Conference in Honor of Joseph Steenbrink

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Deligne's tubular neighborhoods in étale cohomology, after Gabber and Orgogozo

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O. Gabber, *Finiteness theorems for étale cohomology of excellent schemes*, Conference in honor of P. Deligne on the occasion of his 61st birthday, IAS, Princeton, October 2005.

F. Orgogozo, *Modifications et cycles proches sur une base générale*, IMRN, vol. 2006, ID 25315, 38 p., 2006.

PLAN

1. Oriented products (Deligne's tubes)
2. Nearby cycles over general bases
3. Gabber's finiteness and uniformization theorems
4. Tubular cohomological descent
5. What next ?

1. ORIENTED PRODUCTS (DELIGNE'S TUBES)

Recall :

- topos = {sheaves on a site}
- $f : X \to Y$: (f^*, f_*)
- point of X : morphism $x : Pt \to X$

(= fiber functor $x^* : F \mapsto F_x$)

Examples

• X = sober topological space,

$X = Pt(X)$

• X = scheme with étale topology,

$Pt(X)$ = geometric points of X

(= morphisms $\operatorname{Spec} k \to X$ (k sep. closed))

Specialization morphisms

$f, g : X \to Y$ morphisms of toposes

morphism $u : f \to g$ = morphism of functors $f_* \to g_*$

($\Leftrightarrow g^* \to f^*$)

For $s, t \in Pt(X)$, $u : t \to s$ = specialization

from t to s

Examples

• X = scheme, Zariski topology , $t \to s \Leftrightarrow s \in \overline{\{t\}}$

$\Leftrightarrow t \in \operatorname{Spec} \mathcal{O}_{X,s}$

• X = scheme, étale topology

$t \to s \Leftrightarrow t \to X_{(s)} \Leftrightarrow X_{(t)} \to X_{(s)}$

$X_{(s)} = \operatorname{Spec} \mathcal{O}_{X,s}$

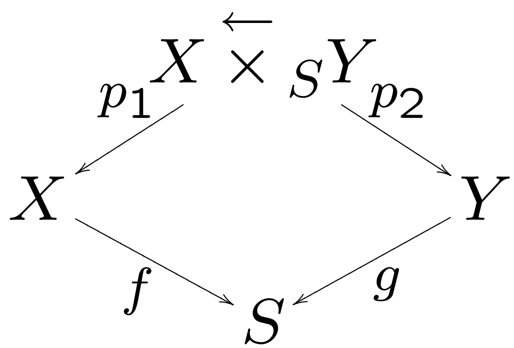
$\mathcal{O}_{X,s}$ = strict henselization at s

Oriented products

Given morphisms of toposes

$f : X \to S$, $g : Y \to S$,

construct universal diagram of toposes :



$$\tau : gp_2 \to fp_1$$

$X \overset{\leftarrow}{\times}_S Y$: Deligne's oriented product

universal property : for any topos T

$\{$ morphisms $T \to X \overset{\leftarrow}{\times}_S Y \} =$

$\{$ triples $(q_1 : T \to X, q_2 : T \to Y, t : gq_2 \to fq_1) \}$

In particular :

points of $X \overleftarrow{\times}_S Y$ = triples

(point x of X, point y of Y,

specialisation $g(y) \to f(x)$)

Defining site and structural maps

$X \overleftarrow{\times}_S Y :=$ { sheaves on site C }

objects of $C = \{(U \to V \leftarrow W)$ above $(X \to S \leftarrow Y)\}$

(U, V, W objects of defining sites

for X, S, Y)

maps : obvious

topology defined by covering families of types :

(a) $$\begin{array}{ccccc} U_i & & & & \\ \downarrow & \searrow & & & \\ U & \longrightarrow & V & \longleftarrow & W \end{array}$$ $((U_i \to U)$ covering)

(b) $$\begin{array}{ccccc} & & & & W_i \\ & & & \swarrow & \downarrow \\ U & \longrightarrow & V & \longleftarrow & W \end{array}$$ $((W_i \to W)$ covering)

(c) $$\begin{array}{ccccc} & & V' & \longleftarrow & W' \\ & \nearrow & \downarrow & & \downarrow \\ U & \longrightarrow & V & \longleftarrow & W \end{array}$$ (cartesian square) presheaf $(U \to V \leftarrow W) \mapsto F(U \to V \leftarrow W)$

$=$ sheaf on C

$\Leftrightarrow$ F satisfies sheaf condition for (a), (b), and

$F(U \to V \leftarrow W) \xrightarrow{\sim} F(U \to V' \leftarrow W')$ for type (c)

$$p_1^{-1}(U) = (U \to S \leftarrow Y)$$

$$p_2^{-1}(W) = (X \to S \leftarrow W)$$

$\tau : (gp_2)_* F \to (fp_1)_* F$

defined by

$F(X \to S \leftarrow g^{-1}(V)) \to F(f^{-1}(V) \to V \leftarrow g^{-1}(V)) \leftarrow$
$F(f^{-1}(V) \to S \leftarrow Y)$

Examples (étale topology)

• S = scheme ; $s \to S$ = geometric point

$s \overset{\leftarrow}{\times}_S S = S_{(s)}$

• X = scheme ; $Y \subset X$ closed, $U = X - Y \subset X$

$Y \overset{\leftarrow}{\times}_X U$ = punctured (étale) tubular neighborhood of Y in X

($= Y \overset{\leftarrow}{\times}_{X'} U'$ for X' étale neighborhood of Y in X, $U' = X' \times_X U$)

- s = Spec k (k = field), X/s,

$X \overleftarrow{\times}_s s = X$

- S = strictly local trait, s closed, η generic, Y/s

$Y \overleftarrow{\times}_S \eta = \{$ sheaves on Y

with continuous action of Gal($\overline{\eta}/\eta$) $\}$

2. NEARBY CYCLES OVER GENERAL BASES

S = scheme, étale topology ; $\Lambda = \mathbb{Z}/n\mathbb{Z}$ (n invertible on S)

For schemes X/S, Y/S, universal property of $X \overleftarrow{\times}_S Y$ gives a morphism of toposes

$$\Psi = \Psi_{X/S} : X \times_S Y \to X \overleftarrow{\times}_S Y,$$

$$\Psi^{-1}(U \to V \leftarrow W) = U \times_V W$$

For $Y = S$,

$$R\Psi_* : D^+(X, \Lambda) \to D^+(X \overleftarrow{\times}_S S, \Lambda)$$

(denoted also $R\Psi$)

called nearby cycles functor

(Deligne, Laumon ; 1981)

Example

S = strictly local trait, s closed, η generic,

$\overline{\eta}$ = generic geometric

$i : X_s \to X$, $\overline{j} : X_{\overline{\eta}} \to X$

$$(R\Psi F)|X_s \overset{\leftarrow}{\times}_S \eta = i^* R\overline{j}_*(F|X_{\overline{\eta}})$$

(usual (= SGA 7 XIII) functor $R\Psi$)

Stalks

point $(x, s \leftarrow t)$ of $X \overleftarrow{\times}_S S$

$(x \to X,\ s \to S =$ geom. pts, $s \leftarrow t =$ specialization$)$

$$(R\Psi F)_{(x,s\leftarrow t)} = R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, F)$$

$$X_{(x)} \supset X_{(x)} \times_{S_{(s)}} S_{(t)} \supset X_{(x)} \times_{S_{(s)}} t$$

(Milnor ball $\supset$ Milnor tube $\supset$ Milnor fiber)

Vanishing cycles

$p_1\Psi = Id_X$ gives map

$$p_1^* \to \Psi_*$$

and distinguished triangle

$$p_1^* F \to R\Psi F \to R\Phi F \to$$

$R\Phi$ = vanishing cycles functor

(for S = strictly local trait,

$(R\Phi F)|X_s \overleftarrow{\times}_S \eta$ = usual (= SGA 7 XIII) $R\Phi F$)

Constructibility

S noetherian, X/S, Y/S finite type

sheaf of Λ-modules F on $X \overleftarrow{\times}_S Y$ constructible if

$X = \cup X_i$, $Y = \cup Y_j$ (finite disjoint unions) and

$F|X_i \overleftarrow{\times}_S Y_j$ locally constant of finite type

$\{$constructible sheaves$\}$ = thick subcategory

$D_c^b(X \overleftarrow{\times}_S Y, \Lambda)$: bounded, constructible cohomology

Main result

THEOREM (F. Orgogozo, 2005)

S noetherian, X/S finite type, $\Lambda = \mathbb{Z}/n\mathbb{Z}$,

n invertible on S ; $F \in D_c^b(X, \Lambda)$

There exists a modification $S' \to S$

such that for $X' = X \times_S S'$,

$R\Psi_{X'/S'}(F|X')$ belongs to $D_c^b(X' \overleftarrow{\times}_{S'} S', \Lambda)$ and

is base change compatible

Remarks

• S = trait : recover Deligne's th. in [SGA 4 1/2, Th. finitude]

• dim(S) $\geq$ 2 : in general, $R\Psi F$ not in D_c^b and not base change compatible :

Example : $f : X \to S$ = blow up of origin in the plane,

L = line through origin,

$R\Psi((\Lambda)|f^{-1}(L))$ moves with L

• isolated singularities

if bad (= non universal local acyclicity) locus of (f, F) quasi-finite / S (e. g. $F = \Lambda$, f smooth outside Σ quasi-finite / S),

then $R\Psi F$ is in D_c^b and base change compatible (no modification of base necessary) main ingredient of proof :

de Jong's th. on plurinodal curves

3. GABBER'S FINITENESS AND UNIFORMIZATION THEOREMS

Recall :

A ring A is quasi-excellent if A noetherian,

formal fibers of A are geometrically regular, and

for any A' of finite type over A, Reg(Spec A') open

A scheme X is quasi-excellent (qe for short)

if $X=$ union of

open affine quasi-excellent schemes

Examples

- A complete, local, noetherian $\Rightarrow$ A qe
- A Dedekind, Frac(A) of char. zero $\Rightarrow$ A qe
- Y qe, X/Y locally of finite type $\Rightarrow$ X qe

THEOREM 3.1 (Gabber, 2005) :

Y noetherian, qe, $f : X \to Y$ f. t.,

$\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n \geq 1$ invertible on Y,

F = constructible Λ-module on X

Then :

(a) $R^q f_* F$ constructible $\forall q$,

(b) $\exists N$ s. t. $R^q f_* F = 0$ for $q \geq N$.

Remarks :

- (a) + (b) $\Leftrightarrow Rf_* : D^b_c(X, \Lambda) \to D^b_c(Y, \Lambda)$
- f proper : Y qe, n invertible on Y superfluous

(finiteness th. [SGA 4 XIV])

- char$(Y) = 0$: Artin [SGA 4 XIX]

- $f = S$-morphism, X, Y f. t. $/S$ regular, dim ≤ 1 : Deligne [SGA 4 1/2, Th. Finitude]
- $f = S$-morphism, X, Y f. t. $/S$ noetherian

$\Rightarrow$ generic constructibility of $R^q f_* F$: Deligne [SGA 4 1/2, Th. Finitude]

- qe not needed for $q = 0$, needed for $q > 0$

General idea of proof :

reduce to absolute purity th. (Gabber, 1994)
via cohomological descent

absolute purity th. $\Rightarrow$

THEOREM 3.2

X regular, locally noetherian

$D = \sum_{i \in I} D_i \subset X$ snc (= strict normal crossings) divisor

$j : U = X - D \to X$

Then :

$$R^q j_* \Lambda = \begin{cases} \Lambda & \text{if } q = 0 \\ \oplus \Lambda_{D_i}(-1) & \text{if } q = 1 \\ \wedge^q R^1 j_* \Lambda & \text{if } q > 1. \end{cases}$$

In particular, $Rj_*\Lambda \in D^b_c(X, \Lambda)$

To prove 3.1 (a), easy reductions $\Rightarrow$

- enough to show : $Rj_*\Lambda \in D^+_c(X, \Lambda)$ for $j : U \to X$ dense open immersion, X qe
- if de Jong available, (e. g. /schemes f. t. $\mathbb{Z}$), i. e. can find $\pi : X' \to X$ proper surjective, X' regular, $U' := \pi^{-1}(U)$ complement of strict dnc,

construct cartesian diagram :

$$(*) \quad \begin{array}{ccc} U_{\cdot} & \xrightarrow{j_{\cdot}} & X_{\cdot} \\ \downarrow & & \downarrow \varepsilon_{\cdot} \\ U & \xrightarrow{j} & X \end{array}$$

with

- $\varepsilon_{\cdot}$ proper hypercovering
- X_n regular $\forall n$
- $j_n : U_n \to X_n$ = inclusion of

complement of strict dnc $\forall n$

cohomological descent for $\varepsilon_.$ $\Rightarrow$

$Rj_*\Lambda = R\varepsilon_{.*}Rj_{.*}\Lambda$

absolute purity $\Rightarrow$ $Rj_{p*}\Lambda$ in D_c^b

ε_p proper$\Rightarrow$ $R^q\varepsilon_{p*}Rj_{p*}\Lambda$ constructible

spectral sequence $(R^q\varepsilon_{p*}Rj_{p*}\Lambda \Rightarrow R^{p+q}j_*\Lambda)$

$\Rightarrow$ $R^ij_*\Lambda$ constructible

Instead of de Jong (not available), use

Gabber's local uniformization theorem

S a scheme

pspf topology on (schemes loc. f. p. / S) :

generated by :

- proper surjective f. p. morphisms
- Zariski open covers

(pspf = propre, surjectif, présentation finie)

pspf finer than étale

S noetherian : pspf $/S$ = Voevodsky's h-topology

= Goodwillie-Lichtenbaum's ph-topology

S pspf local $\Leftrightarrow S = \operatorname{Spec} V$

V valuation ring, Frac(V) alg. closed

THEOREM 3.3 (Gabber, 2005)

X noetherian, qe, $Y \subset X$ nowhere dense closed subset

Then :

$\exists$ finite family $(f_i : X_i \to X)$ $(i \in I)$ s. t. :

- (f_i) pspf covering
- $\forall i$, X_i regular, connected
- $Y_i = f_i^{-1}(Y)$ = support of strict dnc (or $\emptyset$)
- $\forall i$, f_i generically quasi-finite and

sends maximal pts to maximal pts

NB. f_i not necessarily proper

3.3 = local uniformization theorem

compare with

- Hironaka ($/\mathbb{Q}$)
- de Jong (f. t. $/S$ regular, dim. ≤ 1)

which are both global

Rough outline of proof

• reduction to X local henselian

• reduction to X local complete :

uses : Artin-Popescu's th.

$+$ Gabber's new formal approximation technique

• by induction on $\dim(X)$, proof in local complete case relies on :

- Gabber's refined Cohen structure th.
- de Jong's th. on nodal curves
- log regularity and resolution of toric singularities (Kato)

4. TUBULAR COHOMOLOGICAL DESCENT

- enough to show : $Rj_*\Lambda \in D_c^+(X, \Lambda)$ for

$j : U \to X$ dense open immersion, X qe

- using uniformization theorem,

construct

$$\begin{array}{ccc} U_\cdot & \xrightarrow{j_\cdot} & X_\cdot \\ \downarrow & & \downarrow \varepsilon_\cdot \\ U & \xrightarrow{j} & X \end{array}$$

with $\varepsilon_\cdot$ = pspf hypercovering

(and X_n, j_n as above)

pb : ε_n no longer proper

- circumvent this by :

- Deligne's generic constructibility th. ([SGA 4 1/2 Th. fin.])

- Gabber's hyper base change th. [G2]

• by standard criterion of constructibility,

have to show :

(P) $\forall i \geq 0$, $\forall\ g : X' \to X$ closed irreducible subset,

$\exists$ dense open $V \subset X'$ s. t. $g^* R^i j_* \Lambda | V$ constructible

• by Gabber's hyper base change th. (Gabber, 2005)

$g^* R j_* \Lambda = R\varepsilon'_{\cdot *} g_{\cdot}^{*}(R j_{\cdot *} \Lambda)$

where $g_{\cdot}$, $\varepsilon'_{\cdot}$ defined by cartesian diagram

$$\begin{array}{ccc} X'_{\cdot} & \xrightarrow{g_{\cdot}} & X_{\cdot} \\ \downarrow{\scriptstyle \varepsilon'_{\cdot}} & & \downarrow{\scriptstyle \varepsilon_{\cdot}} \\ X' & \xrightarrow{g} & X \end{array}$$

Remark

base change by g for ε_n not OK

as ε_n non proper

only hyper base change works

Proof of constructibility (modulo hyper base change)

- by absolute purity,

$K_p := {g_p}^*(Rj_{p*}\Lambda) \in D^b_c(X'_p, \Lambda)$

- by Deligne's generic constructibility th.

$\exists$ dense open $V_{pq} \subset X'$ s. t.

$R^q\varepsilon'_{p*}K_p|V_{pq}$ constructible

- spectral sequence

$R^q\varepsilon'_{p*}K_p \Rightarrow g^*R^{p+q}j_*\Lambda$

implies $\exists V$ satisfying (P)

Main ingredient for hyper base change :

Tubular cohomological descent

Idea :

Consider punctured tube

$\overleftarrow{U'} = X' \overleftarrow{\times}_X U$:

$$
\begin{array}{ccccc}
 & & \overleftarrow{U'} & & \\
 & \swarrow^{p_1} & & \searrow^{p_2} & \\
X' & & & & U \\
 & \searrow_{g} & & \swarrow_{j} & \\
 & & X & &
\end{array}
$$

general fact : tubular base change holds :

$g^* R j_* F = R p_{1*} p_2^* F$

for $F \in D^+(U, \Lambda)$

Remarks

- base change not OK for $U' = X' \times_X U$!
- tubular base change holds more generally

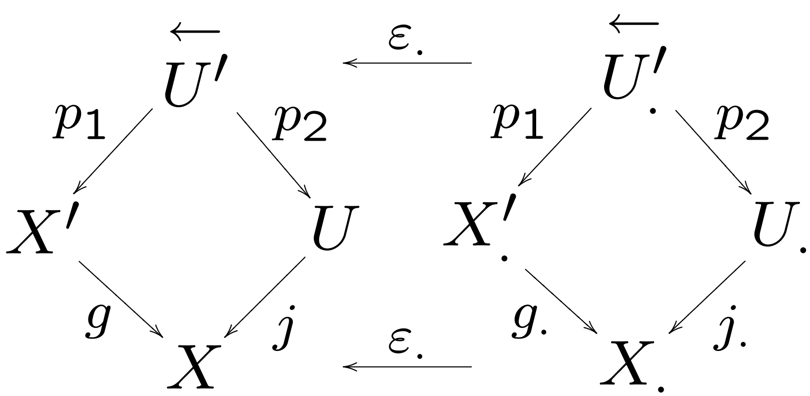
for oriented product $X \overleftarrow{\times}_S Y$

with Y/S quasi-compact and quasi-separated

Similarly, consider simplicial tube

$$\overleftarrow{U'_{\cdot}} = X'_{\cdot} \overleftarrow{\times}_{X_{\cdot}} U_{\cdot}$$

and map



By tubular cohomological descent

$$\Lambda_{\overleftarrow{U'}} = R\varepsilon_{.*}\Lambda_{\overleftarrow{U'_.}}$$

so hyper base change follows from :

$g^* Rj_* \Lambda = Rp_{1*}\Lambda_{\overleftarrow{U'}}$ (tubular base change)

$= Rp_{1*}R\varepsilon_{.*}\Lambda_{\overleftarrow{U'_.}}$ (tubular cohomological descent)

$= R\varepsilon_{.*}Rp_{1*}\Lambda_{\overleftarrow{U'_.}}$ (trivial)

$= R\varepsilon_{.*}g^*_. Rj_{.*}\Lambda_{X'_.}$ (tubular base change)

NB. More general tubular cohomological descent :

- $F = R\varepsilon_{.*}\varepsilon_{.}^{*}F$, $F \in D^{+}(\overleftarrow{U'}, \Lambda)$
- oriented products $X \overleftarrow{\times}_S Y$, Y/S f. p.

Ingredients for proof :

- classical cohomological descent (pspf case)
- tubular base change (easy)
- cohomological invariance of tubes under blow-ups

$$\begin{array}{ccccc} Y' & \longrightarrow & Z' & \nwarrow & \\ \downarrow & & \downarrow f & & \\ Y & \longrightarrow & Z & \longleftarrow & U \end{array}$$

f proper, square cartesian,

$Y \subset Z$, $Y' \subset Z'$ closed, $U = Z - Y = Z' - Y'$

giving map of tubes

$$\overleftarrow{f} : T' = Y' \overleftarrow{\times}_{Z'} U \to T = Y \overleftarrow{\times}_{Z} U$$

Then (cohomological invariance):

$$F = R\overleftarrow{f}_* \overleftarrow{f}^* F \qquad F \in D^+(T, \Lambda)$$

5. WHAT NEXT ?

5.1. Problems in the étale set-up

• More on general nearby cycles

- calculations for specific families,

(e. g. : - semistable reduction along dnc, log smooth maps

- confluences of semistable reduction and quadratic singularities (S. Saito, U. Jannsen))

- discuss iterated monodromies and variations
- compatibility of $R\Psi$ with duality ?
- perversity of $R\Psi$?

- find applications !

(so far : conjugation of vanishing cycles in Lefschetz pencils (Gabber-Orgogozo, 2005))

e. g. : revisit Deligne's approach (1976) to RR pbs via nearby cycles for families of local pencils ? (relation with ramification, variation of Swan conductor, Abbes-K. Kato-T. Saito's work on $\chi(X, F)$)

- Investigate cohomology of tubes

(six operations, finiteness, ...)

5.2. Other set-ups and comparison problems

- Complex analytic case

Pb 1: Define oriented product $\mathcal{X} \overleftarrow{\times}_{\mathcal{S}} \mathcal{Y}$ for maps $\mathcal{X} \to \mathcal{S}$, $\mathcal{Y} \to \mathcal{S}$ of complex analytic spaces, canonical map

$$\varepsilon : X^{\text{an}} \overleftarrow{\times}_{S^{\text{an}}} Y^{\text{an}} \to X \overleftarrow{\times}_S Y$$

for $X \to S$, $Y \to S$ maps of schemes of f. t. $/\mathbb{C}$

with adjunction map

$F \to R\varepsilon_* \varepsilon^* F$

being an isomorphism for $F \in D_c^b(X \overleftarrow{\times}_S Y, \mathbb{Z}/n\mathbb{Z})$

(after possible modification of S ?)

work in progress (D. Treumann) for

stratified topological analogues

(related to MacPherson's theory of exit paths)

Pb 2 : Find common generalization of Orgogozo's th. and Sabbah's th. (1981)

(proper $f : X \to S$

between complex an. spaces

acquires good punctual theory of nearby cycles for constant sheaves after modification of S)

Pb 3 : Find de Rham (or $\mathcal{D}$-modules) analogues, generalize (to higher dimensional bases)

Steenbrink's formula

$$R\Psi\mathbb{C} = \omega^{\cdot}_{X_0}$$

for X semistable / disc

(log variants in [I-Kato-Nakayama])

- Rigid analytic case

Define oriented products $\mathcal{X} \overleftarrow{\times}_{\mathcal{S}} \mathcal{Y}$ for maps $\mathcal{X} \to \mathcal{S}$, $\mathcal{Y} \to \mathcal{S}$ of rigid analytic spaces generalizing Fujiwara's tubes (for X closed in S noetherian, $Y = S - X$), get comparison isomorphism

rigid vs étale

as in Pb 1 above

(work in progress : Gabber, Berkovitch)