# On log flat descent

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**Abstract**

We prove the log flat descent of log étaleness, log smoothness, and log flatness for log schemes.

**Contents**



## Introduction

The aim of this paper is to give proofs of the following results, which were announced by K. Kato in [4].

**Theorem 0.1.** *Let $f\colon X \to Y$ be a morphism of fs log schemes, and let $g\colon Y' \to Y$ be a surjective, kummer, and log flat morphism locally of finite presentation of fs log schemes. Let $X' = X \times_Y Y'$ and let $f'\colon X' \to Y'$ be the morphism induced by $f$.*

*Consider the following three properties of morphisms of fs log schemes* :

(1) *log étale* ;
(2) *log smooth* ;
(3) *log flat.*

*Then, if* **P** *denotes one of these properties, $f$ has the property* **P** *if and only if $f'$ has* **P**.

**Theorem 0.2.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of fs log schemes, and assume that $f$ is surjective and kummer. Let* **P** *be one of the properties in* 0.1.

*If $f$ and $g \circ f$ have the property* **P**, *then $g$ has the property* **P**.

Our proofs of the above theorems use the stack-theoretic characterizations by M. C. Olsson ([8] 4.6) of the property **P** (where **P** is as in 0.1). It seems that the original proofs, not yet written, are different from ours. Precisely, in [4], the above theorems were announced in slightly weaker forms. In particular, some noetherian and local finite presentation assumptions were imposed in the case where **P** is "log flat."

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For variants of 0.1 and 0.2 and a related example, see the last section 3.

In the next section 1, we review the necessary terminology. We prove the theorems in the section 2.

Notation and Terminology. The log structures we are considering are taken for the étale topology. We adopt the usual notation $\overline{x}$ for a geometric point over a point $x$ of a scheme.

For a monoid $P$, the subset of the invertible elements is denoted by $P^\times$, and $P/P^\times$ by $\overline{P}$.

We say that a homomorphism $h\colon P \to Q$ of fs monoids is *kummer* (resp. *exact*) if it is injective and for any $a \in Q$, some power of $a$ belongs to the image (resp. it satisfies $P = (h^{\mathrm{gp}})^{-1}(Q)$ in $P^{\mathrm{gp}}$). A kummer homomorphism of fs monoids is exact. We say that a morphism $f\colon X \to Y$ of fs log schemes is *strict* (resp. *kummer*, resp. *exact*) if $f^{-1}(M_Y/\mathcal{O}_Y^\times) \to M_X/\mathcal{O}_X^\times$ is an isomorphism (resp. is kummer at stalks, resp. is exact at stalks). Here, "stalks" means stalks at geometric points.

Throughout the paper, a chart of an fs log scheme (resp. a morphism of fs log schemes) means a chart by an fs monoid (resp. fs monoids).

# 1 Review of Log flat morphisms

**1.1.** We briefly review log étale, log smooth, and log flat morphisms.

The log étaleness and the log smoothness were introduced and studied by K. Kato in [3].

The log flatness was also introduced in his unpublished [4]. A morphism $f\colon X \to Y$ of fs log schemes is said to be *log flat* if strict fppf locally on $X$ and on $Y$, there is an injective chart $P \to Q$ of $f$ such that the induced morphism of schemes $X \to Y \times_{\operatorname{Spec}\mathbf{Z}[P]} \operatorname{Spec}\mathbf{Z}[Q]$ is flat.

These properties are stable under compositions and base changes in the category of fs log schemes. For the composition of log flat morphisms, see [8] 4.12.

For a strict morphism of fs log schemes, it is log étale (resp. log smooth, resp. log flat) if and only if its underlying morphism of schemes is étale (resp. smooth, resp. flat).

**Lemma 1.2.** *Let $Y$ be an fs log scheme, and let $y$ be a point of $Y$. Let $Y \to \operatorname{Spec}\mathbf{Z}[P]$ be a chart of $Y$ such that the induced map $P \to \overline{M_{Y,\overline{y}}}$ is bijective. Let $P \to Q$ be a kummer homomorphism of fs monoids with $Q^\times = \{1\}$ and let $X = Y \times_{\operatorname{Spec}\mathbf{Z}[P]} \operatorname{Spec}\mathbf{Z}[Q]$. Then, there is only one point, say, $x$, of $X$ lying over $y$. Further, for any open neighborhood $U$ of $x$, there is an open neighborhood $V$ of $y$ whose pullback by $p\colon X \to Y$ is contained in $U$.*

*Proof.* First we prove that the fiber $p^{-1}(y)$ over $y$ by $p$ consists of a single point. Let $k$ be the residue field of $y$. The concerned fiber is isomorphic to the spectrum of the ring $k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$. We show that the homomorphism $h\colon k \to (k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])_{\mathrm{red}}$ is bijective. To see it, let $q \in Q$. By the assumption that $P \to Q$ is kummer, $q^n \in P$ for some $n \geqq 1$. If $q^n = 1$, then $q = 1$ by $Q^\times = \{1\}$, and $q$ is in the image of $h$. Otherwise, i.e., $q^n \in P - \{1\}$, $q^n$ is zero in $k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, and $q$ is zero in $(k \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q])_{\mathrm{red}}$. Therefore, $h$ is surjective, and hence $h$ is an isomorphism, which means that the fiber is a singleton, say, $\{x\}$.

Next, let $U$ be an open neighborhood of $x$. Since the fiber $\{x\}$ over $y$ is contained in $U$, the point $y$ does not belong to $p(U^c)$, where $(-)^c$ means the complement of a set. Again by the kummerness of $P \to Q$, the morphism $p$ is finite. In particular, it is a closed map. Therefore, $V = (p(U^c))^c$ is a desired open neighborhood of $y$. $\square$

We show that we can take some good chart for a kummer and log flat morphism. Compare it with K. Kato's theory of neat charts ([8] 4.13–4.15).

**Proposition 1.3.** *Let $f\colon X \to Y$ be a kummer and log flat morphism of fs log schemes. Let $x$ be a point of $X$ with its image $y$ in $Y$. Then the following hold.*

(1) *Strict fppf locally around $x$ and $y$, there is a kummer chart $P \to Q$ of $f$ such that the induced maps $P \to \overline{M_{Y,\overline{y}}}$ and $Q \to \overline{M_{X,\overline{x}}}$ are bijective, and such that the induced morphism of schemes $X \to Z := Y \times_{\mathrm{Spec}\,\mathbf{Z}[P]} \mathrm{Spec}\,\mathbf{Z}[Q]$ is flat.*

(2) *Further, assume that $f$ is locally of finite presentation. Then, we can take a chart as in* (1) *such that $X \to Z$ is also surjective and locally of finite presentation.*

*Proof.* (1) is proved in a similar way to [2] 5.3. For the readers' convenience, we briefly review the argument. Take a chart as in the definition of the log flatness around $x$ and $y$ (1.1). Localizing $P$ and $Q$, we may assume that the induced maps $\overline{P} \to \overline{M_{Y,\overline{y}}}$ and $\overline{Q} \to \overline{M_{X,\overline{x}}}$ are bijective. We want to replace $P$ with $\overline{P}$ and $Q$ with $\overline{Q}$ respectively. Take an isomorphism between $P$ (resp. $Q$) and the direct sum of $\overline{P}$ (resp. $\overline{Q}$) and $P^\times$ (resp. $Q^\times$). Further localizing $X$ by replacing $Q^\times$ with $(Q^\times)^{\frac{1}{n}}$ for a suitable $n \geqq 1$, we may assume that the map $P \to Q$ is the direct sum of the map $\overline{P} \to \overline{Q}$ and a map $P^\times \to Q^\times$. Hence, we can replace $P$ with $\overline{P}$ and $Q$ with $\overline{Q}$ respectively, as desired.

(2) Take a chart as in (1). Since $f$ is locally of finite presentation, by [1] 1.4.3 (v), $X \to Z$ is also locally of finite presentation. Hence, by [1] 2.4.6, its image $U \subset Z$ is an open subset. By 1.2, there is an open neighborhood $V$ of $y$ in $Y$ such that its pullback by $Z \to Y$ is contained in $U$. Replacing $Y$ with $V$, we may assume that $U = Z$, that is, $X \to Z$ is surjective. $\square$

In the proofs of 0.1 and 0.2, we use the characterization by M. C. Olsson of log étale, log smooth, and log flat morphisms. His result ([8] 4.6) is interpreted by K. Kato and T. Saito ([5] 4.3.1) into the following form without the terms of algebraic stacks.

**Theorem 1.4.** *A morphism $f\colon X \to Y$ of fs log schemes is log étale (resp. log smooth, resp. log flat) if and only if the following condition holds: For any morphism $Y' \to Y$ of fs log schemes, and any log étale morphism of fs log schemes $X'' \to X'$, where $X' = X \times_Y Y'$, such that the composite $f''\colon X'' \to X' \to Y'$ is strict, the underlying morphism of $f''$ is étale (resp. smooth, resp. flat).*

# 2 Proofs

First we prove two lemmas.

**Lemma 2.1.** *Let $h\colon P \to Q$ be a kummer homomorphism of fs monoids. Let $S \to \mathrm{Spec}\,\mathbf{Z}[P]$ be a strict morphism of fs log schemes. Let $p\colon S' := S \times_{\mathrm{Spec}\,\mathbf{Z}[P]} \mathrm{Spec}\,\mathbf{Z}[Q] \to S$ be the projection. Then, the homomorphism $\mathcal{O}_S \to p_*(\mathcal{O}_{S'})$ is injective and its image is a direct summand as an $\mathcal{O}_S$-module.*

*Proof.* It is enough to show that $\mathbf{Z}[P]$ is a direct summand of $\mathbf{Z}[Q]$ as a $\mathbf{Z}[P]$-module. For this, it suffices to show that $P$ is a direct summand of $Q$ as a $P$-set, i.e., the subset $Q-P$ is a $P$-set. Let $q \in Q$ and $p \in P$. Assume $pq \in P$ and we prove $q \in P$. The assumption $pq \in P$ implies $q \in P^{\mathrm{gp}}$. On the other hand, since $h$ is kummer, some power of $q$ is in $P$. Since $P$ is saturated, $q$ is in $P$. $\square$

**Remark 2.2.** The above proof shows that the kummerness assumption of $h$ in 2.1 can be replaced with a weaker assumption that $h$ is exact and injective.

**Lemma 2.3.** *Let $f\colon X \to Y$ be a morphism of schemes, and let $g\colon Y' \to Y$ be an affine morphism of schemes such that the homomorphism $\mathcal{O}_Y \to g_*(\mathcal{O}_{Y'})$ is injective and its image is locally a direct summand as an $\mathcal{O}_Y$-module. Let $X' = X \times_Y Y'$ and let $f'\colon X' \to Y'$ be the morphism induced by $f$.*

*Denote by $\mathbf{P}$ one of the following :*

(1) *étale ;*
(2) *smooth ;*
(3) *flat ;*
(4) *locally of finite type ;*
(5) *locally of finite presentation.*

*Then, $f$ is $\mathbf{P}$ if and only if $f'$ is $\mathbf{P}$.*

*Proof.* We have to prove the if part. First we prove (3). It is enough to show the following: Let $f\colon A \to B$ be a homomorphism of rings, and let $A \to A'$ be an injective homomorphism of rings whose image is a direct summand of $A'$ as an $A$-module. Let $B' = B \otimes_A A'$. Assume that the induced $A' \to B'$ is flat. Then, $f$ is flat. To see this, let $h\colon M_1 \to M_2$ be an injective $A$-homomorphism. Then, $h$ is a direct summand of the $A'$-homomorphism $h \otimes_A A'$ as an $A$-homomorphism. Let $N$ be the image of $h \otimes_A A'$, and let $g\colon N \to M_2 \otimes_A A'$ be the induced injective $A'$-homomorphism. Since $h$ is injective, it is also a direct summand of $g$ as an $A$-homomorphism. Hence, $h_1 := h \otimes_A B$ is a direct summand of the $B'$-homomorphism $g_1 := g \otimes_A B = g \otimes_{A'} B'$ as a $B$-homomorphism. Since $A' \to B'$ is flat and $g$ is injective, $g_1$ is injective, and hence, $h_1$ is also injective. This shows that $f$ is flat.

We prove (4) and (5). Let $A$, $B$, etc. be as above. Assume that $B'$ is finitely generated over $A'$, and we show that $B$ is so over $A$. Write $B$ as $\varinjlim B_i$, where each $B_i$ is finitely generated over $A$. Then, $B' = \varinjlim (B_i \otimes_A A')$. By the assumption, for some $i$, the homomorphism $h\colon B_i \otimes_A A' \to B' = B \otimes_A A'$ is surjective. Since $B_i \to B$ is a direct summand of $h$ as an $A$-homomorphism, it is also surjective. Hence, $B$ is finitely generated over $A$.

Next, assume further that $B'$ is of finite presentation over $A'$. We show that $B$ is so over $A$. Since we already know that $B$ is finitely generated, there is a surjective homomorphism from a polynomial ring $C := A[T_1, \dots, T_n]$ to $B$. Let $K$ be its kernel. Write $K$ as the union of $K_i$, where each $K_i$ is a finitely generated ideal of $C$. Then, the kernel $K'$ of the induced surjection $A'[T_1, \dots, T_n] \to B'$ is the union of the ideals $K'_i$ generated by the image of $K_i$. By the assumption, $K'$ is finitely generated ([1] 1.4.4). Hence, for some $i$, we have $K' = K'_i$, which implies that the map $h\colon (C/K_i) \otimes_A A' = A'[T_1, \dots, T_n]/K'_i \to B' = B \otimes_A A'$ is bijective. Since $C/K_i \to B$ is a direct summand of $h$ as an $A$-homomorphism, it is also bijective. Hence, $B$ is of finite presentation over $A$.

We prove (1) and (2). Assume that $f'$ is étale (resp. smooth), and we prove that $f$ is étale (resp. smooth). Since we already know that $f$ is flat and locally of finite presentation, it remains to show that each fiber of $f$ is étale (resp. smooth) ([1] 17.8.2). Hence, we may assume that $Y$ is the spectrum of a field. In this case, $g$ is flat, and we reduce to the usual flat descent [1] 17.7.3 (ii). □

For a proof of the following lemma, see [6] (2.2.2).

**Lemma 2.4.** *Any base change of a surjective and kummer morphism of fs log schemes is surjective.*

**2.5.** We prove 0.1. Since the log étaleness implies **P**, by 1.4 and 2.4, we may assume that $f$ is strict. Let $x$ be a point of $X$ with the image $y$ in $Y$, and we work around $x$ and $y$. Take a chart for $g$ as in 1.3 (2). Since, if $g$ is strict, the statement is equivalent to the classical flat descent [1] 17.7.4, we may assume that $g$ is a base change of $\operatorname{Spec}\mathbf{Z}[h]$ with a kummer homomorphism $h$ of fs monoids. By 2.1, $g$ satisfies the assumptions in 2.3, and we can apply 2.3 (1)–(3), and conclude that $f$ is **P**.

**2.6.** We prove 0.2. Again by 1.4 and 2.4, we may assume that $g$ is strict. Let $x$ be a point of $X$ with the image $y$ in $Y$ and $z$ in $Z$. We will prove that $g$ is **P** at $y$. For this, we may work around $z$ and may assume that there is a chart $P$ around $z$ such that $P \to \overline{M_{Z,\overline{z}}}$ is bijective. It also gives a chart of $Y$. Let $Q = \overline{M_{X,\overline{x}}}$. Since $P \to Q$ is kummer, there is an $n \geqq 1$ such that for any $a \in Q$, the power $a^n$ belongs to the image of $P$. Consider the base change $X' = X \times_{\mathbf{Z}[P]} \mathbf{Z}[P^{\frac{1}{n}}]$, $Y' = Y \times_{\mathbf{Z}[P]} \mathbf{Z}[P^{\frac{1}{n}}]$, and $Z' = Z \times_{\mathbf{Z}[P]} \mathbf{Z}[P^{\frac{1}{n}}]$ ($P^{\frac{1}{n}}$ is the $P$-monoid which is isomorphic to $n\colon P \to P$), and the induced diagram $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$. Since the map $P^{\frac{1}{n}} \to (Q \oplus_P P^{\frac{1}{n}})^{\mathrm{sat}}/(\text{torsion})$ induced by the cobase change of $P \to Q$ is bijective, $f'$ is strict at a point $x'$ lying over $x$ ([6] 2.1.1). Take open neighborhoods $U'$ of $x'$ and $V'$ of $y' = f'(x')$ such that $f'(U') \subset V'$ and such that the induced morphism $U' \to V'$ is strict.

In the case where **P** is "log étale" (resp. "log smooth"), we may further assume that $U' \to V'$ is surjective because $f'$ is an open map ([1] 2.4.6). Then, applying [1] 17.7.5 (ii) and 17.7.7 to $U' \to V' \to Z'$, the morphism $V' \to Z'$ is étale (resp. smooth). By 1.2, there is an open neighborhood $V$ of $y$ in $Y$ such that its pullback in $Y'$ is contained in $V'$. By 2.1 and 2.3 (1) (resp. (2)), $V \to Z$ is étale (resp. smooth). Hence, $g$ is étale (resp. smooth) at $y$, which completes the proof of the case of log étale (resp. log smooth) morphisms.

In the case where **P** is "log flat", replacing $Y$ and $Z$ with $\operatorname{Spec}\mathcal{O}_{Y,y}$ and $\operatorname{Spec}\mathcal{O}_{Z,z}$ respectively, we may assume that $Y$ and $Z$ are local and that $y$ and $z$ are the closed points. Then, $Y'$ and $Z'$ are also local (cf. 1.2), and $V' = Y'$. Since both $U' \to Y'$ and $U' \to Y' \to Z'$ are flat at $x'$, by [1] 2.2.11 (iv), $Y' \to Z'$ is flat at $y'$, which means that $Y' \to Z'$ is flat. Then, by 2.1 and 2.3 (3), $g\colon Y \to Z$ is flat, which completes the proof of the case of log flat morphisms.

# 3 Variants

We mention some variants of 0.1 and 0.2.

**3.1.** First, in both theorems, the assumption "kummer" can be replaced with "exact." This generalization is reduced to the kummer case by taking quasi-sections. See [7] 3.3 and 3.4.

**3.2.** On the other hand, in the cases of 0.2 where $\mathbf{P}$ is "log étale" or "log smooth", the assumption that $f$ is $\mathbf{P}$ can be replaced by the assumption that $f$ is log flat and locally of finite presentation. For this variant, the same proof in 2.6 works. Further, we can generalize it to the exact case as in 3.1. The resulting statement is :

**Theorem 3.3.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of fs log schemes, and assume that $f$ is surjective, exact, log flat, and locally of finite presentation. If $g \circ f$ is log étale (resp. log smooth), then $g$ is also log étale (resp. log smooth).*

**3.4.** In the absence of the exactness assumption on $f$ the conclusion of 3.3 may fail, as the following example, due to I. Vidal shows*. Let $Q = \mathbf{N}e_1 \oplus \mathbf{N}e_2$, and let $P$ be the submonoid of $Q^{\mathrm{gp}}$ generated by $e_2$ and $e_1 - e_2$. Let $Y$ be the log point Spec $k$ (over a field $k$) with the log structure associated to $Q \to k, e_1 \mapsto 0, e_2 \mapsto 0$. Let $Z$ be the affine line Spec $k[e_1]$, with the log structure associated to $Q \to k[e_1], e_1 \mapsto e_1, e_2 \mapsto 0$. We have a natural strict morphism $g \colon Y \to Z$, charted by the identity on $Q$, whose underlying morphism of schemes is the closed immersion $k[e_1] \to k, e_1 \mapsto 0$. On the other hand, we have an inclusion $u \colon Q \to P$, with $u^{\mathrm{gp}}$ an isomorphism, defining a log étale morphism $h\colon \operatorname{Spec} k[P] \to \operatorname{Spec} k[Q]$. Let $X := Y \times_{\operatorname{Spec} k[Q]} (\operatorname{Spec} k[P], h)$, where $Y$ is sent to Spec $k[Q]$ by the composition $Y \to Z \to \operatorname{Spec} k[Q]$, and let $f \colon X \to Y$ be given by the first projection. Then, in the cartesian square,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Z \times_{\operatorname{Spec} k[Q]} (\operatorname{Spec} k[P], h) \arrow[d] \\ Y \arrow[r, "g"] & Z \end{tikzcd} ,$$

the upper horizontal map is strict, and an isomorphism, corresponding to the ring isomorphism $k[x', y] \otimes_{k[x,y]} k[x] \xrightarrow{\sim} k[x', y] \otimes_{k[x,y]} k = k[x']$, with $(x, y)$ sent to $(x'y, y)$ in $k[x', y]$. So $f$ is log étale and surjective, $g \circ f$ is log étale, but the strict map $g$ is not étale, not even flat.

More geometrically, the above example can be described as follows. Blow up the affine plane $\mathbf{A}^2 = \operatorname{Spec} k[e_1, e_2] = \operatorname{Spec} k[Q]$ at the origin $Y$, and remove the strict transform of the divisor $Z$ defined by $e_2 = 0$. Then, above $Z$, we have only the affine line $X$ obtained from the exceptional divisor by removing a point. Hence, $X \to Z$ factors through $Y$ and it is still log étale, since it is a base change of a partial log blow up.

*(from an email to the first author, dated March 6, 1998)

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