p-adic Geometry and Homotopy Theory

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Review of de Rham-Witt Theory

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INTRODUCTION

k perfect, char. $p > 0$, $W = W(k)$

X/k proper, smooth

$W\Omega^{\cdot}_X$: the de Rham-Witt complex of X/k

an inverse limit of dga $W_n\Omega^{\cdot}_X$ on X,

with $W\Omega^0_X = W\mathcal{O}_X$, $W_1\Omega^{\cdot}_X = \Omega^{\cdot}_{X/k}$

• calculates crystalline cohomology :

$$H^*(X, W\Omega^{\cdot}_X) = H^*(X/W)$$

• relates it to :

- Serre's Witt vector cohomology $H^*(X, W\mathcal{O}_X)$

- Artin-Mazur formal gps Φ^q associated with $H^q(-, \mathbb{G}_m)$
($\Phi^1 = \widehat{\text{Pic}}$, $\Phi^2 = \widehat{\text{Br}}$, etc.)

- Hodge cohomology : $H^j(X, \Omega^i_X)$

- analyzes Frobenius on crystalline cohomology

via operators F, $V : W\Omega^i_X \to W\Omega^i_X$

$FV = VF = p$, $\quad FdV = d$

slope spectral sequence

$$E_1^{ij} = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}(X/W)$$

1974 Bloch : for $p > 2$, $\dim X < p$, constructs

inverse sytem of complexes of

typical curves on symbolic part of Quillen's K groups

$$0 \to TC_n\mathcal{K}_1 \to TC_n\mathcal{K}_2 \to \cdots \to TC_n\mathcal{K}_{q+1} \to \cdots$$

$TC_n\mathcal{K}_{q+1}$ in degree q

1975 Deligne proposes alternate construction :

a universal quotient $(L_n^{\cdot})_{n\geq 1}$ of $(\Omega^{\cdot}_{W_n(\mathcal{O}_X)/(\mathbb{Z}/p^n\mathbb{Z})})_{n\geq 1}$ with operators $V : L_n^i \to L_{n+1}^i$ and relations

- inspired by a former construction of Lubkin

- based on explicit description of

subcomplex of integral forms

(see below)

- no more K-theory

1976 - 1978 I. carries out Deligne's program :

- construction of $W.\Omega^{\cdot}_X$ (any $X/\mathbb{F}_p$)
- comparison with Bloch's construction, and with crystalline cohomology (X/k smooth)
- global geometric applications (e. g. H^* of surfaces)

1979 K. Kato removes restrictions $p > 2$ and $\dim X < p$ in Bloch's construction

1979 - 1983 I., Raynaud, Nygaard, Ekedahl study fine structure of slope spectral sequence

discovery of higher Cartier isomorphisms

$\Rightarrow$ new construction of $W_\cdot\Omega^\cdot_X$

(for X/k smooth)

1988 Hyodo-Kato adapt it to log geometry

(X/k log smooth, Cartier type)

$\Rightarrow$ key tool in construction of Hyodo-Kato isomorphism

2004 extensions of DRW theory to

mixed char. and relative situations

- Hesselholt-Madsen : $W_{\cdot}\Omega^{\cdot}_{A}$, $A/\mathbb{Z}_{(p)}$, $p > 2$
- Langer-Zink : $W_{\cdot}\Omega^{\cdot}_{A/R}$, $R/\mathbb{Z}_{(p)}$

2007 Olsson : stack-theoretic variants

2008 Davis-Langer-Zink : overconvergent variants

PLAN

1. Construction(s) of $W_{\cdot}\Omega^{\cdot}_{X}$
2. Comparison with crystalline cohomology
3. Higher Cartier isomorphisms
4. De Rham-Witt for log schemes
5. The Hyodo-Kato isomorphism

1. CONSTRUCTION(S) OF $W_{\cdot}\Omega^{\cdot}_X$

$X = (X, \mathcal{O}_X)$ ringed topos over $\mathbb{F}_p$

$W_n\mathcal{O}_X$: Witt vectors of length n on $\mathcal{O}_X$

Theorem 1.

(a) There exists an inverse system of dga

$W_n\Omega^{\cdot}_X = (W_n\Omega^0_X \to W_n\Omega^1_X \to \cdots \to W_n\Omega^i_X \to \cdots)$

$(n \geq 1)$

with operators $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$

satisfying

(0) $RV = VR$

$(R : W_{n+1}\Omega^{\cdot}_X \to W_n\Omega^{\cdot}_X)$

(1) $W_n\Omega^0_X = W_n\mathcal{O}_X$, $R = R$, $V = V$

(2) $V(xdy) = (Vx)dVy$

(3) $(Vx)d[y] = V(x[y^{p-1}]d[y])$,

$y \in \mathcal{O}_X$, $[y] = (y, 0, \cdots, 0)$

universal for these properties

(b) The natural map

$$\Omega^{\cdot}_{W_n(\mathcal{O}_X)/(\mathbb{Z}/p^n\mathbb{Z})} \to W_n\Omega^{\cdot}_X$$

is surjective,

an isomorphism for $n = 1$:

$$\Omega^{\cdot}_{X/\mathbb{F}_p} = W_1\Omega^{\cdot}_X.$$

(c) There exist unique operators

$$F : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$$

satisfying :

$$FdV = d$$

$$Fa.Fb = F(ab)$$

$$Fa = RFa,\ a \in W_n\mathcal{O}_X$$

$$Fd[a] = [a]^{p-1}d[a],\ a \in \mathcal{O}_X,\ [a] = (a, 0, \cdots, 0).$$

Proof : (a), (b) : straightforward

(c) relies on

description of $W_{\cdot}\Omega^{\cdot}_{A}$ for $A = \mathbb{F}_p[T_1, \cdots, T_r]$

in terms of the complex $E^{\cdot}$ of integral forms :

$$W_n\Omega^{\cdot}_{A} = E^{\cdot}/(V^n E^{\cdot} + dV^n E^{\cdot}),\ V = pF^{-1},\ FT_i = T_i^p$$

$$E^{\cdot} \subset \Omega^{\cdot}_{C/\mathbb{Q}_p},\ C = \mathbb{Q}_p[T_1^{p^{-\infty}}, \cdots, T_r^{p^{-\infty}}],$$

$\omega \in E^i \Leftrightarrow \omega$ and $d\omega$ integral

(i. e. coefficients in $\mathbb{Z}_p$)

Example : $r = 1,\ A = \mathbb{F}_p[T]$,

$W\Omega^{\cdot}_A :=$ proj.lim. $W_n\Omega^{\cdot}_A$

$WA = \{\sum_{k\in\mathbb{N}[1/p]} a_k T^k, a_k \in \mathbb{Z}_p, \mathrm{den}(k)|a_k\ \forall k,$
$\lim_{k\to\infty} \mathrm{a_k} = 0\}$

$W\Omega^1_A = \{\sum_{k>0,k\in\mathbb{N}[1/p]} a_k T^k (dT/T), a_k \in \mathbb{Z}_p,$
$\lim_{k\to\infty} \mathrm{den(k).a_k} = 0\}$

$W\Omega^i_A = 0, i > 1.$

More formulas

(1) $F : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$ satisfies

$FV = p$, $VF = p$,

$V(aFb) = (Va)b$, $\forall a \in W_n\Omega^i_X, b \in W_{n+1}\Omega^j_X$

(2) $VW_n\mathcal{O}_X \subset W_{n+1}\mathcal{O}_X$ has a canonical pd-structure :

$\gamma_k(Va) = (p^{k-1}/k!)Va^k$ $(k \geq 1)$

and d is a pd-derivation :

$$d\gamma_k(a) = \gamma_{k-1}(a)da, \ (a \in VW_n\mathcal{O}_X)$$

Moreover,

$$Fd : W_{n+1}\mathcal{O}_X \to W_n\Omega^1_X,$$
$$(Fd([a] + Vb) = [a]^{p-1}d[a] + db)$$

is also a pd-derivation.

Structure for X/k smooth

$$0 \to \mathrm{gr}^n\, W.\Omega^{\cdot}_X \to W_{n+1}\Omega^{\cdot}_X \to W_n\Omega^{\cdot}_X \to 0,$$

$$\mathrm{gr}^n = (V_n + dV^n)\Omega^{\cdot}_X$$

locally free of finite type $/\mathcal{O}_X$ in each degree i
(extension of Ω^{i-1}/Z_n by Ω^i/B_n)

New constructions

d, Fd pd-derivations $\Rightarrow$ Langer-Zink's construction of a relative (pro-F-V-) de Rham-Witt complex

$$W_{\cdot}\Omega^{\cdot}_{X/S}$$

($f : X \to S$ a morphism of $\mathbb{Z}_{(p)}$-ringed toposes)

- $W_n\Omega^{\cdot}_{X/S} = f^{-1}W_n(\mathcal{O}_S)$-dga
- $F : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$ built in together with

$V : W_{n-1}\Omega^i_X \to W_n\Omega^i_X$

satisfying above formulas

(except possibly $VF = p$)

- $W_n\Omega^0_{X/S} = W_n\mathcal{O}_X$,

$\Omega^{\cdot}_{W_n\mathcal{O}_X/W_n\mathcal{O}_S,pd} \to W_n\Omega^{\cdot}_{X/S}$ surjective

- $W_{\cdot}\Omega^{\cdot}_{X/S} = W_{\cdot}\Omega^{\cdot}_X$ for $S = \{pt, \mathbf{F}_p\}$

Applications :

- theory of displays (Zink)
- generalized Hyodo-Kato isomorphisms (Olsson)

Langer-Zink's construction competes with :

Hesselholt-Madsen's construction

of an absolute (pro-F-V-) de Rham-Witt complex

$$W_{\cdot}\Omega^{\cdot}_X$$

($X \to S$ a $\mathbb{Z}_{(p)}$-ringed topos)

- hypothesis : $p > 2$
- $W_n\Omega^0_X = W_n\mathcal{O}_X$
- $W_n\Omega^{\cdot}_X = \mathbb{Z}_{(p)}$-dga (not a $W_n(\mathcal{O}_S)$-dga)

again :

- $F : W_n\Omega^i_X \to W_{n-1}\Omega^i_X$ built in with

$V : W_{n-1}\Omega^i_X \to W_n\Omega^i_X$

satisfying above formulas (except possibly $VF = p$)

- $W_{\cdot}\Omega^{\cdot}_X =$ "classical" $W_{\cdot}\Omega^{\cdot}_X$ for $X/\{pt, \mathbf{F}_p\}$

Applications :

- relations with K-theory and cyclic homology
- construction generalizes to log spaces
- relations with p-adic vanishing cycles

(fixed points of F) (Geisser-Hesselholt)

Comparison LZ - HM

canonical map

$$W_n\Omega^{\cdot}_X \to W_n\Omega^{\cdot}_{X/S}$$

- surjective, iso if $p1_S = 0$ and S perfect
- not iso for $S = \{pt, \mathbb{Z}_p\}$

$(W_{\cdot}\Omega^1_{\mathbb{Z}_p/\mathbb{Z}_{(p)}} \otimes \mathbb{Z}/p^n = 0$,
$W_{\cdot}\Omega^1_{\mathbb{Z}_p} \otimes \mathbb{Z}/p^n \supset \mathbb{Z}_p^*/\mathbb{Z}_p^{*p^n})$

• $W_n\Omega^{\cdot}_{X/S} = W_n\Omega^{\cdot}_X/I$,
I = dg ideal generated by image of $f^{-1}W_n\Omega^1_S \to W_n\Omega^1_X$
(analyzed by Hesselholt for X/S smooth, S noetherian, $p1_S$ nilpotent)

Overconvergent de Rham-Witt (Davis-Langer-Zink) gives natural lattices in Berthelot's rigid cohomology of X/k smooth (k perfect, char. p)

2. COMPARISON WITH CRYSTALLINE COHOMOLOGY

k perfect field, $\text{char}(k) = p > 0$, $W_n = W_n(k)$, X/k

Recall :

- $u = u_{X/W_n} : (X/W_n)_{\text{crys}} \to X$ canonical morphism

$(u_*(E)(U) = \Gamma((U/W_n)_{\text{crys}}, E))$

- If $\exists$ embedding, $X \subset Z/W_n$ smooth :

$Ru_{X/W_n *}\mathcal{O}_{X/W_n} = \mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}$,

$(D =$ pd-envelope of $X \subset Z/W_n)$

THEOREM 2.1 Assume X/k smooth.

Then $\exists$ can., functorial inverse sytem of isomorphisms (of $D(X, W_n)$)

$$Ru_{X/W_n *}\mathcal{O}_{X/W_n} \xrightarrow{\sim} W_n\Omega^{\cdot}_X$$

- compatible with multiplicative structures
- compatible with Frobenius

($\Phi : W_n\Omega^{\cdot}_X \to W_n\Omega^{\cdot}_X$, $\Phi | W_n\Omega^i_X = p^i F$).

Proof

• construction of map :

- local case : $X \to W_nX \to D \to Z$,

(Z/W_n smooth)

$\mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n} \to \Omega^{\cdot}_{W_nX/W_n,pd} \to W_n\Omega^{\cdot}_X$

-globalization : cohomological descent

• map = iso : use local structure of DRW

$X = \mathrm{Spec}\,\mathbb{F}_p[\underline{T}]$,

$E^{\cdot}$ = complex of integral forms $\subset \Omega^{\cdot}_{\mathbb{Q}_p[\underline{T}^{p^{-\infty}}]/\mathbb{Q}_p}$

$E^{\cdot} = \Omega^{\cdot}_{\mathbb{Z}_p[\underline{T}]/\mathbb{Z}_p} \oplus$ homotop. triv. complex

$W_n\Omega^{\cdot}_X = E^{\cdot}/(V^n + dV^n)E^{\cdot}$

$= \Omega^{\cdot}_{\mathbb{Z}/p^n[\underline{T}]/(\mathbb{Z}/p^n)} \oplus$ acyclic complex

Slopes of Frobenius

X/k proper and smooth

$R\Gamma(X/W_n) := R\Gamma((X/W_n)_{\text{crys}}, \mathcal{O}_{X/W_n}) = R\Gamma(X, Ru_*\mathcal{O}_{X/W_n})$

2.1 $\Rightarrow R\Gamma(X/W_n) = R\Gamma(X, W_n\Omega^{\cdot}_X)$

$W\Omega^{\cdot}_X :=$ proj.lim. $W_n\Omega^{\cdot}_X$,

$R\Gamma(X/W) := R\Gamma(X, W\Omega^{\cdot}_X)$

2.1 $\Rightarrow R\Gamma(X/W) = R\lim R\Gamma(X/W_n) \in D(W)_{\text{perf}}$

($\Rightarrow H^m(X/W) = H^m R\Gamma(X/W)$ f. g. over W)

absolute Frobenius $F : X \to X$ gives

σ-linear $\Phi : R\Gamma(X/W) \to R\Gamma(X/W)$

$\Phi : H^m(X/W) \to H^m(X/W)$ isomorphism mod torsion

(i. e. $=$ F-crystal)

better : $v : W\Omega^{\cdot}_X \to W\Omega^{\cdot}_X$, $v|W\Omega^i_X = p^{d-i-1}V$ gives

$\Phi v = v\Phi = p^d$

on $W\Omega^{\cdot}_X$ and $R\Gamma(X/W)$,

(X of pure dim. d)

(cf. Berthelot, Berthelot-Ogus)

Slope spectral sequence

$$E_1^{ij} = H^j(X, W\Omega_X^i) \Rightarrow H^{i+j}(X/W)$$

- degenerates at E_1 mod p-torsion
- $K_0 \otimes H^j(X, W\Omega_X^i)$ = part of slope $\subset [i, i+1[$ of $K_0 \otimes H^{i+j}(X/W)$

($K_0 = \mathrm{Frac}(W)$)

- torsion studied by Nygaard, I-Raynaud, Ekedahl

Other applications

• Poincaré duality, Künneth (Ekedahl)

• logarithmic Hodge Witt sheaves $W_n\Omega^i_{X,\log} \subset W_n\Omega^i_X$ étale loc. generated by $\mathrm{dlog}[x_1]\cdots\mathrm{dlog}[x_i]$ (Milne, Bloch-Gabber-Kato, Kato, I-Raynaud, Colliot-Thélène, Gros, ...)

• cycle and Chern classes in crystalline cohomology (Gros)

3. HIGHER CARTIER ISOMORPHISMS

X/k smooth

$F^n : W_{2n}\Omega^i_X \to W_n\Omega^i_X$

induces isomorphism

$$C^{-n} : W_n\Omega^i_X \xrightarrow{\sim} \mathcal{H}^i W_n\Omega^{\cdot}_X,$$

compatible with products,

generalizing standard Cartier isomorphism for $n = 1$:

$$C^{-1} : \Omega^i_X \to \mathcal{H}^i \Omega^{\cdot}_X$$

th. 2.1 $\Rightarrow$: C^{-n} induces W_n-linear isomorphism

(3.1) $$W_n\Omega^i_X \xrightarrow{\sim} \sigma^n_* \mathcal{H}^i(X/W_n)$$

$(\mathcal{H}^i(X/W_n) = R^i u_{X/W_n *}\mathcal{O}_{X/W_n})$

$i = 0$: $W_n\mathcal{O}_X \xrightarrow{\sim} \sigma^n_* \mathcal{H}^0(X/W_n)$

(cf. Fontaine-Messing : $\widetilde{W}^{DP}_n \xrightarrow{\sim} \mathcal{O}^{\text{crys}}_n$
on X_{syn})

X lifted to Z smooth $/W_n$,

$(a_0, \cdots, a_{n-1}) \in W_n\mathcal{O}_X$
$\mapsto b_0^{p^n} + pb_1^{p^{n-1}} + \cdots + p^{n-1}b_{n-1}^p$
$\in \mathcal{H}^0(X/W_n) = \mathcal{H}^0_{dR}(Z/W_n)$
$b_i \in \mathcal{O}_Z \mapsto a_i$

$i = 1:\ W_n\Omega^1_X \xrightarrow{\sim} \sigma^n_*\mathcal{H}^1(X/W_n) = \sigma^n_*\mathcal{H}^1_{dR}(Z/W_n)$
$d(a_0, \cdots, a_{n-1}) \mapsto \sum b_i^{p^{n-i-1}} db_i$
(cf. Serre's map $F^n d : W_n\mathcal{O}_X \to \Omega^1_X$)

Katz's observation : 3.1 $\Rightarrow$

re-construction of the de Rham-Witt complex :

$W_n\Omega^i_X := \sigma^n_* \mathcal{H}^i(X/W_n)$

$d : W_n\Omega^i_X \to W_n\Omega^{i+1}_X =?$

$R : W_{n+1}\Omega^i_X \to W_n\Omega^i_X =?$

$F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X =?$

$V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X =?$

d = Bockstein = coboundary from

$$0 \to \Omega^{\cdot}_{Z_n/W_n} \xrightarrow{p^n} \Omega^{\cdot}_{Z_{2n}/W_{2n}} \to \Omega^{\cdot}_{Z_n/W_n} \to 0$$

(X lifted to Z_{2n} smooth $/W_{2n}$, $Z_n := Z_{2n} \otimes W_n$)

$F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ induced by can. map
$\mathcal{H}^i(X/W_{n+1}) \to \mathcal{H}^i(X/W_n)$

$V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$ induced by
$p : \mathcal{H}^i(X/W_n) \to \mathcal{H}^i(X/W_{n+1})$

$R : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ more delicate,
uses formal smooth lifting Z/W,
plus lifting F of Frobenius, and
$\varphi = p^{-i}F^*$ on $\Omega^i_{Z/W}$
$Ra = b$ if $a =$ class of φb
(cf. Mazur-Ogus gauges th. : $F^* : \Omega^{\cdot}_{Z_n} \to \Omega^{\cdot}_{Z_n,(i \mapsto i)}$ quasi-isomorphism)

$\Rightarrow$ construction of DRW and subsequent formalism (for X/k smooth) independent of the complex of integral forms for the affine space

Applications

• structure of conjugate spectral sequence (I-Raynaud)

(X/k proper and smooth)

$$E_2^{ij} = \text{proj.lim}\, H^i(X, \mathcal{H}^j(X/W_n)) \Rightarrow H^{i+j}(X/W)$$

- degenerates at E_2 mod p-torsion
- $K_0 \otimes \text{proj.lim}\, H^i(X, \mathcal{H}^j(X/W_n))$

$=$ part of slope $\subset]j-1, j]$ in $K_0 \otimes H^{i+j}(X/W)$

- construction works in other contexts :

log schemes (Hyodo-Kato)

algebraic stacks (Olsson)

4. DE RHAM-WITT FOR LOG SCHEMES

$\alpha : L \to k$: a fine log str. on $s = \mathrm{Spec}(k)$

$W_n(L)$: Teichmüller lifting of L to $W_n(s)$:

$W_n(L) = L \oplus \mathrm{Ker}(W_n(k)^* \to k^*)$

$L \to W_n(k) : a \mapsto [\alpha(a)]$

Example : $L = (\mathbb{N} \to k, 1 \mapsto 0)^a$ (standard log point)

$W_n(L) = (\mathbb{N} \to W_n(k), 1 \to 0)^a$

$(X,M)/(s,L)$: fine log scheme, log smooth and Cartier type $/(s,L)$

Cartier type (= integral, relative Frobenius exact) ensures the existence of Cartier isomorphism :

$$C^{-1} : \Omega^q_{(X,M)'/(s,L)} \xrightarrow{\sim} F_*\mathcal{H}^q(\Omega^{\cdot}_{(X,M)/(s,L)})$$

$(F : (X,M) \to ((X,M)' = (s,L) \times_{(s,L)} (X,M)$ relative Frobenius) ,

C^{-1} : $a \mapsto F^*a$, $\mathrm{dlog}\, b \mapsto \mathrm{dlog}\, b$

Typical example : (s, L) standard log point,

(X, M) of semistable type $/(s, L)$:

étale loc. $X = \operatorname{Spec} k[t_1, \cdots, t_d]/(t_1 \cdots t_r)$, with charts

$$\begin{array}{ccc} k[t_1, \cdots, t_d]/(t_1 \cdots t_r) & \longleftarrow & \mathbb{N}^r \\ \uparrow & & \uparrow {\scriptstyle 1 \mapsto (1, \cdots, 1)} \\ k & \xleftarrow{1 \mapsto 0} & \mathbb{N} \end{array}$$

(e. g. special fiber of semistable scheme over trait)

Log crystalline (Hyodo-Kato) cohomology :

$$H^m((X,M)/(W_n,W_n(L)) :=$$
$$H^m(((X,M)/(W_n,W_n(L))_{crys},\mathcal{O})$$
$$= H^m(X, Ru_{(X,M)/(W_n,W_n(L)*}\mathcal{O})$$

$((X,M)/(W_n,W_n(L))_{crys}$: log crystalline site (built with local log pd-thickenings)

comes equipped with

- Frobenius operator

$$\varphi : H^m((X,M)/(W_n, W_n(L)) \to H^m((X,M)/(W_n, W_n(L))$$

defined by (Frobenius on schemes, p on monoids)

$\varphi = \sigma$-linear isogeny (Berthelot-Ogus, Hyodo-Kato)

$\exists\ \sigma^{-1}$-linear ψ, $\varphi\psi = \psi\varphi = p^r$

$(r = \dim(X) = \mathrm{rk}\,\Omega^1_{(X,M)/(s,L)})$

• (for (s, L) = standard log point)
monodromy operator

$$N : H^m((X, M)/(W_n, W_n(L)) \to H^m((X, M)/(W_n, W_n(L))$$

N = residue at $t = 0$ of Gauss-Manin connection on $H^m((X, M)/(W_n < t >, \text{can}))$ rel. to W_n (with trivial log str.)

basic relation :

$$N\varphi = p\varphi N$$

for X/k proper, get (φ, N)-module structure on

$K_0 \otimes H^m((X,M)/(W,W(L)))$,

where

$H^m((X,M)/(W,W(L))) = \text{proj.lim.}\, H^m((X,M)/(W_n,W_n(L))$

a f. g. W-module,

N is nilpotent

Log de Rham-Witt complex

imitate Katz-I-Raynaud's re-construction :

change notations : $Y = (Y, M)$ log smooth, Cartier type $/(k, L)$

$W_n\omega^i_Y = \sigma^n_* R^i u_{(Y,M)/(W_n, W_n(L)^*}\mathcal{O})$,

d : Bockstein

F : given by restriction from W_{n+1} to W_n

V : given by multiplication by p

R : uses Cartier type, variant of Mazur-Ogus gauges th.

get pro-complex

$$W.\omega^{\cdot}_{Y},$$

with operators F, V satisfying standard formulas ($FV = VF = p$, $FdV = d$, etc.)

new feature : $\exists$ can. homomorphism

$$\mathrm{dlog} : M \to W_n\omega^1_Y,$$

$(a \mapsto \mathrm{dlog}\,\tilde{a} \in \mathcal{H}^1(\mathcal{O}_D \otimes \Omega^{\cdot}_{(Z,M_Z)/(W_n,W_n(L))}),\ \tilde{a} \in M_Z \mapsto a \in M$,

$Y \subset (Z, M_Z)/(W_n, W_n(L))$ log smooth embedding,

$D =$ pd-envelope)

satisfying

$F\,\mathrm{dlog} = \mathrm{dlog}$, $[\alpha(a)]\,\mathrm{dlog}\,a = d[\alpha(a)]$

$W_n\omega^0_Y = W_n\mathcal{O}_Y$, and

$W_n\omega^*_Y$ generated as $W_n(\mathcal{O}_Y)$-algebra by $dW_n\mathcal{O}_Y$, $\mathrm{dlog}\,M^{gp}$

comparison th. 2.1 generalizes :

$$Ru_{(Y,M)/(W_n,W_n(L)*}\mathcal{O} \xrightarrow{\sim} W_n\omega^{\cdot}_{Y}$$

$$R\Gamma((Y,M)/(W_n,W_n(L)) \xrightarrow{\sim} R\Gamma(Y,W_n\omega^{\cdot}_{Y})$$

slope spectral sequence, higher Cartier isom., etc. generalize, too

5. THE HYODO-KATO ISOMORPHISM

$S = \operatorname{Spec} A$, A complete dvr, char. $(0, p)$,

$K = \mathrm{Frac}(A)$, $k = A/\pi A$ residue field, perfect

$K_0 = \mathrm{Frac}(W(k))$

X/S semi-stable reduction

$Y = X \otimes k$ the special fiber

goal : for X/S proper, define (K_0, φ, N)-structure on $H^m_{dR}(X_K/K)$
using log crystalline cohomology of Y

• case X/S smooth : Berthelot-Ogus isomorphism

$$K \otimes_W H^m(Y/W) \xrightarrow{\sim} H^m_{dR}(X_K/K)$$

$\Rightarrow$ (K_0, φ)-structure $(N = 0)$

• general case : use log str.

M_X : can. log str. on X, induced by special fiber

$(M_X = \mathcal{O}_X \cap j_* \mathcal{O}^*_{X_K}$, $j : X_K \subset X)$

M_Y : induced log str. on Y

L : $(\mathbb{N} \to k, 1 \mapsto 0)^a$: log str. on Spec k,

induced by standard log str. on S, $(\mathbb{N} \to A, 1 \to \pi)^a$

Put $A_n = A \otimes \mathbb{Z}/p^n\mathbb{Z} = A \otimes_W W_n$, $S_n = \mathrm{Spec}\, A_n$

$X_n = X \otimes \mathbb{Z}/p^n\mathbb{Z} = X \otimes_W W_n$, with induced log str.

Consider projective systems (a) and (b) of

$D^+(X_n, A_n) = D^+(Y, A_n)$:

(a) $A_n \otimes^L_{W_n} Ru_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O}$

(b) $\omega^{\cdot}_{X_n/A_n} := \Omega^{\cdot}_{(X_n,M_{X_n})/(S_n,M_{S_n})}$

Note : (a) $\xrightarrow{\sim} A_n \otimes^L_{W_n} W_n\omega^{\cdot}_{Y/W_n}$

THEOREM 5.1 (Hyodo-Kato)

There exists a canonical isomorphism

$$\rho_\pi : \mathbb{Q} \otimes (A. \otimes^L_{W.} W.\omega^{\cdot}_{Y/W.}) \xrightarrow{\sim} \mathbb{Q} \otimes \omega^{\cdot}_{X./S.}$$

in $\mathbb{Q} \otimes$ proj.sys. $D^+(Y, A_n)$

(for additive cat. C, $\mathrm{Hom}_{\mathbb{Q}\otimes C} = \mathbb{Q} \otimes \mathrm{Hom}_{\mathcal{C}}$,
$\mathbb{Q} \otimes K$: image of K in $\mathbb{Q} \otimes \mathcal{C}$)

COROLLARY 5.2

For X/S proper (and semistable), ρ_π induces an isomorphism :

$$\rho_\pi : K \otimes_W H^m((Y, M_Y)/(W, W(L)) \xrightarrow{\sim} H^m_{dR}(X_K/K)$$

(gives (K_0, φ, N) structure on $H^m_{dR}(X_K/K)$)

Remarks

(a) 5.1 valid for X/S log smooth, Cartier type

(b) ρ_π depends on π :

$\rho_{\pi u} = \rho_\pi \exp(\log(u)N)$, $u \in A^*$

(c) if X/S smooth, then :

$$H^m(Y/W) \xrightarrow{\sim} H^m((Y, M_Y)/(W, W(L)),$$

ρ_π = Berthelot-Ogus isomorphism, independent of π.

Highlights of proof

(Rough) idea : (a), (b) come from

F-crystal on $W < t >$ with log pole at $t = 0$

use Frobenius ($t \mapsto t^p$), an isogeny, contracting the disc, to connect :

(a) = log crys side = fiber at 0,

(b) = dR side = general fiber

Two main steps :

- Use Berthelot-Ogus's method to reduce to rigidity th. $/W_n < t >$ mod bounded p-torsion

- Construction of rigidification (uses de Rham-Witt, lifting of higher Cartier isomorphisms)

Embed Spec A into Spec $W[t]$, $t \to \pi$,

with log str. $\mathbb{N} \to W[t], 1 \to t$

Spec R_n = pd-envelope of Spec A_n in Spec $W_n[t]$

= pd-envelope of $S_1 =$ Spec A_1 in Spec $W_n[t]$

Then :

$$\omega^{\cdot}_{X_n/A_n} = A_n \otimes^{L}_{R_n} Ru_{X_1/R_n *}\mathcal{O}$$

and

$$\varphi : Ru_{X_1/R_n *}\mathcal{O} \to Ru_{X_1/R_n *}\mathcal{O}$$

(F on X_1, S_1, $\varphi t = t^p$ on $W_n[t]$)

is an isogeny : $\exists \psi$, $\psi\varphi = \varphi\psi = p^r$, $r = \dim Y$

As $F^N : X_1/S_1 \to X_1/S_1$ factors

through Y/k for $N >> 0$,

Th. 5.1 follows from rigidity th. :

THEOREM 5.3 (Hyodo-Kato)

(i) There exists a unique homomorphism

$$h : \mathbb{Q}\otimes(W. < t > \otimes_{W.} Ru_{(Y/(W.,W.(L))^*}\mathcal{O}) \longrightarrow \mathbb{Q}\otimes Ru_{(Y/W.<t>)^*}\mathcal{O}$$

compatible with φ,

and whose composition with natural map

$Ru_{(Y/W.<t>)^*}\mathcal{O} \longrightarrow Ru_{(Y/(W.,W.(L))^*}\mathcal{O}$

= can. projection

(ii) h is an isomorphism.

5.3 $\Rightarrow$ 5.1

$N \geq \log_p e$, $e = [K : K_0]$,
5.3 $\Rightarrow$

$$h_\pi = \varphi^N h \varphi^{-N} : \mathbb{Q} \otimes (R. \otimes^L_{W.} Ru_{(Y/(W.,W.(L))*}\mathcal{O}) \xrightarrow{\sim} \mathbb{Q} \otimes Ru_{X_1/R.*}\mathcal{O}$$

and

$$A_n \otimes^L_{R_n} Ru_{X_1/R_n*}\mathcal{O} = \omega^{\cdot}_{X_n/A_n},\ \rho_\pi = \mathbb{Q} \otimes (A. \otimes^L_{R.} h_\pi)$$

Proof of (i) :

use endomorphism φ of triangle

$$I_n \otimes^L_{W_n<t>} Ru_{(Y/W_n<t>)*}\mathcal{O} \to$$
$$Ru_{(Y/W_n<t>)*}\mathcal{O} \to Ru_{(Y/(W_n,W_n(L))*}\mathcal{O}) \to$$

($I_n = \mathrm{Ker}\, W_n < t > \to W_n$) together with :

- $\varphi\psi = \psi\varphi = p^r$ on 3rd term
- $\varphi^i(I_n) \subset (p^i)! I_n$

$\Rightarrow$ unique φ-splitting mod bounded p-torsion

Proof of (ii) :

relies on existence of

lifted higher Cartier isomorphism

If $(Y, M_Y)/(k, L) \subset (Z., M.)/(W.[t], 1 \mapsto t)$

= compatible system of log smooth embeddings

$$R^q u_{Y/W_n<t>*}\mathcal{O} = \mathcal{H}^q(\mathcal{O}_{D_n} \otimes \omega^{\cdot}_{Z_n/W_n<t>})$$

(D_n = pd-envelope of Y in Z_n)

THEOREM 5.4 (Hyodo-Kato)

(a) $\exists$ unique homomorphism of graded algebras

$$c_n : W_n\omega_Y^* \to R^* u_{Y/W_n<t>*}\mathcal{O},$$

$(a_0, \cdots, a_{n-1}) \mapsto \Sigma_{0\le i\le n-1}\, p^i \tilde{a}_i^{p^{n-i}}$,
$d(a_0, \cdots, a_{n-1}) \mapsto \Sigma_{0\le i\le n-1}\, \tilde{a}_i^{p^{n-i-1}} d\tilde{a}_i$, $\operatorname{dlog} b \mapsto \operatorname{dlog} \tilde{b}$,

($\tilde{a}_i \in \mathcal{O}_{D_n}$ (resp. $\tilde{b} \in M_{Z_n}$) lifts a_i (resp. b)).

(b) The composition of c_n with the inverse higher Cartier isom. C^n

$$R^q u_{Y/(W_n, W_n(L)^*} \mathcal{O} \xrightarrow{C^n} W_n \omega_Y^q \xrightarrow{c_n} R^q u_{Y/W_n<t>^*} \mathcal{O}$$

is a φ-equivariant section of the can. projection, and induces a $W_n < t >$-linear isomorphism

$$h_n^q : W_n < t > \otimes_{W_n} R^q u_{Y/(W_n, W_n(L)^*} \mathcal{O} \xrightarrow{\sim} R^q u_{Y/W_n<t>^*} \mathcal{O}.$$

End of proof of (ii) :

One shows :

$$\mathcal{H}^q(h) = \mathbb{Q} \otimes h_{\cdot}^q,$$

with $h_{\cdot}^q$ as in 5.4 (uniqueness of φ-equivariant sections).

Proof of 5.4 :

uses explicit presentations of $W_n\omega_Y^*$ by generators and relations.

Remarks

• Ogus (1995) : Simpler proof, variants and generalizations of 5.2 for log F-crystals

But : 5.1, 5.3 crucial for crystalline interpretation of

$$B_{\mathrm{st}}^{+} \otimes_W H^m((Y, M_Y)/(W, W(L))$$

via Künneth formulas at finite levels :

$$H^0(\overline{S}_1/R_n) \otimes_{R_n} H^m(X_1/R_n) \xrightarrow{\sim} H^m(\overline{X}_1/R_n),$$

where

lhs $= H^0(\overline{S}_1/R_n) \otimes_{W_n} H^m(Y/(W_n, W_n(L))$

mod bounded p-torsion by Hyodo-Kato 5.1

($\overline{X}_1 = X_1 \otimes \mathcal{O}_{\overline{K}}$, etc.,

can. log. str. on $\overline{S}_1$, $\overline{X}_1$, Y, R_n)

and construction of comparison map

$$B_{\mathrm{st}} \otimes H^m(X_{\overline{K}}, \mathbb{Q}_{p)}) \to B_{\mathrm{st}} \otimes H^m((Y, M_Y)/(W, W(L))$$

• Olsson (Astérisque 316, 412 p.) :

variants and generalizations of 5.3, 5.4 for

certain algebraic stacks using

- Langer-Zink relative de Rham-Witt complex

- dictionary :

(log scheme) = (scheme over a certain alg. stack)