

$D^b_c(k)$, $(\mathcal{R}_1 \otimes^L_R P'(n))^i \to (\mathcal{R}_1 \otimes^L_R P(n))^i \to (\mathcal{R}_1 \otimes^L_R P''(n))^i \to$, from which the assertion follows, as

$$\sum_{j,i\leqslant n}(-1)^{i+j}(n-i)h^{i,j}(Q) = \sum_{i\leqslant n}(-1)^i(n-i)(\sum_j(-1)^j(h^{i,j}(Q))$$

for $Q \in D^b_c(\mathcal{R}_k)$.

By (4.2.1) and (4.2.2), we have

$$Z(P,t) = Z(P',t)Z(P'',t),$$

$$\rho_n(P) = \rho_n(P') + \rho_n(P''),$$

hence

$$Z^*(P,q^{-n}) = Z^*(P',q^{-n})Z^*(P'',q^{-n}).$$

Finally, a fiber sequence $P' \to P \to P''$ [14] gives, by (4.2.3), a fiber sequence

$$(R\Gamma(P',\mathbb{Z}_p(n)),u') \to (R\Gamma(P,\mathbb{Z}_p(n)),u) \to (R\Gamma(P'',\mathbb{Z}_p(n)),u''),$$

where u', u, u'' are defined by $1-\gamma$. By Proposition 4.3 and (2.4.2), we have

$$\chi_s(P,\mathbb{Z}_p(n)) = \chi_s(P',\mathbb{Z}_p(n)) + \chi_s(P'',\mathbb{Z}_p(n)),$$

hence

$$\chi_s(P,\mathbb{Z}_p(n))^\times = \chi_s(P',\mathbb{Z}_p(n))^\times \chi_s(P'',\mathbb{Z}_p(n))^\times.$$

By Ekedahl's dévissage recalled in the proof of Proposition 4.3,[15] we may therefore assume that P is of one of the following types:

(a) Type I. P is a $\mathcal{R}^0_k$-module placed in one degree m, which is finitely generated over $W(k)$.

(b) Type II. $P = \mathbf{U}_i[m]$ for some $i \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

Case (a). Up to replacing n by $n-m$, we may assume that $m=0$, i.e., P is a Dieudonné module placed in degree zero. We have

$$Z(P,t) = \det(1 - F^{*a}t, P \otimes K)^{-1},$$

hence

$$Z^*(P,q^{-n}) = (\prod_{\alpha\neq q^n}(1-q^{-n}\alpha))^{-1}, \tag{4.5.2}$$

[14] In the upgraded ∞-version of $D^b_c(\mathcal{R}_k)$. See [36] for this upgrading.

[15] Or rather the upgraded ∞-version of it.

where α runs through the eigenvalues (with multiplicities) of F^{*a} on $P \otimes K$ distinct from q^n.

Suppose first that P is torsion. Then $Z(P,t) = Z^*(P, q^{-n}) = 1$. By 2.5, Proposition 2, we have $\chi_s(P, \mathbb{Z}_p(n))^\times = 1$. By ([48], 2.2 (a)), $h^{i,j}_W(P) = 0$ for all i, j. By Crew–Milne's formula (([48], Corollary 2.9), ([22], IV, Theorem 3.2), ([33], (6.3.5)))

$$\chi(P,n) = \sum_{i \leqslant n} (-1)^i (n-i) (\sum_j (-1)^j h^{i,j}_W(P)), \tag{4.5.3}$$

hence $\chi(P,n) = 0$. One can also argue as follows. One can reduce to the last two cases of the proof of Proposition 4.3 for type I. Using the two step resolution of $\mathcal{R}_1$ by free $\mathcal{R}$-modules ([33], (2.3.3)), one finds that $\mathcal{R}_1 \otimes^L_{\mathcal{R}} P$ is the complex (in grading degree 0 and cohomological degrees 0, -1, -2)

$$k \overset{(F,0)}{\to} k \oplus k \overset{(0,V)}{\to} k,$$

hence $\sum_j (-1)^j h^{i,j}(P) = \sum_j (-1)^j h^{0,j}(P) = 0$.

Therefore, by dévissage, we may assume that P is torsion-free.

Recall that

$$M := R\Gamma(P_{W(\bar{k})}, \mathbb{Z}_p(n)) = \mathrm{Fib}(1 - F : \mathrm{gr}^n P_{W(\bar{k})}[-n] \to \mathrm{gr}^n P_{W(\bar{k})}[-n]).$$

Suppose that $n \neq 0$. Then $M = 0$, so $\chi_s(P, \mathbb{Z}_p(n))^\times = 1$. If $n < 0$, as the slopes of P are $\geqslant 0$, all eigenvalues α are distinct from q^n, and moreover, the q-adic valuation of $q^{-n}\alpha$ for any such α is > 0, hence $|Z^*(P, q^{-n})|_p = 1$. Finally, for $i \leqslant n$, $h^{i,j}(P) = 0$. So (4.5.1) holds. If $n > 0$, as $\mathrm{ord}_q(\alpha) \in [0,1)$, all eigenvalues α are distinct from q^n, and we have $\mathrm{ord}_q(q^{-n}\alpha) < 0$, hence $|q^{-n}\alpha|_p > 1$, and $|1 - q^{-n}\alpha|_p = |q^{-n}\alpha|_p$. By Dieudonné–Manin, we find

$$|\prod \frac{q^n}{\alpha}|_p^{-1} = q^{\sum(n-\alpha)} = q^{e_n(P)}$$

in the notation of ([48], Theorem 2.3), and $q^{e_n(P)} = q^{\chi(P,n)}$ by *loc. cit.* and Crew–Milne's formula (4.5.3)).

We are now left with the case $n = 0$. We have $M = P^{1-F}_{W(\bar{k})}$ (as $1 - F$ is surjective on $P_{W(\bar{k})}$). By Proposition 2 in 2.5 and ([48], 5.5), for $u = 1 - \gamma$,we have

$$\chi_s(P, \mathbb{Z}_p)^\times = z_s(d_u : M^u \to M_u) = |\prod(1-\alpha)|_p$$

where α runs through the nonzero eigenvalues α of F^{*a} on $P_{\bar{k}} \otimes \mathbb{Q}_p$ which are p-adic units (and distinct from 1). The other eigenvalues, which are of positive slope, do not contribute to the p-adic absolute value of $Z^*(P,1)$ (4.5.2). As $\chi(P,0)=0$, (4.5.1) follows.

Case (b). We may assume that $m=0$. As P is killed by p, we have $Z(P,t)=Z^*(P,q^{-n})=1$.

Suppose $n<0$. Then $\mathrm{gr}^n P=0$, hence $M=0$ and $\chi_s(P,\mathbb{Z}_p(n))^\times=1$. For $i\leqslant n$, $h^{i,j}(P)=0$ (4.4.2), hence $\chi(P,n)=0$, and (4.5.1) holds.

Suppose $n=0$. Then $\mathrm{gr}^0 P_{W(\bar{k})}=\bar{k}[[V]]$, and $M=\mathbb{F}_p$, $\mathrm{Fib}(u)=\mathbb{F}_p\oplus\mathbb{F}_p[-1]$, hence

$$\chi_s(\mathrm{Bock}(M,u))=\chi^l(\mathrm{Bock}(M,u))=0$$

Trivially, $\chi(P,0)=0$. So (4.5.1) holds.

Suppose $n=1$. Then $M\xrightarrow{\sim}\bar{k}[-1]$ (as we have seen in the proof of Proposition 4.3, Case II, $m=1$), so, by (2.2 (b)), $\mathrm{Bock}(M,u)=(\mathbb{F}_q\xrightarrow{d_u}0)[-1]$, and, by (2.4.1),

$$\chi_s(\mathrm{Bock}(M,u))^\times=(z_s(d_u))^{-1}=(z(du))^{-1}=q^{-1}.$$

The invariants $T^{r,s}$ of $\mathcal{U}_i$ ([48], p. 210) are zero except for $(r,s)=(0,0)$, with $T^{0,0}=1$. The nonzero Hodge–Witt numbers are $h_W^{0,0}=1$, $h_W^{1,-1}=-2$, $h_W^{2,-2}=1$. By ([48], p. 813, l. 6)) the invariant $e_n(P)=e_1(P)$ is

$$e_1(P)=h_W^{0,0}=1.$$

By Crew–Milne's formula (4.5.3), $\chi(P,1)=q$, hence

$$\chi_s(\mathrm{Bock}(M,u))^\times\cdot\chi(P,1)=1,$$

which gives (4.5.1).

Finally, for $n>1$, $\mathrm{gr}^n P=0$, and $e_n(P)=0$ by ([48], p. 813, l. 6), which completes the proof of Theorem 5.

5. Application to zeta values: the p-adic case via syntomic F-gauges

5.1. Let k be a perfect field of characteristic $p>0$. For a smooth scheme X/k, the category $D_{\mathrm{qc}}(X^{\mathrm{syn}})$ of quasi-coherent complexes on the Bhatt–Lurie's syntomic stack X^{syn} is called the category of *F-gauges over X* (([9], Definition 4.2.1)). The structural map $f:X\to\mathrm{Spec}(k)$ induces a morphism $f^{\mathrm{syn}}:X^{\mathrm{syn}}\to\mathrm{Spec}(k)^{\mathrm{syn}}$ between the corresponding syntomic stacks, and if $\mathcal{O}$ denotes the structural sheaf of X^{syn}, the direct image $Rf_*^{\mathrm{syn}}(\mathcal{O})$ lies in $D_{\mathrm{qc}}(\mathrm{Spec}(k)^{\mathrm{syn}})$. If in addition X is proper, then this direct image lies in the full subcategory $D_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$ of *dualizable* F-gauges, i.e., F-gauges whose underlying complex of $W(k)$-modules is perfect.

By definition ([9], Construction 3.3.1), the *Nygaard stack* $\mathrm{Spec}(k)^{\mathcal{N}}$ is

$$\mathrm{Spec}(k)^{\mathcal{N}} = (\mathrm{Spec}(W(k)[u,t]/(ut-p))/\mathbf{G}_m.$$

The *syntomic stack* $\mathrm{Spec}(k)^{\mathrm{syn}}$ is obtained by gluing the two open substacks of $\mathrm{Spec}(k)^{\mathcal{N}}$ isomorphic to $\mathrm{Spec}(k)^{\Delta}$ by the canonical σ-linear isomorphism between them ([9], Definition 4.1.1). F-gauges M over k are thus described by graded $W(k)[u,t]/(ut-p)$-complexes

$$M^{\bullet} = \bigoplus_{i \in \mathbb{Z}} M^i$$

(the $W(k)$-complex M^i being by definition the ith-step $\mathrm{Fil}^i_{\mathcal{N}} M$ of the *Nygaard filtration* of M), together with linear maps $t : M^{i+1} \to M^i$, $u : M^i \to M^{i+1}$ satisfying $ut = tu = p$, and a σ-linear isomorphism $\tau : \sigma^* M^{\infty} \xrightarrow{\sim} M^{-\infty}$ where $M^{-\infty} = \mathrm{colim}_t M^i$, $M^{\infty} = \mathrm{colim}_u M^i$.

As recalled in 1.1 (c), by pull-back to the prismatic site of k, we recover crystalline cohomology up to a Frobenius twist: one has canonically

$$R\Gamma(X/W(k)) \xrightarrow{\sim} (\sigma^{-1})^* j^*_{\Delta} R f^{\mathrm{syn}}_*(\mathcal{O}). \tag{5.1.1}$$

We will use Mondal's notation

$$M(X) := R f^{\mathrm{syn}}_*(\mathcal{O}) \in D_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}}) \tag{5.1.2}$$

to denote this dualizable F-gauge over k. By (5.1.1) one recovers the F-crystal structure of $R\Gamma(X/W(k))$, namely its σ-linear endomorphism F^* corresponding to $\varphi^*_{X^{\Delta}}$, which becomes an isomorphism over the fraction field K of $W(k)$. In particular, if k is finite, with $q = p^a$ elements, we recover the zeta function of X by

$$Z(X,t) = \prod_i \det(1 - F^{*a} t, H^i(M(X)_K))^{(-1)^{i+1}},$$

$M(X)_K = M(X) \otimes K$ (if X is of dimension d, then $H^i(M(X)_K)) = 0$ of $i \notin [0, 2d]$).

This suggests to define ([51], Definition 3.2), for a dualizable F-gauge M over k, with $k = \mathbb{F}_q$ as above, its zeta function by

$$Z(M,t) := \prod_i \det(1 - F^{*a} t, H^i(M_K))^{(-1)^{i+1}} \tag{5.1.3}$$

as the dualizability condition implies that M_K is a perfect F-isocrystal, in particular, $F^* : M_K \to M_K$ is a (σ-linear) isomorphism ([9], proof of 3.4.11, Proposition 4.3.1).

For the same reasons as for $P_i(P,t)$ (4.2.1), the polynomials

$$P_i(M,t) = \det(1 - F^{*a}t, H^i(M_K))$$

lie in $\mathbb{Q}_p[t]$. For $n \in \mathbb{Z}$, denoting by ρ_n the order of the pole of $Z(M,t)$ at $t = q^{-n}$, one defines the special value $Z^*(M, q^{-n})$ by

$$Z(M,t) = (1 - q^n t)^{-\rho_n} Z^*(M,t), \tag{5.1.4}$$

with $Z^*(M, q^{-n}) \neq 0$.

5.2. There is a canonical line bundle $\mathcal{O}\{-1\}$ on $\mathrm{Spec}(k)^{\mathrm{syn}}$, called the *Breuil–Kisin twist*. By the correspondence ([9], Proposition 4.3.1) between vector bundles on $\mathrm{Spec}(k)^{\mathrm{syn}}$ and (non-necessarily effective) F-crystals on $\mathrm{Spec}(k)$ (i.e., finite projective $W(k)$-modules C equipped with a σ-linear isomorphism $C[1/p] \xrightarrow{\sim} C[1/p]$), $\mathcal{O}\{-1\}$ corresponds to the effective F-crystal $(W(k), p\sigma)$. Confusing as it may look, as explained two lines below ([9], Definition 4.3.4), it corresponds to the line bundle $\mathcal{O}(1)$ on the Nygaard stack $\mathrm{Spec}(k)^{\mathcal{N}}$. In particular, it has Hodge–Tate weight 1 (in the sense of ([51], Definition 2.5), i.e., its pull-back by the (closed) immersion $(B\mathbf{G}_m)_k \to \mathrm{Spec}(k)^{\mathrm{syn}}$ is the graded module k concentrated in degree 1.

For $n \in \mathbb{Z}$, one defines $\mathcal{O}\{n\} := \mathcal{O}\{-1\}^{\otimes(-n)}$, and, for an F-gauge M, $M\{n\} := M \otimes \mathcal{O}\{n\}$, where the tensor products are taken for the natural symmetric monoidal structure on $D_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$. Finally, as in ([51], Definition 2.19), one defines the *weigth n* syntomic cohomology of M by

$$R\Gamma(M, \mathbb{Z}_p(n)) := R\Gamma(\mathrm{Spec}(k)^{\mathrm{syn}}, M\{n\}) \in D(\mathbb{Z}_p). \tag{5.2.1}$$

For $M = M(X)$, we have a formula similar to (4.2.4), namely

$$R\Gamma(M(X), \mathbb{Z}_p(n)) = R\Gamma(X, \mathbb{Z}_p(n)). \tag{5.2.2}$$

This follows from ([9], Proposition 4.4.2, Remark 4.4.9).

We have the following analog of Proposition 4.3:

Proposition 5.3. Let $M \in D_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$ (5.1). Let u be the endomorphism $1 - \gamma$ of $R\Gamma(M_{\overline{k}}, \mathbb{Z}_p(n))$. Then:
(i) $R\Gamma(M_{\overline{k}}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p$ is a perfect complex of $\mathbb{Q}_p$-vector spaces.
(ii) $R\Gamma(M, \mathbb{Z}_p(n))$ ($\xrightarrow{\sim} \mathrm{Fib}(u)$) is a perfect complex of $\mathbb{Z}_p$-modules.

Proof. (i) is more or less a formal consequence of the Dieudonné–Manin classification ([51], Corollary 2.21). (ii) is a result of Bhatt ([9], Proposition 4.5.1).

By 2.3, Proposition 1, we get

Corollary 5.4. For M and n as in Proposition 5.3, the multiplicative stable Bockstein characteristic (Definition 2.4)

$$\chi_s(M, \mathbb{Z}_p(n))^\times := \chi_s(\mathrm{Bock}(R\Gamma(M_{\overline{k}}, \mathbb{Z}_p(n)), 1-\gamma))^\times, \tag{5.4.1}$$

denoted $\mu_{\mathrm{syn}}(M, n)$ in ([51], 6.4), is defined.

Proposition 5.3 and Proposition 4.3 should actually be equivalent, as we will see in section 6.

Finally, one defines ([51], Definition 6.3) the *n-th weighted Hodge–Euler characteristic* (Milne's correcting factor) $\chi(M, n)$ by

$$\chi(M, n) := \sum_{i,j,i \leqslant n} (-1)^{i+j}(n-i)h^{i,j}(M), \tag{5.4.2}$$

where $h^{i,j}(M)$ are the *Hodge numbers* of M, defined as

$$h^{i,j}(M) := \dim_k H^{i+j}(M^{\mathrm{Hodge},i}),$$

where $M^{\mathrm{Hodge}} = \oplus_i M^{\mathrm{Hodge},i}$ is the graded complex of k-vector spaces defined by pull-back by the closed immersion $(B\mathbf{G}_m)_k \to \mathrm{Spec}(k)^{\mathrm{syn}}$.

In the same way as Theorem 4 followed from 4.5, Theorem 5, Theorem 4 will follow from the next theorem.

5.5. **Theorem 6.** ([51], Theorem 6.5) Let M be a dualizable F-gauge over $k = \mathbb{F}_q$, $q = p^a$, and let $n \in \mathbb{Z}$. Then, with the notation (5.1.4), (5.4.1), (5.4.2), we have

$$|Z^*(M, q^{-n})|_p^{-1} = \chi_s(M, \mathbb{Z}_p(n))^\times \cdot q^{\chi(M,n)}. \tag{5.5.1}$$

As for the proof of Theorem 5, the strategy consists in exploiting the additivity of the stable Bockstein characteristic on fibrations to reduce, by dévissage, to a few cases that can be easily handled. The problem is that, in contrast with the case of $D^b_c(\mathcal{R}_k)$ no dévissage for F-gauges similar to that of Ekedahl was available in the literature.

The very first step is to rewrite $R\Gamma(M, \mathbb{Z}_p(n))$ is terms of the *Nygaard filtration*, namely we have a fiber sequence

$$R\Gamma(M, \mathbb{Z}_p(n)) \to \mathrm{Fil}^n M \overset{\varphi_n - \mathrm{can}}{\to} M, \tag{5.5.2}$$

where $\mathrm{Fil}^\bullet$ denotes the Nygaard filtration ([9], Construction 3.3.1, Theorem 3.3.5, Remark 4.4.6) and φ_n a divided Frobenius, cf. ([9], Proposition 4.4.2)), where the proof works similarly for a general F-gauge M. In the second and third terms of (5.5.2), M denotes by abuse the restriction of M to the prismatic site of k (for $M = M(X)$ this restriction is $(\sigma^{-1})^* R\Gamma(X/W(k))$).

The second one is to interpret $\chi(M,n)$ in terms of the Nygaard filtration of M. By Proposition 5.3, the complex of $W(k)$-modules underlying M is perfect, and similarly for any quotient $M/\mathrm{Fil}^r M$. Moreover, by definition of Fil^r, $\mathrm{Fil}^r M \to M$ is an isogeny, ie. $(M/\mathrm{Fil}^r M) \otimes K = 0$, hence all $H^i(M/\mathrm{Fil}^r M)$ are finite length $\mathbb{Z}_p$-modules, so that the finite length Euler characteristic $\chi^l(M/\mathrm{Fil}^r M)$ is defined (2.1.5). Recall that $q = p^a$. We have ([51], Proposition 6.7):

$$\chi^l(M/\mathrm{Fil}^n M) = a\chi(M,n). \tag{5.5.3}$$

The proof is formal, using pull-back by the maps $(B\mathbf{G}_m)_k \to \mathbb{A}^1_k/\mathbb{G}_m \to \mathrm{Spec}(k)^{\mathrm{syn}}$. The dévissage of $\mathrm{gr}^r M$ in terms of Hodge-type invariants is similar to the formula describing the graded for the Nygaard filtration on the de Rham–Witt complex

$$\mathrm{gr}^r_{\mathcal{N}}(W\Omega^\bullet_X) \xrightarrow{\sim} \tau^{\leqslant r} F_*\Omega^\bullet_{X/k}.$$

In the proof of Theorem 6, formula (5.5.3) is a substitute for the (mysterious) Crew–Milne's formula (4.5.3).

In the proof of Theorem 4 in case II, though $R\Gamma(P_{\overline{k}}, \mathbb{Z}_p(n))$ was far from being perfect over $\mathbb{Z}_p$, it turned out that $\mathrm{Fib}(1-\gamma : R\Gamma(P_{\overline{k}}, \mathbb{Z}_p(n)) \to R\Gamma(P_{\overline{k}}, \mathbb{Z}_p(n))$ was, and in fact was of finite length, so that the stable Bockstein characteristic of $(R\Gamma(P_{\overline{k}}, \mathbb{Z}_p(n)), 1-\gamma)$ was just the naive Bockstein characteristic. A substitute is provided by the following result:

Proposition 5.6. ([51], Proposition 6.9). Assume that M is annihilated by p^m for some $m > 0$, i.e., multiplication by p^m on M is homotopic to zero. Then we have

$$\chi_s(M, \mathbb{Z}_p(n))^\times = q^{-\chi(M,n)}. \tag{5.6.1}$$

Proof. In the fiber sequence (5.5.2) all three terms are perfect complexes over $\mathbb{Z}_p$ (by 5.3 (ii) for the left one). The hypothesis implies that they are annihilated by p^m, hence have finite length cohomology groups. Therefore, we have

$$\chi_s(M, \mathbb{Z}_p(n)) = \chi^l(M, \mathbb{Z}_p(n)) = -\chi^l(M/\mathrm{Fil}^n M),$$

which, by (5.5.3), gives (5.6.1).

A substitute for the dévissage of $D^b_c(\mathcal{R}_k)$ into objects of type I and type II is provided by part (a) of the following key result:

Proposition 5.7. ([51], Lemmas 6.10, 6.14). (a) Let M be a (dualizable) F-gauge over k. Then there exists a fiber sequence

$$T \to M \to V,$$

with $V = \bigoplus_{1 \leqslant i \leqslant N} V_i[r_i]$, where V_i is a vector bundle on $\mathrm{Spec}(k)^{\mathrm{syn}}$, $r_i \in \mathbb{Z}$, and T is an F-gauge where multiplication by p^m is homotopic to zero for some $m \in \mathbb{N}$.

(b) Let V be a vector bundle over $\mathrm{Spec}(k)^{\mathrm{syn}}$. Then there exists a fiber sequence

$$T \to V \to \bigoplus_{-m_1 \leqslant i \leqslant m_2} U_i,$$

where m_1, m_2 are nonnegative integers, and for each i, $U_i\{i\}$ is a vector bundle with Hodge–Tate weights in $\{0, 1\}$, and T is an F-gauge where multiplication by p^m is homotopic to zero for some $m \in \mathbb{N}$.

Proof. Part (a) follows from the fact that $M[1/p]$ is an F-isocrystal, and from the description of vector bundles on $\mathrm{Spec}(k)^{\mathrm{syn}}$ as not necessarily effective F-crystals ([9], Proposition 4.3.1).

Part (b) follows from the Dieudonné–Manin classification and the slope decomposition of F-isocrystals over k.

The relevance of vector bundles of Hodge–Tate weights in $\{0, 1\}$ appears in the following (deep) result of Mondal, which, in a sense, shows that such vector bundles behave in the same way as the objects of $D^b_c(\mathcal{R}_k)$ called *Hodge–Witt complexes* ([22], p.28). In particular, over $\overline{k}$, their cohomology groups with coefficients in $\mathbb{Z}_p(n)$ are finitely generated over $\mathbb{Z}_p$.

Proposition 5.8. ([51], Proposition 6.15). Let k be an algebraically closed field (of characteristic $p > 0$). Let V be a vector bundle over $\mathrm{Spec}(k)^{\mathrm{syn}}$, with Hodge–Tate weights in $\{0, 1\}$. Then

$$H^i(V, \mathbb{Z}_p(n)) = 0 \tag{5.8.1}$$

for $i \neq 0$ or $n \notin \{0, 1\}$, and, if $n = 0$ or $n = 1$, $H^0(V, \mathbb{Z}_p(n))$ is finitely generated over $\mathbb{Z}_p$.

Proof. This is the heart of Mondal's paper [51]. The starting point is that the (contravariant) Dieudonné module functor defines an equivalence from the opposite of the category of p-divisible groups over k to the category of vector bundles on $\mathrm{Spec}(k)^{\mathrm{syn}}$ with Hodge–Tate weights in $\{0, 1\}$, i.e., F-crystals V over k such that $pV \subset F(V)$ ([52], (3.4.15)). The equivalence with the latter category is of course due to Dieudonné, however, its reformulation in the language of F-gauges, due to Mondal ([52], Theorem 1.0.13), turns out to be a key ingredient in the proof of Proposition 5.8.

The vanishing (5.8.1) for $i < 0$ or $n < 0$ is trivial. For $n \geqslant 0$, we have

$$H^i(V, \mathbb{Z}_p(n)) = H^i(\mathrm{Fil}^n V \overset{p^{-n}F-\iota}{\to} V)$$

(with ι the canonical map), where the complex on the right hand side is placed in degrees 0 and 1. The vanishing for $i > 1$ is therefore trivial, and it is easy for $i = 0$ and $n > 1$, as well as for $i = 1$ and $n = 0$. The finite generation for $n = 0$ is easy, too. What is not "easy" is:

(i) $H^1(V, \mathbb{Z}_p(n)) = 0$ for $n > 1$;

(ii) finite generation of $H^0(V, \mathbb{Z}_p(1))$;

(iii) $H^1(V, \mathbb{Z}_p(1)) = 0$.

For these three assertions, Mondal uses that by ([52], Theorem 1.0.13, Definition 1.0.11), V corresponds to a Barsotti-Tate group G over k:

$$V = R^2 v_* \mathcal{O}_{BG^{\mathrm{syn}}} \in D_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}}, \mathcal{O})$$

where $v : BG^{\mathrm{syn}} \to \mathrm{Spec}(k)^{syn}$ is the functoriality map. In particular, it implies ([52], Proposition 3.5.16) that

$$H^i(V, \mathbb{Z}_p(1)) \overset{\sim}{\to} H^i_{\mathrm{qsyn}}(\mathrm{Spec}(k), T_p(G^\vee)),$$

where $G^\vee$ is the dual of G, $T_p(G^\vee)$ its Tate module, and the subscript means that cohomology is taken for the quasi-syntomic topology. The proof of (ii) and (iii) relies on this. The proof of (i) uses the Hodge–Tate sequence for V (cf. ([52], Proposition 3.3.8).

Proof of Theorem 6. If M is annihilated by p^m for some $m > 0$, then $M_K = 0$, $Z(M, t) = Z^*(M, q^{-n}) = 1$, and (5.5.1) holds by Proposition 5.6. By Proposition 5.7 (a), (b), we may therefore assume that M is a vector bundle V with Hodge–Tate weights in $\{0, 1\}$. Then, we have

$$Z(V, t) = \det(1 - F^{*a}t, V_K) = (1 - q^n t)^{-\rho_n} Z^*(V, t),$$

with ρ_n the order of the pole at n and $Z^*(V, q^{-n}) \neq 0$, hence

$$|Z^*(V, q^{-n})|_p^{-1} = |\prod_{u_i \neq q^n} (1 - u_i q^{-n})|_p,$$

where $\{u_i\}$ runs through the eigenvalues (with multiplicities) of F^{a*} on V_K (a formula similar to (4.5.2).

By Proposition 5.8, we have

$$R\Gamma(V_{\overline{k}}, \mathbb{Z}_p(n)) = H^0(V_{\overline{k}}, \mathbb{Z}_p(n)),$$

which is a finitely generated $\mathbb{Z}_p$-module, and

$$H^0(V_{\overline{k}}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p) = (\overline{V} \otimes \mathbb{Q}_p)^{F_{\overline{k}} - p^n}$$

where $\overline{V}$ on the right hand side means the underlying $W(\overline{k})$-module induced by V. As in the proof of case (a) of Theorem 5, by Proposition 2 in 2.5 and ([48], 5.5), for $u = 1 - \gamma$, $\gamma = \sigma^a$, we have

$$\chi_s(H^0(V_{\overline{k}}, \mathbb{Z}_p(n))^\times = z_s(d_u : H^0(V_{\overline{k}}, \mathbb{Z}_p(n))^u \to H^0(V_{\overline{k}}, \mathbb{Z}_p(n))_u)$$

$$= |\prod_{u_i \neq q^n, v_p(u_i/q^n) = 0} (1 - \frac{q^n}{u_i})|_p.$$

It follows that

$$|Z^*(V, q^{-n})|_p^{-1} = \chi_s(H^0(V_{\overline{k}}, \mathbb{Z}_p(n))^\times \cdot |\prod_{u_i \neq q^n, v_p(u_i/q^n) < 0} (1 - \frac{q^n}{u_i})|_p$$

To complete the proof, one observes that the formula

$$|\prod_{u_i \neq q^n, v_p(u_i/q^n) < 0} (1 - \frac{q^n}{u_i})|_p = q^{\chi(V,n)},$$

i.e.,

$$\sum_{v_q(u_i) < n} n - v_q(u_i) = \chi(V, n),$$

which replaces Crew–Milne's formula, follows from the interpretation of V as the Dieudonné module of a p-divisible group G over Spec(k), and the Hodge–Tate sequence for G, which implies that the Hodge numbers of V are $h^{0,0} = \dim_k t_{G^\vee} = \dim(G^\vee)$ and $h^{1,-1} = \dim_k \omega_G = \dim(G)$, hence

$$\chi(V, n) = n\dim(G^\vee) + (n-1)\dim(G)$$

for $n \geqslant 1$, and $\chi(V, n) = 0$ for $n \leqslant 0$.

Complements 5.9.

(a) Let M be a dualizable F-gauge over k and $n \in \mathbb{Z}$. Then we have ([51], Proposition 3.5):

(5.9.1) $$\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{rk}\, H^i(M, \mathbb{Z}_p(n)) = 0$$

Proof. The main point is that, by Proposition 5.3.1, for all $i \in \mathbb{Z}$, $H^i(M_{\overline{k}}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p$ is of finite dimension over $\mathbb{Q}_p$. Hence, for $u = \gamma - 1$, $\gamma = \sigma^a \in G_k$, we have

$$\dim(H^i(M_{\overline{k}}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p)_u = \dim(H^i(M_{\overline{k}}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p)^u \tag{5.9.2}$$

(where $(-)_u$ and $(-)^u$ denote cokernel and kernel of u respectively). Formula (5.9.1) follows from the exact sequences
(5.9.3)

$$0 \to (H^{i-1}(M_{\overline{k}}, \mathbb{Z}_p(n)))_u \to H^i(M, \mathbb{Z}_p(n)) \to (H^i(M_{\overline{k}}, \mathbb{Z}_p(n)))^u \to 0.$$

Formula (5.9.1) is similar to assertion (a) in ([48], Theorem 0.1). However, the semisimplicity assumption of the eigenvalue q^n made in *loc. cit.* is not needed. In fact, using Proposition 4.3, one can directly prove assertion (a) *unconditionally.* As we will see in the next section, the two formulations should in fact be equivalent.

(b) Let M be a dualizable F-gauge over k and $n \in \mathbb{Z}$. Let ρ_n be the order of the pole of $Z(M, t)$ at $t = q^{-n}$. Now, *assume that, for all* $i \in \mathbb{Z}$, q^n *is a semisimple eigenvalue of* F^{*a} *acting on* $H^i(M_K)$. Then we have ([51], Proposition 3.9)

$$\rho_n = \sum_{i \in \mathbb{Z}} (-1)^{i+1} i \operatorname{rk} H^i(M, \mathbb{Z}_p(n)) \tag{5.9.4}$$

Proof. Let $\rho_{n,i}$ be the order of the zero of $P_i(M, t) := \det(1 - F^{a^*}t, H^i(M_K))$ at $t = q^{-n}$, equivalently, the multiplicity of the eigenvalue q^n of F^{a*} on $H^i(M_K)$, i.e.,

$$P_i(M, t) = (1 - q^n t)^{\rho_{n,i}} P_i^*(M, t)$$

with $P_i^*(M, q^{-n}) \neq 0$. Thus,

$$\rho_n = \sum_i (-1)^i \rho_{n,i}.$$

The semisimplicity assumption says that, for all i,

$$\rho_{n,i} = \operatorname{rk} H^i(M_{\overline{k}}, \mathbb{Z}_p(n))^{G_k}.$$

By (5.9.2) and (5.9.3) it follows that

$$\operatorname{rk} H^i(M, \mathbb{Z}_p(n)) = \rho_{n,i-1} + \rho_{n,i},$$

which implies (5.9.4).

Formula (5.9.4) is similar to assertion (b) in ([48], Theorem 0.1), and again can be proven directly using Proposition 4.3. And the two formulations should be equivalent.

But in contrast with (a), the semisimplicity assumption is needed here. In the case $M = M(X)$ for X/k a proper and smooth surface, and $n = 1$, only $\rho_{1,2}$ can be nonzero, and $\rho_{1,2} = \rho_1$. The semisimplicity assumption is equivalent to conjecture (iv1) in Theorem 1, of which (5.9.4) is a reformulation.

6. From coherent complexes to syntomic F-gauges

This section is a sketch. Details will hopefully be written elsewhere.

6.1. Recall (5.1) that by ([9], Remark 4.2.8) an object M of $D_{\mathrm{qc}}(\mathrm{Spec}(k)^{\mathrm{syn}})$ corresponds to a p-complete, graded $(W(k)[u,t]/(ut-p))$ complex

$$M^\bullet = \bigoplus_{i \in \mathbb{Z}} M^i,$$

with $W(k)$-linear maps

$$t : M^{i+1} \to M^i, \ u : M^i \to M^{i+1}$$

satisfying $ut = tu = p$,[16] together with an isomorphism

$$\tau : \sigma^* M^\infty \xrightarrow{\sim} M^{-\infty},$$

where $M^{-\infty} = \mathrm{colim}_t M^i$, $M^\infty = \mathrm{colim}_u M^i$ (both p-completed). The F-gauge is *dualizable* (or *perfect*) if t is an isomorphism for $i << 0$, u an isomorphism for $i >> 0$ (so that the maps $u^\infty : M^0 \to M^\infty$ and $t^\infty : M^0 \to M^{-\infty}$ are p-isogenies), and M^∞ (or equivalently $M^{-\infty}$) belongs to $D^b_c(W(k))$ ([9], Remark 3.4.6).

Given an object P of $D^b_c(\mathcal{R}_k)$, Ekedahl ([22], II Definition 3.1) associates to it an F-gauge M defined in the following way: M^i is the object deduced from P by replacing the differential $d : P^{i-1} \to P^i$ by $dV : \sigma_* P^{i-1} \to P^i$, and leaving the other differentials unchanged (except that P^j is replaced by $\sigma_* P^j$ for $j \leqslant i-1$). This modification was first introduced by Nygaard in the de Rham–Witt complex ([56], Definition 1.1) and denoted $P(i,1)$. In this case (X/k smooth), one has canonically

$$W\Omega^\bullet_X(i,1) \xrightarrow{\sim} (p^{i-1} VW\mathcal{O}_X \xrightarrow{d} p^{i-2} VW\Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} VW\Omega^{i-1}_X \xrightarrow{d} W\Omega^i_X \xrightarrow{d} \cdots).$$

[16]The maps u and t are denoted respectively $\widetilde{F}$ and $\widetilde{V}$ in ([22], II Definition 2.1) and f and v in [23].)

The complex on the right hand side is the ith step $\mathrm{Fil}^i_{\mathcal{N}} W\Omega^\bullet_X$ of the Nygaard filtration, and by application of $R\Gamma(X,-)$, when X is in addition proper, gives $\mathrm{Fil}^i_{\mathcal{N}} R\Gamma(X/W(k))$.

The maps

$$u : M^i \to M^{i+1},\ t : M^i \to M^{i-1}$$

are defined as in ([56], Definition 1.2), i.e.,

$$u : P(i,1) \to P(i+1,1)$$

is given by $F : P^i \to \sigma_* P^i$ in degree i (p in degree $\geqslant i+1$ and identity in degree $\leqslant i-1$),

$$t : P(i+1,1) \to P(i,1)$$

is given by $V : \sigma_* P^i \to P^i$ in degree i (identity in degree $\geqslant i+1$ and p in degree $\leqslant i-1$). This construction gives rise to a functor (denoted **S** in ([22], Definition 3.1)), that we will denote

(6.1.1) $$\mathcal{E} : D^b_c(\mathcal{R}_k) \to D^b_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}}),$$

which should be upgraded to a functor between stable derived ∞-categories. Ekedahl's main theorem is the following.

6.2. **Theorem 7.** ([22], III Theorem 5.3) The functor (6.1.1) is an equivalence.

Ekedahl's statement is, of course, for the underlying homotopy categories, but one can reasonably expect that it can be upgraded to the ∞-level. Ekedahl's proof, which is long and technical, consists in constructing a left adjoint to $\mathcal{E}$ and showing that the adjunction maps are isomorphisms. It would be nice to find a simpler, more conceptual proof.

6.3. The categories $D^b_c(\mathcal{R}_k)$ and $D^b_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$ have natural symmetric monoidal structures. For the latter one, it is of course given by usual $\otimes^L$ and $R\mathcal{H}om$. For the former one, its construction is non-trivial. It was done by Ekedahl ([21], I). See ([33], 2.6) and ([48], 3) for a review of the main points. The functor $\mathcal{E}$ should be compatible with these structures.

Both categories have Tate twists: $P \mapsto P(n) = P^{\bullet+n,\bullet-n}$ [45], ([48], 1.7) for $D^b_c(\mathcal{R}_k)$ and $M \to M\{n\}$ (the Breuil–Kisin twist) for $D^b_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$. Both $W(k)(-1)$ and $W(k)\{-1\}$ correspond to the F-crystal $(W(k), p\sigma)$. If, as expected, $\mathcal{E}$ is compatible with the symmetric monoidal structures, then we will have, for $P \in D^b_c(\mathcal{R}_k)$ and all $n \in \mathbb{Z}$,

$$\mathcal{E}(P(n)) \xrightarrow{\sim} \mathcal{E}(P)\{n\}.$$

6.4. Assuming that the above holds, then, for $k = \mathbb{F}_q$, $q = p^a$, $P \in D^b_c(\mathcal{R}_k)$ and $M = \mathcal{E}(P) \in D^b_{\mathrm{perf}}(\mathrm{Spec}(k)^{\mathrm{syn}})$, and $n \in \mathbb{Z}$, we will have:

$$Z(P,t) = Z(M,t)$$

$$\begin{aligned} R\Gamma(P, \mathbb{Z}_p(n)) &= R\Gamma(M, \mathbb{Z}_p(n)), \\ \chi_s(P, \mathbb{Z}_p(n)) &= \chi_s(M, \mathbb{Z}_p(n)), \\ h^{i,j}(P) &= h^{i,j}(M), \\ \chi(P,n) &= \chi(M,n), \end{aligned}$$

and therefore (4.5.1) for P will be equivalent to (5.5.1) for M. A similar equivalence will hold for the complementary results 5.9 (a) and (b).

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