Old and new in étale cohomology
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Luc Illusie

The purpose of these lectures is to provide an introduction to some recent developments concerning certain old questions and conjectures in étale cohomology. We have chosen three main themes :

(a) Galois actions and traces,

(b) vanishing cycles,

(c) finiteness theorems.

In §1 we recall basic definitions and results in étale cohomology. Theme (a) is considered in §2. We explain some variations, due to Serre, on the trace formula of Grothendieck and divisibility of traces by powers of ℓ, with applications to bounding the orders of finite subgroups of reductive groups. Theme (b) is dealt with in §§3, 4, 5. After reviewing classical material in §3, in §4 we present Deligne's construction of oriented products of toposes used for the definition of nearby and vanishing cycles over bases of arbitrary dimension. In §5 we state Orgogozo's theorem on the good behavior of nearby cycles after suitable modifications of the base, as predicted by Deligne, and give some highlights on the proof. §§6, 7, 8 are devoted to theme (c). The central result is Gabber's recent finiteness theorem, on the constructibility of higher direct images of constructible sheaves of torsion prime to the characteristics under finite type morphisms between quasi-excellent schemes, as conjectured by Artin and Grothendieck. The proof relies on the absolute purity theorem, proved earlier by Gabber, and a new, crucial ingredient, a local uniformization theorem, which we discuss in §7. Oriented products reappear in §8 as a useful technical tool in the proof.

These notes overlap with other surveys ([I 4], [I 5], [I 7]), in which the reader will find complements on the topics discussed here and related ones. They cover the contents of talks given at the UMIST between October 25 and November 1, 2006, at the invitation of Prof. Martin Taylor FRS. I wish to thank him heartily for his warm hospitality and generous support. I also thank Kay Rülling for many helpful comments on a first draft.

[*Added in May, 2014*] These notes were written in the fall of 2006, and slightly revised in the spring of 2007. A detailed account of the results and proofs presented in §§ 6, 7, 8 together with several refinements and complements will appear in

(1) *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Séminaire à l'École polytechnique 2006-2008,

dirigé par Luc Illusie, Yves Laszlo et Fabrice Orgogozo, Astérisque 361-362, Société mathématique de France, 2014.

This book is an expanded version of the proceedings of the seminar [ILO], based on [Ga2], [Ga3]. [I6] has become Exp. XI in (1), [I7] the Introduction of (1), [PS], Exp. XIV and XV, [R1] and [R2], Exp. XVI and XVII respectively. The material of [Mo] has been absorbed in Exp. VI and VIII. In [On Gabber's refined uniformization [Ga1], Talks at the Univ. Tokyo, Jan. 17, 22, 31, Feb. 7, 2008] (http://www.math.u-psud.fr/ illusie/) I gave an introduction to a refinement of Th. 7.2, which is the object of Exp. VIII and X, [O1] has become [*Sur la p-dimension des corps*, Gabber O., Orgogozo F., Inventiones mathematicae 174, 1 (2008) 47-80], [O3], Exp. XII A, [Z], [*Sur la cohomologie des faisceaux l-adiques entiers sur les corps locaux*, Bull. Soc. Math. France 136 (2008), no. 3, 465–503].]

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1. A quick review of étale cohomology ([SGA 4], [SGA 5], [SGA 4 1/2])

1.1. A morphism of schemes $f : X \to Y$ is called *étale* if it is locally of finite presentation, flat, and unramified (net, in another terminology), unramified meaning that $\Omega^1_{X/Y} = 0$. There are several other equivalent definitions, see [SGA 1 I], [EGA IV 17]. The (small) *étale site* of a scheme X is the category of étale morphisms $u : U \to X$, endowed with the *étale topology*, i. e. the topology defined by the pretopology whose covering families are the surjective families $(f_i : U_i \to U)_{i \in I}$ (note that any X-morphism between étale X-schemes is étale). The *étale topos* of X is the category of sheaves of sets on the étale site of X. The étale site of X is sometimes denoted by $X_{\text{ét}}$, and often simply by X when no confusion can arise. The structural sheaf $\mathcal{O}_X$ uniquely extends to an étale sheaf of rings on X, still denoted $\mathcal{O}_X$, such that

$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$ for U étale over X.

If $f : X \to Y$ is a morphism of schemes, then f defines a morphism of étale toposes, usually still denoted f, given by a pair of adjoint functors (f^*, f_*), such that $f^*(V) = V \times_Y X$ for V étale over Y. 1.2. A *geometric point* of a scheme X is a morphism $u : x \to X$, where x is the spectrum of a separably closed field. One says that u is *above* (or *localized at*) $u(x) \in X$. The morphism u uniquely factorizes into $x \to \operatorname{Spec} k(u(x)) \to X$, where the second map is the canonical one. One sometimes omits, by abuse, the mention of u and calls x the geometric point u. An *étale neighborhood* of u (or, by abuse, x) is an étale morphism $U \to X$ together with a lifting $v : x \to U$ of u. Etale neighborhoods of x form a decreasing filtering category, and, for an étale sheaf F on X, the *stalk* of F at x, i. e. u^*F, is the inductive limit

$$F_x = \varinjlim F(U),$$

where U runs through the étale neighborhoods of x. The functors $F \mapsto F_x$ thus defined form a conservative system, and any *point* of the topos X (i. e. morphism from the punctual topos to X) is of this form [SGA 4 VIII 7.9]. The stalk of $\mathcal{O}_X$ at x is the *strict henselization* of $\mathcal{O}_{X,x_0}$ at the extension $k(x)$ of $k(x_0)$, where $x_0 = u(x)$ [SGA 4 VIII 4.3]. Its spectrum is called the *strict localization* of X at x and denoted $X_{(x)}$:

$$X_{(x)} = \operatorname{Spec} \mathcal{O}_{X,x}.$$

It is the projective limit of affine étale neighborhoods of x. It can be viewed as the analogue of a small ball around x. We will sometimes call $X_{(x)}$ the *Milnor ball at* x. If F is an étale sheaf on X, then

$$F_x = \Gamma(X_{(x)}, F|X_{(x)}),$$

where $F|X_{(x)}$ denotes the inverse image of F on $X_{(x)}$ by the canonical morphism $X_{(x)} \to X$. 1.3. Let k be a field, $x = \operatorname{Spec} k$, $\overline{k}$ a separable closure of k, $\overline{x} = \operatorname{Spec} \overline{k}$, G the Galois group of $\overline{k}$ over k. If F is a sheaf on x, by transportation of structure G acts continuously on the stalk $F_{\overline{x}}$ (on the left) and the functor $F \mapsto F_{\overline{x}}$ is an equivalence between the étale topos of x and the topos of continuous G-sets. It follows that, if Λ is a ring, then for any sheaf of Λ-modules F on x, the cohomology of x with coefficients in F is identified with the Galois cohomology of $F_{\overline{x}}$:

$$R\Gamma(x, F) = R\Gamma(G, F_{\overline{x}}).$$

1.4. Let $f : X \to Y$ be a morphism of schemes and $y \to Y$ a geometric point of Y. The fiber product

$$X_{(y)} := Y_{(y)} \times_Y X$$

can be thought of as a tubular neighborhood of the fiber $X_y = y \times_Y X$ of f at y, and for this reason is sometimes called the *tube* of X over y. Assume that f is *coherent*, i. e. quasi-compact and quasi-separated. Then, for any sheaf F on X, the stalk of f_*F at y can be calculated as the set of global sections of F on this tube :

$$(f_*F)_y = \Gamma(X_{(y)}, F|X_{(y)}), \tag{1.4.1}$$

where $F|X_{(y)}$ denotes the inverse image of F on $X_{(y)}$. This extends to the cohomology : if Λ is a ring, and $F \in D^+(X, \Lambda)$, then ([SGA 4 VII 5.8, VIII 5.2])

$$Rf_*(F)_y = R\Gamma(X_{(y)}, F|X_{(y)}). \tag{1.4.2}$$

There is a natural closed immersion

$$X_y \to X_{(y)}$$

above the inclusion of the closed point y in $Y_{(y)}$, hence a restriction map

$$(f_*F)_y \to \Gamma(X_y, F) \tag{1.4.3}$$

and similarly

$$Rf_*(F)_y \to R\Gamma(X_y, F) \tag{1.4.4}$$

for $F \in D^+(X, \Lambda)$, where for brevity we have omitted the symbols of restriction. It is not true, in general, that (1.4.3) (resp. (1.4.4)) an isomorphism. By analogy with the analogous case of proper maps between locally compact spaces, this is, however, the case when f is *proper* (resp. when f is proper and Λ is *torsion*). This is the content of the *proper base change theorem*, whose statement we recall in the abelian context :

Theorem 1.5 [SGA 4 XII 5.1] Let

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{1.5.1}$$

be a cartesian square of schemes, with f proper, and let Λ be a torsion ring. Then for any $F \in D^+(X, \Lambda)$, the base change map

$$g^*Rf_*F \to Rf'_*(h^*F)$$

is an isomorphism.

It is easily seen that 1.5 (for all cartesian squares (1.5.1)) is in fact equivalent to (1.4.4) being an isomorphism for all proper maps f and all geometric points y. The conclusion may be wrong if one doesn't assume Λ to be torsion ([SGA 4 XII 2]).

1.6. Let Λ be a torsion ring. From 1.5 one derives a formalism of *direct images with proper support* and *cohomology with compact support*. Namely, let $f : X \to Y$ be a *compactifiable* morphism, i. e. admitting a factorization $f = gj$, with $g : Z \to Y$ proper and $j : X \to Z$ an open immersion. For $F \in D^+(X, \Lambda)$, one defines

$$Rf_!F := Rg_*(j_!F), \tag{1.6.1}$$

where $j_!$ denotes the extension by zero. It follows from 1.5 that, up to a canonical system of isomorphisms, the right hand side of (1.6.1) does not depend on the choice of the factorization. The functor

$$Rf_! : D^+(X, \Lambda) \to D^+(Y, \Lambda)$$

thus obtained is called *direct image with proper support*. It commutes with any base change. It is not the derived functor of $R^0f_!$. When Y is the spectrum of a separably closed field, one writes $R\Gamma_c(X, F)$ for $Rf_!F$. If $y \to Y$ is a geometric point of Y, then one has

$$Rf_!(F)_y = R\Gamma_c(X_y, F) \tag{1.6.2}$$

and if y is a separable closure of its image y_0 in Y, the Galois group $\mathrm{Gal}(k(y)/k(y_0))$ acts continuously on the cohomology groups with compact support $H^q_c(X_y, F)$. If the dimension of the fibers $f^{-1}(y)$ of f is bounded by d, then $Rf_!$ is of *cohomological dimension* $\leq 2d$, i. e. $R^if_!F = 0$ for all sheaves of Λ-modules F and $i > 2d$ [SGA 4 XVII 5.2.8.1]. For Y quasi-compact and quasi-separated, $Rf_!$ admits a right adjoint

$$Rf^! : D^+(Y, \Lambda) \to D^+(X, \Lambda) \tag{1.6.3}$$

([SGA 4 XVIII 3.1]). The pairs of adjoint functors

$$(f^*, Rf_*) \quad , \quad (Rf_!, Rf^!)$$

together with the bifunctors

$$- \otimes^L_\Lambda - \quad , \quad R\mathcal{H}om_\Lambda(-, -)$$

form the so-called *Grothendieck's six operations*.

1.7. If $i : S \to S'$ is a closed immersion defined by a nilideal, then i^* gives an equivalence (with quasi-inverse i_*) from the topos of étale sheaves on S' to that of étale sheaves on S (this holds more generally if i is a universal homeomorphism, this is the so-called *topological invariance* of the étale topos [SGA 4 VIII 1.1]). Thanks to this, one can define a notion of constructibility for étale sheaves. For simplicity, we will limit ourselves to noetherian bases. Let X be a noetherian scheme and Λ a noetherian, torsion ring. A sheaf of Λ-modules F on X is called *constructible* if X is a finite disjoint union of locally closed subsets X_i such that the restriction of F to X_i is locally constant of finite type. Constructible sheaves of Λ-modules form a *thick subcategory* of that of all sheaves of Λ-modules (i. e. constructibility is stable under kernel, cokernel and extension), so that the subcategory

$$D_c(X, \Lambda) \subset D(X, \Lambda)$$

consisting of complexes whose cohomology sheaves are constructible is a triangulated subcategory. Any sheaf of Λ-modules on X is a filtering direct limit of constructible ones. The constructible sheaves of Λ-modules are the *noetherian* objects of the category of sheaves of Λ-modules on X [SGA 4 IX 2.4].

One of the main issues addressed in [SGA 4] is that of the stability of D_c under the six operations. An easy and basic corollary of the proper base change theorem is the following *finiteness theorem* ([SGA 4 XIV 1.1, XVII 5.3.6]) :

Theorem 1.8. Let Λ be a noetherian, torsion ring and $f : X \to Y$ a compactifiable morphism, with Y noetherian. Then $Rf_!$ sends $D_c^+(X, \Lambda)$ into $D_c^+(Y, \Lambda)$ (and $D_c^b(X, \Lambda)$ into $D_c^b(Y, \Lambda)$).

The second assertion comes from the fact that $Rf_!$ is of finite cohomological dimension.

One cannot expect a similar result for Rf_* as is already shown by the Artin-Schreier sequence for the cohomology of $\mathbb{A}^1_k$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$, where k is an algebraically closed field of characteristic $p > 0$. It was hoped by Grothendieck, however, that under mild restrictions on the schemes (such as excellency), finiteness should hold for Rf_* as well, for f of finite type, provided that one would impose to Λ to be annihilated by an integer prime to the residual characteristics. Such a finiteness result was established by Artin in [SGA 4 XIX 5] for Y excellent of characteristic zero, using Hironaka's resolution of singularities. The extension to schemes of positive or mixed characteristics seemed to strongly depend on a generalization of Hironaka's theorem. In 1973, though, Deligne proved the following basic theorem [SGA 4 1/2 Th. finitude, 1.1] :

Theorem 1.9. Let S be a noetherian regular scheme, of dimension ≤ 1, and $f : X \to Y$ an S-morphism between S-schemes of finite type. Let Λ be a noetherian ring annihilated by an integer invertible on S. Then, for any constructible sheaf F of Λ-modules on X, $R^q f_* F$ is constructible for all q, and there exists an integer N such that $R^q f_* F = 0$ for $q > N$. In other words, $Rf_* : D^+(X, \Lambda) \to D^+(Y, \Lambda)$ sends $D^+_c(X, \Lambda)$ into $D^+_c(Y, \Lambda)$, and $D^b_c(X, \Lambda)$ into $D^b_c(Y, \Lambda)$.

The second assertion follows from the fact that under the hypotheses of 1.9, Rf_* is of finite cohomological dimension : one can take N independent of F, but in general N is larger that $2d$, if d bounds the dimension of the fibers of f ([SGA 4 X 3]).

It follows from 1.9 that, for schemes separated and of finite type over a regular base S of dimension ≤ 1, and Λ of torsion invertible on S, constructibility is stable under the six operations ([SGA 4 Th. finitude, 2.9]). However, the problem of extending Artin's finiteness theorem of [SGA 4 XIX] to excellent schemes with no restriction of characteristics remained open, until, quite recently, Gabber solved it positively, see §6.

1.10. The calculation of étale cohomology of "classical" schemes and the proof that, ultimately, ℓ-adic cohomology yields a *Weil cohomology* for schemes of finite type over a field of characteristic $\neq \ell$ rely on two basic results : (a) the structure theorem for the étale cohomology of proper smooth curves over an algebraically closed field (1.11), (b) the local acyclicity of smooth morphisms (1.12).

In what follows we will assume that $\Lambda = \mathbb{Z}/n\mathbb{Z}$, where n is an integer ≥ 1. If n is invertible on the scheme X, the étale sheaf $(\mu_n)_X$ of n^{th}-roots of unity, kernel of $a \mapsto a^n$ on $\mathcal{O}^*_X$, is a locally constant, invertible Λ-module, often denoted $\Lambda(1)$. Its powers $\Lambda(i) := (\Lambda(1))^{\otimes i}$, for $i \in \mathbb{Z}$, are called the *Tate sheaves*. The exact sequence of abelian sheaves on X :

$$1 \longrightarrow \mu_n \longrightarrow \mathcal{O}^*_X \xrightarrow{a \mapsto a^n} \mathcal{O}^*_X \longrightarrow 1 \tag{1.10.1}$$

is called the *Kummer* sequence. From it and Tsen's theorem (cf. [SGA 4 XI] and [SGA 4 1/2, Cohomologie étale : les points de départ]) one deduces :

Theorem 1.11. Let k be an algebraically closed field, n an integer invertible in k, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, X a proper, smooth connected curve of genus g

over k. Then one has :

$$H^q(X,\Lambda) = \begin{cases} \Lambda & \text{if } q=0 \\ {}_nPic^0(X)(-1) & \text{if } q=1 \\ \Lambda(-1) & \text{if } q=2 \\ 0 & \text{if } q>2 \end{cases}$$

where ${}_n(-)$ denotes the kernel of the multiplication by n. Moreover, ${}_nPic^0(X)$ is a free Λ-module of rank $2g$, and the pairing

$$H^q(X,\Lambda) \otimes H^{2-q}(X,\Lambda(1)) \to H^2(X,\Lambda(1)) = \Lambda$$

defined by cup-product is perfect.

(For the assertion about duality (*Poincaré duality*), see [SGA 4 1/2, Cohomologie étale : les points de départ].)

Theorem 1.12. Let $f : X \to S$ be a smooth morphism. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on S. Let x be a geometric point of X, s the geometric point of S image of x. Consider the morphism of Milnor balls induced by f :

$$f_{(s)} : X_{(x)} \to S_{(s)},$$

and let t be a geometric point of $S_{(s)}$. Then

$$R\Gamma(X_{(x)t},\Lambda) = \Lambda,$$

where $X_{(x)t}$ is the fiber of $f_{(s)}$ at t.

In other words, *Milnor fibers* of smooth morphisms are acyclic. For the proof, as well as variants and generalizations, see [SGA 4 XV] and [SGA 4 1/2, Cohomologie étale : les points de départ, Th. finitude].

1.13. The proper base change theorem 1.5 and the two previous theorems 1.11, 1.12 are the pillars over which étale cohomology is built. Here's a list of some of their most important consequences (with again $\Lambda = \mathbb{Z}/n\mathbb{Z}$, n invertible on the schemes considered) :

- *relative purity* for closed immersions $i : Y \to X$ between S-smooth schemes, i. e. the calculation of $Ri^!\Lambda$ as $\Lambda(-d)[-2d]$ where d is the codimension of i [SGA 4 XVI]

- base change by smooth morphisms (*smooth base change theorem*) [SGA 4 XVI]

- *Poincaré* (or *global*) duality, in the form of the construction of a canonical isomorphism between $Rf^!\Lambda$ and $\Lambda(d)[2d]$ for $f : X \to S$ smooth of relative dimension d [SGA 4 XVIII]

- *local* duality, for S-schemes X separated and finite type, with S regular of dimension ≤ 1, in the form of a *biduality isomorphism* $F \xrightarrow{\sim} DDF$, for $F \in D_c^b(X, \Lambda)$, where $D = R\mathcal{H}om(-, Rf^!\Lambda)$ $(f : X \to S)$ [SGA 4 1/2 Dualité]

- for schemes of finite type over $\mathbb{C}$, a *comparison theorem* between étale and Betti cohomology [SGA 4 XVI]

- a *weak* (or *affine*) Lefschetz theorem [SGA 4 XIV]

- the construction of *cycle* and *Chern classes*, calculation of the étale cohomology (with coefficients in Λ) of classical schemes (e. g. standard affine or projective spaces, punctured affine spaces, flag varieties, reductive groups) ([SGA 4 1/2 Cycle], [SGA 5 VII], [SGA 4 1/2 Sommes trig. p. 230])

- a general Lefschetz formula for schemes separated and of finite type over an algebraically closed field (*Lefschetz-Verdier formula*) [SGA 5 III].

1.14. However, in order to deal with arithmetic questions, such as those involving zeta or L functions, étale cohomology with coefficients in $\mathbb{Z}/n\mathbb{Z}$ does not suffice. One needs to obtain vector spaces over fields of characteristic zero. This is the purpose of *ℓ-adic cohomology*, which is derived from étale cohomology with torsion coefficients by taking certain projective limits.

Let ℓ be a prime number. If X is a noetherian scheme over which ℓ is invertible, a *constructible $\mathbb{Z}_\ell$-sheaf* F on X is an inverse system (F_n), $n \geq 1$, of constructible sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on X such that, for each $n \geq 1$, the transition map $\mathbb{Z}/\ell^n\mathbb{Z} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} F_{n+1} \to F_n$ is an isomorphism. One often says "$\mathbb{Z}_\ell$-sheaf" for "constructible $\mathbb{Z}_\ell$-sheaf". One says that F is *lisse* if each F_n is locally constant. Typical examples are the *Tate sheaves* : the $\mathbb{Z}_\ell$-sheaf

$$\mathbb{Z}_\ell(1)_X = (\mathbb{Z}/\ell^n\mathbb{Z}(1))_X,$$

with transition maps $x \mapsto x^\ell : \mu_{\ell^{n+1}} \to \mu_{\ell^n}$, (often denoted simply $\mathbb{Z}_\ell(1)$), is lisse and locally free of rank one, as well as its tensor powers $\mathbb{Z}_\ell(i) := \mathbb{Z}_\ell(1)^{\otimes i}$ for $i \in \mathbb{Z}$. If F is a $\mathbb{Z}_\ell$-sheaf on X, then X is the disjoint union of a finite number of locally closed subsets X_i over which F is lisse. If F, G are $\mathbb{Z}_\ell$-sheaves on X, the group of homomorphisms from F to G is defined as

$$\mathrm{Hom}(F, G) = \varprojlim \mathrm{Hom}(F_n, G_n)$$

(this is the same as the group of homomorphisms of projective systems). $\mathbb{Z}_\ell$-sheaves on X form an abelian category. If X is connected, and x is a geometric point of X, then the *fiber functor* $F \mapsto F_x$ gives an equivalence between the category of lisse $\mathbb{Z}_\ell$-sheaves on X and that of continuous representations of the fundamental group $\pi_1(X, x)$ (in the sense of [SGA 1 V 7]) in $\mathbb{Z}_\ell$-modules of finite type. The category of $\mathbb{Q}_\ell$-sheaves has for objects the

$\mathbb{Z}_\ell$-sheaves and the group of morphisms from a $\mathbb{Q}_\ell$-sheaf L into a $\mathbb{Q}_\ell$-sheaf M is defined as $\mathbb{Q}_\ell \otimes \mathrm{Hom}(L, M)$. One usually writes $F \otimes \mathbb{Q}_\ell$ for the $\mathbb{Q}_\ell$-sheaf defined by a $\mathbb{Z}_\ell$-sheaf F. There are variants with $\mathbb{Q}_\ell$ replaced by a finite extension of $\mathbb{Q}_\ell$ and $\mathbb{Z}_\ell$ by its rings of integers, and also with $\mathbb{Q}_\ell$ replaced by an algebraic closure $\overline{\mathbb{Q}}_\ell$. For all this, see [SGA 5 V, VI], [SGA 4 1/2 Rapport], [Weil II].

It is not easy to extend the cohomological formalism from $\mathbb{Z}/\ell^n\mathbb{Z}$ to $\mathbb{Z}_\ell$. In [SGA 5 VI] higher direct images with proper support $R^q f_!$ are discussed, but no derived category formalism is introduced. One solution is proposed in [Weil II], and discussed in more details in [BBD, 2.2.16]. It suffices, when one works with schemes of finite type over a finite field or an algebraically closed field. A satisfactory general definition was proposed by Gabber (unpublished). A variant is developed by Ekedahl [Ek] (and independently, Gabber (unpublished)). The theory he constructs is valid for schemes X separated and of finite type over a base scheme S, which is regular of dimension ≤ 1, and over which ℓ is invertible. With such an X is associated a *triangulated category*

$$D^b_c(X, \mathbb{Z}_\ell),$$

equipped with a t-structure whose heart is the category of $\mathbb{Z}_\ell$-sheaves. Its objects are not complexes of $\mathbb{Z}_\ell$-sheaves, but projective systems of complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules satisfying certain conditions, and morphisms are defined by a certain localization procedure. Moreover, there is a compatible family of *reduction mod* ℓ^n functors

$$D^b_c(X, \mathbb{Z}_\ell) \to D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z}), \quad F \mapsto F_n,$$

which are triangulated and induce the "usual" reduction functor $F \mapsto F \otimes^L \mathbb{Z}/\ell^n\mathbb{Z}$ on the category of $\mathbb{Z}_\ell$-sheaves ; each F_n is of finite tor-dimension. An important property is that the reduction mod ℓ functor, $F \mapsto F_1$ is *conservative*. For E, F in $D^b_c(X, \mathbb{Z}_\ell)$, one has a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Ext}^{-1}(E_n, F_n) \to \mathrm{Hom}(E, F) \to \varprojlim \mathrm{Hom}(E_n, F_n) \to 0.$$

It follows that, when, for each n, and each E, F in $D^b_c(X, \mathbb{Z}_\ell)$, $\mathrm{Hom}(E_n, F_n)$ is finite, for example, when S is the spectrum of a finite or algebraically closed field, the reduction mod ℓ^n functors define an equivalence between $D^b_c(X, \mathbb{Z}_\ell)$ and the category defined by Deligne in (*loc. cit.*). In general, for schemes separated and of finite type over S, Ekedahl constructs a formalism of Grothendieck's six operations in $D^b_c(-, \mathbb{Z}_\ell)$, compatible with those in $D^b_c(-, \mathbb{Z}/\ell^n\mathbb{Z})$ via the reduction mod ℓ^n functors. There is a variant of this formalism with $\mathbb{Z}_\ell$ replaced by $\mathbb{Q}_\ell$, or E_λ (a finite extension of $\mathbb{Q}_\ell$), or $\overline{\mathbb{Q}}_\ell$

(see [BBD 2.2.18]). In view of Gabber's recent results [Ga 3], this formalism should extend to quasi-excellent schemes (or even algebraic stacks) over which ℓ is invertible. The case of algebraic stacks of finite type over an S as above has been treated by Laszlo-Olsson [La-Ol].

1.15. Let $\overline{k}$ be a separable closure of a field k and X a scheme separated and of finite type over k. For $E \in D^b_c(X, \mathbb{Z}_\ell)$, the cohomology groups $H^q(X_{\overline{k}}, E|X_{\overline{k}})$ (resp. $H^q_c(X_{\overline{k}}, E|X_{\overline{k}})$) are finitely generated over $\mathbb{Z}_\ell$ and zero for q large. One has

$$H^q(X_{\overline{k}}, E|X_{\overline{k}}) = \varprojlim H^q(X_{\overline{k}}, E_n|X_{\overline{k}})$$

(resp.

$$H^q_c(X_{\overline{k}}, E|X_{\overline{k}}) = \varprojlim H^q_c(X_{\overline{k}}, E_n|X_{\overline{k}})).$$

The Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ acts continuously on them : for $g \in G_k$, g defines an isomorphism $X_{\overline{k}} \xrightarrow{\sim} X_{\overline{k}}$, still denoted g, above $\mathrm{Spec}\, g : \mathrm{Spec}\, \overline{k} \xrightarrow{\sim} \mathrm{Spec}\, \overline{k}$, hence an isomorphism

$$g^* : H^q(X_{\overline{k}}, E|X_{\overline{k}}) \xrightarrow{\sim} H^q(X_{\overline{k}}, E|X_{\overline{k}})$$

(resp.

$$g^* : H^q_c(X_{\overline{k}}, E|X_{\overline{k}}) \xrightarrow{\sim} H^q_c(X_{\overline{k}}, E|X_{\overline{k}})),$$

sometimes still denoted g. For the image $\mathbb{Q}_\ell \otimes E$ of E in $D^b_c(X, \mathbb{Q}_\ell)$, one has

$$H^q(X_{\overline{k}}, \mathbb{Q}_\ell \otimes E|X_{\overline{k}}) = \mathbb{Q}_\ell \otimes H^q(X_{\overline{k}}, E|X_{\overline{k}})$$

and a similar formula for H^q_c. When E is a $\mathbb{Z}_\ell$-sheaf (i. e. is cohomologically concentrated in degree zero), the preceding groups are zero for $q > 2\dim(X)$.

For $g \in G_k$, and $L \in D^b_c(X, \mathbb{Q}_\ell)$, one can consider the eigenvalues (in $\overline{\mathbb{Q}_\ell}$) of g^* on the finite dimensional $\mathbb{Q}_\ell$-vector spaces $H^i(X_{\overline{k}}, L)$ (we write L instead of $L|X_{\overline{k}}$ for short) (resp. $H^i_c(X_{\overline{k}}, L)$) (they are integral over $\mathbb{Z}_\ell$), the traces $\mathrm{Tr}(g, H^i(X_{\overline{k}}, L))$ (resp. $\mathrm{Tr}(g, H^i_c(X_{\overline{k}}, L))$, or their alternate sums

$$\mathrm{Tr}(g, H^*(X_{\overline{k}}, L)) = \sum (-1)^i \,\mathrm{Tr}(g, H^i(X_{\overline{k}}, L))$$

(resp.

$$\mathrm{Tr}(g, H^*_c(X_{\overline{k}}, L)) = \sum (-1)^i \,\mathrm{Tr}(g, H^i_c(X_{\overline{k}}, L)),$$

which are in $\mathbb{Z}_\ell$. The properties of these numbers and longstanding conjectures about them have been the focus of several recent works on which I will give a few glimpses. See [I 4] for a more detailed survey.

1.16. Before coming to this, let me recall a fundamental result, *Grothendieck's trace formula.* Let k be a finite field $\mathbb{F}_q$, of characteristic p, $\overline{k}$ an algebraic closure of k, ℓ a prime $\neq p$. For $n \geq 1$, denote by k_n ($\simeq \mathbb{F}_{q^n}$) the extension of degree n of k in $\overline{k}$. The Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ is isomorphic to $\hat{\mathbb{Z}}$ and has a distinguished topological generator, the *Frobenius substitution* or *arithmetic Frobenius*, $a \mapsto a^q$. Its inverse, denoted F, is called the *geometric Frobenius.* Let X be a scheme separated and of finite type over k. The set $X(\overline{k})$ of its $\overline{k}$-points (or rational, or closed points of $X_{\overline{k}}$) is acted on by G_k, and, for $n \geq 1$, those fixed by F^n are exactly those rational over k_n ; their number is finite. Grothendieck's trace formula is the following :

Theorem 1.17. (Grothendieck [SGA 5 XIV]) With the notations of 1.16, let $L \in D^b_c(X, \mathbb{Q}_\ell)$. Then, for all $n \geq 1$, one has :

(1.17.1) $$\mathrm{Tr}(F^n, H^*_c(X_{\overline{k}}, L)) = \sum_{x \in X(k_n)} \mathrm{Tr}(F^n, L_x).$$

On the right hand side, the geometric point $x \in X(\overline{k})$ is fixed by F^n, hence F^n acts on the stalk of L at x, and by definition,

$$\mathrm{Tr}(F^n, L_x) = \sum (-1)^i \,\mathrm{Tr}(F^n, \mathcal{H}^i(L_x)).$$

The left hand side of (1.17.1) can also be viewed as the trace of the *cohomological correspondence* defined by $F^n_X : X \to X$ and the Frobenius isomorphism $F^n_X : (F^n_X)^* L \to L$. Here $F_X : X \to X$ is the morphism which is the identity on the underlying space and $a \mapsto a^q$ on $\mathcal{O}_X$; the Frobenius isomorphism F_X extends the inverse of the natural isomorphism $E \xrightarrow{\sim} F_X^{-1}(E)$ for a representable étale sheaf E (see [SGA 5 XIV], [SGA 4 1/2, Rapport]). When X is not proper, there is no analogous formula for $H^*(X_{\overline{k}}, L)$, as *contributions at infinity* must be taken into account.

In particular :

Corollary 1.18.

(1.18.1) $$\mathrm{Tr}(F^n, H^*_c(X_{\overline{k}}, \mathbb{Q}_\ell)) = |X(k_n)|,$$

where $|\ |$ denotes a cardinality.

Consider the zeta function of X

$$Z(X/k, t) = \prod (1 - t^{\deg(x)})^{-1},$$

where the product is extended to the closed points x of X and $\deg(x) = \deg(k(x)/k)$. Formula (1.18.1) implies the famous cohomological expression for $Z(X/k,t)$ as a rational function of t, namely :

Corollary 1.19.

$$Z(X/k,t) = \prod \det(1 - Ft, H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}. \tag{1.19.1}$$

It follows from (1.19.1) that the *Euler-Poincaré characteristic* of $X_{\overline{k}}$ with compact support :

$$\chi_c(X_{\overline{k}}, \mathbb{Q}_\ell) = \sum (-1)^i \dim H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)$$

can be extracted from the zeta function as

$$\chi_c(X_{\overline{k}}, \mathbb{Q}_\ell) = -\text{degree } Z(X/k,t), \tag{1.19.2}$$

and, in particular, is *independent of* ℓ. By a standard *spreading out argument*, this property extends to arbitrary fields of characteristic p (write such a field K as the inductive limit of its finitely generated $\mathbb{F}_p$-sub-algebras A_i, take a model of X over a suitable A_i, i. e. a scheme $\mathcal{X}$ separated and of finite type over $\operatorname{Spec} A_i$ such that $\operatorname{Spec} K \times_{\operatorname{Spec} A_i} \mathcal{X} = X$, and apply (1.6.2) and 1.8).

We can also consider the Euler-Poincaré characteristic (with no supports)

$$\chi(X_{\overline{k}}, \mathbb{Q}_\ell) = \sum (-1)^i \dim H^i(X_{\overline{k}}, \mathbb{Q}_\ell).$$

By a result of Laumon [La 1] (and independently, Gabber), first proved by Grothendieck in characteristic zero, these two characteristics are the same :

$$\chi_c(X_{\overline{k}}, \mathbb{Q}_\ell) = \chi(X_{\overline{k}}, \mathbb{Q}_\ell). \tag{1.19.3}$$

Laumon's result is in fact more general, and works for $\mathbb{Q}_\ell$ replaced by an object of $D^b_c(X, \mathbb{Q}_\ell)$ (or $D^b_c(X, E_\lambda)$ for E_λ a finite extension of $\mathbb{Q}_\ell$), and in a relative setting (an equality of f_* and $f_!$ in suitable K-groups).

1.20. When X is *proper and smooth* over k, then, by the fundamental theorem of Deligne ([Weil I] for X projective, [Weil II] in general), each characteristic polynomial in (1.19.1)

$$\det(1 - Ft, H^i(X_{\overline{k}}, \mathbb{Q}_\ell))$$

belongs to $\mathbb{Z}[t]$, is independent of ℓ, and its reciprocal roots (eigenvalues of F) are *pure of weight* i, i. e. are algebraic numbers whose complex conjugates are all of absolute value $q^{i/2}$. In particular, for each i, the Betti number $b_i = \dim H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ is independent of ℓ. By the spreading out argument sketched above, this independence holds for any field k of characteristic p.

Gabber has generalized Deligne's theorem to *proper*, equidimensional schemes X over k, but with $H^*(-, \mathbb{Q}_\ell)$ replaced by *intersection cohomology.*

This is derived from a general result of stability of rationality and independence of ℓ under the six operations for families of objects of $D_c^b(-, \overline{\mathbb{Q}}_\ell)$, see [F 2], and [I 4] for a brief survey. As before, this implies that the intersection ℓ-adic Betti numbers of such an X, over a finite field, or more generally over any field of characteristic p, are independent of ℓ.

Outside of Gabber's theorem, not much is known about the independence of ℓ of Betti numbers, or the rationality and independence of ℓ of traces of endomorphisms, or cohomological correspondences. See [I 4] for a discussion of some problems and partial results. Let us mention that Zheng [Z 2] has quite recently proved the analogue of Gabber's theorem of stability of rationality and independence of ℓ for schemes separated and of finite type over local fields with finite residue fields.

2. ℓ-divisibility : Serre's bounds [Se 2]

2.1. Let k be a field. Then k is the filtering inductive limit of its $\mathbb{Z}$-subalgebras of finite type R. If X is a scheme over k, separated and of finite type, there exists an R and a scheme $\mathcal{X}$ over $S = \operatorname{Spec} R$, separated and of finite type, such that $\mathcal{X} \times_S \operatorname{Spec} k = X$. Such an $\mathcal{X}/S$ is called a *model* of X/k over R (or S). Any two models $\mathcal{X}_1/S_1$, $\mathcal{X}_2/S_2$ become isomorphic over an S above both S_1 and S_2. When k is of characteristic $p > 0$, models of X over finitely generated sub-$\mathbb{F}_p$-algebras of k form a cofinal family.

Theorem 2.2. (Serre [I 4, 2.3]) Let k be a field, $\overline{k}$ a separable closure of k, $G_k = \operatorname{Gal}(\overline{k}/k)$, ℓ a prime number different from the characteristic of k, n an integer ≥ 1, X a scheme over k, separated and of finite type. The following conditions are equivalent :

(i) There exists a model $\mathcal{X}/S$, $S = \operatorname{Spec} R$, of X/k having the following property : for all points $s : \operatorname{Spec} k' \to S$ of S with value in a finite field k' of characteristic $\neq \ell$, one has

$$|\mathcal{X}(s)| \equiv 0 \mod \ell^n$$

(where $\mathcal{X}(s)$ denotes the set of points of $\mathcal{X}$ above s, i. e. rational points over k' of the pull-back of $\mathcal{X}$ by s).

(ii) For all $g \in G_k$, one has

$$\operatorname{Tr}(g, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell)) \equiv 0 \mod \ell^n.$$

Recall (1.15) that $\operatorname{Tr}(g, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell))$ belongs to $\mathbb{Z}_\ell$, so that its divisibility by ℓ^n makes sense.

2.3. There are two main ingredients in the proof :

(a) *Grothendieck's trace formula* (1.18.1)

(b) *Chebotarev's density theorem*, in the following form ([Se 1, th. 7], [P, App. B]) :

Let S be an integral, normal scheme of finite type over $\mathbb{Z}$, t a geometric point of S. Then, if G is a finite quotient of the fundamental group $\pi_1(S,t)$, any conjugacy class in G contains the image of a geometric Frobenius $F_s \in \mathrm{Gal}(k(\overline{s})/k(s))$ associated to a point $s \to S$ of S with value in a finite field, an algebraic closure $k(\overline{s})$ of $k(s)$ and a path from $\overline{s}$ to t.

Recall ([SGA 1 V] that a *path* c from $\overline{s}$ to t is an isomorphism from the fiber functor at t (on the category of finite étale covers of S) to that at $\overline{s}$, and that the induced isomorphism $\pi_1(S,\overline{s}) \xrightarrow{\sim} \pi_1(S,t)$ does not depend on c up to conjugacy by an element of $\pi_1(S,t)$; if t is localized at the generic point of S, such a path can be given by a lifting of t to the strict localization of S at $\overline{s}$.

Note that the above statement is implied by that in [Se 1, th. 7] when $\dim S \geq 1$, and if S is of dimension zero, i. e. is the spectrum of a finite field k_0, just says that the powers of the geometric Frobenius of k_0 are dense in its Galois group.

Here is a sketch of the proof of 2.2. By an easy dévissage, using the fact that $D^b_c(-,\mathbb{Z}_\ell)$ is stable under $Rf_!$ (1.14), one is reduced to proving the following lemma :

Lemma 2.4. ([I 4, 7.1]) Let S be an integral, normal scheme, separated and of finite type over $\mathbb{Z}$, $f : \mathcal{X} \to S$ a morphism which is separated and of finite type, ℓ a prime number invertible on S, and n an integer ≥ 1. Assume that all the ℓ-adic sheaves $R^i f_! \mathbb{Q}_\ell$ are lisse. Let η be the generic point of S, $\overline{\eta}$ a geometric point over it. The following conditions are equivalent :

(1) For every point s of S with value in a finite field, we have $|\mathcal{X}(s)| \equiv 0 \mod \ell^n$.

(2) For every $g \in \pi_1(S,\overline{\eta})$, we have $\mathrm{Tr}(g, H^*_c(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_\ell)) \equiv 0 \mod \ell^n$.

The implication (2) $\Rightarrow$ (1) follows immediately from Grothendieck's trace formula (1.18.1) and the smoothness of the sheaves $R^q f_! \mathbb{Z}_\ell$. For the converse, assume that there exists $g \in \pi_1(S,\overline{\eta})$ such that $t \not\equiv 0 \mod \ell^n$, where $t = \mathrm{Tr}(g, H^*_c(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_\ell))$. As the map $\pi_1(S,\overline{\eta}) \to \mathbb{Z}/\ell^n\mathbb{Z}$, $\sigma \mapsto \mathrm{Tr}(\sigma, H^i_c(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_\ell)) \mod \ell^n$ is continuous, there exists an open invariant subgroup H of $\pi_1(S,\overline{\eta})$ such that $\mathrm{Tr}(gh, H^i_c(Z_{\overline{\eta}},\mathbb{Q}_\ell)) = t \mod \ell^n$ for all $h \in H$. By Chebotarev's density theorem (b), there exists a point s of S with value in a finite field such that, after choosing a path c as above, the image of the geometric Frobenius of s in $\pi_1(S,\overline{\eta})/H$ is conjugate to the image of g. By Grothendieck's trace formula again (and the smoothness of the sheaves $R^q f_! \mathbb{Z}_\ell$) we then get a contradiction.

Remark 2.4.1. In [I 4, 7.1] the assumption *normal* is missing. Counterexamples to the implication (1) $\Rightarrow$ (2) of (*loc. cit.*) in the non geometrically unibranch case can be given, see [I 8]. However, the proof of the main result [I 4, 2.3] uses only the normal case.

Corollary 2.5. With the notations of 2.2, assume that a finite group A acts freely on X/k (by k-automorphisms). Then one has :

$$v_\ell(A) \leq \inf_{g\in G_k} v_\ell(\mathrm{Tr}(g, H^*_c(X_{\overline{k}}, \mathbb{Q}_\ell))), \tag{2.5.1}$$

where v_ℓ denotes the ℓ-adic valuation of an integer (or ℓ-adic integer), and $v_\ell(A)$ stands for $v_\ell(|A|)$.

We may assume A of order ℓ^n, $n \geq 1$. One can choose a model $\mathcal{X}/S$ of X/k on which A acts freely by S-automorphisms. Condition 2.2 (i) is then satisfied, so the conclusion follows from 2.2.

Remark 2.6. If X/k is smooth and equidimensional of dimension d, by Poincaré duality one has, for $g \in G_k$,

$$\mathrm{Tr}(g, H^*(X_{\overline{k}}, \mathbb{Q}_\ell)) = \chi(g^{-1})^d \mathrm{Tr}(g^{-1}, H^*_c(X_{\overline{k}}, \mathbb{Q}_\ell)),$$

where $\chi : \mathrm{Gal}(\overline{k}/k) \to \mathbb{Z}^*_\ell$ is the cyclotomic character. Therefore in 2.5 one gets the bound

$$v_\ell(A) \leq \inf_{g\in G_k} v_\ell(\mathrm{Tr}(g, H^*(X_{\overline{k}}, \mathbb{Q}_\ell))). \tag{2.6.1}$$

2.7. If X is a group scheme over k and A a finite subgroup of $X(k)$, A acts freely on X/k (by translations). When X is *reductive*, the cohomology of $X_{\overline{k}}$ with coefficients in $\mathbb{Q}_\ell$ is known (cf. [SGA 4 1/2, Somme trig. 8.2, p. 230]) and the action of the Galois group G_k on it can be made explicit. The bounds (2.6.1) thus obtained are studied in detail in [Se 2] (and established by alternate, non cohomological methods).

Suppose, for example, that $X = GL_n$. Then, by the formula of [SGA 4 1/2, Sommes trig. 8.2, p. 230], we get

$$H^*(X_{\overline{k}}, \mathbb{Q}_\ell) = \Lambda V,$$

where V is the graded $\mathbb{Q}_\ell$-vector space

$$V = \oplus_{1\leq i\leq n} V_{2i-1}, \quad V_{2i-1} = \mathbb{Q}_\ell(-i).$$

Therefore

$$\mathrm{Tr}(g^{-1}, H^*(X_{\overline{k}}, \mathbb{Q}_\ell)) = (-1)^n \det(1 - g^{-1}, V) = (-1)^n \prod_{1\leq i\leq n} (1 - \chi(g)^i),$$

where, as above, $\chi : \mathrm{Gal}(\overline{k}/k) \to \mathbb{Z}_\ell^*$ is the cyclotomic character. Suppose for simplicity that $\ell \neq 2$. Then, following Serre ([Se 2, 4.1], [I 4, 9]), let us write

$$\mathrm{Im}(\chi) = C_t \times (1 + \ell^m \mathbb{Z}_\ell), \tag{2.7.1}$$

where C_t is a cyclic group of order t dividing $\ell - 1$ and m an integer ≥ 1 (or ∞, with $\ell^\infty = 0$). Then

$$\inf_{g \in G_k} v_\ell(\mathrm{Tr}(g, H^*(X_{\overline{k}}, \mathbb{Q}_\ell))) = \inf_{g \in G_k} \sum v_\ell(1 - \chi(g)^i),$$

and an elementary calculation [I 4, 7.10] shows that the right hand side is equal to

$$\sum_{t|i} (m + v_\ell(i)).$$

For $k = \mathbb{Q}$, χ is surjective : $t = \ell - 1$, $m = 1$. After some rewriting, one recovers Minkowski's bound [Se 2, 1.1] for finite subgroups A of $GL_n(\mathbb{Q})$:

$$v_\ell(A) \leq [\frac{n}{\ell - 1}] + [\frac{n}{\ell(\ell - 1)}] + [\frac{n}{\ell^2(\ell - 1)}] + \cdots , \tag{$*$}$$

where $[x]$ denotes the integral part of a real number x. See [Se 2 §1] for the presentation of Minkowski's original argument (using Dirichlet's theorem on primes in arithmetic progressions, a particular case of Chebotarev's theorem). The previous bound also holds for $\ell = 2$, and can be obtained similarly. It is a sharp bound : there exists a finite ℓ-subgroup A of $GL_n(\mathbb{Q})$ for which $v_\ell(A)$ is equal to the right hand side of (*), see (*loc. cit.*).

If X is a *torus* over k with character group C ($\simeq \mathbb{Z}^n$), then

$$H^*(X_{\overline{k}}, \mathbb{Q}_\ell) = \Lambda(C \otimes \mathbb{Q}_\ell(-1)).$$

The action of G_k on $X_{\overline{k}}$ defines a representation $\sigma : G_k \to \mathrm{Aut}(C)$ and

$$\mathrm{Tr}(g, H^*(X_{\overline{k}}, \mathbb{Q}_\ell) = \det(1 - \sigma(g)\chi(g^{-1}), C \otimes \mathbb{Q}_\ell).$$

The bound

$$v_\ell(A) \leq \inf_{g \in G_k} v_\ell(\det(1 - \sigma(g)\chi(g^{-1}), C \otimes \mathbb{Q}_\ell))$$

given by (2.6.1) is elementary in this case (see [Se 2, 5.2, Lemma 5], where ρ is the contragredient representation of σ). Unraveling the right hand side gives the bound (*loc. cit.*) :

$$v_\ell(A) \leq m[\frac{\dim X}{\varphi(t)}],$$

where t and m are the integers defined in (2.7.1) for $\ell \neq 2$ (and in (loc. cit., 4.2) for $\ell = 2$), [-] denotes an integral part, and φ is Euler's indicator.

As a last example, let X be a *semisimple* group over k, with maximal torus T and Weyl group W. Let r be the rank of T and $1 \leq d_1 \leq \cdots \leq d_r$ be the *invariant degrees* of W acting on the symmetric algebra $S(C \otimes \mathbb{Q})$, where C, considered as homogeneous of degree 1, is the character group of T, i. e. the d_i's are the degrees of homogeneous polynomials $P_1, \cdots, P_r$ such that $S(C \otimes \mathbb{Q})^W = \mathbb{Q}[P_1, \cdots, P_r]$. Now, let $C \otimes \mathbb{Q}_\ell(-1)[-2]$ be the $\mathbb{Q}_\ell$-graded module concentrated in degree 2 defined by $C \otimes \mathbb{Q}_\ell(-1)$. Let I be the augmentation ideal of $S(C \otimes \mathbb{Q}_\ell(-1)[-2])^W$. The *indecomposable quotient* I/I^2 is a graded module, concentrated in *even* degrees :

$$I/I^2 = \oplus_{d \in D} L_{2d}(-d),$$

where

$$D = \{d_1, \cdots, d_r\}$$

is the *set* of invariant degrees. Here L_{2d} is the homogeneous part of degree d of the corresponding indecomposable quotient of $S(C \otimes \mathbb{Q})^W$, tensored with $\mathbb{Q}_\ell$; its dimension is the number of indices i for which $d_i = d$. The action of G_k respects this grading. By [SGA 4 1/2, Sommes trig., 8.2, p. 230], the Hopf algebra $H^*(X_{\overline{k}}, \mathbb{Q}_\ell)$ is the (graded) exterior algebra of its *primitive part* V :

$$H^*(X_{\overline{k}}, \mathbb{Q}_\ell) = \Lambda V.$$

Moreover, the graded module V is endowed with a G_k-equivariant isomorphism

$$V \xrightarrow{\sim} I/I^2[1]$$

(this is an ℓ-adic analogue of the *transgression* studied by Borel, cf. [Bo, 13.1, 19.1]). In other words, V is a graded module, concentrated in *odd* degrees,

$$V = \oplus_{d \in D} V_{2d-1},$$

with $V_{2d-1} = L_{2d}(-d)$. In particular, the 1-dimensional, *top* cohomology group is $H^n(X_{\overline{k}}, \mathbb{Q}_\ell) = \Lambda^n V$, with

$$n = \sum_{d \in D} (2d-1) \dim V_{2d-1} = \sum_{1 \leq i \leq r} (2d_i - 1) = \dim X.$$

The action of G_k on V_{2d-1} is given by $\varepsilon_d \otimes \chi^{-d}$, where χ is the cyclotomic character and $\varepsilon_d : G_k \to \mathrm{Aut}(L_{2d})$ is the representation deduced from the action of G_k on the Dynkin diagram of X. Therefore, one has

$$\mathrm{Tr}(g, H^*(X, \mathbb{Q}_\ell)) = \prod_{d \in D} \det(1 - \varepsilon_d(g) \otimes \chi(g)^{-d}, L_{2d}),$$

and by (2.6.1), for finite subgroups A of $X(k)$ one gets the bound [Se 2, 6.6. th. 6'] :

$$v_\ell(A) \leq \inf_{g \in G_k} \sum_{d \in D} v_\ell(\det(1 - \varepsilon_d(g) \otimes \chi(g)^{-d}, L_{2d})).$$

When X is of *inner type*, i. e. when G_k acts trivially on the Dynkin diagram (so that G_k acts by inner automorphisms on X), the above formula simplifies :

$$v_\ell(A) \leq \inf_{g \in G_k} \sum_{1 \leq i \leq r} v_\ell(\chi(g)^{d_i} - 1),$$

and, unraveling the right hand side, one gets, for $\ell \neq 2$,

$$v_\ell(A) \leq \sum_{t | d_i} (m + v_\ell(d_i)),$$

with m and t as in (2.7.1), cf. [Se 2, 6.1 th. 6, prop. 4].

2.8. Let k be a *finite field* with q elements, $\overline{k}$ an algebraic closure of k, $F \in G_k = \mathrm{Gal}(\overline{k}/k)$ the geometric Frobenius, X/k a scheme separated an of finite type, and ℓ a prime different from the characteristic of k. By definition of $H^*_c(X_{\overline{k}}, \mathbb{Q}_\ell)$, the ℓ-adic numbers $\mathrm{Tr}(F, H^*_c(X_{\overline{k}}, \mathbb{Q}_\ell))$ are in $\mathbb{Z}_\ell$, and this holds more generally for F replaced by any $g \in G_k$. For F a stronger result holds : by a theorem of Deligne [SGA 7 XIX, App.], for each $i \in \mathbb{Z}$, the *eigenvalues* of F on $H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)$ (in $\overline{\mathbb{Q}}_\ell$) are *algebraic integers*. It makes sense, then, to study their *divisibility by powers of* q. This question has been reconsidered recently by several authors, and significant results of arithmetic or geometric flavor have been obtained. See [I 4, 4] for a survey.

Deligne's result in (*loc.cit.*) is a corollary of the stability of integrality under $Rf_!$. A $\overline{\mathbb{Q}}_\ell$-sheaf L on X is called *integral* if for each closed point x of X the eigenvalues of the geometric Frobenius $F_x \in \mathrm{Gal}(k(\overline{x})/k(x))$ acting on $L_{\overline{x}}$ ($k(\overline{x})$ an algebraic closure of $k(x)$) are algebraic integers. A complex $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is called *integral* if its cohomology sheaves $\mathcal{H}^i(K)$ are integral. Deligne's theorem is that for $f : X \to Y$ a k-morphism between k-schemes separated and of finite type, then if $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is integral, so is $Rf_!K$. Other stabilities have been recently established by Zheng [Z 1], for example the stability of integrality under Rf_*.

3. Basics on classical étale nearby cycles

3.1. Let S be a *henselian trait*, i. e. the spectrum of a henselian discrete valuation ring, with closed point s and generic point η. Let $\overline{s}$ be a geometric point above s, $S_{(\overline{s})}$ the corresponding strict localization, and $\overline{\eta}$ a geometric generic point of $S_{(\overline{s})}$ (thus defining a geometric point of S above η). We will assume $\overline{s}$ (resp. $\overline{\eta}$) algebraic over s (resp. η). Fix some coefficients ring Λ, which for simplicity we choose to be $\mathbb{Z}/\ell^{\nu}$ (ℓ a prime invertible on S, $\nu \geq 1$) (there are variants with $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}_\ell}$). Let $f : X \to S$ be a morphism of schemes. For $F \in D^+(X_\eta, \Lambda)$, the complex of *nearby cycles* $R\Psi F$ (of (F,f)) is a complex on the geometric special fiber $X_{\overline{s}}$ defined by

$$R\Psi F = \overline{i}^* R\overline{j}_*(F|X_{\overline{\eta}}), \tag{3.1.1}$$

where $\overline{i} : X_{\overline{s}} \to X_{(\overline{s})}$ and $\overline{j} : X_{\overline{\eta}} \to X_{(\overline{s})}$ are deduced by pull-back from the maps $\overline{i} : \overline{s} \to S_{(\overline{s})}$, $\overline{j} : \overline{\eta} \to S_{(\overline{s})}$, with $S_{(\overline{s})}$ the strict localization of S at $\overline{s}$. If $\overline{x}$ is a geometric point of X over $\overline{s}$, and $X_{(\overline{x})}$ denotes the strict localization of X at $\overline{x}$ (a Milnor ball), the fibre $(X_{(\overline{x})})_{\overline{\eta}}$ of $X_{(\overline{x})} \to S_{(\overline{s})}$ at $\overline{\eta}$ plays the role of a *Milnor fiber*, and

$$(R\Psi F)_{\overline{x}} = R\Gamma((X_{(\overline{x})})_{\overline{\eta}}, F). \tag{3.1.2}$$

The complex $R\Psi F$ comes equipped with more structure : if G is the Galois group of $\overline{\eta}$ over η, $R\Psi F$ underlies a complex of sheaves of Λ-modules endowed with a continuous action of G compatible with its action on $X_{\overline{s}}$ (an object of $D^+(X_s \times_s \eta, \Lambda)$ in Deligne's notations in [SGA 7 XIII]). This action plays the role of the monodromy action of π_1 of the punctured disc on the cohomology of a Milnor fiber in the analytic context.

For $F \in D^+(X, \Lambda)$, the adjunction map defines an equivariant triangle

$$F|X_{\overline{s}} \to R\Psi(F|X_\eta) \to R\Phi F \to, \tag{3.1.3}$$

where $R\Phi F$ is by definition the complex of *vanishing cycles* of (F,f). This complex measures the non local acyclicity of (F,f) : by definition, the stalk of $R\Phi F$ at $\overline{x}$ vanishes if and only if (F,f) is locally acyclic at $\overline{x}$; this is the case, for example, if F is lisse and f is smooth at $\overline{x}$.

3.2. The functor $R\Psi$ enjoys the following basic properties :

(a) Let $f : X \to S$, $g : Y \to S$, $h : X \to Y$ be S-morphisms, with $gh = f$.

(i) If h is *proper*, then, for $F \in D^+(X_\eta, \Lambda)$,

$$R\Psi(R(h_\eta)_* F) \xrightarrow{\sim} R(h_{\overline{s}})_* R\Psi F. \tag{3.2.1}$$

This is an immediate consequence of the proper base change theorem [SGA 4 XI] (cf. 1.5).

(ii) If h is *smooth*, then, for $F \in D^+(Y_\eta, \Lambda)$,

$$h_{\overline{s}}^* R\Psi F \xrightarrow{\sim} R\Psi(h_\eta^* F). \tag{3.2.2}$$

(b) Suppose f is of finite type. Then $R\Psi$ preserves constructibility (1.7), i. e. sends $D_c^+(X_\eta, \Lambda)$ (resp. $D_c^b(X_\eta, \Lambda)$) into $D_c^+(X_{\overline{s}}, \Lambda)$ (resp. $D_c^b(X_{\overline{s}}, \Lambda)$).

Moreover, the formation of $R\Psi F$ commutes with any dominant change of traits $S' \to S$ ([SGA 4 1/2, Th. finitude, 3.2, 3.7]).

(c) Suppose f is separated and of finite type. Then $R\Psi$ commutes with *duality*, in the following sense. Let $D_{\overline{s}} = R\mathcal{H}om(-, Rf_{\overline{s}}^{!}\Lambda)$ (resp. $D_\eta = R\mathcal{H}om(-, Rf_\eta^!\Lambda)$) be the *dualizing functor* on $X_{\overline{s}}$ (resp. X_η) (cf. [SGA 4 1/2 Th. finitude, 4] (if D is $D_{\overline{s}}$ (resp. D_η), D sends D_c^b to D_c^b and $Id \xrightarrow{\sim} DD$). Then, for $F \in D_c^b(X_\eta, \Lambda)$, of finite tor-dimension, there is a natural isomorphism [I 2, 4.2]

$$R\Psi(D_\eta F) \xrightarrow{\sim} D_{\overline{s}}(R\Psi F). \tag{3.2.3}$$

This implies that $R\Psi$ preserves perversity, i. e. is *t-exact* [I 2, 4.5]. In (*loc. cit.*) it is shown along the same lines that $R\Psi$ *commutes with external products.*

3.3. As a typical example, in which nearby cycles can be explicitly calculated, consider the case where $f : X \to S$ has *strict semistable reduction.* This means that X is regular, X_η is smooth and the special fiber X_s is a divisor with strict normal crossings (see 6.3 for the definition). Write $Y = X_s$ as the sum of its irreductible components $Y = \sum_{1 \le i \le r} Y_i$. Then, $R^0\Psi\Lambda = \Lambda_Y$, there is a canonical exact sequence

$$0 \to \Lambda_Y \to \bigoplus \Lambda_{Y_i} \to R^1\Psi\Lambda(1) \to 0,$$

where the first map is the diagonal, and the cup-product defines isomorphisms

$$\Lambda^q R^1\Psi\Lambda \xrightarrow{\sim} R^q\Psi\Lambda.$$

The inertia subgroup I of G (3.1) *acts trivially* on the sheaves $R^q\Psi\Lambda$, and *unipotently* on $R\Psi\Lambda$ (as an object of $D_c^b(X_{\overline{s}}, \Lambda)$). These results were first proven by Rapoport-Zink [RZ], see [I 3, 3.3, 3.4] for an alternate exposition.

3.4. In general, complexes of nearby cycles do not behave well in families, as is shown by the following elementary example, discussed by Deligne [D]. Let Y be the affine plane $\mathbb{A}^2_{\mathbb{C}}$, $f : X \to Y$ the blow-up of the origin in Y, $E = f^{-1}(0) = \mathbb{P}^1_{\mathbb{C}}$ the exceptional divisor. Lines in Y passing through 0 are parametrized by E ; for t in E, let D_t be the corresponding line. Fix $t \in E$. Let U be an open neighborhood of $t \in E$ in X, sent by f into some open neighborhood V of the origin in Y. Then $R(f|U)_*\mathbb{Z}|V = (f|U)_*\mathbb{Z}|V$ is $\mathbb{Z}$ on some sector around $D_t \cap V$ and zero elsewhere. These sectors shrink as U, V do. Therefore the cohomologies of the (generalized) Milnor fibers around the origin in Y do not form a nice family : the inductive system $R(f|U)_*\mathbb{Z}|V$ around 0 in Y, is not essentially constant (and, in addition, each member is, in general, not analytically constructible, because of the shape of these sectors).

The morphism f above is not flat, but other examples with f flat and with the same bad properties were given by Deligne [D] and Lê [Lê]. Lê (*loc. cit*) showed that morphisms "without blow-ups", i. e. admitting suitable Thom-Whitney stratifications, had a good theory of "punctual" nearby cycles, i. e. the inductive systems considered above were constructible and essentially constant. Deligne [D] asked whether such good properties could be obtained after a suitable modification of the base (for instance, in the example discussed above, the modification by f itself works). This was proven by Sabbah [S], at least for f proper.

It was not at all clear how to proceed in the étale set-up. While trying to prove the product formula for the constants of the functional equations of L functions for function fields, in the tamely ramified case, Deligne conceived a theory of nearby and vanishing cycles valid over general bases. A short summary, without proofs, was written by Laumon [La 2]. This topic long remained untouched, because Laumon's proof of the product formula using the ℓ-adic Fourier transform [La 3] rendered Deligne's approach useless. But it has been recently revisited by Orgogozo [O 1], who (with the help of Gabber for certain points) proved an analogue of Sabbah's theorem in the étale context. We will outline this in section 5. In the next section, we summarize the abstract constructions needed to state the results. Some of them turned out to have quite different, unexpected applications, see 8.5-8.7.

4. Oriented products and vanishing toposes

4.1. Let $f : X \to S$, $g : Y \to S$ be morphisms of toposes defined by functors f^*, g^* between small defining sites having finite projective limits, such that f^*, g^* are continuous and commute with finite projective limits. Here "small" refers to a fixed universe $\mathcal{U}$. Unless otherwise stated all sites (resp. toposes) will be $\mathcal{U}$-sites (resp. toposes). In practise, f and g will be morphisms of étale toposes defined by morphisms of schemes.

A basic construction, due to Deligne [La 2], associates with the pair (f,g) a topos, denoted

$$X \overleftarrow{\times}_S Y, \tag{4.1.1}$$

called the *oriented product* of f and g, together with morphisms of toposes

$$p_1 : X \overleftarrow{\times}_S Y \to X, \ p_2 : X \overleftarrow{\times}_S Y \to Y,$$

and a *morphism*

$$\tau : gp_2 \to fp_1,$$

i. e. a morphism of functors $(gp_2)_* \to (fp_1)_*$ (or equivalently $(fp_1)^* \to (gp_2)^*$), having the following universal property : let T be a topos equipped with morphisms $a : T \to X$, $b : T \to Y$, $t : gb \to fa$. Then there exists a unique triple $(h : T \to \tilde{X} \overset{\leftarrow}{\times}_S Y, \alpha : p_1 h \simeq a, \beta : p_2 h \simeq b)$ making τh equal to t.

The topos $X \overset{\leftarrow}{\times}_S Y$ is constructed as the topos of sheaves on the following site C. Let C be the category of pairs of morphisms $U \to V \leftarrow W$ above $X \to S \leftarrow Y$, where $U \to V$ (resp. $V \leftarrow W$) means a morphism $U \to f^*V$ (resp. $g^*V \leftarrow W$) of the defining site of X (resp. Y) and V is an object of the defining site of S. Endow C with the topology generated by covering families $(U_i \to V_i \leftarrow W_i) \to (U \to V \leftarrow W)$ $(i \in I)$ of the following types :

(a) $V_i = V$, $W_i = W$ for all i and $(U_i \to U)$ is covering,

(b) $U_i = U$, $V_i = V$ for all i and $(W_i \to W)$ is covering,

(c) $(U' \to V' \leftarrow W') \to (U \to V \leftarrow W)$, where $U' = U$ and $W' \to W$ is obtained by base change from a map $V' \to V$ of the defining site of S.

The projections p_1, p_2 are defined by $p_1^*(U) = (U \to e_S \leftarrow e_Y)$, $p_2^*(W) = (e_X \to e_S \leftarrow W)$, where e_X, e_S, e_Y are the final objects of the defining sites of X, S, Y respectively.

To define τ, observe first that if F is a sheaf on C, i. e. an object of $X \overset{\leftarrow}{\times}_S Y$, then for any covering $(U \to V' \leftarrow W') \to (U \to V \leftarrow W)$ of type (c), the restriction map

$$F(U \to V \leftarrow W) \to F(U \to V' \leftarrow W') \tag{$*$}$$

is bijective (this follows from the fact that the morphism

$$(U \to V' \leftarrow W') \to (U \to V' \times_V V' \leftarrow W' \times_W W')$$

given by the diagonal maps is a covering of type (c)). Then τ is given by the following morphism of functors $\tau : (gp_2)_* \to (fp_1)_*$: for a sheaf F on C, and an object V of S,

$$\tau : (gp_2)_* F(V) \to (fp_1)_* F(V)$$

is the composition

$$F(e_X \to e_S \leftarrow g^*V) \to F(f^*V \to V \leftarrow g^*V) \leftarrow F(f^*V \to e_S \leftarrow e_Y),$$

where the second map is an isomorphism of the form (*).

The verification that $(X \overset{\leftarrow}{\times}_S Y, p_1, p_2, \tau)$ satisfies the required universal property is straightforward [I 6, 1.2].

4.2. In most applications, $f : X \to S$, $g : Y \to S$ will be the morphisms of étale toposes associated with morphisms of schemes. In this case, however, the oriented product $X \overset{\leftarrow}{\times}_S Y$ is not, in general, the étale topos of a scheme. Geometric, topological or cohomological properties of such objects are far from being well understood. Here are a few elementary facts and examples (see [I 6]).

(a) By the universal property, *points* of $T = X \overset{\leftarrow}{\times}_S Y$, i. e. morphisms from the punctual topos to T are triples (x, y, u), where x (resp. y) is a geometric point of X (resp. Y) (1.2) and u is a morphism from gy to fx, i. e. a lifting of $gy : \operatorname{Spec} k \to S$ (k a separably closed field) to the strict localization $S_{(fx)}$ of S at the image of x by f, in other words, a *specialization morphism* from gy to fx in the sense of [SGA 4 VIII 7.2]. it follows from a general result of Deligne [SGA 4 VI 9.0] that if f and g are *coherent*, i. e. quasi-compact and quasi-separated, then T has enough points.

(b) By the universal property again, there is a unique morphism

$$\Psi : X \times_S Y \to X \overset{\leftarrow}{\times}_S Y \tag{4.2.1}$$

such that $p_i\Psi = pr_i$ ($i = 1,\ 2$) and $\tau\Psi = Id : gpr_2 \to fpr_1$, where the left hand side is the étale topos of the fiber product $X \times_S Y$. With the notations of 4.1, one has

$$\Psi^*(U \to V \leftarrow W) = U \times_V W,$$

and for a sheaf F on $X \times_S Y$, the stalk of $\Psi_* F$ at (x, y, u) is $\Gamma(X_{(x)} \times_{S_{(fx)}} Y_{(y)}, F)$, where $Y_{(y)} \to S_{(fx)}$ is the composition $Y_{(y)} \to S_{(gy)} \to S_{(fx)}$, the second map being given by u. This identification is derived into an isomorphism

$$R\Psi_* F_{(x,y,u)} \simeq R\Gamma(X_{(x)} \times_{S_{(fx)}} Y_{(y)}, F) \tag{4.2.2}$$

for $F \in D^+(X \times_S Y, \Lambda)$, Λ a ring. We will abbreviate (x, y, u) to (x, y) when no confusion can arise. For brevity, we will usually write $R\Psi$ instead of $R\Psi_*$.

If S is the spectrum of a field (or more generally a finite scheme), then Ψ is an equivalence. In general, Ψ is far from being an equivalence. For example, if Y is a closed subscheme of a scheme S, with open complement U, the fiber product $Y \times_S U$ is empty, while the oriented product $Y \overset{\leftarrow}{\times}_S U$ is an interesting object, playing the role of a *punctured tubular neighborhood* of Y in S, see 8.6.

4.3. Let $f : X \to S$ be a morphism of schemes, and take $g : Y \to S$ to be the identity morphism of S. The corresponding oriented product (for the associated morphisms of étale toposes)

$$X \overset{\leftarrow}{\times}_S S \tag{4.3.1}$$

is called the *vanishing topos* of f. We have the following relations

$$X \times_S S = X \quad , \quad p_1\Psi = Id_X \quad , \quad p_2\Psi = f, \tag{4.3.2}$$

and there is a structural morphism $\tau : p_2 \to fp_1$.

Fix a ring Λ as in 3.1, but for the beginning not assuming ℓ invertible on S. The functor

$$R\Psi : D^+(X, \Lambda) \to D^+(X \overset{\leftarrow}{\times}_S S, \Lambda)$$

is called the functor of *nearby cycles* of f. We denote it sometimes by $R\Psi_f$ or $R\Psi_X$ to avoid confusion. Let $(x, t) = (x, t, u : t \to fx)$ be a point of $X \overset{\leftarrow}{\times}_S S$. Consider the fiber product

$$X_{(x,t)} = X_{(x)} \times_{S_{(s)}} S_{(t)}, \tag{4.3.3}$$

which one might call the *Milnor tube* at (x, t), with center the *Milnor fiber* $(X_{(x)})_t$. It follows from the definitions that for a sheaf F on X, the stalk of ΨF at (x, t) is given by

$$(\Psi F)_{(x,t)} = \Gamma(X_{(x,t)}, F),$$

and this is derived into

$$R\Psi F_{(x,t)} = R\Gamma(X_{(x,t)}, F), \tag{4.3.4}$$

for $F \in D^+(X, \Lambda)$. Note that the restriction to the Milnor fiber

$$R\Gamma(X_{(x,t)}, F) \to R\Gamma((X_{(x)})_t, F)$$

is not an isomorphism in general, see 5.2 for more about this problem.

By construction, $p_1\Psi = Id$ (4.3.2). The map

$$p_{1*} \to \Psi^* \tag{4.3.5}$$

obtained by applying p_{1*} to the adjunction map $Id \to \Psi_*\Psi^*$ is an isomorphism [I 6, 3.2]. On the other hand, the identity of F, for $F \in D^+(X, \Lambda)$, gives a canonical map

$$p_1^*F \to R\Psi F,$$

and a triangle

$$p_1^*F \to R\Psi F \to R\Phi F \to . \tag{4.3.6),}$$

which can be made functorial by the usual techniques of filtered complexes (cf. [SGA 7 XIII]). The functor $R\Phi$ is called the functor of *vanishing cycles* of f. When S is the spectrum of a field, the morphism Ψ is an equivalence (4.2 (b)), hence $R\Phi F = 0$ for all $F \in D^+(X, \Lambda)$.

When S is a trait, as in 3.1, the topos denoted $X_s \times_s S$ (resp. $X_s \times_s \eta$) by Deligne in [SGA 7 XIII] can be identified with the sub-topos $X_s \overleftarrow{\times}_S S$ (resp. $X_s \overleftarrow{\times}_S \eta$) of $X \overleftarrow{\times}_S S$, and (4.3.6) induces the usual triangle relating (classical) nearby and vanishing cycles.

The vanishing toposes and the functors Ψ, Φ satisfy various formal properties with respect to composition and base change, whose description we will omit, see [I 5, 2.4] for some of them.

4.4. On oriented products of schemes there are good notions of finiteness and constructibility ([O 1, 7]). Assume, for simplicity, that S is noetherian and X and Y are of finite type over S. A sheaf F of Λ-modules on $X \overleftarrow{\times}_S Y$ is called *constructible* if there exist finite partitions of X and Y into disjoint locally closed subsets : $X = \cup X_i$, $Y = \cup Y_j$, such that, for all (i, j), the restriction of F to the sub-topos $X_i \overleftarrow{\times}_S Y_j$ is locally constant of finite type. The constructible sheaves of Λ-modules form a thick subcategory of the category of all sheaves of Λ-modules, so that the full subcategory $D^b_c(X \overleftarrow{\times}_S Y, \Lambda)$ consisting of complexes with bounded, constructible cohomology sheaves is a triangulated subcategory. Constructible sheaves are the noetherian objects of the category of Λ-modules and any sheaf of Λ-modules is a filtering inductive limit of constructible sheaves.

5. Main results on nearby cycles over general bases

5.1. Properties 3.2 (a) are easy to generalize. Let S be a scheme and $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$, with ℓ a prime invertible on S and $\nu \geq 1$.

(i) Let $f : X \to Y$ be a morphism of S-schemes. We then get a commutative diagram

(5.1.1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Psi_X"] & Y \arrow[d, "\Psi_Y"] \\ X \overleftarrow{\times}_S S \arrow[r, "\overleftarrow{f}"] & Y \overleftarrow{\times}_S S \end{tikzcd}$$

which induces an isomorphism

$$R\overleftarrow{f}_* R\Psi_X F \xrightarrow{\sim} R\Psi_Y Rf_* F$$

for $F \in D^b_c(X, \Lambda)$. Moreover, if f is *proper*, it follows from the proper base change theorem 1.5 that $R\overleftarrow{f}_*$ commutes with any "base change" $X' \to X$,

$S' \to S$, so that, taking into account the compatibility (4.3) between the general $R\Psi$ and the classical one when S is a trait, we recover 3.2 (a) (i).

(ii) If f in (5.1.1) is *smooth*, it follows from the local acyclicity of smooth maps (1.12) that the base change map associated with (5.1.1)

$$\overleftarrow{f}^* R\Psi_Y F \to R\Psi_X f^* F$$

is an isomorphism. This generalizes 3.2 (a) (ii).

Properties (b) of 3.2 turn out to be false in general. As shown by Orgogozo [O 1, 9], in the case of the blow-up discussed in 3.4, $R\Psi_f \Lambda$ has not constructible cohomology, and its formation does not commute with the base change by f itself. Additional assumptions are necessary.

We call a morphism $g : S' \to S$ a *modification* (resp. an *alteration*) if g is proper, surjective and induces an isomorphism (resp. a finite morphism) over an everywhere dense open subscheme, with the property that each maximal point of S' is sent to a maximal point of S. Orgogozo proved the following theorem, conjectured by Deligne :

Theorem 5.2. [O 1, 1.1, 5.1, 6.1] Let S be a noetherian scheme and $f : X \to S$ be a morphism of finite type. Let $F \in D^b_c(X, \Lambda)$. Then there exists a modification $g : S' \to S$ such that if f' (resp. F') is deduced from f (resp. F) by base change by g, $R\Psi_{f'} F'$ belongs to $D^b_c(X' \overleftarrow{\times}_{S'} S', \Lambda)$ and the formation of $R\Psi_{f'} F'$ commutes with any base change $S'' \to S'$.

In particular, after base change by g, the cohomology of the Milnor tube restricts isomorphically to that of the Milnor fiber : for any point (x, y) of $X' \overleftarrow{\times}_{S'} S'$, the restriction map

$$R\Psi_{f'} F'_{(x,y)} \to R\Gamma(X'_{(x)} \times_{S'_{(f(x))}} y, F') \tag{5.2.1}$$

is an isomorphism.

It follows that when S is regular of dimension ≤ 1, $R\Psi F$ is already constructible (i. e. belongs to D^b_c) and commutes with any base change. For S a trait, one recovers Deligne's results 3.2 (b). For S of dimension zero, 5.2 says that $R\Phi F$ is universally zero (cf. (4.3.6), i. e. after any base change $S' \to S$, the cone of the restriction to any Milnor fiber of the pull-back f',

$$F'_x \to R\Gamma((X'_{(f'(x))})_t, F'), \tag{5.2.2}$$

is zero : one recovers Deligne's universal local acyclicity theorem [SGA 4 1/2 Th. finitude, 2.16].

There is an important case in which no modification of the base is needed to make $R\Psi F$ constructible and commuting with base change, namely the case of *isolated singularities*. More precisely, 5.2 implies the following result, generalizing that stated in [La 2, 3.3.5] :

Corollary 5.3. [O 1, 5.1] Let S be a noetherian scheme and $f : X \to S$ be a separated and of finite type morphism. Let $F \in D_c^b(X, \Lambda)$. Let Σ be the complement in X of the largest open subset U such that $F|U$ is universally locally acyclic over S. Assume that Σ is quasi-finite over S. Then $R\Psi F$ (resp. $R\Phi F$) belongs to $D_c^b(X \overset{\leftarrow}{\times}_S S, \Lambda)$ and its formation commutes with any base change $S' \to S$ (in particular, for any point (x, y) of $X \overset{\leftarrow}{\times}_S S$, the cohomology of the Milnor tube at (x, y) restricts isomorphically to the cohomology of the Milnor fiber). Moreover, $R\Phi F$ is concentrated on $\Sigma \overset{\leftarrow}{\times}_S S$.

Here "universally locally acyclic" means that, after any base change, the restriction maps of type (5.1.3) are isomorphisms [SGA 4 1/2 Th. finitude, 2.12]. This is the case, for example, when U is smooth over S and $F|U$ is a locally constant sheaf (1.12).

5.4. Here are some glimpses on the proof of 5.2, see [O 1] for the details and [I 5] for an outline. Standard arguments reduce to proving the following assertion : (5.4.1) For S of finite type over $\mathbb{Z}$, there exists an *alteration* $g : S' \to S$ such that after base change by g, $R\Psi F$ becomes constructible and for any S'-morphism $T' \to T$, the cone $K_{TT'} R\Psi_f F$ of the base change map

$$R\Psi_{f'_T}(F_T)|T' \to R\Psi_{f'_{T'}}(F_{T'})$$

vanishes, where $f' : X' \to S'$ is deduced from f by base change by g. That one can replace "modification" by "alteration" follows from Gruson-Raynaud's flattening theorem [GR].

Orgogozo proves (5.4.1) by a rather intricate induction on the triples of integers $t = (\delta, r, d)$, $\delta \geq 0$, $r \geq -2$, $d \geq 0$, lexicographically ordered. He proves by induction on t that for $\dim S \leq \delta$, $\dim f \leq d$ and $F \in D_c^b(X, \Lambda)$ with $\mathcal{H}^q F = 0$ for $q < 0$, there exists an alteration g as above such that $\tau_{\leq r} R\Psi_{X'} F'$ is constructible and for any S'-morphism $T' \to T$, $\tau_{\leq r} K_{TT'} R\Psi_f F = 0$.

For $t = (0, -2, 0)$ (more generally any $t = (\delta, -2, d)$) the assertion is trivial. That this inductive procedure proves (5.4.1) follows from the fact that $R\Psi_f$ is of cohomological dimension $\leq 2 \dim f$, a nontrivial point, based on an old result of Artin on the joins of henselian rings [A, 3.4], to the effect that if x, y are geometric points of an affine noetherian scheme X, the connected components of the fiber product $X_{(x)} \times_X X_{(y)}$ are strictly local.

The key ingredient in the proof of (5.4.1) is de Jong's main theorem [dJ 2, 5.10], namely that for $f : X \to S$ a proper morphism between noetherian, integral, excellent (e. g. of finite type over $\mathbb{Z}$, cf. 6.4) schemes, there exists a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S, \end{tikzcd}$$

where g and h are alterations, with Z integral, and f' is plurinodal. A morphism is called *plurinodal* [dJ 2, 5.8] if it is a finite composition of proper nodal curves, i. e. proper, flat morphisms of relative dimension 1 whose geometric fibers have at most ordinary quadratic singularities (this definition is slightly less restrictive than that in (*loc. cit.*) as we don't require the curves to be quasi-split nor have sections).

In the induction procedure, a typical, crucial step is the following one, which we reproduce from [I 5, 4.3.1] :

Suppose that f can be factored as $f = ba$, where $b : Y \to S$ is proper, of relative dimension $\leq d-1$, and $a : X \to Y$ is a proper nodal curve. Let us show that after a suitable alteration $g : S' \to S$, we have $\tau_{\leq r} K_{TT'} R\Psi_f \Lambda = 0$ for every S'-morphism $T' \to T$.

Let $U \subset X$ be the open subset of smoothness of a. The complement $\Sigma = X - U$ is finite over Y. As b is of relative dimension $\leq d-1$, by the induction assumption we may assume, up to base changing by an alteration $S' \to S$, that $\tau_{\leq r} K_{TT'} R\Psi_b \Lambda = 0$. By the basic property 5.1 (ii), as $a_U = a|U$ is smooth, we then have

$$0 = \overleftarrow{a_U}^* \tau_{\leq r} K_{TT'} R\Psi_b \Lambda = \tau_{\leq r} K_{TT'} R\Psi_f \Lambda | U.$$

Hence $\tau_{\leq r} K_{TT'} R\Psi_f \Lambda$ is concentrated on Σ, and it suffices to show that

$$\overleftarrow{a}_{\Sigma *} \tau_{\leq r} K_{TT'} R\Psi_f \Lambda = 0.$$

Now

$$\overleftarrow{a}_{\Sigma *} \tau_{\leq r} K_{TT'} R\Psi_f \Lambda = \tau_{\leq r} R \overleftarrow{a}_* K_{TT'} R\Psi_f \Lambda,$$

and by the basic property 5.1 (i) we have

$$R \overleftarrow{a}_* K_{TT'} R\Psi_f \Lambda = K_{TT'} R\Psi_b (Ra_* \Lambda).$$

As $Ra_*\Lambda$ is in $D^b_c(Y, \Lambda)$ and cohomologically concentrated in nonnegative degrees, by the induction assumption again, we may assume, up to base changing S by an alteration, that $\tau_{\leq r} K_{TT'} R\Psi_b(Ra_*\Lambda) = 0$, hence $\tau_{\leq r} K_{TT'} R\Psi_f \Lambda = 0$ as required.

5.5. The above results 5.1, 5.2 extend 3.2 (a) and (b) in the best possible way. For the moment, however, no generalization of 3.2 (c) is known. Besides, no explicit calculation of nearby or vanishing cycles over a base of dimension ≥ 2 has been performed as yet. This is related to the problem of generalizing the variation and monodromy operators of the classical theory. For constant coefficients, the examples of nodal curves, or of semistable reduction along a divisor with normal crossings (in the sense of [I 1, 1.1]), or more generally, borrowing notions from log geometry, morphisms underlying log smooth and exact morphisms of log schemes, should be accessible and would shed light on these questions.

Despite this lack of understanding, the mere existence of a formalism, together with 5.3, enabled Gabber and Orgogozo to solve questions raised in [Weil II] concerning the vanishing cycles associated with Lefschetz pencils, namely, in all cases, the conjugacy of vanishing cycles under the monodromy group of the pencil, and the constancy of this group when the pencil varies, see [O 1, 11.2] and, for an outline, [I 5, 5.1, 5.2].

6. Gabber's recent results on étale cohomology

Gabber has recently solved fundamental problems and conjectures left open in [SGA 4] and [SGA 5], namely :

(i) Grothendieck's absolute purity conjecture

(ii) constructibility of higher direct images of constructible sheaves of torsion prime to the characteristics by finite type morphisms between noetherian quasi-excellent schemes

(iii) affine Lefschetz type theorems for excellent schemes

(iv) existence of dualizing complexes on excellent schemes.

The main results and a sketch of the key ideas in their proofs are presented in [Ga 2], [Ga 3]. A more detailed account is in preparation [ILO]. In this section we give a short overview of the statements.

Theorem 6.1. (Gabber) Let X be a regular, locally noetherian scheme, $Y \subset X$ a regular divisor, $j : U = X - Y \to X$ the inclusion, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, with $n > 0$ invertible on X. Then :

$$R^q j_* \Lambda = \begin{cases} \Lambda & \text{if } q = 0 \\ \Lambda_Y(-1) & \text{if } q = 1 \\ 0 & \text{if } q > 1 \end{cases}$$

The isomorphism $\Lambda \xrightarrow{\sim} R^1 j_* \Lambda(1)$ sends the unit section to the section of $R^1 j_* \Lambda(1)$ over Y which is locally given by the opposite of the class of the torsor of n^{th}-roots of a local equation of Y. The composition of this isomorphism with the canonical isomorphism $R^1j_*\Lambda(1) \xrightarrow{\sim} \mathcal{H}^2_Y(\Lambda(1))$ sends the unit section to the cohomology class of Y, $c(Y) \in H^0(Y, \mathcal{H}^2_Y(\Lambda(1)) = H^2_Y(X, \Lambda(1))$, according to the conventions of [SGA 4 1/2, Cycle].

This theorem had been conjectured by Grothendieck in the oral seminar of [SGA 5], under the assumption that X is excellent (which turned out to be superfluous). It was proven in [SGA 4 XVI] for X, Y smooth over a field, and in [SGA 4 XIX] for X of characteristic zero. Gabber's first proof of 6.1, in 1994 ([Ga 1], see [F 3] for a written account) used techniques of algebraic K-theory developed by Thomason (who had himself proved 6.1 in 1984 [Th] under some restrictive hypotheses). Gabber's new proof [Ga 3] no longer uses K-theory (see [R1]).

Corollary 6.2. Let X and Λ be as in 6.1, and let Y be a closed, regular subscheme of X, of pure codimension d. Then :

$$\mathcal{H}^q_Y(\Lambda) = \begin{cases} 0 & \text{if } q \neq 2d \\ \Lambda_Y(-d) & \text{if } q = 2d \end{cases}$$

The isomorphism $\Lambda_Y \xrightarrow{\sim} \mathcal{H}^{2d}_Y(\Lambda)(d)$ sends the unit section to the section $c(Y) \in H^0(Y, \mathcal{H}^{2d}_Y(\Lambda)(d))$, which is characterized by the property that, if Y is the transverse intersection of regular divisors Y_i $(1 \leq i \leq d)$, $c(Y)$ is the product of the classes $c(Y_i)$ of 2.1.

Recall that on a regular noetherian scheme X, a divisor D is called a *simple (or strict) normal crossings divisor* if D is the sum of a locally finite family $(D_i)_{i\in I}$ of regular divisors crossing transversally, i. e. such that, at each point x of the support of D, if $J(x)$ is the (finite) set of indices $i \in I$ such that x belongs to the support of D_i, then, for all subset J of $J(x)$, the intersection of the D_i's for $i \in J$ is, locally around x, a regular closed subscheme of X of codimension $|J|$. A divisor D is called a *normal crossings divisor* if, étale locally, D is a strict normal crossings divisor.

Corollary 6.3. Let X and Λ be as in 6.1, and let $D = \sum_{i\in I} D_i$ be a strict normal crossings divisor on X. Let $j : U = X - D \to X$ be the inclusion. Then :

$$R^qj_*\Lambda = \begin{cases} \Lambda & \text{if } q = 0 \\ \oplus\Lambda_{D_i}(-1) & \text{if } q = 1 \\ \Lambda^q R^1j_*\Lambda & \text{if } q \geq 1 \end{cases}$$

The isomorphism $\oplus\Lambda_{D_i} \xrightarrow{\sim} R^1j_*\Lambda(1)$ sends the unit section of Λ_{D_i} to the image by restriction of $c(D_i) \in \Gamma(X, R^1j_{i*}\Lambda(1))$ into $\Gamma(X, R^1j_*\Lambda(1))$, where

$j_i : X - D_i \to X$ is the inclusion. The isomorphism $\Lambda^q R^1 j_* \Lambda \xrightarrow{\sim} R^q j_* \Lambda$ is induced by the cup-product. It defines an isomorphism

$$R^q j_* \Lambda = \oplus \Lambda_{D_J}(-|J|),$$

where J runs through the subsets of I with q elements and D_J is the intersection of the D_i's for $i \in J$. It is formal to deduce 6.3 from 6.2, see [I 3].

The K-theoretic free proof of 6.1 given by Gabber in [Ga 3] uses some arguments of the original proof [F 3] together with some new, delicate techniques of canonical desingularization and logarithmic geometry, see [Mo] and [R] for some details.

6.4. Recall that a ring A is called *quasi-excellent* if A is noetherian and satisfies the following conditions :

(i) for all $x \in X = \operatorname{Spec} A$, the fibers $f^{-1}(y)$ of the canonical morphism $f : \operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ are geometrically regular, i. e. are regular and remain so after any finite extension of $k(y)$.

(ii) for any A-algebra of finite type A', the set of regular points of $\operatorname{Spec} A'$ is open.

When A is local, (i) implies (ii) [EGA IV 7.8.3 (i)]. A ring A is called *excellent* if it is quasi-excellent and moreover, is *universally catenary*, i. e. any A-algebra of finite type A' is catenary (i. e. the chain condition on codimensions for chains of irreducible closed subsets $X_1 \subset X_2 \subset X_3$ of $\operatorname{Spec} A'$ is satisfied).

A scheme X is called *quasi-excellent* (resp. *excellent*) if it admits an open cover by the spectra of quasi-excellent (resp. excellent) rings. For example, the spectrum of a complete noetherian local ring, a scheme of finite type over a Dedeking ring of mixed characteristics is excellent. Any scheme locally of finite type over a quasi-excellent (resp. excellent) scheme is itself quasi-excellent (resp. excellent). See [EGA IV 7.8] for details (the terminology *quasi-excellent* cannot be found in (*loc.cit.*) but seems to have become standard).

Theorem 6.5. (Gabber) Let Y be a noetherian, quasi-excellent scheme and let $f : X \to Y$ be a morphism of finite type. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$, with $n \geq 1$ invertible on Y. Let F be a constructible sheaf of Λ-modules on X. Then :

(a) $R^q f_* F$ is constructible for all q,

(b) there exists an integer N such that $R^q f_* F = 0$ for $q \geq N$.

The conjunction of (a) and (b) is equivalent to saying that Rf_* sends $D^b_c(X, \Lambda)$ into $D^b_c(Y, \Lambda)$. Recall (1.8, 1.9) that, if f is proper, (a) and (b) hold even if n is not invertible on Y, and that (a) and (b) were proven by

Artin for Y of characteristic zero [SGA 4 XIX], or, by Deligne [SGA 4 1/2 Th. finitude] if f is a morphism of S-schemes of finite type, with S noetherian, regular of dimension ≤ 1 (but not necessarily quasi-excellent) it's possible, however, to reduce the non quasi-excellent case to the quasi-excellent one (Gabber)). Besides, Gabber has shown that the constructibility of f_*F holds even if Y is not quasi-excellent, but he has constructed a counter-example in the non quasi-excellent case for $q = 1$ (compare with the analogous facts concerning base change for a proper morphism, 1.4, 1.5). He has also proven an analogue of 6.5 in the non abelian setting [Ga 2].

Gabber gave two proofs of 6.5 (a). Both rely on the absolute purity theorem and use uniformization theorems that we will discuss in the next section. An outline will be given in section 8. The second proof gives (a) and (b) at the same time. It uses some of the techniques alluded to above for the proof of the purity theorem.

The same ingredients (absolute purity, uniformization) are essential in the proofs of the next results in 6.6 and 6.7.

6.6. *Affine Lefschetz.* A classical theorem of Artin and Grothendieck [SGA 4 XIV] asserts that if $f : X \to Y$ is an *affine* morphism between schemes of finite type over a field k and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n prime to the characteristic of k, then for any sheaf F of Λ-modules on X and all $q \in \mathbb{Z}$, one has $\dim \operatorname{supp}(R^q f_* F) \leq \dim \operatorname{supp}(F) - q$.

Gabber generalized this to quasi-excellent schemes as follows :

Theorem 6.6.1. (Gabber [Ga 3]) Let $f : X \to Y$ be an *affine* morphism of finite type between noetherian, quasi-excellent schemes. Assume Y admits a dimension function δ_Y and let δ_X be the corresponding dimension function on X, defined by $\delta_X(x) = \delta_Y(f(x)) + \mathrm{tr.deg}(k(x)/k(f(x)))$. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on Y. Then for any sheaf F of Λ-modules on X and all $q \in \mathbb{Z}$, one has $\delta_Y(R^q f_* F) \leq \delta_X(F) - q$.

Here by a *dimension function* on Y we mean a function $\delta : Y \to \mathbb{Z}$ such that, for any immediate specialization $\overline{y} \to \overline{x}$ of geometric points above $y \to x$, $\delta(x) = \delta(y) - 1$. Such a dimension function exists étale locally. If Y is a closed irreducible subscheme of a regular scheme, then $y \mapsto -\dim \mathcal{O}_{Y,y}$ is a dimension function. By $\delta(G)$ in 6.6.1 we mean $\sup_{G_x \neq 0} \delta(x)$.

It follows from 6.6.1 that if X is the spectrum of a strictly local, noetherian, integral, excellent ring A with fraction field K, then for any prime ℓ invertible on X, one has $\mathrm{cd}_\ell(K) = \dim(X)$. This was conjectured by Artin in [SGA 4 X]. A variant for $\ell = \mathrm{char}(k)$, taking into account the logarithmic differentials of the residue field, has recently been proven by Gabber and Orgogozo, solving a conjecture of Kato. We will discuss this briefly in 7.4.

6.7. *Dualizing complexes.* In [SGA 5 I] Grothendieck conjectured that if X is a regular, noetherian, excellent scheme of finite Krull dimension, and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on X, then Λ_X is a dualizing complex.

By a *dualizing complex* on X we mean an object K of $D_c^b(X, \Lambda)$ such that there exists an integer N for which $\mathcal{E}xt^i(F, K) = 0$ for all $i \geq N$ and all constructible sheaves F of Λ-modules on X, the functor $D = R\mathcal{H}om(-, K)$ sends $D_c^b(X, \Lambda)$ into itself and the biduality map $F \to DDF$ is an isomorphism for any $F \in D_c^b(X, \Lambda)$.

Grothendieck's conjecture was proven in [SGA 5 I] for X of dimension ≤ 1, or of characteristic zero (using Hironaka's resolution). Gabber proved the conjecture in general (see [R2]). He also proved the existence of dualizing complexes on any noetherian, excellent scheme admitting a dimension function and of finite Krull dimension. This existence had been proven by Deligne in [SGA 4 1/2, Th. finitude] for X of finite type over a regular scheme of dimension ≤ 1 (no further assumption is needed in this case).

7. Gabber's uniformization theorems

7.1. Let X be a scheme. Denote by (pf/X) the category of schemes locally of finite presentation over X. The *pspf-topology* (pspf for "propre surjectif, de présentation finie") on (pf/X) (or on X, for short) is the topology generated by covering families of the following types : (i) $T' \to T$ proper surjective, of finite presentation (ii) open Zariski covers $(T_i \to T)_{i \in I}$. The corresponding site is called the *pspf-site* of X and denoted X_{pspf}. This topology is close to that considered by Suslin-Voevodsky [SV, 10], called h-topology. It coincides with it when X is noetherian [GL]. The pspf-topology on (pf/X) is finer than the étale topology.

Theorem 7.2. (Gabber [Ga 3]) Let X be a noetherian quasi-excellent scheme and Y a nowhere dense closed subset. There exists a finite family of finite type morphisms $(f_i : X_i \to X)_{i \in I}$ having the following properties :

(i) the family (f_i) is covering for the pspf topology ;

(ii) each X_i is regular and connected ;

(iii) for each $i \in I$, $Y_i := f_i^{-1}(Y)$ is either empty or the inclusion of the support of a divisor with strict normal crossings ;

(iv) for each $i \in I$, f_i is quasi-finite over a dense open subset of X, and if η_i is the generic point of X_i, $f_i(\eta_i)$ is a maximal point of X.

In particular, a pair $(Y \subset X)$ as above locally for the pspf topology looks like the inclusion of the support of a divisor with strict normal crossings (or of the empty space) in a regular, noetherian (quasi-excellent) scheme. For this reason, one can call 7.2 a *local uniformization* theorem. This is a (weak) form of resolution à la Hironaka. This is also a *local* generalization of part of de Jong's theorem [dJ 1, 8.2], namely the statement that if X is an integral scheme, separated, of finite type and flat over the spectrum S of a (non-necessarily excellent) Dedeking ring, and Y is a nonempty proper closed subset of X, then, after a suitable finite extension of S, there exists an *alteration* $f : X' \to X$, with X' regular and $f^{-1}(Y)$ the support of a strict normal crossings divisor. Note that, in 7.2, the morphism $f : \coprod X_i \to X$ sum of the f_i's is not necessarily proper. Using Gruson-Raynaud's flattening theorem [GR] one can show that the property for the family (f_i) to be covering for the h-topology is equivalent to the fact that f acquires a section after a base change of the form $g = g_1 g_2 g_3 g_4$, where g_1 is a closed nilpotent immersion, g_2 a modification, g_1 a finite flat map, g_1 a finite Zariski cover.

7.3. The main steps of the proof are the following :

(1) *One may assume X local henselian.*

This is the "easy" part : the statement is local for the étale topology, quasi-excellency is preserved by henselization [EGA IV 18.7.6], and by passing to the limit arguments, data (f_i) over the henselization of X at a point x can be descended to an étale neighborhood of x.

Therefore, one may proceed by induction on the integer d, assuming the theorem established for all X as in 7.2 which are *local* of dimension $< d$ (hence for any X as in 7.2 which is of finite dimension $< d$).

(2) *One may assume X complete, local* (of dimension $\leq d$).

Let $\hat{X}$ be the completion of the local henselian scheme X at its closed point. Having found morphisms $(f_i : X'_i \to \hat{X})_{i \in I}$ as in 7.2 for $\hat{X}$, one has to descend them to X, preserving their properties. Let $h : \hat{X} \to X$ be the canonical morphism. Choose a filtering projective system of affine morphisms $h_\alpha : X_\alpha \to X$ of finite type such that $h = \operatorname{inv.lim} h_\alpha$. By *Popescu's theorem* [Po 1, 1.3] (see also [Po 2], and [Sw] for an independent exposition of the proof of the main theorem in [Po 1]), for each α and any integer $n \geq 0$, h_α has a *section* $s_{\alpha,n} : X \to X_\alpha$, whose restriction to the n^{th}-infinitesimal neighborhood X_n of the closed point of X coincides with the restriction of $\hat{X} \to X_\alpha$ to X_n. The morphisms f_i can be descended to some X_α, and then pulled back to X via $s_{\alpha,n}$. The problem is to show that, for suitable α, n, the morphisms thus obtained satisfy the properties required in 7.2. This is delicate. The proof relies on a new technique of formal approximation developed by Gabber [Ga 3].

(3) *The conclusion of 7.2 holds for X complete, local, of dimension $\leq d$* (under the induction assumption made above).

This is the longest part of the proof. There are three main steps.

(a) The starting point is a refinement of a theorem of Cohen [EGA 0_{IV} 19.8.8 (ii)] :

Theorem 7.3.1. (Gabber [Ga 2, 8.1]) Let $Z = \operatorname{Spec} A$ be a noetherian, complete, local, reduced scheme, equidimensional of dimension r, and of equicharacteristics. There exists a formal power series ring of the form $B = k[[t_1, \cdots, t_r]]$, where k is isomorphic to the residue field of A, and a local morphism $g : Z \to T = \operatorname{Spec} B$, such that A is finite over B, torsionfree and generically étale.

The improvement on *loc.cit.* is that X is not assumed integral and, for Z of characteristic $p > 0$, that g is generically étale. The proof is in the spirit of that of Nagata's jacobian criterion [EGA 0_{IV}22.7].

(b) The next step is a fibration theorem :

Theorem 7.3.2. (Gabber [Ga 3]) Let X be a noetherian, complete, normal, local scheme of dimension $d \geq 2$, and Y a proper closed subscheme. There exists :

(i) a noetherian, complete, regular, local scheme S of dimension $d-1$

(ii) a noetherian, normal, local scheme X_1, a local morphism $f : X_1 \to S$, which is essentially of finite type and of relative dimension 1, and a proper closed subscheme Y_1 of X_1

(iii) a local, finite and surjective morphism $g : \hat{X}_1 \to X$, where $\hat{X}_1$ is the completion of X_1 at its closed point, such that the pull-back of Y by g is the completion of Y_1 at its closed point.

To deduce 7.3.2 from 7.3.1 one proceeds as follows.

Suppose first that X is of equicharacteristics. Applying 7.3.1 to $Z = X$, one finds a local, finite, and generically étale morphism $h : X \to S[[t_d]]$, where $S = \operatorname{Spec} R$, $R = k[[t_1, \cdots, t_{d-1}]]$. Using the Weierstrass preparation theorem, one may assume, up to a change of coordinates, that h is étale above the complement of $V(P)$ where $P \in R[t_d]$ is a unitary polynomial, vanishing at the closed point x_0 of $S[[t_d]]$. One then applies Elkik's algebraization theorem [El, th. 5 p. 577] to the morphism h and the henselian pair cut off by $(S[t_d], V(P))$ at the henselization of $S[t_d]$ at x_0. Additional work is required to algebraize Y. The morphism obtained from h by algebraization descends to a morphism $h' : X' \to T'$, where T' is an étale neighborhood of x_0 in $S[t_d]$, such that the composition $X' \to T' \to S$ yields (by localization) the desired morphism f.

Suppose now that the closed point x of X is of characteristic $p > 0$ and the generic point of X is of characteristic zero. Then X is above $\operatorname{Spec} \mathbb{Z}_p$. Assume first that the special fiber $X_0 = V(p)$ is reduced. Then, applying 7.3.1 to $Z = X_0$, one finds a finite, local, generically étale morphism h_0 :

$X_0 \to \operatorname{Spec} k[[t_1, \cdots, t_{d-1}]]$ (with k isomorphic to $k(x)$), which one lifts to $h : X \to \operatorname{Spec} W[[t_1, \cdots, t_{d-1}]]$, where W is a Cohen ring for k. One then proceeds as above, this time taking $S = \operatorname{Spec} R$, $R = W[[t_1, \cdots, t_{d-2}]]$. If the special fiber X_0 is not reduced, one uses a theorem of Epp [Ep] to show that up to replacing $X \to \operatorname{Spec} \mathbb{Z}_p$ by $X' \to \operatorname{Spec} W'$, where X' is finite over X and W' is a finite extension of the Witt ring of a suitable perfect subfield of $k(x)$, the special fiber X'_0 of $X' \to \operatorname{Spec} W'$ is reduced. One then continues as before, with X replaced by X' and X_0 by X'_0. (c) The last step uses techniques of logarithmic geometry and (a particular case of) de Jong's theorem on alterations and nodal curves, namely :

Theorem 7.3.3. (de Jong) [dJ 2, 2.4] Let $f : X \to S$ be a proper morphism of integral excellent schemes, whose generic fiber X_η is smooth, geometrically irreducible, and of dimension 1, and let $Y \subset X$ be a proper closed reduced subscheme such that Y_η is étale. Then there exists a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

where X', S' are integral, g and h are alterations, f' is a proper nodal curve, and the components of $h^{-1}(Y)$ dominating S' are disjoints sections of f' contained in the smooth locus of f'.

Using 7.3.2 one may assume that there exist a local, normal scheme X_1, with $\hat{X}_1 = X$, a local, essentially of finite type morphism $f : X_1 \to S$, of relative dimension 1, with S local, complete, regular and of dimension $d-1$, and a closed reduced subscheme Y_1 of X_1 such that $\hat{Y}_1 = Y$. (Note that X_1 is excellent hence $\hat{Y}_1$ is reduced.) It is enough to construct a pspf covering (f_i) of X_1 satisfying the properties of 7.2 with respect to Y_1. Indeed, as X_1 is excellent, they will be preserved by pull-back to $\widehat{X}_1 (= X)$. Up to replacing X_1 by some affine model whose X_1 is the localization and base changing by a finite normal extension of S, one may assume that f is of finite type, the generic fiber of f is smooth and that the components of Y_1 dominating S are generically étale. One then applies 7.3.3 : after alterating S and X_1, one obtains X' and S', excellent integral, with S' of dimension $\leq d-1$, and a nodal curve $f' : X' \to S'$, such that the components of the inverse image Y' of Y in X' dominating S' are étale and contained in the smooth locus of f'. Applying the induction assumption to S' (and suitable proper closed subsets), one can assume that S' is regular, f' smooth outside a divisor with strict normal crossings T' and $Y' = D \cup E$, where D is a divisor contained in the smooth locus of f' and étale over S', and E the support of the inverse image of a union of components of T'. By the local structure theorem for nodal curves [dJ 1, 2.23, 3.3], it turns out that for the log structure on X' defined by Y' (push-out of the trivial one on the complement), X' is *log regular* [K 2, 2.1]. Finally, by Kato's log desingularization theorem [K 2, 10.4], a modification of X' makes X' regular and Y' a strict normal crossings divisor.

7.4. *Application of 7.3.2 to the p-dimension of fields.*

Let k be a field and p a prime number. One defines the *p-dimension* (or cohomological p-dimension) of k,

$$\dim_p(k), \tag{7.4.1}$$

in the following way. If $\mathrm{char}(k) \neq p$, $\dim_p(k)$ is the cohomological dimension of $\mathrm{Spec}\, k$ for the étale topology and p-torsion coefficients. If $\mathrm{char}(k) = p$, one defines the *p-rank* of k, $p - \mathrm{rk}(k)$, as the dimension of $\Omega^1_{k/\mathbb{F}_p}$ over k. Let $r = p - \mathrm{rk}(k)$. If $r = \infty$, then $\dim_p(k) = \infty$. Suppose that $r < \infty$. If, for all finite extensions k' of k, $H^1(\mathrm{Spec}\, k', \Omega^r_{k'/\mathbb{F}_p,\log}) = 0$, then $\dim_p(k) = r$. If not, then $\dim_p(k) = r + 1$. Here $\Omega^i_{k'/\mathbb{F}_p,\log}$ is the abelian subsheaf of $\Omega^i_{k'/\mathbb{F}_p}$ generated étale locally by the logarithmic differentials, or, equivalently, the kernel of $1 - C^{-1} : \Omega^i \to \Omega^i/d\Omega^{i-1}$, where C^{-1} is the Cartier isomorphism. Thus

$$H^1(\mathrm{Spec}\, k', \Omega^r_{k'/\mathbb{F}_p,\log}) = \mathrm{Coker}(1 - C^{-1} : \Omega^r_{k'/\mathbb{F}_p} \to \Omega^r_{k'/\mathbb{F}_p}/d\Omega^{r-1}_{k'/\mathbb{F}_p}).$$

The following result, conjectured by Kato, was recently proved by Orgogozo :

Theorem 7.4.2. [O 2] Let A be a local, noetherian, henselian, excellent, integral ring, with residue field k of characteristic $p > 0$ and fraction field K. Then

$$\dim_p(K) = \dim_p(k) + \dim(A). \tag{7.4.2.1}$$

Kato had proven 7.4.2 for A a discrete valuation ring. Orgogozo's proof uses this particular case, and relies in an essential way on Gabber's fibration theorem 7.3.2.

If A satisfies the assumptions of 7.4.2 but its residue field k is of characteristic $p' \neq p$, and, if $p = 2$, k cannot be ordered, then (7.4.2.1) still holds, as a consequence of [SGA 4 X 2.4] and a recent affine Lefschetz theorem of Gabber [Ga 3], see [PS] for details.

8. On the proof of the constructibility theorem

The purpose of this section is to give an outline of the proof of part (a) of Gabber's theorem 6.5.

If $j : U \to Z$ is the inclusion of the complement of a divisor with strict normal crossings in a regular scheme Z, then, for all q, $R^q j_* \Lambda$ is constructible by corollary 6.3 of Gabber's absolute purity theorem 6.1. The general idea is to reduce to this case using Gabber's uniformization theorem 7.2 and cohomological descent. There is, however, a serious difficulty in implementing this method, arising from the fact that the morphisms f_i occuring in 7.2 are not necessarily proper. In particular, if $j : U \to Z$ is as above, $f : Z \to Y$ is a member of a pspf covering family of type 7.2, and G is a constructible Λ-module on Z, such as $R^q j_* \Lambda$, we do not know whether the sheaves $R^i f_* G$ are constructible. To deal with this issue the strategy is to use noetherian induction and Deligne's *generic constructibility* theorem :

Theorem 8.1. (Deligne) [SGA 4 1/2, Finitude, 1.9] Let S be a noetherian scheme, $f : X \to Y$ a morphism of S-schemes of finite type, $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on S, and F a constructible Λ-module on X. Then there is a dense open subset U of S such that above U the $R^q f_* F$ are constructible (and zero except for a finite number of them), and their formation commutes with any base change $S' \to U \subset S$.

One wants to apply this to the morphism $f : Z \to X$ above. A second problem appears here. To show constructibility of $R^i f_* G$, by [SGA 4 IX 2.4 (v)] it is enough to show that for any irreducible closed subscheme $g : X' \to X$ of X, there is a nonempty open subset U' of X' such that the restriction of $g^* R^i f_* G$ to U' is locally constant of finite type (or, equivalently, constructible). The problem is that, as f is not necessarily proper, $R^i f_*$ does not necessarily commute with base change, in particular the base change map $g^* R^i f_* G \to R^i f'_* G'$ is non necessarily an isomorphism, where $f' : Z' \to X'$ is deduced from f by base change by g and G' is the inverse image of G on X'. Therefore, applying 8.1 to (f', G') yields no information on $g^* R^i f_* G$. However, when one uses cohomological descent, one is not interested in a single pspf covering family $f_i : Z_i \to X$ but on pspf *hypercoverings* $\varepsilon. : Z. \to X$. For these, the situation is better. First of all, we have a general result of cohomological descent :

Theorem 8.2. [SGA 4 Vbis] Let X be a scheme and $\varepsilon_. : X_. \to X$ be a hypercovering for the pspf topology. Let Λ be a torsion ring. Then, for any $F \in D^+(X, \Lambda)$, the adjunction map

$$F \to R\varepsilon_{.*}\varepsilon_.^* F$$

is an isomorphism.

Recall that the assumption that $\varepsilon_.$ is a pspf hypercovering means that, for all $n \in \mathbb{N}$, the natural map $X_n \to \mathrm{cosk}_{n-1}(X_{\leq n-1})_n$ is covering for the pspf topology, where, for $n = 0$, $\mathrm{cosk}_{n-1}(X_{\leq n-1})_n = X$ and the natural map is ε_0. As proper surjective morphisms and Zariski open covers are both of universal 1-cohomological descent, the same is true by [SGA 4 Vbis 3.3.1] for coverings for the pspf topology, and 8.2 follows from [SGA 4 Vbis 3.3.3].

Now a key technical point is that, in the situation of 8.2, for *certain* complexes $G. \in D^+(X_., \Lambda)$, the formation of $R\varepsilon_{.*}G_.$ commutes with base change $X' \to X$. Namely, we have the following result, which can be thought of as a theorem of *refined cohomological descent* (or cohomological descent *with base change*) :

Theorem 8.3. (Gabber) Let $\varepsilon_. : X. \to X$ and Λ be as in 8.2, with X noetherian. In addition, let $f : U \to X$ and $g : X' \to X$ be morphisms of finite type. Let $f_. : U_. \to X_.$ be the pull-back of f by $\varepsilon_.$, Let $F \in D^+(U, \Lambda)$, $F_.$ its inverse image on $U_.$ and $G_. := Rf_{.*}F_. \in D^+(X_., \Lambda)$. Consider the cartesian square

(8.3.1)
$$\begin{tikzcd} X'_. \arrow[r, "g."] \arrow[d, "\varepsilon'_."] & X_. \arrow[d, "\varepsilon."] \\ X' \arrow[r, "g"] & X \end{tikzcd} .$$

Then the base change map

$$g^*R\varepsilon_{.*}G_. \to R\varepsilon'_{.*}g_.^*G_.$$

associated with (8.3.1) is an isomorphism.

8.4. We will sketch the proof of 8.3 in 8.5. Let us first show how, putting together the absolute purity theorem, the uniformization theorem, and 8.2, 8.3, one can prove the constructibility theorem. To prove that for any $f : X \to Y$ and F as in 6.5, the direct images R^qf_*F are constructible, it suffices to show that for any *open and dense immersion* $f : U \to X$, with X noetherian quasi-excellent, the direct images $R^qf_*\Lambda$ are constructible. This follows from standard dévissages. By Deligne's step by step procedure [SGA 4 V^{bis}, 5.1], using 7.2, one can construct a pspf hypercovering

$$\varepsilon_. : X_. \to X$$

such that X_n is a regular scheme for all n and if we form the cartesian square

(8.4.1)
$$\begin{tikzcd} U_. \arrow[r, "f."] \arrow[d, "\eta."] & X_. \arrow[d, "\varepsilon."] \\ U \arrow[r, "f"] & X \end{tikzcd} ,$$

each f_n is, on any connected component of X_n, either an isomorphism or the inclusion of the complement of a strict normal crossings divisor. By 8.2, we have

$$\Lambda_U = R\eta_{.*}\Lambda_{U_.},$$

hence

$$Rf_*\Lambda = R\varepsilon_{.*}(Rf_{.*}\Lambda).$$

Let $G_. := Rf_{.*}\Lambda$. By 6.3, we know that $G_. \in D^+_c(X_., \Lambda)$, i. e. each $G_n \in D^+(X_n, \Lambda)$ has constructible cohomology sheaves. In order to prove that the sheaves $L_q := R^q\varepsilon_{.*}G_.$ are constructible, by the criterion recalled above, we have to show that for any closed irreducible subscheme $g : X' \to X$, there is a nonempty open subset V of X' such that the restriction of g^*L_q to V is locally constant of finite type. Consider the corresponding cartesian square (8.3.1), and let $G'_. = g_.^*G_.$, which belongs to $D^+_c(X'_., \Lambda)$. By 8.3, we have

$$g^*L_q \xrightarrow{\sim} R^q\varepsilon'_{.*}G'_..$$

By generic constructibility (8.1) we know that, for each (i, j), there is a dense open subset V_{ij} of X' such that the restriction of $R^j\varepsilon'_{i*}G'_i$ to V_{ij} is locally constant of finite type. The conclusion then follows from the spectral sequence

$$E_1^{ij} : R^j\varepsilon'_{i*}G'_i \Rightarrow R^{i+j}\varepsilon'_{.*}G'_..$$

8.5. Let us now sketch the proof of 8.3. Consider the cartesian square (8.4.1) defined by f and $\varepsilon_.$.

Here is a naive attempt to prove 8.3. Let $\eta'_. : U'_. \to U'$ be the fiber product of $\varepsilon'_.$ and $\eta_.$ over $\varepsilon_.$, so that we get a cube with cartesian faces. The bottom horizontal face, say, is

(1)
$$\begin{tikzcd} U' \arrow[r, "h"] \arrow[d, "f'"] & U \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd},$$

and the top one

(2)
$$\begin{tikzcd} U'_. \arrow[r, "h_."] \arrow[d, "f'_."] & U_. \arrow[d, "f_."] \\ X'_. \arrow[r, "g_."] & X_. \end{tikzcd},$$

with vertical maps from (2) to (1) given by $\varepsilon_.$, $\varepsilon'_.$, $\eta_.$, $\eta'_.$. As $\eta'_.$ is deduced by base change from $\varepsilon_.$, $\eta'_.$ is a pspf hypercovering, and cohomological descent in the form of 8.2 holds for $\eta'_.$. On the other hand, make the following assumption :

(*) *base change by g (resp. $g_.$) holds in (1) (resp. (2)) for Rf_*F (resp. $Rf_{.*}F_.$) where $F_. = \eta_.^*F$.*

Then the conclusion of 8.3 follows formally. Indeed, modulo trivial compatibilities between base change maps, we have :

$$\begin{aligned} g^*R\varepsilon_{.*}Rf_{.*}F_. &= g^*Rf_*F \quad \text{(descent for } \eta_.) \\ &= Rf'_*h^*F \quad \text{(base change in (1))} \\ &= Rf'_{.*}R\eta'_{.*}\eta'^*_.h^*F \quad \text{(descent for } \eta'_.) \\ &= R\varepsilon'_{.*}Rf'_{.*}h_.^*F_. \quad \text{(transitivity)} \\ &= R\varepsilon'_{.*}g_.^*Rf_{.*}F_. \quad \text{(base change in (2)).} \end{aligned}$$

Unfortunately, assumption (*) does not hold in general, as the case where g is a closed immersion and f the inclusion of the complement already shows. To circumvent this difficulty, one replaces the fiber products U' and $U'_.$ by the *oriented products*

$$\overleftarrow{U'} = X' \overleftarrow{\times}_X U \quad , \quad \overleftarrow{U'_.} = X'_. \overleftarrow{\times}_{X_.} U_.,$$

and form the analogous cube, with squares (1)', (2)', and $\eta'_.$ defined by the oriented product

$$\overleftarrow{\eta'_.} = \varepsilon'_. \overleftarrow{\times}_{\varepsilon_.} \eta_..$$

Then the analogue (*)' of (*) becomes true, and is essentially tautological. Cohomological descent for $\overleftarrow{\eta'_.}$ and h^*F also holds, but is more delicate. Granted this, the proof follows the same formal lines as before. To conclude the proof of 8.3 it suffices therefore to establish the following general results 8.6, 8.7.

The first one is an *oriented base change* theorem :

Theorem 8.6. (Gabber-Orgogozo) [I 6, 3.7] Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes. Assume that f is coherent, i. e. quasi-compact and quasi-separated. Let $T = Y \overleftarrow{\times}_S X$ be the oriented product of the corresponding étale toposes, with projections $p_1 : T \to Y$, $p_2 : T \to X$, and canonical map $\tau : fp_2 \to gp_1$ (4.1). Let Λ be a ring. For any $F \in D^+(X, \Lambda)$, the base change map (in $D^+(X, \Lambda)$)

$$g^*Rf_*F \to Rp_{1*}p_2^*F \tag{8.6.1}$$

associated with τ is an isomorphism.

As we claimed in 4.2, when g is a closed immersion and f is the complementary open immersion, the usual fiber product $Y \times_S X$ is empty, but T is not, and in fact plays the role of a *punctured tubular neighborhood* of Y in S. The analogous formula in differential geometry, for Y a closed submanifold of a manifold S, $X = S - Y$, T a suitable punctured tubular neighborhood of Y in S, and F locally constant near Y, is immediate. The proof of 8.6 is not much more difficult. For example, if g is the inclusion of a closed point y such that $k(y)$ is separably closed, and f the complementary open immersion, then T is the *punctured Milnor ball* $S_{(y)} - \{y\}$, and the conclusion of 8.6 follows from the calculation of the stalks of direct images by coherent maps : $Rf_*(F)_y = R\Gamma(S_{(y)} - \{y\}, F)$. In the general case, the main point is that for X, Y, S strictly local, and g a local map (but f not necessarily local), if t is the point of T defined by the closed points of X, Y, S, then T is a *local topos of center* t, i. e. for any sheaf G on T, the natural map $\Gamma(T, G) \to G_t$ is an isomorphism.

Gabber has generalized 8.6 to morphisms of toposes. On the other hand, for S noetherian, g a closed immersion, and f the complementary open immersion, one can view 8.6 as an essentially trivial analogue of a theorem of Fujiwara on rigid tubular neighborhoods [F 1, 6].

The second result is an *oriented cohomological descent* theorem :

Theorem 8.7. (Gabber) Let f, g, T be as in 8.6. Let $\varepsilon_{.} : S_{.} \to S$ be a pspf hypercovering. Let $\eta_{.} : T_{.} \to T$ be the augmented simplicial topos defined by base change by $\varepsilon_{.}$, i. e.

$$T_{.} = Y_{.} \overset{\leftarrow}{\times}_{S_{.}} X_{.},$$

where $X_{.}$ (resp. $Y_{.}$) is deduced from X (resp. Y) by base change by $\varepsilon_{.}$. Let Λ be a torsion ring. Then, for any $F \in D^+(T, \Lambda)$, the adjunction morphism

$$F \to R\eta_{.*}\eta_{.}^*F$$

is an isomorphism.

See [O 3] for the proof. Here are the main steps.

(a) Suppose that X, S, Y are strictly local, and g local, so that, as was observed above, T is local. Then one shows that $p_2 : T \to X$ has a unique section s sending the closed point of X to that of T, and that, moreover, for any Λ-module F on T, the natural map $p_{2*}F = s^*p_2^*p_{2*}F \to s^*F$ is an isomorphism.

(b) Let $\xi_{.} : X. \to X$ be a pspf hypercovering and $\theta_{.} : T. \to T$ the augmented simplicial topos deduced from $\xi_{.}$ by base change. Then for $F \in D^+(T, \Lambda)$, the adjunction map $F \to R\theta_{.*}\theta_{.}^*F$ is an isomorphism. To show this, using (a) and the proper base change theorem one reduces to the case where $F = p_2^* E$, for $E \in D^+(X, \Lambda)$. The conclusion then follows from 8.2.

(c) (*the key point*) Assume that $f : X \to S$ has a factorization

$$X \xrightarrow{f'} S' \xrightarrow{a} S ,$$

with *a proper*. Let $Y' = Y \times_S S'$, $T' = Y' \overleftarrow{\times}_{S'} X$, and

$$h : T' \to T$$

be the map induced by a. Then, for any $F \in D^+(T, \Lambda)$, the adjunction map

$$F \to Rh_* h^* F$$

is an isomorphism.

Note that (c) is *not a cohomological descent assertion*. When S is noetherian, $g : Y \to S$ (resp. $g' : Y' \to S'$) is a closed immersion and $f : X \to S$ (resp. $f' : X \to S'$) the complementary open immersion, so that a induces an isomorphism over X, one can think of (c) as a cohomological equivalent of the (built-in) invariance of Fujiwara's punctured rigid tubular neighborhood (of Y in S) under an admissible modification of S. In fact, Gabber's original approach to 8.6 was via Fujiwara's tubular neighborhoods instead of oriented products. The proof of (c) is easy, using the proper base change theorem and 8.6.

(d) Finally, to prove 8.7, let us factorize $\eta_. : T_. \to T$ into

$$T_. = Y_. \overleftarrow{\times}_{S_.} X_. \xrightarrow{\alpha_.} Y \overleftarrow{\times}_S X_. \xrightarrow{\theta_.} T = Y \overleftarrow{\times}_S X ,$$

where $\theta_.$ is deduced from the projection $X_. \to X$ as in (b), and $\alpha_.$ given by the $\alpha_n : Y_n \overleftarrow{\times}_{S_n} X_n \to Y \overleftarrow{\times}_S X_n$ defined by the projections $S_n \to S$ and $Y_n \to Y$ as in (c). The adjunction map relative to $\theta_.$ is an isomorphism by (b), while that relative to $\alpha_.$ is an isomorphism by (c) (applied to each α_n).

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