

PURELY INSEPARABLE GALOIS THEORY I: THE FUNDAMENTAL THEOREM

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ABSTRACT. We construct a Galois correspondence for finite purely inseparable field extensions F/K , generalising a classical result of Jacobson for extensions of exponent one (where $x^p \in K$ for all x).

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1. INTRODUCTION

A finite extension of fields F/K of characteristic p is *purely inseparable* if for each $x \in F$, there is some n such that $x^{p^n} \in K$. Any finite field extension F/k can be broken down into a separable part K/k and a purely inseparable part F/K . While the intermediate extensions of K/k can be understood using classical Galois theory [Gal46], the situation is more subtle for the purely inseparable extension F/K . For instance if F/K is purely inseparable, the classical Galois group satisfies $\text{Gal}(F/K) = 0$.

If F/K has exponent one, that is, if x^p belongs to K for all $x \in F$, then Jacobson [Jac44] classified intermediate extensions of F/K in terms of the restricted Lie algebra $\text{Der}_K(F)$ of derivations. Later, Sweedler [Swe68], Gerstenhaber–Zaromk [GZ70], and Chase [Cha71] extended Jacobson’s correspondence to modular subextensions of modular extensions, by using the higher derivations of Hasse and Schmidt [SH37].

In Theorem 1.11 of this article, we establish a Galois correspondence for general finite purely inseparable field extensions by using the methods of derived algebraic geometry.

1.1. Background. We begin by introducing the main objects of study:

Definition 1.1 (Purely inseparable extensions). Let F/K be a field extension in characteristic $p > 0$.

- (1) The extension is said to be *purely inseparable* if for any $x \in F$, there is i such that x^{p^i} belongs to K .
- (2) The extension has *exponent* n if $x^{p^n} \in K$ for all $x \in F$, and n is minimal with this property.

Given a finite purely inseparable field extension F/K , we can consider the F -vector space

$\text{Der}_K(F)$ of K -linear derivations of F , that is, of K -linear maps $D : F \rightarrow F$ satisfying

$$D(xy) = D(x)y + xD(y) \quad \text{for all } x, y \in F$$

for all $x, y \in F$. Observe that this forces $D(a) = 0$ for all $a \in K$.

The F -vector space $\text{Der}_K(F)$ is equipped with a K -bilinear Lie bracket

$$[-, -] : \text{Der}_K(F) \times \text{Der}_K(F) \rightarrow \text{Der}_K(F)$$

$$(D_1, D_2) \mapsto [D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

and a self map, called the restriction, which is given by

$$(-)^{[p]} : \text{Der}_K(F) \rightarrow \text{Der}_K(F)$$

$$D \mapsto D^{[p]} := \underbrace{D \circ \dots \circ D}_p.$$

Denoting by $\text{ad}(x)$ the adjoint representation $\text{Der}_K(F) \rightarrow \text{Der}_K(F)$, $y \mapsto [x, y]$, we have for all derivations $D_1, D_2 \in \text{Der}_K(F)$ and all scalars $\lambda \in K$:

- (1) $(\lambda D_1)^{[p]} = \lambda^p D_1^{[p]}$;
- (2) $\text{ad}(D_1^{[p]}) = \text{ad}(D_1)^p$;
- (3) $(D_1 + D_2)^{[p]} = D_1^{[p]} + \sum_{i=1}^{p-1} s_i(D_1, D_2) + D_2^{[p]}$

Here $s_i(D_1, D_2)$ is the coefficient of t^{i-1} in $\text{ad}(tD_1 + D_2)^{p-1}(D_1)$, and so is a linear combination of Lie brackets of D_1 and D_2 .

A Lie algebra with a self-map $(-)^{[p]}$ satisfying these three conditions on is called a *restricted Lie algebra*. For more details on restricted Lie algebras see [Jac41].

More generally, if $K \subset E \subset F$ is an intermediate field, we can consider the F -linear injection

$$(1) \quad \rho : \text{Der}_E(F) \hookrightarrow \text{Der}_K(F).$$

Again, $\text{Der}_E(F)$ carries a Lie bracket and a restriction satisfying the above axioms. The *anchor map* ρ is compatible with both bracket and restriction, and satisfies the following equation for all $\phi \in F$:

$$[X, \phi Y] = \phi[X, Y] + \rho(X)(\phi) \cdot Y$$

One might therefore think of $(\text{Der}_E(F) \hookrightarrow \text{Der}_K(F))$ as an F/K -*restricted Lie algebroid* whose anchor map happens to be injective.

If $x^p \in K$ for all elements $x \in F$, this construction induces the following correspondence [Jac44]:

Theorem 1.2 (Jacobson). *Let F/K be a finite purely inseparable field extension of exponent one. There is an inclusion-reversing bijection between F/K -restricted Lie algebroids with injective anchor map ($\mathfrak{g} \hookrightarrow \text{Der}_K(F)$) and intermediate field extensions $K \subset E \subset F$.*

The Lie algebroid corresponding to an intermediate field E is given by $(\text{Der}_E(F) \hookrightarrow \text{Der}_K(F))$.

The field corresponding to an algebroid $(\mathfrak{g} \hookrightarrow \text{Der}_K(F))$ is $\{x \in F \mid D(x) = 0 \text{ for } D \in \mathfrak{g}\} \subset F$.

Jacobson's theory extends from the generic fibre to give a geometric correspondence [RS76, Eke87]:

Theorem 1.3. *Let X be a normal variety over an algebraically closed field k of characteristic $p > 0$. Then there is an inclusion-reversing bijection between saturated subsheaves of \mathcal{T}_X which are closed under Lie bracket and p -powers (called foliations), and morphisms*

$$X \rightarrow Y$$

with Y normal factoring the absolute Frobenius $\text{Fr} : X \rightarrow X$.

This extension of Jacobson's Galois theory to varieties has led to many further geometric applications, for example [SB91], [She10], [DCF15], [Lan15], [PW], and [JW19].

A major drawback of Jacobson's theory is that it can only be applied to exponent one extensions. Indeed, if F/K has exponent larger than one, the assignment $E \mapsto (\text{Der}_E(F) \hookrightarrow \text{Der}_K(F))$ can no longer distinguish between all intermediate fields, and Jacobson's correspondence breaks down:

Example 1.4. Let F be a field of characteristic p . Then $\text{Der}_{F^{p^n}}(F) = \text{Der}_{F^p}(F) = \text{Der}_{\mathbb{F}_p}(F)$, as for any derivation $D \in \text{Der}_{\mathbb{F}_p}(F)$, the Leibnitz rule implies that for all $x \in F$, we have

$$D(x^p) = px^{p-1}D(x) = 0.$$

Hence derivations cannot tell the extensions F/F^p and F/F^{p^n} apart. Note that if X is a positive dimensional variety and $F = K(X)$ is its function field, F/F^{p^n} is a purely inseparable extension of exponent exactly n , and so F^{p^n} are distinct fields for distinct n . On the other hand, if F is a perfect field, a similar argument shows that $\text{Der}_{\mathbb{F}_p}(F) = 0$.

Jacobson's theory has been extended to modular subextensions, i.e. subextensions of the form

$$F = E[X_1]/(X_1^{p^{n_1}} - a_1) \otimes \dots \otimes E[X_k]/(X_k^{p^{n_k}} - a_k).$$

The key property of these extensions was established by Sweedler [Swe68], who proved that F/E is modular if and only if E is the fixed field of a set of Hasse–Schmidt derivations (D_0, \dots, D_m) of F . This fundamental fact was later used to study the Galois theory of modular extensions by Chase [Cha71], Heerema–Deveney [HD74], Gerstenhaber–Zaromp [GZ70], and others.

However, not all purely inseparable field extensions are modular – we recall the following simple example by Sweedler (cf. [Swe68, Example 1.1]):

Example 1.5. For $K = \mathbb{F}_p(x^p, y^p, z^{p^2})$ and $F = K[xz + y, z]$, the extension F/K is not modular.

1.2. Techniques. In this article, we establish a Galois correspondence for arbitrary finite purely inseparable field extensions. To retain the information lost by Jacobson's functor

$$(K \subset E \subset F) \quad \mapsto \quad (\text{Der}_E(F) \hookrightarrow \text{Der}_K(F)),$$

we will construct a refinement using the cotangent complex formalism in derived algebraic geometry.

Recall that for any map of rings $A \rightarrow B$, the *cotangent complex* $L_{B/A}$ is the complex of B -modules obtained by first resolving B by a free simplicial A -algebra P_\bullet , then applying the Kähler differentials functor $\Omega^1_{-/A}$ in each simplicial degree, and finally applying the Dold–Kan correspondence. The zeroth homology group $\pi_0(L_{B/A})$ is given by $\Omega^1_{B/A}$. Note that here and everywhere else in this paper, we write $\pi_i(M)$ for the i^{th} homology group of a chain complex, which is equal to the i^{th} homotopy group of the associated module spectrum.

If F/K is a finite field extension, pick $x_1, \dots, x_n \in F$ such that the following map is surjective with kernel I :

$$K[X_1, \dots, X_n] \xrightarrow{X_i \mapsto x_i} F.$$

As we will see in Proposition 4.2, the cotangent complex $L_{F/K}$ is then concentrated in two degrees, and given by

$$L_{F/K} = \left(\dots \rightarrow 0 \rightarrow I/I^2 \xrightarrow{[i] \mapsto di \otimes 1} \Omega^1_{K[X_1, \dots, X_n]/K} \otimes_{K[X_1, \dots, X_n]} F \right).$$

If F/K is separable, then $L_{F/K}$ vanishes, analogously to the vanishing of the usual Galois group for purely inseparable extensions. Hence our Galois theory is perpendicular to the classical Galois theory of separable extensions, and from now on, we shall assume that F/K is purely inseparable.

To refine Jacobson's functor (1), we will consider the assignment

$$(2) \quad (K \subset E \subset F) \quad \mapsto \quad (L_{F/E}^\vee[1] \rightarrow L_{F/K}^\vee[1]),$$

where $(-)^{\vee}$ denotes the F -linear dual of a chain complex and $[1]$ is a homological shift by $+1$.

It is of course not enough to consider this assignment (2) merely as a functor to $(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$, the ∞ -category of F -linear chain complexes with a map to $L_{F/K}^\vee[1]$, just like Jacobson's functor did not just record $(\text{Der}_E(F) \rightarrow \text{Der}_K(F))$ as a map of F -vector spaces.

Instead, we equip $(L_{F/E}^\vee[1] \hookrightarrow L_{F/K}^\vee[1])$ with a kind of derived restricted Lie algebroid structure.

To this end, we elaborate on the theory of *partition Lie algebras*, which was introduced in [BM19] to study infinitesimal deformations in characteristic p . More precisely, for any field F , there is an equivalence between Moduli_F , the ∞ -category of formal moduli problems, and $\text{Alg}_{\text{Lie}_F^\pi}$, the ∞ -category of partition Lie algebras, cf. [BM19, Theorem 1.11]. In [op.cit.], an additional subscript $(-)_\Delta$ highlights that we work in the setting of simplicial commutative rings (rather than \mathbb{E}_∞ -rings).

Remark 1.6. In characteristic zero, partition Lie algebras are simply (shifted) differential graded Lie algebras, and [BM19, Theorem 1.11] recovers the Lurie–Pridham theorem, cf. [Lur11][Pri10].

Partition Lie algebras are defined by a monad Lie_F^π on Mod_F satisfying the following properties:

Construction 1.7 (Partition Lie algebras).

- (1) Lie_F^π commutes with sifted colimits, i.e. with filtered colimits and geometric realisations.
- (2) If $V \in \text{Mod}_F$ is coconnective and $\pi_i(V)$ is finite-dimensional for all i , then $\text{Lie}_F^\pi(V)$ is given by $L_{F/F \oplus V^\vee}^\vee[1]$, where $F \oplus V^\vee$ denotes the trivial square-zero extension of F by V^\vee .

Note that any $W \in \text{Mod}_F$ is a sifted colimit of chain complexes V of the above form.

- (3) If V^\bullet is a cosimplicial F -vector space with totalisation $V = \text{Tot}(V^\bullet)$, then

$$\text{Lie}_F^\pi(V) \simeq \bigoplus_n \text{Tot} \left(\tilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, F) \otimes (V^\bullet)^{\otimes n} \right)^{\Sigma_n}.$$

Here $\tilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, F)$ are the F -valued cosimplices of the n^{th} partition complex (cf. e.g. [AB21]), the functor $(-)^{\Sigma_n}$ takes strict fixed points, and the tensor product is computed in cosimplicial F -modules.

- (4) The tangent fibre $\text{cot}_F^\vee(R) = L_{F/R}^\vee[1]$ of any augmented simplicial commutative F -algebra $R \in \text{SCR}_F^{\text{aug}}$ carries a canonical Lie_F^π -algebra structure. Note that there is a natural equivalence $\text{cot}_F^\vee(R) \simeq (F \otimes_R L_{R/F})^\vee$ as the composite $F \rightarrow R \rightarrow F$ induces a fibre sequence $F \otimes_R L_{R/F} \rightarrow L_{F/F} \rightarrow L_{F/R}$, and $L_{F/F} \simeq 0$.

Remark 1.8. Under the natural grading conventions we adopt, the homotopy groups $\pi_*(\mathfrak{g})$ of any partition Lie algebra form a *shifted* Lie algebra, which means that the bracket

$$[-, -] : \pi_i(\mathfrak{g}) \times \pi_j(\mathfrak{g}) \rightarrow \pi_{i+j-1}(\mathfrak{g})$$

preserves $\pi_1(\mathfrak{g})$ and satisfies $[x, y] = (-1)^{|x||y|}[y, x]$, as well as the usual graded Jacobi identity. Shifted Lie brackets are familiar from homotopy theory, where they appear as Whitehead products.

Example 1.9. For any field extension $K \subset F$, we can construct two different partition Lie algebras:

- (1) The tangent fibre of the representable F -formal moduli problem $\mathrm{Spf}(F \otimes_K F)$ (given by the functor of points, see Example 2.21), which encodes deformations of the diagonal, is a partition Lie algebra over F with underlying chain complex

$$L_{F/K}^\vee \simeq \mathrm{cot}(F \otimes_K F)^\vee.$$

- (2) Infinitesimal deformations of the K -scheme $\mathrm{Spec}(F)$ are encoded by a Kodaira–Spencer formal moduli problem; the corresponding partition Lie algebra over K has underlying chain complex

$$L_{F/K}^\vee[1].$$

1.3. Statement of Results. Fix a finite purely inseparable field extension F/K .

We will construct a monad $\mathrm{LieAlgd}_{F/K}^\pi$ acting on the ∞ -category $(\mathrm{Mod}_F)_{/L_{F/K}^\vee[1]}$ of arrows $M \rightarrow L_{F/K}^\vee[1]$. We refer to $\mathrm{LieAlgd}_{F/K}^\pi$ -algebras as F/K -partition Lie algebroids, and denote the resulting ∞ -category by $\mathrm{Alg}_{\mathrm{LieAlgd}_{F/K}^\pi}$. Given $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \in \mathrm{Alg}_{\mathrm{LieAlgd}_{F/K}^\pi}$, we call ρ the *anchor map*.

F/K -partition Lie algebroids are derived generalisations of the classical F/K -restricted Lie algebroids on p. 2. Construction 1.14 below will list the key properties of $\mathrm{LieAlgd}_{F/K}^\pi$ – for now, let us simply record that for any simplicial commutative K -algebra B with a map to F , the basic arrow

$$\left(L_{F/B}^\vee[1] \rightarrow L_{F/K}^\vee[1] \right)$$

can be equipped with a canonical F/K -partition Lie algebroid structure $\mathfrak{D}(B)$.

Definition 1.10 (Galois partition Lie algebroids). Given an intermediate field E of the finite purely inseparable field extension F/K , the *Galois partition Lie algebroid* is given by $\mathfrak{gal}_{F/K}(E) := \mathfrak{D}(E)$; its underlying object is given by the arrow of chain complexes $(L_{F/E}^\vee[1] \rightarrow L_{F/K}^\vee[1])$.

Our main result is the following generalisation of Jacobson’s correspondence to arbitrary exponents:

Theorem 1.11 (Fundamental theorem of purely inseparable Galois theory).

Let F/K be a finite purely inseparable field extension. Then there is a contravariant equivalence between the poset of intermediate field extensions

$$K \subset E \subset F$$

and the ∞ -category of F/K -partition Lie algebroids

$$\left(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1] \right)$$

satisfying the following conditions:

- (1) *Injectivity:* the anchor map ρ induces an injection $\pi_1(\mathfrak{g}) \hookrightarrow \pi_1(L_{F/K}^\vee[1]) \cong \mathrm{Der}_K(F)$.
- (2) *Vanishing:* $\pi_k(\mathfrak{g}) = 0$ for $k \neq 0, 1$.
- (3) *Balance:* $\dim_F(\pi_0(\mathfrak{g})) = \dim_F(\pi_1(\mathfrak{g})) < \infty$.

The partition Lie algebroid $\mathfrak{gal}_{F/K}(E) = \mathfrak{D}(E)$ corresponding to an intermediate field E has underlying object $(L_{F/E}^\vee[1] \rightarrow L_{F/K}^\vee[1]) \in (\mathrm{Mod}_F)_{/L_{F/K}^\vee[1]}$, while the field $F^\mathfrak{g}$ corresponding to a partition Lie algebroid satisfying (1)–(3) is given by its Chevalley–Eilenberg complex $C^(\mathfrak{g})$, cf. Construction 3.14.*

Remark 1.12. Note that we in particular assert that the full subcategory of $\text{Alg}_{\text{Lie}_{F/K}^\pi}$ spanned by all objects satisfying conditions (1) – (3) in Theorem 1.11 is the nerve of an ordinary category, which is in fact a poset.

The proof of the correspondence proceeds as follows. First we define partition Lie algebroids as algebras over a monad coming from an adjunction in Section 3.1. We then show that after restriction to certain subcategories, the adjunction induces an equivalence between certain partition Lie algebroids and intermediate complete local Noetherian objects in Section 3.2. Finally we determine when these are fields using the conditions (1), (2), and (3) in Section 4.

By applying the theorem to the generic point of a variety, we immediately get a classification of purely inseparable morphisms of normal varieties.

Corollary 1.13. *Let X be a normal variety over a perfect field k , with function field F . Then purely inseparable k -morphisms to a normal variety $\pi : X \rightarrow Y$ of exponent at most n are in bijection with F/F^{p^n} -partition Lie algebroids satisfying the conditions (1), (2) and (3) from Theorem 1.11.*

We will now record several key properties of the monad $\text{LieAlgd}_{F/K}^\pi$:

Construction 1.14 (Partition Lie algebroids).

- (1) The functor $\text{LieAlgd}_{F/K}^\pi$ commutes with filtered colimits and geometric realisations.
- (2) For any simplicial commutative K -algebra B with a map to F , the basic arrow

$$\left(L_{F/B}^\vee[1] \rightarrow L_{F/K}^\vee[1] \right)$$

can be equipped with a canonical F/K -partition Lie algebroid structure $\mathfrak{D}(B)$.

- (3) The forgetful functor $\text{Alg}_{\text{LieAlgd}_{F/K}^\pi} \rightarrow (\text{Mod}_K)_{/L_{F/K}^\vee[1]}$ sending $(\mathfrak{g} \rightarrow L_{F/K}^\vee[1])$ to the underlying object in $(\text{Mod}_K)_{/L_{F/K}^\vee[1]}$ lifts canonically to a sifted-colimit-preserving functor

$$U : \text{Alg}_{\text{LieAlgd}_{F/K}^\pi} \rightarrow (\text{Alg}_{\text{Lie}_K^\pi})_{/L_{F/K}^\vee[1]},$$

where $L_{F/K}^\vee[1]$ is the K -partition Lie algebra of Example 1.9 (2).

- (4) The fibre functor $\text{fib} : \text{Alg}_{\text{LieAlgd}_{F/K}^\pi} \rightarrow (\text{Mod}_F)_{/L_{F/K}^\vee}$ sending $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ to $\text{fib}(\rho)$ lifts canonically to a sifted-colimit-preserving functor

$$\text{Alg}_{\text{LieAlgd}_{F/K}^\pi} \rightarrow (\text{Alg}_{\text{Lie}_F^\pi})_{/L_{F/K}^\vee},$$

where $L_{F/K}^\vee$ is the F -partition Lie algebra of Example 1.9 (1).

Very informally, the anchor map ρ measures the failure of the bracket on \mathfrak{g} to be F -linear.

- (5) We say that an object $(V \rightarrow L_{F/K}^\vee[1]) \in (\text{Mod}_F)_{/L_{F/K}^\vee[1]}$ has a *vanishing anchor map* if the associated map $(V \rightarrow L_{F/K}^\vee[1])$ in Mod_F is nullhomotopic (in Mod_F).

Given an object $(V \xrightarrow{0} L_{F/K}^\vee[1]) \in (\text{Mod}_F)_{/L_{F/K}^\vee[1]}$ with vanishing anchor map, there is an equivalence

$$\text{LieAlgd}_{F/K}^\pi(V \xrightarrow{0} L_{F/K}^\vee[1]) \simeq (\text{Lie}_F^\pi(V) \rightarrow L_{F/K}^\vee[1]).$$

Note that any $(W \rightarrow L_{F/K}^\vee[1])$ in $(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$ is the geometric realisation of a simplicial diagram of objects $(V \xrightarrow{0} L_{F/K}^\vee[1])$ with vanishing anchor maps and V coconnective.

Remark 1.15. It might be surprising that nonzero maps can be colimits of zero maps. However, this phenomenon is standard in homotopy theory. For instance, the identity map on S^2 appears as a (homotopy) pushout of a map from the diagram $* \leftarrow S^1 \rightarrow *$ to the constant diagram $S^2 \leftarrow S^2 \rightarrow S^2$, in spite of the maps $* \rightarrow S^2$ and $S^1 \rightarrow S^2$ being nullhomotopic.

A finite purely inseparable field extension F/K is simple precisely if

$$\dim_F(\Omega_{F/K}^1) = 1.$$

Using this, we show that modular extensions can be characterised using their partition Lie algebroids.

Proposition 1.16. *Let F/K be a finite purely inseparable extension. Then F/K is modular precisely if there are finitely many F/K -partition Lie algebroids*

$$\rho_i : \mathfrak{g}_i \rightarrow L_{F/K}^\vee[1]$$

such that the following conditions hold:

- (1) each \mathfrak{g}_i satisfies conditions (1) – (3) of Theorem 1.11;
- (2) $\dim_F(\pi_0(\text{fib}(\rho_i))) = 1$ for each i ;
- (3) the canonical map $L_{F/K}^\vee \rightarrow \oplus_i \text{fib}(\rho_i)$ is an equivalence in Mod_F .

Our construction of F/K -partition Lie algebroids is higher categorical in nature, but one can also construct simplicial-cosimplicial models by generalising [BCN21, Theorem 5.33] to Lie algebroids. In our forthcoming paper [BW], we will study these concrete models in more detail, and explore geometric applications of our Galois correspondence to algebraic foliations and Brauer groups.

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2. PRELIMINARIES

To set the stage, we briefly review some basic notions of derived algebraic geometry, including chain complexes, simplicial commutative rings, and André–Quillen’s cotangent complex formalism. We will not attempt to give a comprehensive treatment – the aim of this section is merely to introduce the reader to several necessary ideas, and provide references which allow them to find the details, in particular within the references [Lur09], [Lur11], [Lur16, Chapter 25], and [Lur17].

The language of ∞ -categories, also known as quasi-categories, provides an essential tool for us, which we will use freely. These objects were first defined by Boardman and Vogt [BV73], explored further by Joyal [Joy02], and treated comprehensively in the seminal work of Lurie [Lur09]. In particular, we will use the terms “limit” and “colimit” in the ∞ -categorical sense. Hence, these correspond to “homotopy limits” and “homotopy colimits” in the more classical, model categorical sense.

2.1. Chain complexes. Let R be an ordinary commutative ring.

Notation 2.1 (Chain complexes). We write Mod_R for the derived ∞ -category of the ring R . Its objects are represented by chain complexes of R -modules $\dots \rightarrow M_1 \rightarrow M_0 \rightarrow M_{-1} \rightarrow \dots$ (cf. [Lur17, Definition 1.3.5.8]), or equivalently by R -module spectra [Lur17, Remark 7.1.1.16]).

Given $M \in \mathrm{Mod}_R$, let $\pi_i(M)$ be the i^{th} homology group of the corresponding chain complex, or equivalently the i^{th} homotopy group of the corresponding R -module spectrum.

Remark 2.2 (Basic properties of Mod_R). We recall several basic facts from [Lur17, Chapter 1, 7].

- (1) The homotopy category $h\mathrm{Mod}_R$ of Mod_R is equivalent to the classical derived category of R . In particular, Mod_R should not be confused with the category of ordinary R -modules. However, Mod_R can be equipped with a natural t -structure, the heart of which recovers ordinary R -modules as chain complexes concentrated in degree zero (cf. [Lur17, Definition 7.1.1.13]).
- (2) The full subcategory $\mathrm{Mod}_{R, \geq 0}$ of connective objects for this t -structure consists of all M with $\pi_i(M) = 0$ for $i < 0$, i.e. by all chain complexes with vanishing homology in negative degrees.
- (3) In fact, $\mathrm{Mod}_{R, \geq 0}$ can be obtained by freely adjoining sifted colimits, i.e. filtered colimits and geometric realisations, to the ordinary category of finitely generated free R -modules. To this end, we use the \mathcal{P}_Σ -construction, which sends an ∞ -category \mathcal{C} to the ∞ -category $\mathcal{P}_\Sigma(\mathcal{C})$ of finite-product-preserving functors from $\mathcal{C}^{\mathrm{op}}$ to spaces, c.f. [Lur09, Section 5.5.8].
- (4) The ∞ -category Mod_R admits a symmetric monoidal structure denoted by \otimes_R (cf. [Lur17, Section 4.5.2]), which computes the derived tensor product of chain complexes over R , and is traditionally denoted by \otimes_R^L . This product preserves small colimits in each entry.

We will often need two finiteness properties of chain complexes. For simplicity, we will only spell them out in the generality needed for our later arguments.

Definition 2.3 (Finiteness properties in Mod_R). Let R be an ordinary Noetherian commutative ring. A chain complex $M \in \mathrm{Mod}_R$ is said to be

- (1) *perfect* if it can be represented by a finite complex of finitely generated free R -modules (cf. [Lur17, Definition 7.2.4.1]);
- (2) *almost perfect* if it is bounded below and each $\pi_i(M)$ is a finitely generated R -module (cf. [Lur17, Definition 7.2.4.10 and Proposition 7.2.4.17] or also [BGI⁺, I. 2] or [The, 066E], where this notion is called ‘pseudo-coherent’);
- (3) *of finite type* if each $\pi_i(M)$ is a finitely generated R -module.

Write $\mathrm{Perf}_R, \mathrm{APerf}_R, \mathrm{Mod}_R^{\mathrm{ft}} \subset \mathrm{Mod}_R$ for the full subcategories spanned by these families of modules.

Remark 2.4. One can also develop this theory for R equal to the sphere spectrum, in which case we recover the symmetric monoidal ∞ -category Sp of spectra. Any ordinary ring gives rise to a commutative algebra object in Sp , its Eilenberg-MacLane spectrum, and Mod_R is equivalent to the ∞ -category of modules over this \mathbb{E}_∞ -ring. We will not need this perspective in our work.

2.2. Simplicial commutative rings. The affine objects in derived algebraic geometry are given by simplicial commutative rings. Their homotopy theory was first studied by Quillen [Qui], who constructed a cofibrantly generated model structure on this category. The ∞ -category obtained by inverting weak equivalences admits a concise presentation, c.f. [Lur16, Definition 25.1.1.1]:

Definition 2.5 (Simplicial Commutative Rings). Let R be an ordinary commutative ring, and write Poly_R for (the nerve of) the category of finitely generated polynomial R -algebras $R[x_1, \dots, x_n]$. The ∞ -category of simplicial commutative R -algebras is given by

$$\mathrm{SCR}_R := \mathcal{P}_\Sigma(\mathrm{Poly}_R^{\mathrm{op}}),$$

that is, by the ∞ -category of all finite-product-preserving functors from $\mathrm{Poly}_R^{\mathrm{op}}$ to spaces.

A detailed ∞ -categorical treatment of simplicial commutative rings is given in [Lur16, Chapter 25].

Remark 2.6 (Basic properties of SCR_R). We review several basic facts.

- (1) The ∞ -category SCR_R is obtained from Poly_R by formally adjoining sifted colimits, i.e. filtered colimits and geometric realisations (c.f. [Lur09, Section 5.5.8]). It is presentable, and the Yoneda embedding $\mathrm{Poly}_R \hookrightarrow \mathrm{SCR}_R$ preserves coproducts. The image of this embedding consists of compact objects, which generate SCR_R under sifted colimits.
- (2) Every $B \in \mathrm{SCR}_R$ has an underlying (connective) spectrum, which can in fact be equipped with the structure of an \mathbb{E}_∞ - R -algebra in a canonical way.
- (3) Hence, we can take the homotopy groups $\pi_*(B)$ of any $B \in \mathrm{SCR}_R$, and each $\pi_i(B)$ is a module over the ordinary commutative ring $\pi_0(B)$. If we model B by an ordinary simplicial commutative R -algebra, $\pi_i(B)$ is the i^{th} homology of the chain complex obtained by applying the Dold-Kan correspondence to B , thought of as a simplicial B -module.
- (4) If $R = \mathbb{Z}$, we refer to $\mathrm{SCR} := \mathrm{SCR}_{\mathbb{Z}}$ as the ∞ -category of simplicial commutative rings; for any ring R , there is a natural forgetful functor $\mathrm{SCR}_R \rightarrow \mathrm{SCR}$.
- (5) More generally, given a map of rings $R \rightarrow S$, there is a forgetful functor $\mathrm{SCR}_S \rightarrow \mathrm{SCR}_R$. Its left adjoint is computed by $R \otimes_S (-)$ on the level of modules.

Notation 2.7. Given $S \in \mathrm{SCR}_R$, write $\mathrm{SCR}_{R//S} := (\mathrm{SCR}_R)_{/S}$ for the overcategory (cf. [Lur09, Section 1.2.9]) consisting of simplicial commutative R -algebras with a map to S . When $R = S$, we obtain the ∞ -category of augmented simplicial commutative R -algebras $\mathrm{SCR}_R^{\mathrm{aug}} := \mathrm{SCR}_{R//R}$.

The following class of simplicial commutative rings will play a key role in our arguments:

Definition 2.8 (Complete local Noetherian objects). A simplicial commutative ring $B \in \mathrm{SCR}$ is *complete local Noetherian* if $\pi_0(B)$ is a complete local Noetherian ring and $\pi_i(B)$ is a finitely generated $\pi_0(B)$ -module for all $i \geq 0$. Write $\mathrm{SCR}^{\mathrm{cN}} \subset \mathrm{SCR}$, $\mathrm{SCR}_R^{\mathrm{cN}} \subset \mathrm{SCR}_R$, $\mathrm{SCR}_{R//S}^{\mathrm{cN}} \subset \mathrm{SCR}_{R//S}$ for the full subcategories spanned by complete local Noetherian rings.

2.3. Algebraic André–Quillen homology. The homology of simplicial commutative rings was introduced by André [And74] and Quillen [Qui70], and may be thought of as the nonabelian derived functor of the construction which sends an augmented algebra to its indecomposables.

To give a more formal construction, we first introduce its right adjoint:

Definition 2.9 (Trivial square-zero extensions). Let R be an ordinary commutative ring. The *trivial square-zero extension functor* sqz_R is the unique sifted-colimit-preserving functor

$$\text{sqz}_R : \text{Mod}_{R, \geq 0} \rightarrow \text{SCR}_R^{\text{aug}}$$

sending a finite free R -module N to the chain complex $R \oplus N$ with multiplication

$$(r_1, n_1) \cdot (r_2, n_2) = (r_1 r_2, r_1 n_2 + r_2 n_1).$$

Note that the underlying chain complex of $\text{sqz}_R(M)$ is always given by $R \oplus M$.

This construction can be extended to a trivial square-zero extension functor for connective modules over simplicial commutative rings, see [Lur16, Section 25.3.1]. To this end, we consider the ∞ -category $\text{SCRMod}^{\text{cn}}$ whose objects are pairs (R, M) where R is a simplicial commutative ring and M is a connective R -module (cf. [Lur16, Notation 25.2.1.1]), and the full subcategory $\mathcal{C} \subset \text{SCRMod}^{\text{cn}}$ spanned by pairs consisting of a polynomial ring $A = \mathbb{Z}[x_1, \dots, x_n]$ and a finite free A -module. By [Lur16, Proposition 25.2.1.2], there is an equivalence $\text{SCRMod}^{\text{cn}} \simeq \mathcal{P}_\Sigma(\mathcal{C})$ between the sifted completion of \mathcal{C} (cf. [Lur09, Definition 5.5.8.8]) and $\text{SCRMod}^{\text{cn}}$. Using the universal property of the \mathcal{P}_Σ -construction in [Lur09, Proposition 5.5.8.15], one obtains a sifted-colimit-preserving functor $\text{SCRMod}^{\text{cn}} \rightarrow \text{SCR}$, $(R, M) \mapsto \text{sqz}_R(M)$ which extends the classical square-zero extension functor.

More generally, let $R \rightarrow S$ be a map of ordinary commutative rings, the *relative trivial square-zero extension functor* $\text{sqz}_{R//S} : \text{Mod}_{S, \geq 0} \rightarrow \text{SCR}_{R//S}$ sends $M \in \text{Mod}_{S, \geq 0}$ to $\text{sqz}_S(M)$, considered as a simplicial commutative R -algebra via restriction along the map $R \rightarrow S$.

Here, we have used the universal property of $\text{Mod}_{R, \geq 0}$ discussed in Remark 2.2(3).

Notation 2.10. We will often write $\text{sqz}_R(M) = R \oplus M$ and $\text{sqz}_{R//S}(M) = S \oplus M$.

We can now introduce André–Quillen’s homology functor:

Definition 2.11 (Cotangent fibre). For $R \in \text{SCR}$, the *cotangent fibre functor*

$$\text{cot}_R : \text{SCR}_R^{\text{aug}} \rightarrow \text{Mod}_{R, \geq 0}$$

is given by the left adjoint to the trivial square-zero extension functor sqz_R .

Given a map $R \rightarrow S$, the left adjoint of $\text{sqz}_{R//S}$ gives a relative version of this functor denoted by

$$\text{cot}_{R//S} : \text{SCR}_{R//S} \rightarrow \text{Mod}_{S, \geq 0}.$$

Since $\text{sqz}_{R//S}$ is a composite of right adjoints, we can write $\text{cot}_{R//S}$ as a composite of left adjoints

$$\text{cot}_{R//S}(B) \simeq \text{cot}_S(S \otimes_R B).$$

We will describe cotangent fibres in terms of cotangent complexes in the following Section 2.4.

Remark 2.12. The cotangent fibre functor above is often decorated with a subscript $(-)_\Delta$, to indicate that we are working over simplicial commutative rings, rather than \mathbb{E}_∞ -rings. As we only use one of these versions, we will drop this subscript from our notation.

The following result links two finiteness conditions for modules in Definition 2.3 and rings in Definition 2.8, and follows from [Lur04, Proposition 3.1.5] and [Lur04, Proposition 3.2.14]:

Proposition 2.13. *If $A \in \text{SCR}_F^{\text{aug}}$ is Noetherian, then $\text{cot}(A) \in \text{Mod}_{F, \geq 0}^{\text{ft}}$ is connective of finite type.*

2.4. The algebraic cotangent complex. The cotangent complex formalism is of central importance in deformation theory, as was illustrated, for example, in the pioneering work of Illusie [Ill71]. Given a ring map $A \rightarrow B$, the (algebraic) cotangent complex is a derived version of the module of Kähler differentials $\Omega_{B/A}^1$, and its construction parallels the classical definition of $\Omega_{B/A}^1$. For more details, we refer to the modern treatment in [Lur16, Sections 25.3.1, 25.3.2].

Definition 2.14 (Derivations). For a simplicial commutative ring R and connective R -module M , we define the *space of derivations of R into M* as the following mapping space:

$$\mathrm{Der}(R, M) = \mathrm{Map}_{(\mathrm{SCR})/R}(R, R \oplus M).$$

One can prove that there exists an R -module L_R and a universal derivation $\eta \in \mathrm{Der}(R, L_R)$ such that for any connective R -module $M \in \mathrm{Mod}_{R, \geq 0}$, the natural map $\mathrm{Map}_{\mathrm{Mod}_R}(L_R, M) \simeq \mathrm{Der}(R, M)$ is an equivalence. The pair (L_R, η) is uniquely determined up to equivalence.

Definition 2.15 (The algebraic cotangent complex).

- (1) The R -module L_R is called the *the absolute (algebraic) cotangent complex of R* .
- (2) For a morphism of simplicial commutative rings $R \rightarrow S$, the *relative (algebraic) cotangent complex $L_{S/R}$* is the cofibre of the natural map $S \otimes_R L_R \rightarrow L_S$.

The relative cotangent complex also satisfies a universal property; for any S -module M , we have

$$\mathrm{Map}_{\mathrm{Mod}_S}(L_{S/R}, M) \simeq \mathrm{Map}_{(\mathrm{SCR}_R)/S}(S, S \oplus M).$$

Unraveling the definitions, we see that given a morphism $R \rightarrow S$ and some $B \in \mathrm{SCR}_{R//S}$, we have

$$\mathrm{cot}_{R//S}(B) \simeq S \otimes_B L_{B/R}.$$

Taking $R = S$, we deduce that for augmented R -algebra $B \in \mathrm{SCR}_R^{\mathrm{aug}}$, we have $\mathrm{cot}_R(B) \simeq R \otimes_B L_{B/R}$.

Remark 2.16. The algebraic cotangent complex is often decorated with a superscript $(-)^{\mathrm{alg}}$ to distinguish it from the topological cotangent complex of the underlying \mathbb{E}_∞ -rings.

The most important computational tool in dealing with the cotangent complex is the fundamental cofibre sequence, which extends the classical relative cotangent sequence:

Proposition 2.17 (The fundamental cofibre sequence). *Given maps of simplicial commutative rings $f : A \rightarrow B$ and $g : B \rightarrow C$, there is a canonical cofibre sequence in Mod_C given by*

$$C \otimes_B L_{B/A} \rightarrow L_{C/A} \rightarrow L_{C/B}.$$

Note that the cofibre sequence allows us to write the functor $\mathrm{cot}_{R//S}$ in the equivalent form

$$\mathrm{cot}_{R//S}(B) = \mathrm{cofib}(L_{S/R}[-1] \rightarrow L_{S/B}[-1])$$

as explained in [Lur17, Remark 1.1.1.7].

This construction of the cofibre sequence is functorial in the following sense:

Proposition 2.18. *There is a functor of ∞ -categories $\mathrm{SCR}_{A//C} \rightarrow \mathrm{Fun}(\Delta^2, \mathrm{Mod}_C)$ sending an object $(A \rightarrow B \rightarrow C) \in \mathrm{SCR}_{A//C}$ to the canonical cofibre sequence $C \otimes_B L_{B/A} \rightarrow L_{C/A} \rightarrow L_{C/B}$ from Proposition 2.17 and a morphism $(A \rightarrow B_1 \rightarrow B_2 \rightarrow C) \in \mathrm{Fun}(\Delta^1, \mathrm{SCR}_{A//C})$ to the canonical*

map of cofibre sequences

$$\begin{array}{ccccc} C \otimes_{B_1} L_{B_1/A} & \longrightarrow & L_{C/A} & \longrightarrow & L_{C/B_1} \\ \downarrow & & \downarrow & & \downarrow \\ C \otimes_{B_2} L_{B_2/A} & \longrightarrow & L_{C/A} & \longrightarrow & L_{C/B_2} \end{array}$$

Proof. Write $\text{Fun}'(\Delta^1, \text{SCR}_{A//C}) \subset \text{Fun}(\Delta^1, \text{SCR}_{A//C})$ for the full subcategory of arrows sending $1 \in \Delta^1$ to $(A \rightarrow C \xrightarrow{\text{id}} C) \in \text{SCR}_{A//C}$. This object is final, which means that evaluation at $0 \in \Delta^1$ provides an equivalence $\text{Fun}'(\Delta^1, \text{SCR}_{A//C}) \simeq \text{SCR}_{A//C}$. Picking an inverse from a contractible space of choices gives rise to a functor $\text{SCR}_{A//C} \rightarrow \text{Fun}(\Delta^1, \text{SCR}_{A//C})$ sending $(A \rightarrow B \rightarrow C)$ to $(A \rightarrow B \rightarrow C \xrightarrow{\text{id}} C)$.

Postcomposing it with the cotangent fibre functor $\text{cot}_{A//C}$ from Definition 2.11 gives the auxiliary functor $\text{SCR}_{A//C} \rightarrow \text{Fun}(\Delta^1, \text{Mod}_C)$ sending $(A \rightarrow B \rightarrow C)$ to the arrow $C \otimes_B L_{B/A} \rightarrow L_{C/A}$.

Finally, let us write $\text{Fun}'(\Delta^2, \text{Mod}_C) \subset \text{Fun}(\Delta^2, \text{Mod}_C)$ for the full subcategory spanned by all cofibre sequences. Restriction to $\Delta^1 \simeq \Delta^{\{0,1\}} \subset \Delta^2$ defines an equivalence $\text{Fun}'(\Delta^2, \text{Mod}_C) \xrightarrow{\simeq} \text{Fun}(\Delta^1, \text{Mod}_C)$ sending a cofibre sequence $(M \rightarrow M' \rightarrow M'')$ to $(M \rightarrow M')$.

Postcomposing with its inverse, chosen again from a contractible space of choices as in [Lur17, Remark 1.1.1.7], we obtain the desired functor $\text{SCR}_{A//C} \rightarrow \text{Fun}(\Delta^2, \text{Mod}_C)$ sending $(A \rightarrow B \rightarrow C)$ to $(C \otimes_B L_{B/A} \rightarrow L_{C/A} \rightarrow L_{C/B})$, where we have identified the cofibre of $C \otimes_B L_{B/A} \rightarrow L_{C/A}$ with $L_{C/B}$ using Proposition 2.17. \square

2.5. Formal moduli problems. Any reasonably geometric deformation problem over a field K gives rise to a formal moduli problem. To formalise this notion, we need a family of augmented simplicial commutative K -algebras corresponding to the derived infinitesimal thickenings of $\text{Spec}(K)$:

Definition 2.19 (Artinian K -algebras). A simplicial commutative K -algebra A is *Artinian* if

- (1) The K -algebra $\pi_0(A)$ is an Artinian ring with residue field, as a K algebra, being K ;
- (2) The K -vector space $\pi_*(A) = \bigoplus_{n=0}^{\infty} \pi_n(A)$ is finite-dimensional.

We write $\text{SCR}_K^{\text{art}} \subset \text{SCR}_K^{\text{aug}}$ for the full subcategory spanned by all (simplicial) Artinian K -algebras A , along with the canonical augmentation map given by the composite $A \rightarrow \tau_{\leq 0}(A) = \pi_0(A) \rightarrow K$.

We can now recall Lurie's higher categorical framework for formal deformation functors, cf. [Lur11]:

Definition 2.20. A *formal moduli problem* is a functor $X : \text{SCR}_K^{\text{art}} \rightarrow \mathcal{S}$ such that

- (1) $X(K) \simeq *$ is contractible;
- (2) For any pullback square

$$\begin{array}{ccc} A_3 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array}$$

where $\pi_0(A_2) \rightarrow \pi_0(A)$ and $\pi_0(A_1) \rightarrow \pi_0(A)$ are surjective, applying X gives a pullback

$$\begin{array}{ccc} X(A_3) & \longrightarrow & X(A_2) \\ \downarrow & & \downarrow \\ X(A_1) & \longrightarrow & X(A_0). \end{array}$$

We denote the ∞ -category of formal moduli problems over K by Moduli_K .

Example 2.21. Let A be a commutative ring which is an augmented K -algebra. Then A gives a formal moduli problem via the functor of points: $\mathrm{Spf}(A) : \mathrm{SCR}_{K//K}^{\mathrm{art}} \rightarrow \mathcal{S}$, $R \mapsto \mathrm{Map}_{\mathrm{SCR}_{K//K}}(A, R)$.

In characteristic zero, Lurie [Lur11] and Pridham [Pri10] showed that formal moduli problems are controlled by differential graded Lie algebras. Partition Lie algebras were introduced in [BM19] to generalise this statement to base fields of arbitrary characteristic. Note that the link between Lie algebras and formal deformation theory only becomes fully realised if one works in the setting of derived algebraic geometry.

We will now describe the relation between formal moduli problems and Lie algebras in more detail. To begin with, recall that given a formal moduli problem X over a field K , we can associate its tangent fibre $T_X \in \mathrm{Mod}_K$, which, as a spectrum, satisfies

$$\Omega^{\infty-n}(T_X) = X(K \oplus K[n]).$$

Here $K \oplus K[n]$ is the trivial square-zero extension of K by a copy of K in (homological) degree n .

In fact, $T_X \in \mathrm{Mod}_K$ is characterised by a natural equivalence

$$(3) \quad \mathrm{Map}_K(V^*, T_X) \simeq X(K \oplus V),$$

where V varies over perfect connective K -modules, and $(-)^*$ denotes K -linear duality.

The key observation is that T_X can be endowed the structure of a *partition Lie algebra*, that is, an algebra over a certain monad $\mathrm{Alg}_{\mathrm{Lie}_K^\pi}$ on the derived ∞ -category Mod_K of chain complexes over K . We will describe this monad in more detail in Section 2.6 below. The main result of [BM19] uses this construction to generalise the Lurie–Pridham theorem to arbitrary characteristics:

Theorem 2.22 ([BM19], Theorem 1.11). *If K is a field, the functor $\mathrm{Moduli}_K \rightarrow \mathrm{Mod}_K$ sending a formal moduli problem X to its tangent fibre T_X refines to an equivalence of ∞ -categories*

$$\mathrm{Moduli}_K \simeq \mathrm{Alg}_{\mathrm{Lie}_K^\pi}.$$

2.6. Extending monads and functors. We will briefly outline the higher categorical construction of the partition Lie algebra monad Lie_K^π presented in [BM19], which we will generalise to the case of Lie algebroids in Section 3.1 of the main text. For an alternative, more explicit, construction of partition Lie algebras using point set models, we refer to [BCN21, Theorem 5.33, Construction 5.34].

To construct the monad Lie_K^π , we first note that the (contravariant) tangent fibre functor $\mathrm{SCR}_{K//K}^{\mathrm{op}} \rightarrow \mathrm{Mod}_{K, \leq 0}$, $A \mapsto \mathrm{cot}(A)^\vee = (k \otimes_A L_{A/k})^\vee$ from augmented simplicial commutative K -algebras to chain complexes over K is part of an adjunction.

The associated monad T on $\mathrm{Mod}_{K, \leq 0}$ sends $V \in \mathrm{Mod}_K$ to $L_{K/K \oplus V}^\vee[1] \simeq (K \otimes_{K \oplus V} L_{K \oplus V/K})^\vee$.

However, T is *not* the right monad for the purposes of deformation theory, as it does not preserve sifted colimits. To overcome this obstacle, we replace T by a more well-behaved monad Lie_K^π .

Construction 2.23 (Partition Lie algebra monad). We briefly outline the construction of Lie_K^π , which appears, in a more abstract language, in the proof of [BM19, Corollary 5.46].

- (1) First, we check that the monad T preserves the full subcategory $\mathrm{Mod}_{K, \leq 0}^{\mathrm{ft}} \subset \mathrm{Mod}_K$ of chain complexes V which are coconnective and of finite type (cf. e.g. [Lur04, Proposition 3.2.14]). Hence $T|_{\mathrm{Mod}_{K, \leq 0}^{\mathrm{ft}}}$ acquires the structure of a monad.

The proof of [BM19, Proposition 5.49] gives a description of this restriction $T|_{\mathrm{Mod}_{K, \leq 0}^{\mathrm{ft}}}$: if V^\bullet is a cosimplicial K -module whose associated chain complex $\mathrm{Tot}(V^\bullet)$ is of finite type, then

$$T|_{\mathrm{Mod}_{K, \leq 0}^{\mathrm{ft}}}(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot} \left(\tilde{\mathcal{C}}^\bullet(\Sigma|\Pi_n|^\diamond, K) \otimes (V^\bullet)^{\otimes n} \right)^{\Sigma_n}.$$

Here $\tilde{C}^\bullet(-, K)$ are the K -valued cosimplices of the (doubly suspended) n^{th} partition complex (cf. e.g. [AB21]), the functor $(-)^{\Sigma_n}$ takes strict fixed points, and the tensor product is computed in cosimplicial K -modules.

- (2) Using this explicit description, it is not hard to check that the functor $T|_{\text{Mod}_{K, \leq 0}^{\text{ft}}}$ is
- *right complete* which means that the canonical map $\text{colim}_n T(\tau_{\leq -n} V) \xrightarrow{\cong} T(V)$ is an equivalence for all $V \in \text{Mod}_{K, \leq 0}^{\text{ft}}$;
 - *preserves finite coconnective geometric realisations*, which means that if V_\bullet is a simplicial object in $\text{Mod}_{K, \leq 0}^{\text{ft}}$ which is m -skeletal (for some m) and satisfies with $|V_\bullet| \in \text{Mod}_{K, \leq 0}^{\text{ft}}$, then the canonical map $|T(V_\bullet)| \xrightarrow{\cong} T(|V_\bullet|)$ is an equivalence.
- (3) Next, we show in [BM19, Proposition 3.16] that restriction induces an equivalence

$$\text{End}_\Sigma(\text{Mod}_K) \xrightarrow{\cong} \text{Fun}'_\sigma(\text{Mod}_{K, \leq 0}^{\text{ft}}, \text{Mod}_K)$$

between the full subcategory $\text{End}_\Sigma(\text{Mod}_K) \subset \text{End}(\text{Mod}_K)$ of sifted-colimit-preserving endofunctors of Mod_K and the full subcategory $\text{Fun}'_\sigma(\text{Mod}_{K, \leq 0}^{\text{ft}}, \text{Mod}_K) \subset \text{Fun}(\text{Mod}_{K, \leq 0}^{\text{ft}}, \text{Mod}_K)$ of right complete functors which preserve finite coconnective geometric realisations.

Let us write $\text{End}_\Sigma^{\text{Mod}_{K, \leq 0}^{\text{ft}}} \subset \text{End}(\text{Mod}_K)$ for the full subcategory of sifted-colimit-preserving endofunctors of Mod_K which preserve $\text{Mod}_{K, \leq 0}^{\text{ft}}$, and let $\text{End}'_\sigma(\text{Mod}_{K, \leq 0}^{\text{ft}}) \subset \text{End}(\text{Mod}_{K, \leq 0}^{\text{ft}})$ be the full subcategory of right complete endofunctors of $\text{Mod}_{K, \leq 0}^{\text{ft}}$ which preserve finite coconnective geometric realisations.

Then the above equivalence implies that the following restriction functor is an equivalence as well (cf. [BM19, Corollary 3.17]):

$$\text{End}_\Sigma^{\text{Mod}_{K, \leq 0}^{\text{ft}}}(\text{Mod}_K) \xrightarrow{\cong} \text{End}'_\sigma(\text{Mod}_{K, \leq 0}^{\text{ft}})$$

- (4) Using this equivalence, we extend the monad $T|_{\text{Mod}_{K, \leq 0}^{\text{ft}}}$ to obtain the monad Lie_K^π on Mod_K . This monad Lie_K^π preserves filtered colimits and geometric realisations, and if V^\bullet is a cosimplicial K -module with associated chain complex $\text{Tot}(V^\bullet)$, then

$$\text{Lie}_K^\pi(\text{Tot}(V^\bullet)) \simeq \bigoplus_n \text{Tot} \left(\tilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, K) \otimes (V^\bullet)^{\otimes n} \right)^{\Sigma_n}.$$

3. PARTITION LIE ALGEBROIDS

Let F/K be a finite purely inseparable field extension. Write $L_{F/K}^\vee[1]$ for the shifted dual of its relative cotangent complex, so that $\pi_1(L_{F/K}^\vee[1]) \cong \text{Der}_K(F)$ consists of K -linear derivations of F . In this section, we will define F/K -partition Lie algebroids and establish some of their key properties, drawing inspiration from the partition Lie algebras introduced in [BM19].

3.1. Constructing partition Lie algebroids. Partition Lie algebroids will consist of arrows

$$(\mathfrak{g} \rightarrow L_{F/K}^\vee[1])$$

with additional structure. More formally, there will be a monad $\text{LieAlg d}_{F/K}^\pi$ on the overcategory

$$(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$$

whose algebras will be the desired F/K -partition Lie algebroids. These algebraic structures will later allow us to classify intermediate fields $K \subset E \subset F$. To construct our monad, we will prove:

Theorem 3.1 (The F/K -partition Lie algebroid monad).

(1) *The relative tangent fibre*

$$\text{cot}_{K//F}^\vee : \text{SCR}_{K//F}^{\text{op}} \rightarrow \text{Mod}_{F, \leq 0}$$

$$B \mapsto \text{fib}(L_{F/B}^\vee[1] \rightarrow L_{F/K}^\vee[1])$$

admits a left adjoint $V \mapsto F \oplus V^\vee$. Let $\bar{T} : \text{Mod}_{F, \leq 0} \rightarrow \text{Mod}_{F, \leq 0}$ be the induced monad.

- (2) *There is a unique monad T on Mod_F which agrees with \bar{T} on $\text{Mod}_{F, \leq 0}^{\text{ft}}$ and moreover preserves filtered colimits and geometric realisations. Write \mathcal{C} for the ∞ -category of T -algebras.*
- (3) *The forgetful functor $\mathcal{C} \rightarrow \text{Mod}_F$ factors over the fibre functor $\text{fib} : (\text{Mod}_F)_{/L_{F/K}^\vee[1]} \rightarrow \text{Mod}_F$ via a canonical monadic right adjoint*

$$\mathcal{C} \rightarrow (\text{Mod}_F)_{/L_{F/K}^\vee[1]},$$

which gives rise to a sifted-colimit-preserving monad $\text{LieAlg d}_{F/K}^\pi$ on $(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$.

Assuming this result, we can introduce the main definition of this paper:

Definition 3.2 (Partition Lie algebroids). An F/K -partition Lie algebroid is an algebra for the monad $\text{LieAlg d}_{F/K}^\pi$ on $(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$. Write $\text{Alg}_{\text{LieAlg d}_{F/K}^\pi} \simeq \mathcal{C}$ for the resulting ∞ -category. The underlying object of a partition Lie algebroid is usually denoted by $(\mathfrak{g} \rightarrow L_{F/K}^\vee[1])$.

In our proof of Theorem 3.1 and elsewhere, we will need to compute limits and colimits in undercategories (and dually overcategories). This is possible by the following standard result:

Lemma 3.3 (Limits and colimits in undercategories). *Let x be an object in an ∞ -category \mathcal{C} . The forgetful functor $\mathcal{C}_{x/} \rightarrow \mathcal{C}$ creates small colimits indexed by contractible categories and all small limits.*

Proof. The claim about limits follows by the dual of [Lur09, Proposition 1.2.13.8]. The claim about colimits follows from [Lur09, Proposition 4.4.2.9] and [Lur09, Proposition 2.1.2.1]. \square

To prove Theorem 3.1, we will first establish a variant of [BM19, Example 6.14], which will then allow us to deduce the theorem from corresponding results in [op.cit.].

Proposition 3.4. *There is a natural cofibre sequence of functors $\mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}} \rightarrow \mathrm{Mod}_F$ given by*

$$L_{F/K}^\vee \rightarrow \bar{T} \rightarrow \mathrm{Lie}_F^\pi,$$

for $L_{F/K}^\vee$ the constant functor and Lie_F^π the partition Lie algebra monad, cf. [BM19, Definition 5.47].

Proof. Given $V \in \mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}$, write $B = F \oplus V^\vee$ for the trivial square-zero extension of F by V^\vee , considered as an object of $\mathrm{SCR}_{K//F}$. Extending the fundamental cofibre sequence in Proposition 2.17 to the left, we obtain a cofibre sequence $L_{F/B}[-1] \rightarrow (F \otimes_B L_{B/K}) \rightarrow L_{F/K}$. It induces the asserted natural cofibre sequence after taking the F -linear duals, because we have $\bar{T}(V) \simeq (F \otimes_B L_{B/K})^\vee$ and $\mathrm{Lie}_F^\pi(V) \simeq (L_{F/B}[-1])^\vee$ by construction of these two monads. \square

Proof of Theorem 3.1. Statement (1) follows directly from the definition of $\mathrm{cot}_{K//F}$.

For (2), we first show that the functor \bar{T} preserves the subcategory $\mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}} \subset \mathrm{Mod}_F$, commutes with finite coconnective geometric realisations and is right complete (cf. Construction 2.23(2)).

Indeed, this follows from the cofibre sequence in Proposition 3.4 and the corresponding claims for the functor Lie_F^π , which are established in [BM19, Corollary 5.46]. We then deduce from the second equivalence in Construction 2.23(3) (cf. [BM19, Corollary 3.17]) that $\bar{T}|_{\mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}}$ extends uniquely to a sifted-colimit-preserving monad T on Mod_F .

For (3), note that the cofibre sequence in Proposition 3.4 induces a transformation of functors

$$L_{F/K}^\vee \rightarrow T.$$

by applying the first equivalence in Construction 2.23(3) (cf. [BM19, Proposition 3.16]). Hence, every T -algebra receives a canonical map from $L_{F/K}^\vee$. We obtain a factorisation of the forgetful functor $\mathcal{C} := \mathrm{Alg}_T \rightarrow \mathrm{Mod}_F$ over the forgetful functor $U : (\mathrm{Mod}_F)_{L_{F/K}^\vee/} \rightarrow \mathrm{Mod}_F$, $(L_{F/K}^\vee \rightarrow V) \mapsto V$.

By Lemma 3.3, the forgetful functor U creates small limits, filtered colimits, and geometric realisations, because filtered categories and $\mathbf{\Delta}^{\mathrm{op}}$ are contractible categories. Since the forgetful functor $\mathcal{C} \rightarrow \mathrm{Mod}_F$ is conservative and preserves small limits and sifted colimits, this implies that the same holds true for $\mathcal{C} \rightarrow (\mathrm{Mod}_F)_{L_{F/K}^\vee/}$. The (crude) Barr–Beck–Lurie theorem (cf. [Lur17, Theorem 4.7.0.3]) therefore implies that

$$\mathcal{C} \rightarrow (\mathrm{Mod}_F)_{L_{F/K}^\vee/}$$

is a sifted-colimit-preserving monadic right adjoint.

The fibre functor $\mathrm{fib} : (\mathrm{Mod}_F)_{L_{F/K}^\vee/} \rightarrow \mathrm{Mod}_F$, $(f : M \rightarrow L_{F/K}^\vee[1]) \mapsto \mathrm{fib}(f)$ lifts, by [Lur17, Theorem 1.1.2.14], to a canonical equivalence

$$\begin{aligned} (\mathrm{Mod}_F)_{L_{F/K}^\vee/} &\simeq (\mathrm{Mod}_F)_{L_{F/K}^\vee/} \\ (f : M \rightarrow L_{F/K}^\vee[1]) &\mapsto (L_{F/K}^\vee \rightarrow \mathrm{fib}(f)) \\ (\mathrm{cofib}(g) \rightarrow L_{F/K}^\vee[1]) &\leftarrow (g : L_{F/K}^\vee \rightarrow N). \end{aligned}$$

We obtain the following square:

$$(4) \quad \begin{array}{ccc} \mathcal{C} & \longrightarrow & (\mathrm{Mod}_F)_{L_{F/K}^\vee/} \\ \downarrow & \swarrow \simeq & \downarrow \\ (\mathrm{Mod}_F)_{L_{F/K}^\vee/} & \longrightarrow & \mathrm{Mod}_F \end{array}$$

The top horizontal map is therefore a sifted-colimit-preserving monadic right adjoint, defining the sifted-colimit-preserving monad $\text{LieAlgd}_{F/K}^\pi$ on the ∞ -category $(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$. \square

We now extend the natural cofibre sequence from Proposition 3.4:

Corollary 3.5. *There is a natural cofibre sequence*

$$L_{F/K}^\vee \rightarrow T \rightarrow \text{Lie}_F^\pi.$$

Proof. Using the equivalence $\text{End}_\Sigma(\text{Mod}_K) \xrightarrow{\simeq} \text{Fun}'_\sigma(\text{Mod}_{K, \leq 0}^{\text{ft}}, \text{Mod}_K)$ described in Construction 2.23(3) (cf. [BM19, Proposition 3.16]), we can extend the cofibre sequence $L_{F/K}^\vee \rightarrow \bar{T} \rightarrow \text{Lie}_F^\pi$ from Proposition 3.4. Here, we use that the involved functors are right-complete and preserve finite coconnective geometric realisations by [BM19, Corollary 5.46] and Theorem 3.1(2), respectively.

This gives a sequence $L_{F/K}^\vee \rightarrow T \rightarrow \text{Lie}_F^\pi$ of sifted-colimit-preserving endofunctors of Mod_K . We write C for the cofibre of $L_{F/K}^\vee \rightarrow T$ in $\text{End}(\text{Mod}_K)$ and note that as a colimit of sifted-colimit-preserving functors, C also preserves sifted colimits. The universal property of C gives a natural transformation $C \rightarrow \text{Lie}_F^\pi$. Restricting the cofibre sequence $L_{F/K}^\vee \rightarrow T \rightarrow C$ to $\text{Mod}_{K, \leq 0}^{\text{ft}}$ gives a cofibre sequence of functors $\text{Mod}_{K, \leq 0}^{\text{ft}} \rightarrow \text{Mod}_K$, and so the map $C \rightarrow \text{Lie}_F^\pi$ restricts to an equivalence on $\text{Mod}_{K, \leq 0}^{\text{ft}}$. But by the equivalence in Construction 2.23(3), this implies that the map $C \rightarrow \text{Lie}_F^\pi$ is an equivalence on all of Mod_K . \square

Property (1) of Construction 1.14 in the introduction follows by construction, as $\text{LieAlgd}_{F/K}^\pi$ preserves filtered colimits and geometric realisations. For property (5), we prove:

Proposition 3.6. *On objects with vanishing anchor map, the functor $\text{LieAlgd}_{F/K}^\pi$ satisfies*

$$\text{LieAlgd}_{F/K}^\pi \left(V \xrightarrow{0} L_{F/K}^\vee[1] \right) \simeq \left(\text{Lie}_F^\pi(V) \rightarrow L_{F/K}^\vee[1] \right).$$

Proof. Recall that we say that an anchor map vanishes if it is nullhomotopic (in Mod_F). By the universal property of fibres, we observe that $\text{fib} : (\text{Mod}_F)_{/L_{F/K}^\vee[1]} \rightarrow \text{Mod}_F$ admits a left adjoint, which sends a chain complex $V \in \text{Mod}_F$ to the zero morphism $(V \xrightarrow{0} L_{F/K}^\vee[1])$. Writing $\text{LieAlgd}_{F/K}^\pi(V \xrightarrow{0} L_{F/K}^\vee[1]) \simeq (X \rightarrow L_{F/K}^\vee[1])$, we observe from (4) that X is the cofibre of the canonical morphism $L_{F/K}^\vee \rightarrow T(V)$, which is $\text{Lie}_F^\pi(V)$ by Corollary 3.5. \square

Remark 3.7. For $F = K$, we have $L_{F/K} = 0$, and the resulting equivalence $\text{Mod}_F \simeq (\text{Mod}_F)_{/L_{F/K}^\vee[1]}$, $V \mapsto (V \rightarrow 0)$ identifies the monads Lie_F^π on Mod_F and $\text{LieAlgd}_{F/K}^\pi$ on $(\text{Mod}_F)_{/L_{F/K}^\vee[1]}$. Hence the theory of F/K -partition Lie algebroids reduces to the theory of partition Lie algebras for $F = K$.

3.2. Koszul duality. As before, let F/K be a finite purely inseparable field extension. The main aim of this section is to construct an adjunction

$$\mathfrak{D} : \text{SCR}_{K//F}^{\text{op}} \rightleftarrows \text{LieAlgd}_{F/K}^\pi : C^*$$

implementing Koszul duality, and show it restricts to an equivalence on suitably finite objects.

First, we check that the conditions of Definition 2.8 are unchanged by base change along $K \subset F$.

Lemma 3.8. *Let F/K be a finite extension of fields. Then $B \in \text{SCR}_K$ is Noetherian if and only if $R = F \otimes_K B \in \text{SCR}_F$ is Noetherian.*

Proof. First we reduce to the case where B is discrete. Since $K \rightarrow F$ is flat, we have $\pi_i(F \otimes_K B) \cong F \otimes_K \pi_i(B)$. As faithfully flat maps are closed under base change and $K \rightarrow F$ is faithfully flat, we see that $\pi_0(B) \rightarrow F \otimes_K \pi_0(B)$ is faithfully flat as well. By [The, 03C4], this implies that $F \otimes_K \pi_i(B)$ is a finite $F \otimes_K \pi_0(B)$ -module if and only if $\pi_i(B)$ is a finite $\pi_0(B)$ -module. So from now on we may assume that B is discrete.

Assume first that B is discrete and Noetherian. We can write $F = K[X_1, \dots, X_n]/I$ for some suitably chosen ideal I , using that F/K is finite. The Hilbert basis theorem therefore implies that $F \otimes_K B$ receives a surjection from a Noetherian ring, and is therefore itself Noetherian. Conversely, for any ascending chain of ideals $I_0 \subset I_1 \subset \dots$ in B , the chain $F \otimes_K I_0 \subset F \otimes_K I_1 \subset \dots$ in $F \otimes_K B$ must stabilise. Since $K \hookrightarrow F$ is faithfully flat, this implies that the original chain must stabilise too, hence B is Noetherian. \square

Corollary 3.9. *If F/K is a finite extension of fields, and $B \in \text{SCR}_{K//F}$ is Noetherian, then*

$$F \otimes_B L_{B/K} \in \text{Mod}_{F, \geq 0}^{\text{ft}}$$

is connective and of finite type.

Proof. We have an equivalence $F \otimes_B L_{B/K} \simeq \text{cot}(F \otimes_K B) = L_{F/(F \otimes_K B)}[-1]$. Since $F \otimes_K B$ is a Noetherian augmented F -algebra, Proposition 2.13 implies the result. \square

Proposition 3.10. *Let F/K be a finite purely inseparable field extension. An object $B \in \text{SCR}_{K//F}$ is complete local Noetherian if and only if the augmented F -algebra $R = F \otimes_K B \in \text{SCR}_{F//F}$ has this property.*

Proof. It follows from Lemma 3.8 that B is Noetherian if and only if R is Noetherian. Therefore it remains to prove that $\pi_0(B)$ is a complete local (discrete) ring if and only if $\pi_0(R) \simeq F \otimes_K \pi_0(B)$ is a complete local ring. Here we used the faithful flatness of $K \rightarrow F$. Therefore we may assume that B and $F \otimes_K B$ are both discrete.

The main theorem of [Swe75] shows that R is local if and only if B and $F \otimes_K (B/\mathfrak{m})$ are both local, where \mathfrak{m} is the maximal ideal of B . So it remains to show that if B is local then $F \otimes_K (B/\mathfrak{m})$ is local, which follows from [Swe75, Proposition] since F/K is purely inseparable.

Finally we deal with completeness. Consider the exact sequence

$$0 \rightarrow B \rightarrow \hat{B} \rightarrow M \rightarrow 0,$$

where B is complete if and only if $M = 0$. Tensoring this sequence with F we obtain

$$0 \rightarrow R \rightarrow \varprojlim_k (R/\mathfrak{m}_B^k R) \rightarrow F \otimes_K M \rightarrow 0.$$

Note that $M = 0$ if and only if $F \otimes_K M = 0$ by faithful flatness of F/K . This implies that R is \mathfrak{m}_B -complete if and only if B is \mathfrak{m}_B -complete. Next, we claim that the two systems of ideals $(\mathfrak{m}_B R)^k$ and \mathfrak{m}_R^k are cofinal. Then it will follow that R is \mathfrak{m}_B -complete if and only if it is \mathfrak{m}_R -complete by [The, 0319], and we are done.

To show the cofinality, let e be the exponent of F/K . Given $x = \sum_{i=1}^n f_i \otimes b_i \in \mathfrak{m}_R$, we can write x^{p^e} as $\sum_{i=1}^n f_i^{p^e} \otimes b_i^{p^e}$, which belongs to $\mathfrak{m}_R^{p^e} \cap (K \otimes_K B) \subset K \otimes_K \mathfrak{m}_B \subset \mathfrak{m}_B R$. Hence $\mathfrak{m}_R^{p^e} \subset \mathfrak{m}_B R$, and more generally $(\mathfrak{m}_R^k)^{p^e} \subset (\mathfrak{m}_B R)^k$. Conversely, it is clear that $(\mathfrak{m}_B R)^k \subset \mathfrak{m}_R^k$ for all k since \mathfrak{m}_R is maximal. \square

To construct the Koszul duality functor on simplicial commutative K -algebras over F , we will follow the strategy of [BM19, Construction 4.50]. First, let us record a simple categorical observation:

Proposition 3.11. *Let c be an object in a compactly generated ∞ -category \mathcal{C} . Then an object $(x \rightarrow c) \in \mathcal{C}_{/c}$ is compact if and only if $x \in \mathcal{C}$ is compact. Moreover, $\mathcal{C}_{/c}$ is compactly generated.*

Proof. First, observe that if $D : I \rightarrow \mathcal{C}_{/c}$, $i \mapsto (y_i \rightarrow c)$ is a diagram in $\mathcal{C}_{/c}$, the colimit of D is given by the canonical morphism $(\operatorname{colim}_I y_i \rightarrow c)$ out of the colimit of the diagram $\tilde{D} : I \rightarrow \mathcal{C}$, $i \mapsto y_i$ in \mathcal{C} .

Now assume that we are given $(x \xrightarrow{c} c) \in \mathcal{C}_{/c}$ with $x \in \mathcal{C}$ is compact. Given any other object $(y \rightarrow c) \in \mathcal{C}_{/c}$, there is an equivalence $\operatorname{Map}_{\mathcal{C}_{/c}}(x \rightarrow c, y \rightarrow c) \simeq \operatorname{fib}_c(\operatorname{Map}_{\mathcal{C}}(x, y) \rightarrow \operatorname{Map}_{\mathcal{C}}(x, c))$. As filtered colimits commute with finite limits in spaces, this implies that $(x \rightarrow c)$ is compact.

Conversely, assume that $(x \rightarrow c) \in \mathcal{C}_{/c}$ is compact. Given a filtered diagram $D : I \rightarrow \mathcal{C}$, $i \mapsto y_i$, we define a filtered diagram $\tilde{D} : I \rightarrow \mathcal{C}_{/c}$, $i \mapsto (y_i \times c \rightarrow c)$ and obtain the commutative square

$$\begin{array}{ccc} \operatorname{colim}_I \operatorname{Map}_{\mathcal{C}_{/c}}(x \rightarrow c, y_i \times c \rightarrow c) & \xrightarrow{\simeq} & \operatorname{colim}_I \operatorname{Map}_{\mathcal{C}}(x, y_i) \\ \downarrow \simeq & & \downarrow \\ \operatorname{Map}_{\mathcal{C}_{/c}}(x \rightarrow c, \operatorname{colim}_I (y_i \times c \rightarrow c)) & \xrightarrow{\simeq} & \operatorname{Map}_{\mathcal{C}}(x, \operatorname{colim}_I y_i). \end{array}$$

To see that the lower horizontal map is an equivalence, we use that filtered colimits commute with finite limits in compactly generated ∞ -categories. As D was arbitrary, we see that $x \in \mathcal{C}$ is compact.

To see that $\mathcal{C}_{/c}$ is compactly generated, let us fix an object $(x \rightarrow c)$ and write $\mathcal{D}_0 \subset \mathcal{C}_{/c}$ and $\tilde{\mathcal{D}}_0 \subset (\mathcal{C}_{/c})_{/(x \rightarrow c)}$ for the full subcategories of compact objects in \mathcal{C} and $\mathcal{C}_{/c}$ mapping to x and $(x \rightarrow c)$, respectively. We note that $\operatorname{colim}_{(y \rightarrow c) \in \tilde{\mathcal{D}}_0} (y \rightarrow c) \simeq ((\operatorname{colim}_{y \in \mathcal{D}_0} y) \rightarrow c) \simeq (x \rightarrow c)$. \square

We may therefore deduce:

Corollary 3.12. *The ∞ -category $\operatorname{SCR}_{K//F}$ is compactly generated. An object $B \in \operatorname{SCR}_{K//F}$ is compact if and only if, as a K -algebra, B is a retract of a finitely presented K -algebra.*

Notation 3.13. Write $\operatorname{SCR}_{K//F}^{\operatorname{wafp}} \subset \operatorname{SCR}_{K//F}$ for the full subcategory spanned by all B with

$$\operatorname{cot}_{K//F}(B) \in \operatorname{Mod}_{F, \geq 0}^{\operatorname{ft}}.$$

Combining Corollary 3.12 and [Lur04, Proposition 3.2.14], we see that $\operatorname{SCR}_{K//F}^{\operatorname{wafp}}$ contains all compact objects of $\operatorname{SCR}_{K//F}$, and by Corollary 3.9, it also contains all Noetherian objects.

Construction 3.14 (Koszul duality adjunction).

First, note that by construction of the monad \bar{T} in Theorem 3.1 (1), there is a canonical functor

$$\operatorname{SCR}_{K//F}^{\operatorname{op}} \rightarrow \operatorname{Alg}_{\bar{T}}, \quad B \mapsto \operatorname{cot}_{K//F}(B)^\vee.$$

Taking the opposite gives a functor $\operatorname{SCR}_{K//F} \rightarrow \operatorname{Alg}_{\bar{T}}^{\operatorname{op}}$.

As the above functor sends objects in $\operatorname{SCR}_{K//F}^{\operatorname{wafp}}$ to modules in $\operatorname{Mod}_{F, \leq 0}^{\operatorname{ft}}$, and since the monads \bar{T} and T are canonically equivalent on $\operatorname{Mod}_{F, \leq 0}^{\operatorname{ft}}$, restriction gives rise to a functor

$$\mathfrak{D} : \operatorname{SCR}_{K//F}^{\operatorname{wafp}} \rightarrow \operatorname{Alg}_T^{\operatorname{op}} \simeq \operatorname{Alg}_{\operatorname{LieAlgd}_{F/K}^\pi}^{\operatorname{op}}.$$

Since $\operatorname{SCR}_{K//F}$ is compactly generated (cf. Corollary 3.12) and $\operatorname{cot}_{K//F}$ preserves filtered colimits, the functor $\operatorname{cot}_{K//F}$ and its restriction to $\operatorname{SCR}_{K//F}^{\operatorname{wafp}}$ are both left Kan extended from compact objects. As T preserves filtered colimits, this implies that the functor \mathfrak{D} above is also left Kan extended

from these compact objects. We can therefore left Kan extend further to all of $\mathrm{SCR}_{K//F}$ to obtain a colimit-preserving functor

$$\mathfrak{D} : \mathrm{SCR}_{K//F} \rightarrow \mathrm{Alg}_{\mathrm{LieAlgd}_{F/K}}^{\mathrm{op}}.$$

Its right adjoint is called the *Chevalley–Eilenberg complex* and written as

$$C^* : \mathrm{Alg}_{\mathrm{LieAlgd}_{F/K}}^{\mathrm{op}} \rightarrow \mathrm{SCR}_{K//F}.$$

Koszul duality does not lose any information on complete local Noetherian K -algebras over F :

Theorem 3.15. *The adjunction $(\mathfrak{D} \dashv C^*)$ from Construction 3.14 restricts to a contravariant equivalence*

$$\mathrm{SCR}_{K//F}^{\mathrm{cN}} \simeq (\mathcal{D}_0)^{\mathrm{op}}$$

between the ∞ -category of complete local Noetherian objects in $\mathrm{SCR}_{K//F}$ and the full subcategory

$$\mathcal{D}_0 \subset \mathrm{Alg}_{\mathrm{LieAlgd}_{F/K}}^{\mathrm{ft}}$$

consisting of all $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^{\vee}[1])$ for which $\mathrm{fib}(\rho) \in \mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}$ is coconnective and of finite type.

Proof. First, we prove that the adjunction

$$\mathrm{cot}_{K//F} : \mathrm{SCR}_{K//F}^{\mathrm{cN}} \rightleftarrows \mathrm{Mod}_{F, \geq 0}^{\mathrm{ft}} : \mathrm{sqz}_{K//F}$$

is comonadic by verifying the conditions of the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5]; note that the associated comonad is related to the monad \bar{T} on $\mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}$ (cf. Theorem 3.1) via linear duality. To this end, assume that $B^\bullet \in \mathrm{SCR}_{K//F}^{\mathrm{cN}}$ is a cosimplicial diagram for which

$$\mathrm{cot}_{K//F}(B^\bullet) \simeq \mathrm{cot}_F(F \otimes_K B^\bullet) \in \mathrm{Mod}_{F, \geq 0}^{\mathrm{ft}}$$

admits a splitting. Applying [BM19, Theorem 4.20.(1)] to the setup of $\mathrm{SCR}_F^{\mathrm{aug}}$ (as explained in [BM19, Section 5.2]), we see that the limit $\mathrm{Tot}(F \otimes_K B^\bullet) \in \mathrm{SCR}_F^{\mathrm{aug}}$ exists, is computed in Mod_F , and belongs to $\mathrm{SCR}_F^{\mathrm{cN}}$. Moreover, we obtain a canonical equivalence

$$\mathrm{cot}_F(\mathrm{Tot}(F \otimes_K B^\bullet)) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{cot}_F(F \otimes_K B^\bullet)).$$

The map $F \otimes_K \mathrm{Tot}(B^\bullet) \rightarrow \mathrm{Tot}(F \otimes_K B^\bullet)$ is an equivalence because the totalisation preserves finite colimits and F is a finite free K -module. We deduce from Proposition 3.10 that $\mathrm{Tot}(B^\bullet) \in \mathrm{SCR}_{K//F}^{\mathrm{cN}}$ is complete local Noetherian, computed in Mod_K , and satisfies

$$\mathrm{cot}_F(F \otimes_K \mathrm{Tot}(B^\bullet)) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{cot}_F(F \otimes_K B^\bullet)).$$

Finally, the functor $\mathrm{cot}_F(F \otimes_K (-)) : \mathrm{SCR}_{K//F}^{\mathrm{cN}} \rightarrow \mathrm{Mod}_F$ is conservative as this is evidently true for $F \otimes_K (-) : \mathrm{SCR}_{K//F}^{\mathrm{cN}} \rightarrow \mathrm{SCR}_{F//F}^{\mathrm{cN}}$ and also holds for $\mathrm{cot}_F : \mathrm{SCR}_{F//F}^{\mathrm{cN}} \rightarrow \mathrm{Mod}_F$ by the proof of [BM19, Theorem 4.20] applied to the setting in Section 5.2 of [op.cit.]. Note that this is where completeness is crucially used.

To finish the proof, let us observe the following triangle of adjunctions:

$$\begin{array}{ccc}
\mathrm{SCR}_{K//F} & \begin{array}{c} \xrightarrow{\mathfrak{D}} \\ \xleftarrow{C^*} \end{array} & \mathrm{Alg}_{\mathrm{LieAlg}d_{F/K}}^{\mathrm{op}} \\
& \searrow \mathrm{cot}_{K//F}^{\vee} & \uparrow \mathrm{Free} \\
& \mathrm{sqz}_{K//F}(-^{\vee}) & \mathrm{Mod}_F^{\mathrm{op}} \\
& & \downarrow \mathrm{Forget}
\end{array}$$

For $V \in \mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}$, the counit of the adjunction $\mathfrak{D} \dashv C^*$ gives a map $\mathrm{Free}(V) \rightarrow \mathfrak{D}(C^*(\mathrm{Free}(V)))$ in $\mathrm{Alg}_{\mathrm{LieAlg}d_{F/K}}^{\mathrm{op}}$, which is an equivalence because applying the conservative forgetful functor gives the equivalence $\mathrm{Forget}(\mathrm{Free}(V)) = T(V) \xrightarrow{\cong} \bar{T}(V) = \mathrm{cot}_{K//F}^{\vee}(\mathrm{sqz}_{K//F}^{\vee}(V))$. Since the composite $C^*(\mathrm{Free}(V)) \rightarrow C^*(\mathfrak{D}(C^*(\mathrm{Free}(V)))) \rightarrow C^*(\mathrm{Free}(V))$ of unit and counit is always an equivalence, and we may therefore deduce that the unit

$$B \rightarrow C^*(\mathfrak{D}(B))$$

is an equivalence for all $B = \mathrm{sqz}_{K//F}(V^{\vee})$ with $V \in \mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}$.

For a general $B \in \mathrm{SCR}_{K//F}^{\mathrm{cN}}$, we combine the comonadicity established above with the equivalence $(-)^{\vee} : \mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}} \simeq (\mathrm{Mod}_{F, \geq 0}^{\mathrm{ft}})^{\mathrm{op}}$ to write $B \simeq \mathrm{Tot}(B^{\bullet})$ as a totalisation preserved by \mathfrak{D} of a diagram of trivial square-zero extensions. Here, we use the cobar resolution coming from the comonadic adjunction $\mathrm{cot}_{K//F} : \mathrm{SCR}_{K//F}^{\mathrm{cN}} \rightleftarrows \mathrm{Mod}_{F, \geq 0}^{\mathrm{ft}} : \mathrm{sqz}_{K//F}$, cf. [Lur17, Proposition 4.7.3.3].

We may therefore deduce that the unit

$$\eta_B : B \rightarrow C^*(\mathfrak{D}(B))$$

is an equivalence for all $B \in \mathrm{SCR}_{K//F}^{\mathrm{cN}}$, which in turn implies that $\mathfrak{D}|_{\mathrm{SCR}_{K//F}^{\mathrm{cN}}}$ is fully faithful. Indeed, this last implication follows from a well-known categorical argument which we recall for the reader's convenience. Given $B_1, B_2 \in \mathrm{SCR}_{K//F}^{\mathrm{cN}}$, naturality of the unit shows that the composite

$$\begin{aligned}
\mathrm{Map}_{\mathrm{SCR}_{K//F}}(B_1, B_2) & \xrightarrow{\mathfrak{D}(-)} \mathrm{Map}_{\mathrm{Alg}_{\mathrm{LieAlg}d_{F/K}}^{\mathrm{op}}}(\mathfrak{D}(B_2), \mathfrak{D}(B_1)) \\
& \xrightarrow{C^*(-)} \mathrm{Map}_{\mathrm{SCR}_{K//F}}(C^*(\mathfrak{D}(B_1)), C^*(\mathfrak{D}(B_2))) \\
& \xrightarrow{- \circ \eta_{B_1}} \mathrm{Map}_{\mathrm{SCR}_{K//F}}(B_1, C^*(\mathfrak{D}(B_2)))
\end{aligned}$$

is given by postcomposition with the unit $\eta_{B_2} : B_2 \rightarrow C^*(\mathfrak{D}(B_2))$ and therefore an equivalence. As the second two arrows compose to an equivalence by the defining property of adjunctions (cf. [Lur09, Section 5.2.2]), the first map is an equivalence and so $\mathfrak{D}|_{\mathrm{SCR}_{K//F}^{\mathrm{cN}}}$ is fully faithful.

To identify the essential image of $\mathfrak{D}|_{\mathrm{SCR}_{K//F}^{\mathrm{cN}}}$, we unravel the definitions to factor this functor as a chain of equivalences, once more using the comonadicity result established above:

$$\mathrm{SCR}_{K//F}^{\mathrm{cN}} \simeq \mathrm{coAlg}_{\mathrm{cot}_{K//F} \circ \mathrm{sqz}_{K//F}}(\mathrm{Mod}_{F, \geq 0}^{\mathrm{ft}}) \simeq \mathrm{Alg}_T(\mathrm{Mod}_{F, \leq 0}^{\mathrm{ft}}) \simeq \mathcal{D}_0.$$

□

3.3. An interlude on hypercoverings. The aim of this section and the next is to relate our partition Lie algebroids with partition Lie algebras and formal moduli problems via two natural functors described in Construction 1.14 (3) and (4). Since this requires more knowledge of derived algebraic geometry than we have used thus far, the reader interested in the Galois correspondence may wish to skip to section 4 in which we finish the proof of Theorem 1.11.

To construct functors on partition Lie algebroids, we will use the theory of hypercoverings, which allows us to proceed in two steps. First, we define a functor on certain small Lie algebroids which admit an interpretation in terms of rings, second, we extend it via distinguished simplicial resolutions known as hypercoverings.

The theory of hypercoverings originated in the work of Verdier [AGV72], and have since been revisited by many authors, most recently also in a higher categorical setting (cf. e.g. [DHI04], [TV05, Section 3.2], [Lur09, Section 6.5.3] [Lur17, Section 7.2.1], or [BM19, Appendix]).

Rather than delving into the general theory, we shall only discuss hypercoverings in the context of partition Lie algebroids, relying on the more general results established in [BM19, Appendix].

Given a finite purely inseparable field extension F/K , we will use two distinguished classes of objects to build general Lie algebroids.

Notation 3.16. Write $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C} := \text{Alg}_{\text{LieAlgd}_{F/K}^\pi}$ for the full subcategories of free algebras

$$\text{LieAlgd}_{F/K}^\pi \left(V \xrightarrow{0} L_{F/K}^\vee[1] \right),$$

where $V \in \text{Mod}_F$ is assumed to be perfect coconnective for \mathcal{C}_0 and coconnective for \mathcal{C}_1 , respectively.

We introduce the class of morphisms which one can think of as some kind of covers:

Definition 3.17 (Coconnective anchor surjections). A map of F/K -partition Lie algebroids

$$\begin{array}{ccc} \mathfrak{g}_1 & \longrightarrow & \mathfrak{g}_2 \\ & \searrow \rho_1 & \downarrow \rho_2 \\ & & L_{F/K}^\vee[1] \end{array}$$

is a *coconnective anchor surjection* if it induces a surjection $\pi_i(\text{fib}(\rho_1)) \rightarrow \pi_i(\text{fib}(\rho_2))$ for all $i \leq 0$.

We define a distinguished family of simplicial resolutions:

Definition 3.18 (Hypercoverings). Let $Z = (\mathfrak{g} \rightarrow L_{F/K}^\vee[1])$ be an F/K -partition Lie algebroid. A *hypercovering* for Z is an augmented simplicial F/K -partition Lie algebroid

$$X_\bullet \rightarrow Z$$

satisfying the following conditions:

- (1) Each object X_n belongs to \mathcal{C}_1 ;
- (2) Each map $X_i \rightarrow Z$ is a coconnective anchor surjection;
- (3) All matching objects $M_n(X_\bullet)$ and all latching objects $L_n(X_\bullet)$ exist in $(\mathcal{C}_{/Z})'$, the ∞ -category of all F/K -partition Lie algebroids mapping to Z via an anchor surjection (cf. [BM19, Definition 8.1] for a definition of matching and latching objects).
- (4) Each natural map $X_n \rightarrow M_n(X_\bullet)$ is a coconnective anchor surjection. Each natural map $L_n(X_\bullet) \rightarrow X_n$ expresses X_n as a coproduct of $L_n(X_\bullet)$ with an object in the subcategory $(\mathcal{C}_{1/Z})' \subset (\mathcal{C}_{/Z})'$, spanned by all $X \rightarrow Z$ with $X \in \mathcal{C}_1$.

Such hypercoverings exist in abundance:

Proposition 3.19 (Hypercovers and left Kan extensions). *Any $Z \in \text{Alg}_{\text{LieAlg}_{F/K}^\pi}$ admits a hypercovering $X_\bullet \rightarrow Z$, which is in fact a colimit diagram. If \mathcal{D} is a presentable ∞ -category and $F : \text{Alg}_{\text{LieAlg}_{F/K}^\pi} \rightarrow \mathcal{D}$ is left Kan extended from \mathcal{C}_0 , we have a natural equivalence $|F(X_\bullet)| \simeq F(Z)$. Moreover, any sifted-colimit-preserving functor $\text{Alg}_{\text{LieAlg}_{F/K}^\pi} \rightarrow \mathcal{D}$ is left Kan extended from \mathcal{C}_0 .*

Proof. Recall the sifted-colimit-preserving monad T on Mod_F constructed in Theorem 3.1 (2). Starting with the set \mathcal{F}_0 of all T -algebras of the form $T(V)$ with $V \in \text{Perf}_{F, \leq 0}$ perfect coconnective, [BM19, Construction 8.9] gives a weakly orthogonal pair $(\mathcal{F}_1, \mathcal{S})$ in the sense of Definition 8.4 in [op.cit.]. Here \mathcal{F}_1 is the class of objects which are coproducts of objects in \mathcal{F}_0 ; hence \mathcal{F}_1 consists of all T -algebras of the form $T(V)$ with $V \in \text{Mod}_{F, \leq 0}$. The class of morphisms \mathcal{S} consists of all $f : X_1 \rightarrow X_2$ in Alg_T for which the induced map $\pi_0 \text{Map}_{\text{Alg}_T}(F, X_1) \rightarrow \pi_0 \text{Map}_{\text{Alg}_T}(F, X_2)$ is surjective for all $F \in \mathcal{F}_0$. Unravelling the definitions, we see that \mathcal{S} is given by the family of morphisms of T -algebras $X_1 \rightarrow X_2$ which induce surjections on π_i for all $i \leq 0$.

By Lemma 8.6 in [op.cit.], any $B \in \text{Alg}_T$ admits an $(\mathcal{F}_1, \mathcal{S})$ -hypercovering $A_\bullet \rightarrow B$ in the sense of Definition 8.7 in [op.cit.]. We explain in Example 8.12 in [op.cit.] that $A_\bullet \rightarrow B$ is a colimit diagram, that any $G : \text{Alg}_T \rightarrow \mathcal{D}$ which is left Kan extended from \mathcal{F}_0 satisfies $|G(A_\bullet)| \simeq G(B)$, and that any sifted-colimit-preserving functor $F : \text{Alg}_T \rightarrow \mathcal{D}$ is left Kan extended from \mathcal{F}_0 .

Finally, we recall the equivalence $\text{Alg}_{\text{LieAlg}_{F/K}^\pi} \simeq \text{Alg}_T(\text{Mod}_F)$ from Theorem 3.1 (3), lifting the fibre functor $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \mapsto \text{fib}(\rho)$. Unravelling the definitions, we use diagram (4) in the proof of Theorem 3.1 to observe that $\mathcal{C}_0 \subset \mathcal{C}_1$ and \mathcal{S} correspond to $\mathcal{F}_0 \subset \mathcal{F}_1$ and \mathcal{S} under this equivalence, which implies the various assertions. \square

We can now deduce a convenient extension result:

Proposition 3.20. *Let $U : \mathcal{D} \rightarrow \mathcal{E}$ be a functor of presentable ∞ -categories creating sifted colimits. Let $\mathcal{C}_0 \subset \mathcal{C}$ be as in Notation 3.16, and assume we are given a functor $G_0 : \mathcal{C}_0 \rightarrow \mathcal{D}$ for which the composite $U \circ G_0 : \mathcal{C}_0 \rightarrow \mathcal{E}$ admits a sifted-colimit-preserving extension $H : \mathcal{C} \rightarrow \mathcal{E}$. Then G_0 admits a unique sifted-colimit-preserving extension $G : \mathcal{C} \rightarrow \mathcal{D}$ filling the following diagram:*

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{G_0} & \mathcal{D} \\ \downarrow & \nearrow G & \downarrow U \\ \mathcal{C} & \xrightarrow{H} & \mathcal{E} \end{array}$$

Proof. The left Kan extension $G := \text{Lan}_{\mathcal{C}_0}^{\mathcal{C}}(G_0)$ to \mathcal{C} exists by [Lur09, 4.3.2.14] as \mathcal{D} is presentable. We will now verify that G satisfies the conclusion in the proposition, thereby proving existence.

As $H : \mathcal{C} \rightarrow \mathcal{E}$ preserves sifted colimits, it is left Kan extended from \mathcal{C}_0 by Proposition 3.19, and the equivalence $H|_{\mathcal{C}_0} \simeq U \circ G_0$ extends to a natural transformation $\alpha : H \rightarrow U \circ G$. As G is left Kan extended from its values on compact objects, it preserves filtered colimits, and the same holds true for U and H . Since any object in \mathcal{C}_1 is a filtered colimit of objects in \mathcal{C}_0 , we deduce that α is an equivalence on \mathcal{C}_1 . Given a general $Z \in \mathcal{C}$. We use Proposition 3.20 to pick a hypercovering $X_\bullet \rightarrow Z$, which is also a colimit diagram. We then consider the following commuting square:

$$\begin{array}{ccc} |H(X_\bullet)| & \longrightarrow & H(Z) \\ \downarrow & & \downarrow \\ |(U \circ G)(X_\bullet)| & \longrightarrow & (U \circ G)(Z) \end{array}$$

The top horizontal map is an equivalence since H preserves geometric realisations, the bottom since U preserves realisations and $|G(X_\bullet)| \simeq G(Z)$ by Proposition 3.19, and the left vertical map since α is an equivalence on all objects in \mathcal{C}_1 by our previous considerations. Hence $\alpha : H \xrightarrow{\simeq} U \circ G$ is an equivalence, which implies that G preserves sifted colimits as H preserves and U creates them.

The functor G is unique as any sifted-colimit-preserving functor $\mathcal{C} \rightarrow \mathcal{D}$ must be left Kan extended from \mathcal{C}_0 by Proposition 3.19. \square

3.4. Functors on partition Lie algebroids. The main goal of this section is to construct two functors $\text{Alg}_{\text{LieAlg}d_{F/K}^\pi} \rightarrow (\text{Alg}_{\text{Lie}^\pi_F})_{L_{F/K}^\vee}$, and $\text{Alg}_{\text{LieAlg}d_{F/K}^\pi} \rightarrow (\text{Alg}_{\text{Lie}^\pi_K})_{/L_{F/K}^\vee[1]}$ on partition Lie algebroids which enhance the fiber functor and forgetful functor as described in Construction 1.14 (4) and (3), respectively. This illustrates that partition Lie algebroids really do behave like a derived version of classical Lie algebroids. We start with the fibre functor described in Construction 1.14 (4):

Proposition 3.21 (Fibre of the anchor). *Consider the functor $\text{fib} : \text{Alg}_{\text{LieAlg}d_{F/K}^\pi} \rightarrow (\text{Mod}_F)_{L_{F/K}^\vee}$ sending $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ to $\text{fib}(\rho)$. Then fib admits a canonical sifted-colimit-preserving lift*

$$\text{Alg}_{\text{LieAlg}d_{F/K}^\pi} \rightarrow (\text{Alg}_{\text{Lie}^\pi_F})_{L_{F/K}^\vee},$$

where $(\text{Alg}_{\text{Lie}^\pi_F})_{L_{F/K}^\vee}$ is the F -partition Lie algebra of Example 1.9 (1)

Proof. By Theorem 3.15, the assignment $R \mapsto \mathfrak{D}(R) = (L_{F/R}^\vee[1] \rightarrow L_{F/K}^\vee[1]) \in \text{Alg}_{\text{LieAlg}d_{F/K}^\pi}$ induces a canonical equivalence $(\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \xrightarrow{\simeq} \mathcal{D}_0$ with inverse $\mathfrak{g} \mapsto C^*(\mathfrak{g})$. Here \mathcal{D}_0 consists of all F/K -partition Lie algebroids $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \in \text{Alg}_{\text{LieAlg}d_{F/K}^\pi}$ which satisfy $\text{fib}(\rho) \in \text{Mod}_{F, \leq 0}^{\text{ft}}$. Note that \mathcal{D}_0 contains the full subcategory $\mathcal{C}_0 \subset \mathcal{D}_0$ of free algebroids on objects $(V \xrightarrow{0} L_{F/K}^\vee[1])$ with $V \in \text{Perf}_{F, \leq 0}$, cf. Notation 3.16. Let us now consider the following composite functor G_0 :

$$\mathcal{C}_0 \hookrightarrow \mathcal{D}_0 \xrightarrow{\simeq} (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \xrightarrow{B \mapsto F \otimes_K B} \left((\text{SCR}_{F//F}^{\text{cN}})_{/F \otimes_K F} \right)^{\text{op}} \xrightarrow{R \mapsto (L_{F/F}^\vee \otimes_K F[1] \rightarrow L_{F/R}^\vee[1])} (\text{Alg}_{\text{Lie}^\pi_F})_{L_{F/K}^\vee},$$

where the final map is defined using the equivalence contained in [BM19, Corollary 5.46].

Write $\text{Forget} : (\text{Alg}_{\text{Lie}^\pi_F})_{L_{F/K}^\vee} \rightarrow \text{Mod}_F, (L_{F/K}^\vee \rightarrow M) \mapsto M$ for the forgetful functor. Using Proposition 2.18, we observe that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} \mathcal{C}_0 \subset \mathcal{D}_0 & \xrightarrow[\simeq]{C^*} & (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} & \xrightarrow{B \mapsto F \otimes_K B} & \left((\text{SCR}_{F//F}^{\text{cN}})_{/F \otimes_K F} \right)^{\text{op}} & \longrightarrow & (\text{Alg}_{\text{Lie}^\pi_F})_{L_{F/K}^\vee} \\ & \searrow \text{id} & \downarrow \mathfrak{D} \simeq & \searrow B \mapsto \text{cot}_{K//F}(B)^\vee \simeq \text{cot}_F(F \otimes_K B)^\vee \simeq (F \otimes_B L_{B/K})^\vee & \searrow R \mapsto \text{cot}_F(R)^\vee = L_{F/R}^\vee[1] & & \downarrow \text{Forget} \\ & & \mathcal{D}_0 & \xrightarrow{B \mapsto (L_{F/B}^\vee[1] \rightarrow L_{F/K}^\vee[1])} & (\text{Mod}_F)_{/L_{F/K}^\vee[1]} & \xrightarrow{\text{fib}} & \text{Mod}_F \end{array}$$

Hence $\text{Forget} \circ G_0$ simply takes the fibre of the anchor map, and therefore extends to a sifted-colimit-preserving functor $H : \text{Alg}_{\text{LieAlg}d_{F/K}^\pi} \rightarrow \text{Mod}_F$ sending $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ to $\text{fib}(\rho)$. Proposition 3.20 then gives the desired sifted-colimit-preserving lift. \square

To construct the forgetful functor $U : \text{Alg}_{\text{LieAlgd}_{F/K}^{\pi}} \rightarrow (\text{Alg}_{\text{Lie}_K^{\pi}})_{/L_{F/K}^{\vee}[1]}$ appearing in Construction 1.14 (3), we will consider Kodaira–Spencer type formal moduli problems for K -schemes. To this end, we follow the arguments of [Lur16, 19.4] in a derived (rather than spectral) setting.

More precisely, let

$$\text{Var}_{\simeq}^+ : \text{SCR} \rightarrow \mathcal{S}$$

denote the functor sending $B \in \text{SCR}$ to the underlying Kan complex $\text{Var}_{\simeq}^+(B)$ of $\text{Var}^+(B)$, the (essentially small) ∞ -category of maps of derived Deligne–Mumford stacks

$$Z \rightarrow \text{Spec}(B)$$

which are proper, flat, and locally of almost finite presentation. Here a morphism $B_1 \rightarrow B_2$ is sent to the functor $\text{Var}^+(B_1) \rightarrow \text{Var}^+(B_2)$ obtained by pulling back a given $(Z \rightarrow \text{Spec}(B_1)) \in \text{Var}^+(B_1)$ along the morphism $\text{Spec}(B_2) \rightarrow \text{Spec}(B_1)$.

Given $B \in \text{SCR}$ and a point $\eta \in \text{Var}_{\simeq}^+(B)$ corresponding to $Z \rightarrow \text{Spec}(B)$, we may encode derived deformations of Z by the functor

$$(5) \quad \text{Def}_{\eta}^+ : \text{SCR}_{/B} \rightarrow \mathcal{S}, \quad A \mapsto \text{Var}_{\simeq}^+(A) \times_{\text{Var}_{\simeq}^+(B)} \{\eta\}.$$

This functor is cohesive (cf. [Lur04, Definition 3.4.1]) as this holds for the functor Var_{\simeq}^+ by (the derived version of) [Lur16, Theorem 19.4.0.2]. Note that the forgetful functor $\text{SCR}_{/B} \rightarrow \text{SCR}$ creates pullbacks.

Any finite purely inseparable field extension $K \subset F$ gives a map of schemes $\text{Spec}(F) \rightarrow \text{Spec}(K)$ which is proper, flat, and locally of almost finite presentation, and hence gives a point $\eta \in \text{Var}_{\simeq}^+(K)$.

Restricting Def_{η}^+ to the ∞ -category $\text{SCR}_K^{\text{art}}$ from Definition 2.19, we obtain a functor

$$\text{Def}_{F/K}(A) := \text{Var}_{\simeq}^+(A) \times_{\text{Var}_{\simeq}^+(K)} \{\eta\},$$

which is the formal moduli problem (over K) encoding infinitesimal deformations of the K -scheme $\text{Spec}(F)$, cf. [Lur16, Remark 19.4.4.1]. The cited remark also proves the well-known fact that the tangent fibre of $\text{Def}_{F/K}$ is given by $L_{F/K}^{\vee}[1]$. By [BM19], Theorem 1.11], this tangent fibre is moreover equipped with the structure of a partition Lie algebra controlling the formal moduli problem $\text{Def}_{F/K}$, as was already asserted in Example 1.9 (2). With this in hand, we can show:

Proposition 3.22 (Forgetful functor). *The forgetful functor $\text{Alg}_{\text{LieAlgd}_{F/K}^{\pi}} \rightarrow (\text{Mod}_K)_{/L_{F/K}^{\vee}[1]}$ sending an algebroid $(\mathfrak{g} \rightarrow L_{F/K}^{\vee}[1])$ to its underlying object in $(\text{Mod}_K)_{/L_{F/K}^{\vee}[1]}$ lifts canonically to a sifted-colimit-preserving functor*

$$U : \text{Alg}_{\text{LieAlgd}_{F/K}^{\pi}} \rightarrow (\text{Alg}_{\text{Lie}_K^{\pi}})_{/L_{F/K}^{\vee}[1]},$$

where $L_{F/K}^{\vee}[1]$ is the K -partition Lie algebra of Example 1.9 (2).

Proof. We will again use the equivalence in Theorem 2.22. First, we will construct a functor

$$\text{Def}_{F/\bullet/K} : (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow (\text{Moduli}_K)_{/\text{Def}_{F/K}}$$

which sends a given $B \in \text{SCR}_{K//F}^{\text{cN}}$ with maximal ideal \mathfrak{m} to the formal moduli problem (over K) encoding compatible families of deformations of the morphisms $\text{Spec}(F) \rightarrow \text{Spec}(B/\mathfrak{m}^n)$ which hold the targets $\text{Spec}(B/\mathfrak{m}^n)$ fixed.

To formalise this, let us write $\mathrm{SCR}_{K//F}^{\mathrm{art}} \subset \mathrm{SCR}_{K//F}^{\mathrm{cN}}$ for the full subcategory spanned by all B with $\dim_K(\pi_*(B)) < \infty$. Let us fix some $B \in \mathrm{SCR}_{K//F}^{\mathrm{art}}$ and write $\theta_B \in \mathrm{Var}_{\simeq}^+(K)^{\Delta^1}$ for the K -morphism $\mathrm{Spec}(F) \rightarrow \mathrm{Spec}(B)$; note that $\mathrm{Spec}(F)$ and $\mathrm{Spec}(B)$ indeed belong to $\mathrm{Var}_{\simeq}^+(K)$.

Write $\eta_B \in \mathrm{Var}_{\simeq}^+(K)$ for the K -morphism $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(K)$ and

$$\mathrm{triv}_B(A) = \mathrm{Spec}(A) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(B) \in \mathrm{Var}_{\simeq}^+(A) \times_{\mathrm{Var}_{\simeq}^+(K)} \{\eta_B\}$$

for the trivial deformation of $\mathrm{Spec}(B)$ to A . We note that the object $\mathrm{triv}_B(A)$ is picked out by the map $* \simeq \mathrm{Var}_{\simeq}^+(K) \times_{\mathrm{Var}_{\simeq}^+(K)} \{\eta_B\} \xrightarrow{\mathrm{Var}_{\simeq}^+(K \rightarrow A) \times \mathrm{id}} \mathrm{Var}_{\simeq}^+(A) \times_{\mathrm{Var}_{\simeq}^+(K)} \{\eta_B\}$, so it depends functorially on A .

We now consider the functor

$$\mathrm{Def}_{F/B/K} : \mathrm{SCR}_K^{\mathrm{art}} \rightarrow \mathcal{S}$$

$$A \mapsto \mathrm{fib}_{\mathrm{triv}_B(A)} \left(\mathrm{Var}_{\simeq}^+(A)^{\Delta^1} \times_{\mathrm{Var}_{\simeq}^+(K)^{\Delta^1}} \{\theta_B\} \xrightarrow{\mathrm{ev}_1} \mathrm{Var}_{\simeq}^+(A) \times_{\mathrm{Var}_{\simeq}^+(K)} \{\eta_B\} \right).$$

Informally, $\mathrm{Def}_{F/B/K}$ sends $A \in \mathrm{SCR}_K^{\mathrm{art}}$ to the space of all pullback diagrams

$$\begin{array}{ccc} \mathrm{Spec}(F) & \longrightarrow & \tilde{Z} \\ \downarrow \theta_B & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(B) \end{array} .$$

We now claim that $\mathrm{Def}_{F/B/K}$ satisfies the axioms of a formal moduli problem over K .

First, we note that $\mathrm{Def}_{F/B/K}(K)$ is evidently a contractible space. Next, fix a pullback square

$$\begin{array}{ccc} A_3 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array}$$

in $\mathrm{SCR}_K^{\mathrm{art}}$ for which the morphisms $\pi_0(A_1) \rightarrow \pi_0(A_0)$ and $\pi_0(A_2) \rightarrow \pi_0(A_0)$ are surjective. As $\mathrm{Var}_{\simeq}^+(-)$ is cohesive by (the derived version of) [Lur16, Theorem 19.4.0.2], applying $\mathrm{Var}_{\simeq}^+(-)$ to the above square gives a pullback in spaces. Since the functor $\mathrm{Def}_{F/B/K}(-)$ is built from $\mathrm{Var}_{\simeq}^+(-)$ by operations which preserve pullbacks, we see that the following square is a pullback in \mathcal{S} :

$$\begin{array}{ccc} \mathrm{Def}_{F/B/K}(A_3) & \longrightarrow & \mathrm{Def}_{F/B/K}(A_2) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{F/B/K}(A_1) & \longrightarrow & \mathrm{Def}_{F/B/K}(A_0). \end{array}$$

Hence $\mathrm{Def}_{F/B/K}$ is a formal moduli problem.

The assignment $B \mapsto \text{Def}_{F/B/K} \in \text{Moduli}_K \subset \text{Fun}(\text{SCR}_K^{\text{art}}, \mathcal{S})$ is contravariantly functorial in B . Indeed, this follows from the defining pullback diagram

$$\begin{array}{ccc} \text{Def}_{F/B/K} & \longrightarrow & \text{Var}_{\simeq}^+(K) \times_{\text{Var}_{\simeq}^+(K)} \{\eta_B\} \\ \downarrow & & \downarrow \text{Var}_{\simeq}^+(K \rightarrow A) \times \text{id} \\ \text{Var}_{\simeq}^+(A)^{\Delta^1} \times_{\text{Var}_{\simeq}^+(K)^{\Delta^1}} \{\theta_B\} & \xrightarrow{\text{ev}_1} & \text{Var}_{\simeq}^+(A) \times_{\text{Var}_{\simeq}^+(K)} \{\eta_B\}. \end{array}$$

since θ_B and $\eta_B = \text{ev}_1(\theta_B)$ depend contravariantly functorially on B .

Hence we obtain a functor $\text{Def}_{F/\bullet/K} : \text{SCR}_{K//F}^{\text{art,op}} \rightarrow \text{Moduli}_K$

We now consider the ∞ -category $\text{Pro}(\text{SCR}_{K//F}^{\text{art}})^{\text{op}} \subset \text{Fun}(\text{SCR}_{K//F}^{\text{art}}, \mathcal{S})$ of finite-limit-preserving functors $\text{SCR}_{K//F}^{\text{art}} \rightarrow \mathcal{S}$. Given $B \in \text{SCR}_{K//F}^{\text{cN}}$, the functor $\text{Map}_{\text{SCR}_{K//F}}(B, -) : \text{SCR}_{K//F}^{\text{art}} \rightarrow \mathcal{S}$ belongs to $\text{Pro}(\text{SCR}_{K//F}^{\text{art}})^{\text{op}}$, and this assignment gives a functor $Y : (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow \text{Pro}(\text{SCR}_{K//F}^{\text{art}})^{\text{op}}$.

Since $\text{Pro}(\text{SCR}_{K//F}^{\text{art}})^{\text{op}} = \text{Ind}(\text{SCR}_{K//F}^{\text{art,op}})$, we can use the universal property of the Ind-construction (cf. [Lur09, Proposition 5.3.5.10]) to extend the functor $\text{Def}_{F/\bullet/K} : \text{SCR}_{K//F}^{\text{art,op}} \rightarrow \text{Moduli}_K$ in a filtered-colimit-preserving way to a functor $\text{Pro}(\text{SCR}_{K//F}^{\text{art}})^{\text{op}} \rightarrow \text{Moduli}_K$. Precomposing with the Yoneda functor $Y : (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow \text{Pro}(\text{SCR}_{K//F}^{\text{art}})^{\text{op}}$ gives an extension $(\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow \text{Moduli}_K$ of $\text{Def}_{F/\bullet/K}$ from $\text{SCR}_{K//F}^{\text{art,op}}$ to $(\text{SCR}_{K//F}^{\text{cN}})^{\text{op}}$. As this functor sends K to $\text{Def}_{F/K}$, we obtain a lift

$$\text{Def}_{F/\bullet/K} : (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow (\text{Moduli}_K)_{/\text{Def}_{F/K}}.$$

Note that we have slightly abused notation by also using the name $\text{Def}_{F/\bullet/K}$ for this new functor.

We will now show that the tangent fibre of the formal moduli problem $X := \text{Def}_{F/B/K}$ is $L_{F/B}^{\vee}[1]$. Tangent fibres of formal moduli problems deforming morphisms under constraints are well-known to experts. We outline the main steps of the computation for the reader's convenience, and refer to [Nui18, Proposition 6.4.19] or [PY20, Proposition 3.11] for further details.

First, let us assume that $B \in \text{SCR}_{K//F}^{\text{art}}$ is Artinian. Recall from (3) that the tangent fibre $T_X \in \text{Mod}_K$ is characterised by a natural equivalence

$$\text{Map}_K(V^*, T_X) \simeq X(K \oplus V), \quad V \in \text{Perf}_{K, \geq 0}$$

where $(-)^*$ denotes K -linear duality. In what follows below, will write $(-)^{\vee}$ for F -linear duality. By Corollary 3.9 and the fundamental cofibre sequence, $L_{F/B}$ is of finite type. We therefore obtain a chain of natural equivalences for any $V \in \text{Perf}_{K, \geq 0}$:

$$\begin{aligned} \text{Map}_K(V^*, L_{F/B}^{\vee}[1]) &\simeq \text{Map}_F((F \otimes_K V)^{\vee}, L_{F/B}^{\vee}[1]) \\ &\simeq \Omega^{\infty} \left((F \otimes_K V) \otimes_F L_{F/B}^{\vee}[1] \right) \\ &\simeq \text{Map}_F(L_{F/B}, F \otimes_K V[1]) \end{aligned}$$

To show that $T_X \simeq L_{F/B}^{\vee}[1]$, it therefore suffices to construct a natural equivalence of spaces

$$\text{Map}_F(L_{F/B}, F \otimes_K V[1]) \simeq \text{Def}_{F/B/K}(K \oplus V), \quad V \in \text{Perf}_{K, \geq 0}.$$

Indeed, using the universal property of the cotangent complex and the fact that $B \rightarrow F$ is a map over K , we can identify the space of F -linear maps $L_{F/B} \rightarrow F \otimes_K V[1]$ with the fibre of the map $\text{Map}_{\text{SCR}_K}(F, F \oplus (F \otimes_K V[1])) \rightarrow \text{Map}_{\text{SCR}_K}(B, F \oplus (F \otimes_K V[1])) \times_{\text{Map}_{\text{SCR}_K}(B, F)} \text{Map}_{\text{SCR}_K}(F, F)$

over the point $(B \xrightarrow{(\epsilon, 0)} F \oplus (F \otimes_K V[1]), \text{id}_F)$; here $\epsilon : B \rightarrow F$ is the structure morphism of B . Equivalently, the space of F -linear maps $L_{F/B} \rightarrow F \otimes_K V[1]$ can be identified with the space of maps $\alpha : \text{Spec}(F \oplus (F \otimes_K V[1])) \rightarrow \text{Spec}(F)$ rendering commutative the following diagram:

$$\begin{array}{ccccc} \text{Spec}(F) & \longrightarrow & \text{Spec}(F \oplus (F \otimes_K V[1])) & \xrightarrow{\alpha} & \text{Spec}(F) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(B) & \longrightarrow & \text{Spec}(B \oplus (B \otimes_K V[1])) & \longrightarrow & \text{Spec}(B) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(K) & \longrightarrow & \text{Spec}(K \oplus V[1]) & \longrightarrow & \text{Spec}(K). \end{array}$$

Here horizontal composites are identity maps, and the middle and lower horizontal maps on the right correspond to $(\text{id}_B, 0)$ and $(\text{id}_K, 0)$, respectively.

The top right square is a homotopy pullback, and we therefore obtain two equivalences

$$\text{Spec}(B \oplus (B \otimes_K V[1])) \times_{\text{Spec}(B)} \text{Spec}(F) \xleftarrow{(\text{id}_F, 0)} \text{Spec}(F \oplus (F \otimes_K V[1])) \xrightarrow{\alpha} \text{Spec}(B \oplus (B \otimes_K V[1])) \times_{\text{Spec}(B)} \text{Spec}(F)$$

from which we obtain a point in the space $X(K) \times_{X(K \oplus V[1])} X(K) \simeq \Omega(X(K \oplus V[1])) \simeq X(K \oplus V)$, of automorphisms of the trivial deformation.

The resulting map $\text{Map}_F(L_{F/B}, F \otimes_K V[1]) \rightarrow \Omega(X(K \oplus V[1]))$ is an equivalence, and we deduce that $T_{\text{Def}_{F/B/K}} \simeq L_{F/B}^\vee[1]$. A diagram chase now shows that the map

$$\text{Def}_{F/B/K} \rightarrow \text{Def}_{F/K/K} \simeq \text{Def}_{F/K}$$

induces the natural morphism $L_{F/B}^\vee[1] \rightarrow L_{F/K}^\vee[1]$ on tangent fibres for all $B \in \text{SCR}_{K//F}^{\text{art}}$ Artinian. But this in fact holds true for all $B \in \text{SCR}_{K//F}^{\text{cN}}$, because Theorem 3.15 implies that the functor $(\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow (\text{Mod}_{F, \leq 0}^{\text{ft}})_{/L_{F/K}^\vee[1]}$, $B \mapsto (L_{F/B}^\vee[1] \rightarrow L_{F/K}^\vee[1])$ preserves filtered colimits.

Returning to the statement of the theorem, we will now construct the forgetful functor

$$U : \text{Alg}_{\text{LieAlg}_{F/K}^\pi} \rightarrow (\text{Alg}_{\text{Lie}_K}^\pi)_{/L_{F/K}^\vee[1]}.$$

As in Proposition 3.21, let \mathcal{D}_0 contain all $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \in \text{Alg}_{\text{LieAlg}_{F/K}^\pi}$ with $\text{fib}(\rho) \in \text{Mod}_{F, \leq 0}^{\text{ft}}$. Let $\mathcal{C}_0 \subset \mathcal{D}_0$ consist of all free algebroids on objects $(V \xrightarrow{0} L_{F/K}^\vee[1])$ with $V \in \text{Perf}_{F, \leq 0}$. Recall that if $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \in \mathcal{D}_0$ corresponds to $B \in \text{SCR}_{K//F}^{\text{cN}}$ under the equivalence in Theorem 3.15, then the underlying object of $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ is $(L_{F/B}^\vee[1] \xrightarrow{\rho} L_{F/K}^\vee[1])$. We form the composite functor G_0 :

$$\mathcal{C}_0 \subset \mathcal{D}_0 \simeq (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \xrightarrow{\text{Def}_{F/\bullet/K}} (\text{Mod}_K)_{/ \text{Def}_{F/K}} \xrightarrow{[\text{BM19, 1.11}]} (\text{Alg}_{\text{Lie}_K}^\pi)_{/L_{F/K}^\vee[1]}$$

On the other hand, the composite

$$\mathcal{D}_0 \simeq (\text{SCR}_{K//F}^{\text{cN}})^{\text{op}} \rightarrow (\text{Alg}_{\text{Lie}_K}^\pi)_{/L_{F/K}^\vee[1]} \xrightarrow{\text{Forget}} (\text{Mod}_F)_{/L_{F/K}^\vee[1]}$$

sends $B \in \mathcal{D}_0$ to the object $(L_{F/B}^\vee[1] \xrightarrow{\rho} L_{F/K}^\vee[1]) \in (\text{Mod}_F)_{/L_{F/K}^\vee[1]}$, by our above computation of the map of tangent complexes $T_{\text{Def}_{F/B/K}} \rightarrow T_{\text{Def}_{F/K}}$. Hence $\text{Forget} \circ G_0$ naturally extends to a sifted-colimit-preserving functor $H : \text{Alg}_{\text{LieAlg}_{F/K}^\pi} \rightarrow (\text{Mod}_K)_{/L_{F/K}^\vee[1]}$ and Proposition 3.20 provides the desired sifted-colimit-preserving extension G of G_0 . \square

4. THE FUNDAMENTAL THEOREM

To establish a Galois correspondence for finite purely inseparable field extensions F/K , we will use partition Lie algebroids (cf. Definition 3.2) as a natural substitute for the restricted Lie algebroids appearing in Jacobson's exponent one correspondence in Theorem 1.2.

In Theorem 3.15, we have seen that the natural tangent fibre functor

$$\mathfrak{D} : \text{SCR}_{K//F}^{\text{op}} \rightarrow \text{Alg}_{\text{LieAlg}d_{F/K}^{\pi}}$$

lifting the assignment $B \mapsto (L_{F/B}^{\vee}[1] \rightarrow L_{F/K}^{\vee}[1])$ becomes fully faithful after restriction to $(\text{SCR}_{K//F}^{\text{cN}})^{\text{op}}$, the subcategory of complete local Noetherian objects in $(\text{SCR}_{K//F})^{\text{op}}$.

As intermediate fields $K \subset E \subset F$ are in particular objects of $\text{SCR}_{K//F}^{\text{cN}}$, we obtain a description of intermediate fields in terms of partition Lie algebroids. To complete the proof of Theorem 1.11, it therefore suffices to characterise the essential image of $\mathfrak{D}|_{\text{Fields}_{K//F}}$, the restriction of \mathfrak{D} to the full subcategory $\text{Fields}_{K//F} \subset \text{SCR}_{K//F}$ spanned by intermediate fields. Note that as homomorphisms between fields are injective, $\text{Fields}_{K//F}$ in fact forms a poset.

4.1. The homotopical algebra of field extensions. Before characterising this essential image, we will review several elementary facts concerning the cotangent complex of field extensions F/K .

We start by recalling a well-known and fundamental computational tool, cf. e.g. [Sta20, Tag 08SJ]:

Lemma 4.1. *If $f : A \rightarrow B$ is a surjective map of commutative rings whose kernel I is generated by a regular sequence, then the relative cotangent complex is given by*

$$L_{B/A} \simeq I/I^2[1].$$

With this lemma, we can easily compute the relative cotangent complex for finite field extensions, as these are complete intersections. We obtain the following classical result:

Proposition 4.2. *Let F/K be a finite field extension. Pick $x_1, \dots, x_n \in F$ such that the map $\phi : K[X_1, \dots, X_n] \xrightarrow{X_i \mapsto x_i} F$ is surjective, and write I for the kernel of ϕ . There is an equivalence*

$$L_{F/K} \simeq \left(\dots \rightarrow 0 \rightarrow I/I^2 \rightarrow \Omega_{K[X_1, \dots, X_n]/K}^1 \otimes_{K[X_1, \dots, X_n]} F \right),$$

where the boundary map sends a class $[i] \in I/I^2$ to the element $di \otimes 1$.

Proof. For $k \geq 0$, we write $K[x_1, \dots, x_k] \subset F$ for the subring generated by x_1, \dots, x_k . Since $K[x_1, \dots, x_k]$ is a finite domain over K , it is in fact a field. We can inductively pick polynomials

$$P_1, \dots, P_n \in K[X_1, \dots, X_n]$$

such that P_i belongs to $R = K[X_1, \dots, X_i]$, is monic in X_i (over $K[X_1, \dots, X_{i-1}]$), and maps to the minimal polynomial of x_i over $K[x_1, \dots, x_{i-1}]$. As the polynomials are monic, P_1, \dots, P_n form a regular sequence generating $I = \ker(K[X_1, \dots, X_n] \xrightarrow{X_i \mapsto x_i} F)$. Combining Lemma 4.1 with the cofibre sequence

$$F \otimes_R L_{R/K} \rightarrow L_{F/K} \rightarrow L_{F/R},$$

we see that $L_{F/K}$ is the cofibre of a map $I/I^2 \rightarrow \Omega_{K[X_1, \dots, X_n]/K}^1 \otimes_{K[X_1, \dots, X_n]} F$. The boundary map can be identified with $[i] \mapsto di \otimes 1$ using the classical conormal sequence, see for example [Eis95, Proposition 16.3]. \square

Hence for any finite field extension F/K , the homology of $L_{F/K}$ is concentrated in two degrees.

The difference of the nonzero homology groups measures how far F/K is from being algebraic. The following result of Cartier, which appears in [Gro64, Théorème 0.21.7.1] or [Sta20, Tag 07E1], will play an important role in our main argument:

Lemma 4.3 (Cartier's equality). *Let F/K be a finitely generated field extension. Then the module of Kähler differentials $\Omega_{F/K}^1 = \pi_0(L_{F/K})$ and the module of imperfection $\Upsilon_{F/K} = \pi_1(L_{F/K})$ are finite-dimensional, and satisfy*

$$\dim_F(\Omega_{F/K}^1) - \dim_F(\Upsilon_{F/K}) = \text{trdeg}_K(F),$$

where the right hand side denotes the transcendence degree of F over K .

Note that in view of Proposition 4.2, for a finite extension, the integer which appears in Cartier's equality is actually given by the Euler characteristic $\chi(L_{F/K})$.

Remark 4.4. The *module of imperfection* was originally defined as $\Upsilon_{F/K} = \ker(\Omega_K^1 \otimes_K F \rightarrow \Omega_F^1)$, cf. e.g. [Gro64, Théorème 0.21.7.1], where $\Omega_R^1 = \Omega_{R/\mathbb{Z}}^1$ is the module of absolute Kähler differentials. It is well-known that $\Upsilon_{F/K} \cong \pi_1(L_{F/K})$ for fields (cf. [Sai20, Lemma 1.1.2] for a modern reference), and that $\Upsilon_{F/K}$ vanishes precisely if F/K is separable, cf. [Gro64, Proposition 0.20.6.3].

4.2. The essential image theorem. Finally, we come to the main result of this section, in which we characterise the essential image of the functor from intermediate field extensions $K \subset E \subset F$ to F/K -partition Lie algebroids. This will allow us to complete the proof of the fundamental theorem of purely inseparable Galois theory, Theorem 1.11.

Theorem 4.5 (Essential Image Theorem). *Let F/K be a finite purely inseparable field extension. An F/K -partition Lie algebroid $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \in \text{Alg}_{\text{LieAlgd}_{F/K}^\pi}$ is equivalent to one of the form*

$$\mathfrak{D}(E) = (L_{F/E}^\vee[1] \xrightarrow{\rho} L_{F/K}^\vee[1])$$

for some intermediate field $K \subset E \subset F$ if and only if the following conditions are satisfied:

- (1) *Injectivity:* the anchor map ρ induces an injection $\pi_1(\mathfrak{g}) \hookrightarrow \pi_1(L_{F/K}^\vee[1]) \cong \text{Der}_K(F)$.
- (2) *Vanishing:* $\pi_k(\mathfrak{g}) = 0$ for $k \neq 0, 1$.
- (3) *Balance:* $\dim_F(\pi_0(\mathfrak{g})) = \dim_F(\pi_1(\mathfrak{g})) < \infty$.

Proof. First, suppose that $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ is equivalent to $\mathfrak{D}(E) = (L_{F/E}^\vee[1] \xrightarrow{\rho} L_{F/K}^\vee[1])$ for some intermediate field E . Condition (1) then follows as we can identify the map $\pi_1(\mathfrak{g}) \hookrightarrow \pi_1(L_{F/K}^\vee[1])$ with Jacobson's inclusion $\text{Der}_E(F) \hookrightarrow \text{Der}_K(F)$ of E -linear derivations into K -linear derivations. Condition (2) follows from the computation of the cotangent complex of finite field extensions in Proposition 4.2. Finally, condition (3) follows from Cartier's equality in Lemma 4.3, since F/E is an algebraic extension.

For the converse implication, we assume that $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1]) \in \text{Alg}_{\text{LieAlgd}_{F/K}^\pi}$ satisfies (1) – (3). First, we show that there is an $R \in \text{SCR}_{K//F}^{\text{cN}}$ with $\mathfrak{D}(R) \simeq (\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ as partition Lie algebroids. Indeed, by Theorem 3.15, it is enough to check that $\text{fib}(\rho)$ is coconnective and of finite type. Conditions (2) and (3) together imply that \mathfrak{g} is of finite type, and the same holds true for $L_{F/K}^\vee[1]$. As $\text{fib}(\rho)$ fits into a cofibre sequence with \mathfrak{g} and $L_{F/K}^\vee[1]$, it is of finite type as well.

To show that $\text{fib}(\rho)$ is coconnective, we look at the exact sequences

$$\pi_{n+1}(L_{F/K}^\vee[1]) \rightarrow \pi_n(\text{fib}(\rho)) \rightarrow \pi_n(\mathfrak{g}) \rightarrow \pi_n(L_{F/K}^\vee[1]).$$

For $n \geq 2$, the group $\pi_n(\text{fib}(\rho))$ is nested between two vanishing modules, so must itself be zero. For $n = 1$, $\pi_1(\text{fib}(\rho))$ vanishes since $\pi_2(L_{F/K}^\vee[1]) = 0$ and $\pi_1(\mathfrak{g}) \hookrightarrow \pi_1(L_{F/K}^\vee[1])$ is injective by (1).

We may therefore assume that $(\mathfrak{g} \xrightarrow{\rho} L_{F/K}^\vee[1])$ is equivalent to $\mathfrak{D}(R) = (L_{F/R}^\vee[1] \xrightarrow{\rho} L_{F/K}^\vee[1])$ for some complete local Noetherian object $R \in \text{SCR}_{K//F}^{\text{cN}}$. It remains to prove that R is in fact a field, i.e. a discrete simplicial commutative ring which is regular, local, and of dimension zero.

Observe that since F is a field, we may reformulate conditions (2) and (3) as

- (2') Vanishing: $\pi_k(L_{F/R}) = 0$ for $k \neq 0, 1$.
- (3') Balance: $\dim_F \pi_0(L_{F/R}) = \dim_F \pi_1(L_{F/R}) < \infty$.

Denote the residue field of $\pi_0(R)$ by E . We first reduce to the case of $F = E$ by showing that the three conditions for $L_{F/R}$ imply the three conditions for $L_{E/R}$. First consider the sequence

$$L_{E/R} \otimes_E F \rightarrow L_{F/R} \rightarrow L_{F/E}$$

It follows that $L_{E/R}$ has non-zero homology only in degrees 0 and 1 since it is connective, and the other two terms of the sequence also vanish outside these degrees. For $L_{F/R}$ this is by assumption while for $L_{F/E}$ it is because this is a finite purely inseparable extension of fields. This verifies the vanishing condition. Furthermore, the balance condition follows from

$$0 = \chi(L_{F/R}) = \chi(L_{E/R}) + \chi(L_{F/E}) = \chi(L_{E/R}).$$

To check that the injectivity condition $\pi_1(L_{E/R}^\vee[1]) \rightarrow \pi_1(L_{F/K}^\vee[1])$ holds true, it is enough to verify that $\pi_2((L_{R/K} \otimes_R E)^\vee[1]) = 0$, but this follows from the corresponding fact about $K \rightarrow R \rightarrow F$ and the fact that $E \rightarrow F$ is faithfully flat.

Now we show that $\pi_0(R)$ is regular. The maps $R \rightarrow \pi_0(R) \rightarrow E$ induce an exact sequence

$$\pi_2(L_{E/R}) \rightarrow \pi_2(L_{E/\pi_0(R)}) \rightarrow \pi_1(E \otimes_{\pi_0(R)} L_{\pi_0(R)/R}).$$

As the fibre of $R \rightarrow \pi_0(R)$ is 1-connective, $L_{\pi_0(R)/R}$ is 2-connective by [Lur16, Corollary 25.3.6.4], which implies that $\pi_1(E \otimes_{\pi_0(R)} L_{\pi_0(R)/R}) = 0$. Since we have also proven that $\pi_2(L_{E/R}) = 0$, we deduce that $\pi_2(L_{E/\pi_0(R)}) = 0$. By [Qui, Corollary 10.5], we conclude that $\pi_0(R)$ is regular. Note that Lemma 4.1 implies that $L_{E/\pi_0(R)} \simeq (\mathfrak{m}/\mathfrak{m}^2)[1]$, where \mathfrak{m} is the maximal ideal of $\pi_0(R)$. In particular, we see that $\pi_i(L_{E/\pi_0(R)})$ vanishes for all $i \neq 1$.

We next show that R is discrete, for which it suffices by [Lur16, Corollary 25.3.6.6] to show that the relative cotangent complex $L_{\pi_0(R)/R}$ vanishes. We have already seen that $L_{\pi_0(R)/R}$ is 2-connective, i.e. that $\pi_n(L_{\pi_0(R)/R}) = 0$ for $n \leq 1$. So fix $n \geq 2$, and assume we have already checked $\pi_k(L_{\pi_0(R)/R}) = 0$ for all $k < n$. The maps $R \rightarrow \pi_0(R) \rightarrow E$ give rise to an exact sequence

$$\pi_{n+1}(L_{E/\pi_0(R)}) \rightarrow \pi_n(E \otimes_{\pi_0(R)} L_{\pi_0(R)/R}) \rightarrow \pi_n(L_{E/R}).$$

We have already seen that the two terms on the outside vanish for $n > 1$ and therefore we deduce that $\pi_n(E \otimes_{\pi_0(R)} L_{\pi_0(R)/R}) \cong E \otimes_{\pi_0(R)} \pi_n(L_{\pi_0(R)/R}) = 0$. The isomorphism here comes from the fact n is the lowest non-vanishing homology, so this Tor group is just a tensor product. Nakayama's lemma then implies that $\pi_n(L_{\pi_0(R)/R}) = 0$. To show that Nakayama applies, note that $\pi_0(R)$ is almost of finite presentation over R by [Lur04, Proposition 3.1.5] since R is Noetherian, which implies by Proposition 3.2.14 in [op.cit.] that $L_{\pi_0(R)/R}$ is an almost perfect $\pi_0(R)$ -module spectrum, which in turn shows that $\pi_n(L_{\pi_0(R)/R})$ is finitely generated by Proposition 2.5.10 in [op.cit.].

We deduce that $R \simeq \pi_0(R)$ is a discrete regular local ring with maximal ideal \mathfrak{m} . On the other hand since $\pi_0(R) \rightarrow E$ is surjective, we know that $\pi_0(L_{E/R}) = 0$ and hence, by balance, we see that

$\pi_1(L_{E/R}) = 0$. Hence $\mathfrak{m}/\mathfrak{m}^2 \cong \pi_1(L_{E/R}) = 0$ and so R is a discrete regular local ring of dimension zero, i.e. a field. \square

We can now prove the main theorem:

Proof of Theorem 1.11. By Theorem 3.15, the adjunction $\mathfrak{D} : \text{SCR}_{K//F}^{\text{op}} \rightleftarrows \text{Alg}_{\text{LieAlg}_{F/K}^{\pi}} : C^*$ from Construction 3.14 restricts to an equivalence between $(\text{SCR}_{K//F}^{\text{cn}})^{\text{op}}$ and the full subcategory of F/K -partition Lie algebroids for which the fibre of the anchor map belongs to $\text{Mod}_{F, \leq 0}^{\text{ft}}$. The Essential Image Theorem 4.5 shows that further restricting $(\mathfrak{D} \dashv C^*)$ to the subcategory $\text{Fields}_{K//F}^{\text{op}}$ of intermediate fields therefore gives an equivalence between $\text{Fields}_{K//F}^{\text{op}}$ and the full subcategory of F/K -partition Lie algebroids satisfying the conditions (1) – (3) appearing in the theorem. In particular, this subcategory of F/K -partition Lie algebroids is (equivalent to) a poset. \square

Proof of Corollary 1.13. Let X be a normal variety over a perfect field k with fraction field F . There is an equivalence of categories between intermediate fields $F/K/F^{p^n}$ and towers of finite k -morphisms $X \rightarrow Y \rightarrow X^{p^n}$ with Y normal. The forward direction is given by taking the normalization of $\mathcal{O}_{X^{p^n}}$ in K , while the reverse is given by taking the fraction field. It follows that the correspondence with normal varieties follows immediately from the correspondence with field extensions. \square

4.3. Modular extensions. We will now characterise simple and modular extensions in terms of their associated partition Lie algebroids. We begin with some elementary lemmas.

Lemma 4.6. *Let F/K be a finite purely inseparable field extension, and let $\alpha \in F$. Then $F = K(\alpha)$ if and only if $F = (F^p K)(\alpha)$.*

Proof. If $F = K(\alpha)$, then F is also generated by α over $F^p K$ since $K \subset F^p K$.

Conversely, suppose that $F = F^p K(\alpha)$, and consider the tower of extensions

$$F \supseteq F^p K \supseteq F^{p^2} K \supseteq \cdots \supseteq F^{p^n} K = K.$$

We show inductively on $e \in \mathbb{N}_{>0}$ that the extension $F/F^{p^e} K$ is generated by α . This holds for $e = 1$ by assumption. Now assume that the statement holds for e . Since taking p^{th} -powers commutes with addition, we have $F^p = F^{p^{e+1}} K^p(\alpha^p)$ and hence $F^p K = F^{p^{e+1}} K(\alpha^p)$. We can therefore conclude $F = F^p K(\alpha) = (F^{p^{e+1}} K(\alpha^p))(\alpha) = F^{p^{e+1}} K(\alpha)$. Thus the statement holds for $e + 1$. The result follows since the tower of extensions above must terminate since F/K is finite. \square

Recall that given a finite purely inseparable extension F/K , a subset $\{x_i\} \subset F$ is a p -basis of F/K if $\{dx_i\}$ form a basis of $\Omega_{F/K}$. By [The, 07P2], this is equivalent to the elements $\{\Pi_i x_i^{k_i} \mid 0 \leq k_i \leq p - 1\}$ forming a basis of F over KF^p . The latter is often given as the definition of p -basis, particularly when F/K has exponent one.

Lemma 4.7. *Let F/K be a finite purely inseparable field extension. Then F/K is a non-trivial simple extension if and only if $\dim_F \pi_0(L_{F/K}) = \dim_F \pi_1(L_{F/K}) = 1$.*

Proof. By Lemma 4.3, it is sufficient to show that F/K is a non-trivial simple extension if and only if $\dim_F \Omega_{F/K}^1 = 1$.

If F/K has exponent 1, then F/K is a non-trivial simple extension if and only if it has a p -basis consisting of a single element by [Sta20, 07P2].

If F/K has exponent $n > 1$, we note that since $\Omega_{F/K}^1 \cong \Omega_{F/F^p K}^1$, it suffices to show that F/K is simple if and only if the exponent 1 extension $F/F^p K$ is simple by Lemma 4.6, and F/K is non-trivial if and only if $F/F^p K$ is simple by Lemma 4.6 applied to $\alpha = 1$. \square

Remark 4.8. We note that Lemma 4.7 is consistent with our main theorem, despite the fact that it shows that if we have a chain of non-trivial finite simple extensions

$$K = E_n \subset E_{n-1} \subset \dots \subset E_1 \subset E_0 = F$$

where there is a single element $\alpha \in F$ such that $E_{i-1} = E_i(\alpha^{p^i})$, then the homotopy groups of the corresponding partition Lie algebroids $\mathfrak{gal}_{F/K}(E_i)$ are all isomorphic as F -modules.

Indeed, consider the following exact sequence consists of one dimensional F -modules:

$$\begin{aligned} 0 \rightarrow \pi_1(L_{F/E_{i-1}}^\vee[1]) \rightarrow \pi_1(L_{F/E_i}^\vee[1]) \rightarrow \pi_1(F \otimes_{E_{i-1}} L_{E_{i-1}/E_i}^\vee[1]) \\ \rightarrow \pi_0(L_{F/E_{i-1}}^\vee[1]) \rightarrow \pi_0(L_{F/E_i}^\vee[1]) \rightarrow \pi_0(F \otimes_{E_{i-1}} L_{E_{i-1}/E_i}^\vee[1]) \rightarrow 0 \end{aligned}$$

By the one-dimensionality, the first injective map is an isomorphism. Hence following through the sequence we find that $\pi_0(L_{F/E_{i-1}}^\vee[1]) \rightarrow \pi_0(L_{F/E_i}^\vee[1])$ is the zero map, and hence there is no isomorphism of partition Lie algebroids even though the underlying F -modules are isomorphic. A similar argument applies to a sequence of iterated Frobenius maps

$$K = F^{p^n} \subset F^{p^{n-1}} \subset \dots \subset F^p \subset F,$$

for an F -finite field F , where the homotopy groups have dimension $\dim_F(\Omega_{F/F^p}^1)$.

Now we reach the characterisation of modular extensions:

Proposition 4.9. *Let F/K be a finite purely inseparable extension. Then F/K is modular precisely if there are finitely many F/K -partition Lie algebroids*

$$\rho_i : \mathfrak{g}_i \rightarrow L_{F/K}^\vee[1]$$

such that the following conditions hold:

- (1) each \mathfrak{g}_i satisfies conditions (1) – (3) of Theorem 1.11;
- (2) $\dim_F(\pi_0(\text{fib}(\rho_i))) = 1$ for each i ;
- (3) the canonical map $L_{F/K}^\vee \rightarrow \bigoplus_i \text{fib}(\rho_i)$ is an equivalence in Mod_F .

Proof. Suppose that F/K is modular. Then by definition it can be expressed as a tensor product of simple extensions $F \cong \otimes_K E_i$, where $E_i = K(\alpha_i) \subset F$. Setting $\mathfrak{gal}_{F/K}(E_i) = (\mathfrak{g}_i \xrightarrow{\rho_i} L_{F/K}^\vee[1])$, we note that (1) holds by Theorem 1.11 and (2) holds by Lemma 4.7. For (3), we use [Sta20, 09DA] to conclude that $L_{F/K} \cong \bigoplus_i (F \otimes_{E_i} L_{E_i/K})$, as required.

Conversely suppose that we have $\rho_i : \mathfrak{g}_i \rightarrow L_{F/K}^\vee[1]$ satisfying the given conditions. Then by Theorem 1.11 there are intermediate fields $K \subset E_i \subset F$ such that $\mathfrak{gal}_{F/K}(E_i) = (\mathfrak{g}_i \xrightarrow{\rho_i} L_{F/K}^\vee[1])$, and E_i/K are simple by Lemma 4.7, again using the fact that $\text{fib}(\rho_i) \cong (F \otimes_{E_i} L_{E_i/K})^\vee$. Let $R = \otimes_K E_i$, and let $f : R \rightarrow F$ be the natural map, which we must show is an isomorphism. Again by [Sta20, 09DA], we have

$$L_{R/K} \simeq \bigoplus_i (R \otimes_{E_i} L_{E_i/K}).$$

The fibre sequence associated with $K \rightarrow R \rightarrow F$ is $F \otimes_R L_{R/K} \rightarrow L_{F/K} \rightarrow L_{F/R}$, and by assumption, the left hand map is an equivalence, and hence $L_{F/R} \simeq 0$.

Note that $R \rightarrow F$ is of finite presentation since F is a finite extension of the residue field of R , so it follows that $R \rightarrow F$ is étale. Now by [The, 025G], $\text{Spec}(F) \rightarrow \text{Spec}(R)$ is an open immersion, and therefore an isomorphism since F is a field. □

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