

∞ -Categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

LECTURE 4. THE BARR–BECK–LURIE THEOREM

Last lecture, we introduced monoidal ∞ -categories $\mathcal{C}^{\otimes} \rightarrow \mathbf{N}(\Delta^{op})$, defined the relative nerve construction $(J \rightarrow \mathbf{sSet}) \rightsquigarrow (\mathbf{N}_F(J) \rightarrow \mathbf{N}(J))$, and used it to equip ∞ -categories of endofunctors $\mathcal{C} = \mathbf{End}(\mathcal{D})$ with monoidal structures. This allowed us to define monads and their algebras in an ∞ -categorical setting.

Today, we will explain Lurie’s higher categorical variant of Barr–Beck’s monadicity theorem from Lecture 1. This generalisation is a key tool in constructing equivalences between higher categories, and will be of great value in our later applications.

4.1. A reflection on adjunctions. We wish to construct monads from adjunctions.

For ordinary categories, this was straightforward: given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with unit η and counit ϵ , the triple $(T = GF, G\epsilon_F : TT \rightarrow T, \eta : \text{id}_{\mathcal{C}} \rightarrow T)$ evidently satisfies the axioms of a monad (cf. Definition 1.30 of Lecture 1).

The corresponding construction for ∞ -categories is more complicated, as we must supply an infinite amount of coherence data to specify a monad.

Adjunctions of ∞ -categories can be defined in several ways:

- a) Most efficiently, we can define an adjunction as a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ for which the corresponding coCartesian fibration $\mathcal{M} \rightarrow \mathbf{N}(\Delta^1)$ has the property of also being Cartesian.
- b) Slightly less efficiently, we could also specify both functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$. However, this datum alone is overdetermined. To fix this, we must specify a unit natural transformation $u : \text{id}_{\mathcal{C}} \rightarrow GF$ verifying that F and G are indeed adjoint, which means that $\text{Map}_{\mathcal{D}}(FX, Y) \rightarrow \text{Map}_{\mathcal{C}}(GF X, GY) \xrightarrow{u \circ -} \text{Map}_{\mathcal{C}}(X, GY)$ is a weak equivalence for all X, Y .
- c) Even less efficiently, we could also specify two functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ and two natural transformations $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$, $\epsilon : FG \rightarrow \text{id}_{\mathcal{D}}$ satisfying the natural conditions for a unit and counit. Again, this quadruple alone would be overdetermined, which we can fix by also specifying a 2-simplex $\Delta^2 \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$:

$$\begin{array}{ccc}
 F & \xrightarrow{F\eta} & FGF \\
 & \searrow & \downarrow \epsilon_F \\
 & & F
 \end{array}$$

- d) ...

Continuing in this fashion, we obtain infinitely many definitions of what an adjunction is; one can prove that all these notions are equivalent up to a contractible space of choices.

Exercise 4.1. Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ as in a) above, construct a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and a natural transformation $u : \text{id}_{\mathcal{C}} \rightarrow GF$ satisfying the conditions specified in b).

For most applications, the most economical definition a) is entirely sufficient. However, the infinitely many coherences required for a monad force us to use the “least” efficient definition of adjunctions, which we will explain in the following sections.

4.2. The $(\infty, 2)$ -category $\{\mathcal{C}, \mathcal{D}\}$. Given an ∞ -category \mathcal{C} , we have constructed a monoidal structure $\text{End}(\mathcal{C})^{\otimes} \rightarrow \mathbb{N}(\Delta^{op})$ on $\text{End}(\mathcal{C})$ in Definition 3.11 of the preceding lecture.

Shifting perspective, we may think of $\text{End}(\mathcal{C})$ as a model for the full subcategory $\{\mathcal{C}\}$ of the $(\infty, 2)$ -category of $(\infty, 1)$ -categories and (not necessarily invertible) natural transformations.

To construct monads from adjunctions, we will need a similar description of the full subcategory $\{\mathcal{C}, \mathcal{D}\}$ spanned by two ∞ -categories \mathcal{C}, \mathcal{D} . To this end, we proceed in three steps:

a) *Define a labelled version $\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}$ of Δ^{op} .* We have seen that Δ^{op} is modelled by diagrams

$$[n] = (- \bullet \bullet \dots \bullet +),$$

with morphisms corresponding to order-preserving maps sending $-$ to $-$ and $+$ to $+$.

The objects of $\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}$ are given by such diagrams with gaps labelled by either the symbol \mathcal{C} or the symbol \mathcal{D} . For example, we have the following object:

$$(- \mathcal{C} \bullet \mathcal{C} \bullet \mathcal{D} \bullet \mathcal{D} \bullet \mathcal{C} \bullet \mathcal{D} +),$$

More formally, objects of $\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}$ are given by pairs $([n] \in \Delta^{op}, c : [n] \rightarrow \{\mathcal{C}, \mathcal{D}\})$.

Morphisms in $\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}$ are order-preserving maps sending $-$ to $-$ and $+$ to $+$, which have the additional property that all gaps between two arrows carry the same label:

$$(1) \quad \begin{array}{cccccccccccc} - & \mathcal{C} & \bullet & \mathcal{D} & \bullet & \mathcal{C} & \bullet & \mathcal{C} & \bullet & \mathcal{D} & \bullet & \mathcal{D} & \bullet & \mathcal{C} & + \\ & \searrow & \downarrow & & \searrow & & \searrow & \downarrow & \\ & & \downarrow & & & \downarrow & \\ - & \mathcal{D} & \bullet & \mathcal{D} & \bullet & \mathcal{C} & \bullet & \mathcal{C} & \bullet & \mathcal{D} & \bullet & \mathcal{D} & \bullet & \mathcal{C} & + \end{array}$$

b) *Define a functor $F : \Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op} \rightarrow \mathbf{sSet}$.* On objects, we define

$$(- c_0 \bullet c_1 \bullet \dots \bullet c_{n-1} \bullet c_n +) \mapsto \text{Fun}(c_0, c_1) \times \dots \times \text{Fun}(c_{n-1}, c_n).$$

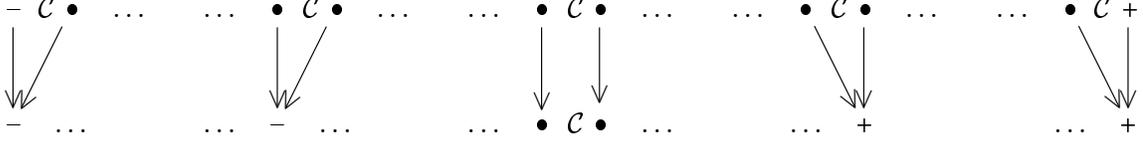
On morphisms, this functor is defined by composing functors and inserting the identities as dictated from the arrows. For example, the morphism (1) above sends an element $(\mathcal{C} \xrightarrow{F_0} \mathcal{D} \xrightarrow{F_1} \mathcal{C} \xrightarrow{F_2} \mathcal{C} \xrightarrow{F_3} \mathcal{D} \xrightarrow{F_4} \mathcal{D} \xrightarrow{F_5} \mathcal{C})$ to the element $(\mathcal{D} \xrightarrow{\text{id}_{\mathcal{D}}} \mathcal{D} \xrightarrow{F_1} \mathcal{C} \xrightarrow{\text{id}_{\mathcal{C}}} \mathcal{C} \xrightarrow{F_4 \circ F_3 \circ F_2} \mathcal{D})$.

c) *Unstraighten.* We apply the relative nerve construction introduced in Definition 4.28 of last class to obtain a coCartesian fibration $p : \text{End}(\mathcal{C}, \mathcal{D})^{\otimes} \rightarrow \mathbb{N}(\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op})$.

This construction will allow us access all functors between \mathcal{C} and \mathcal{D} , and all natural transformation between such functors, in an effective way.

4.3. Adjunction data. We can now keep track of all higher coherence data of adjunctions. We need the following auxiliary definition:

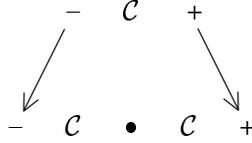
Definition 4.2. A morphism $([n], c) \rightarrow ([m], d)$ is said to be \mathcal{C} -inert if any \mathcal{C} -label in the domain $([n], c)$ sits in one of the following five configurations:



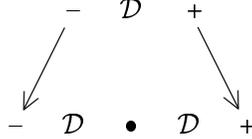
Definition 4.3 (Adjunction data). An *adjunction datum* for a pair of ∞ -categories $(\mathcal{C}, \mathcal{D})$ consists of a section $s : N(\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}) \rightarrow \text{End}(\mathcal{C}, \mathcal{D})^{\otimes}$ of $p : \text{End}(\mathcal{C}, \mathcal{D})^{\otimes} \rightarrow N(\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op})$ sending \mathcal{C} -inert morphisms to p -coCartesian edges. Write $\text{Adj}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\text{Fun}_{N(\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op})}(N(\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}), \text{End}(\mathcal{C}, \mathcal{D})^{\otimes})$ spanned by such sections.

We now unravel the information contained in an adjunction datum $N(\Delta_{\{\mathcal{C}, \mathcal{D}\}}^{op}) \xrightarrow{s} \text{End}(\mathcal{C}, \mathcal{D})^{\otimes}$.

- First, define two functors $F := s(- \mathcal{C} \bullet \mathcal{D} +)$ and $G := s(- \mathcal{D} \bullet \mathcal{C} +)$; these will serve as left and right adjoint, respectively.
- Next, define two endofunctors $T := s(- \mathcal{C} \bullet \mathcal{C} +)$ and $\iota := s(- \mathcal{D} \bullet \mathcal{D} +)$; while T will be the monad induced by the adjunction, ι will just be a version of $\text{id}_{\mathcal{D}}$.
- We obtain a natural transformation $\text{id}_{\mathcal{C}} \rightarrow T$ from the following morphism:



- We obtain a natural equivalence $\text{id}_{\mathcal{C}} \xrightarrow{\simeq} \iota$ from the following \mathcal{C} -inert morphism:



- The triangle $s(- \mathcal{C} \bullet \mathcal{D} \bullet \mathcal{C} +)$ gives two functors $F' : \mathcal{C} \rightarrow \mathcal{D}$ and $G' : \mathcal{D} \rightarrow \mathcal{C}$.

Exercise. Use \mathcal{C} -inert morphisms to produce equivalences $F' \simeq F$, $G' \simeq G$, $G'F' \simeq T$.

- The triangle $s(- \mathcal{D} \bullet \mathcal{C} \bullet \mathcal{D} +)$ gives functors $G'' : \mathcal{D} \rightarrow \mathcal{C}$ and $F'' : \mathcal{C} \rightarrow \mathcal{D}$.

Exercise. Use \mathcal{C} -inert morphisms to produce equivalences $F'' \simeq F$, $G'' \simeq G$, and find a non- \mathcal{C} -inert morphism inducing a natural transformation $\epsilon : F''G'' \rightarrow \text{id}_{\mathcal{D}}$; this will be the counit.

We have produced functors F, G and natural transformations $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$, $\epsilon : FG \rightarrow \text{id}_{\mathcal{D}}$. A more elaborate argument, which appears as [Lur07, Lemma 3.2.9], then shows that these satisfy the axioms of an adjunction between the homotopy categories $h\mathcal{C}$ and $h\mathcal{D}$.

4.4. The Barr–Beck–Lurie theorem. Fix two ∞ -categories \mathcal{C} and \mathcal{D} , and consider the following maximally efficient definition of adjunctions:

Definition 4.4. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *left adjoint* if the corresponding coCartesian fibration over Δ^1 is also Cartesian. Write $\text{Fun}'(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ for the subcategory whose objects are left adjoints and whose morphisms are natural equivalences.

It is not hard to show that any adjunction datum $s \in \text{Adj}(\mathcal{C}, \mathcal{D})$ determines a left adjoint $s(-\mathcal{C} \bullet \mathcal{D}+)$, and we can ask how much information is lost in this process. The following hard theorem of Lurie (cf. [Lur07, Theorem 3.2.10]) shows that the infinitely many higher coherences present in an adjunction datum can be added in an essentially unique way:

Theorem 4.5 (Adjunction data from adjunctions). Evaluation gives a trivial Kan fibration

$$\begin{aligned} \text{Adj}(\mathcal{C}, \mathcal{D}) &\rightarrow \text{Fun}'(\mathcal{C}, \mathcal{D}) \\ s &\mapsto s(-\mathcal{C} \bullet \mathcal{D}+) \end{aligned}$$

Now let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a left adjoint. Using Theorem 4.5 above, we pick a preimage $s \in \text{Adj}(\mathcal{C}, \mathcal{D})$ from a contractible space. Write $G = s(-\mathcal{D} \bullet \mathcal{C}+)$ for the corresponding right adjoint. Restricting s to the full subcategory $\text{N}(\Delta^{op}) \simeq \text{N}(\Delta_{\mathcal{C}}^{op})$ of all diagrams $(-\mathcal{C} \bullet \dots \bullet \mathcal{C}+)$ labelled only by \mathcal{C} gives rise to an algebra object T in $\text{End}(\mathcal{C})$.

Just like in the 1-categorical case discussed in Lecture 1, we can construct a diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\tilde{G}} & \text{Alg}_T(\mathcal{C}) \\ & \searrow G & \downarrow U \\ & & \mathcal{C} \end{array}$$

For a formal construction of the functor \tilde{G} , we refer to [Lur07, Section 3.3].

We are finally ready to state the ∞ -categorical monadicity theorem:

Theorem 4.6 (Barr–Beck–Lurie, crude version). Assume that

- (1) \mathcal{D} admits and G preserves geometric realisations, i.e. $\text{N}(\Delta^{op})$ -shaped colimits;
- (2) G is conservative, i.e. if $G(f)$ is an equivalence in \mathcal{C} , then so is f in \mathcal{D} .

Then the functor $\tilde{G} : \mathcal{D} \xrightarrow{\simeq} \text{Alg}_T(\mathcal{C})$ is an equivalence of ∞ -categories.

In this higher categorical result, geometric realisations play an analogous role to the reflexive coequalisers appearing in the ordinary crude Barr–Beck theorem.

To give a sharp criterion, we need a higher categorical generalisation of split coequalisers. To this end, we introduce the following enlargement of the simplex category:

Definition 4.7. The category $\Delta_{-\infty}$ has objects the finite linearly ordered sets

$$[-1] = \{ \} , \quad [0] = \{0\} , \quad [1] = \{0 < 1\} , \quad [2] = \{0 < 1 < 2\} , \quad \dots$$

Morphisms $[n] \rightarrow [m]$ are given by order-preserving maps $[n] \cup \{-\infty\} \rightarrow [m] \cup \{-\infty\}$ which send $-\infty$ to $-\infty$; here $-\infty$ is defined as the least element.

Exercise 4.8.

- a) Exhibit the simplex category Δ and the augmented simplex category Δ_+ as subcategories of $\Delta_{-\infty}$.
- b) Show that any $\Delta_{-\infty}$ -indexed diagram in an ordinary category gives a split coequaliser.

Definition 4.9 (Split simplicial objects).

- a) A simplicial object $X : N(\Delta^{op}) \rightarrow \mathcal{C}$ in an ∞ -category \mathcal{C} is *split* if it extends to $N(\Delta_{-\infty}^{op})$.
- b) Given a functor $G : \mathcal{D} \rightarrow \mathcal{C}$, a simplicial object $X : N(\Delta^{op}) \rightarrow \mathcal{D}$ is said to be *G-split* if the simplicial object $G \circ X : N(\Delta^{op}) \rightarrow \mathcal{C}$ is split.

Remark 4.10. If X is a split simplicial diagram, then the restriction of X to Δ_+ is a colimit diagram; in other words, $X([-1])$ is the geometric realisation of $X|_{N(\Delta^{op})}$.

Theorem 4.11 (Barr–Beck–Lurie, precise version). Given a left adjoint $F : \mathcal{C} \rightarrow \mathcal{D}$ as above, the induced functor $\mathcal{D} \rightarrow \text{Alg}_T(\mathcal{C})$ is an equivalence if and only if the following conditions hold:

- (1) \mathcal{D} admits and G preserves colimits of G -split simplicial diagrams in \mathcal{D} ;
- (2) G is conservative.

In practical applications, (1) is usually much harder to check than (2). In the next weeks, we will give several concrete applications of this result.

REFERENCES

- [Lur07] Jacob Lurie, *Derived algebraic geometry II: Noncommutative algebra*, Preprint from the author's web page (2007).