∞-Categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

Lecture 9: Koszul duality for Formal moduli problems

In the last two lectures, we set up the Koszul duality functor

$$\mathfrak{D} : \mathrm{SCR}_k^{\mathrm{aug}} \longrightarrow \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}^\pi_k},\ \ R \mapsto \cot(R)^\vee$$

from augmented simplicial commutative k-algebras to *partition Lie algebras*; we introduced these new objects in Definition 8.3 of last lecture. In Proposition 8.6, we then showed that in characteristic zero, partition Lie algebras are equivalent to (shifted) differential graded Lie algebras.

Differential graded Lie algebras have numerous applications, ranging from deformation theory in geometry to rational homotopy theory and configuration space theory in topology. Today, we will outline how partition Lie algebras relate to deformations in characteristic p.

9.1. **A reminder on Kodaira–Spencer theory.** To begin with, let us review the role of differential graded Lie algebras in deformation theory over the complex numbers and introduce several key definitions along the way.

Let Z be a smooth and proper complex algebraic variety. We fix a first order deformation $\widetilde{Z}$ of Z, i.e. a pullback square

$$\begin{array}{ccc} Z & \longrightarrow & \widetilde{Z} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \end{array}$$

where the right vertical map is smooth and proper. The underlying space of $\widetilde{Z}$ is simply Z, and all interesting information is concentrated in the deformation of the structure sheaf $\mathcal{O}_Z$.

To capture this geometric situation algebraically, we pick a cover Z by open affines $\mathrm{Spec}(B_i)$ with open affine intersections $\mathrm{Spec}(B_i) \cap \mathrm{Spec}(B_j) \cong \mathrm{Spec}(B_{ij})$. On each of those affines, we can trivialise our first order deformation, and comparing trivialisations on overlaps gives rise to derivations $B_{ij} \to B_{ij}$. i.e. sections in $\Gamma(\mathrm{Spec}(B_{ij}), T_Z)$. These sections in turn glue to an element in $x_{\widetilde{Z}} \in \mathrm{H}^1(Z, T_Z)$, which is in fact independent of the choices we made. Kodaira–Spencer theory [KS58] then asserts that the assignment

$$\widetilde{Z} \mapsto x_{\widetilde{Z}}$$

defines a bijection between first order deformations of Z and elements of $\mathrm{H}^1(Z, T_Z)$.

We can now ask:

Question 1. Does a given first order deformation $\widetilde{Z}$ extend to a higher order deformation?

To formulate an algebraic answer, we use the commutator of vector fields to define a graded Lie algebra structure on $\mathrm{H}^*(Z, T_Z)$. Then $\widetilde{Z}$ extends to $\mathbb{C}[\epsilon]/\epsilon^3$ if and only if $[x_{\widetilde{Z}}, x_{\widetilde{Z}}] = 0$.

In fact, $\mathrm{H}^*(Z, T_Z)$ is the homology of a *differential graded Lie algebra* $\mathfrak{g}_Z$, i.e. a chain complex with a bilinear bracket satisfying the Jacobi identity, antisymmetry, and the Leibniz rule, all in a graded sense. Concretely, $\mathfrak{g}_Z$ is given by the Dolbeault complex

$$\mathfrak{g}_Z = \left(\mathcal{A}^{0,0}(T_Z) \to \mathcal{A}^{0,1}(T_Z) \to \mathcal{A}^{0,2}(T_Z) \to \dots\right).$$

Miraculously, this differential graded Lie algebra in fact controls *all* infinitesimal deformations of Z. Indeed, if A is a local Artin $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}_A$, one can construct a bijection

$$\left\{ \begin{array}{c} \text{Deformations } \widetilde{Z} \text{ of } Z \\ \text{over } \operatorname{Spec}(A) \end{array} \right\} \Big/ {\scriptstyle \substack{\text{isomorphism} \\ \text{restricting to } \mathrm{id}_Z}} \cong \left\{ \begin{array}{c} \text{Maurer–Cartan elements} \\ x \in (\mathfrak{g}_Z)_{-1} \otimes \mathfrak{m}_A \, : \, dx + \frac{1}{2}[x,x] = 0 \end{array} \right\} \Big/ {\scriptstyle \substack{\text{gauge} \\ \text{equivalence}}}.$$

Here $x, y \in (\mathfrak{g}_Z)_{-1} \otimes \mathfrak{m}_A$ are called *gauge equivalent* if there is some $a \in (\mathfrak{g}_Z)_0 \otimes \mathfrak{m}_A$ satisfying

$$y = x + \sum_{n=0}^{\infty} \frac{[a,-]^{\circ n}}{(n+1)!}([a,x] - da).$$

9.2. **Towards derived algebraic geometry.** There are numerous other algebro-geometric objects Y over $\mathbb{C}$ whose infinitesimal deformations are governed by some differential graded Lie algebra $\mathfrak{g}_Y$, for inxtance subschemes, vector bundles, and representations.

It is natural to ask:

Question 2. Given an algebro-geometric object Y over $\mathbb{C}$, how can we construct the differential graded Lie algebra $\mathfrak{g}_Y$ controlling its infinitesimal deformations?

Unfortunately, many non-equivalent differential graded Lie algebras can control deformations of the same object, and it is not possible to pick a preferred one. We give a well-known example of this phenomenon, cf. e.g. [Toë14]:

Example 9.1. Given a closed immersion of smooth complex varieties

$$Z_0 \subset Z,$$

the infinitesimal deformations of Z_0 inside Z are governed by two differential graded Lie algebras: one interprets Z_0 as a point in the Hilbert scheme of Z, the other interprets it as a point in the Quot scheme of $\mathcal{O}_Z$.

In a visionary letter [Dri] from 1988, Drinfel'd suggested that this issue would disappear once we also took *derived* infinitesimal deformations into account, i.e. deformations over simplicial local Artin $\mathbb{C}$-algebras

$$A = \left(\dots \substack{\rightarrow\\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow} A_1 \substack{\rightarrow \\ \leftarrow \\ \rightarrow} A_0\right).$$

Here $A \in \mathrm{SCR}^{\mathrm{aug}}_{\mathbb{C}}$ a local Artin if $\dim(\pi_*(A)) < \infty$ and $\pi_0(A)$ is local (with residue field $\mathbb{C}$).

Derived infinitesimal deformation functors are most naturally axiomatised using the language of ∞-categories. These are generalisations of categories where the collection of maps between any two objects forms a space rather than merely a set; we refer to Lecture 2 for further details. One then defines:

Definition 9.2. A *formal moduli problem over* $\mathbb{C}$ is a functor $X : \mathrm{SCR}_{\mathbb{C}}^{\mathrm{art}} \to \mathcal{S}$ from the ∞-category $\mathrm{SCR}_{\mathbb{C}}^{\mathrm{art}}$ of simplicial local Artin $\mathbb{C}$-algebras to the ∞-category $\mathcal{S}$ of spaces satisfying:

(1) Normalisation: The space $X(\mathbb{C})$ is contractible.
(2) Gluing: Applying X to a pullback square

$$\begin{array}{ccc} \widetilde{A} & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A & \longrightarrow & A'' \end{array}$$

with $\pi_0(A') \to \pi_0(A'')$, $\pi_0(A) \to \pi_0(A'')$ surjective gives another pullback square.

We will write $\mathrm{Moduli}_{\mathbb{C}} \subset \mathrm{Fun}(\mathrm{SCR}_{\mathbb{C}}^{\mathrm{art}}, \mathcal{S})$ for the ∞-category of formal moduli problems.

Example 9.3. Given a smooth and proper complex variety Z as above, we can enhance its classical deformation functor to a formal moduli problem $\mathrm{Def}_Z : \mathrm{SCR}_{\mathbb{C}}^{\mathrm{art}} \to \mathcal{S}$ by sending a simplicial local Artin $\mathbb{C}$-algebra A to the space of pullback diagrams

$$\begin{array}{ccc} Z & \longrightarrow & \widetilde{Z} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array} ,$$

where $\tilde{Z}$ is a smooth and proper *derived* scheme over $\mathrm{Spec}(A)$, cf. [Lur16, Section 19.4].

9.3. **The Lurie–Pridham theorem.** We can now outline the relation between derived infinitesimal deformation functors, i.e. formal moduli problems, and differential graded Lie algebras over $\mathbb{C}$.

Given a differential graded Lie algebra $\mathfrak{g}$ and a simplicial local Artin $\mathbb{C}$-algebra A with maximal ideal $\mathfrak{m}_A$, Goldman-Millson [GM88] and Hinich [Hin01] constructed a *space* of Maurer–Cartan elements

$$\mathrm{MC}(\mathfrak{m}_A \otimes \mathfrak{g}),$$

which is equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(A), \mathfrak{g}).$$

Varying A, we obtain a functor

$$\Psi : \mathrm{dgla}_{\mathbb{C}} \to \mathrm{Moduli}_{\mathbb{C}},$$
$$\mathfrak{g} \mapsto \mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(-), \mathfrak{g})$$

from differential graded Lie algebras to formal moduli problems. The following result by Lurie [Lur10] and Pridham [Pri10], which generalises earlier work of many others including Kontsevich–Soibelman [KS02] and Manetti [Man09], asserts that differential graded Lie algebras control derived infinitesimal deformations:

Theorem 9.4 (Lurie, Pridham)**.** The functor

$$\Psi : \mathrm{dgla}_{\mathbb{C}} \to \mathrm{Moduli}_{\mathbb{C}},\ \mathfrak{g} \mapsto \mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(-), \mathfrak{g})$$

defines an equivalence between the ∞-categories of formal moduli problems and differential graded Lie algebras over $\mathbb{C}$.

Given a formal moduli problem $X \in \mathrm{Moduli}_{\mathbb{C}}$, it is easy to describe the underlying spectrum of the associated differential graded Lie algebra. Up to a shift, it is given by the *tangent fibre* T_X of X, which is obtained by assembling the sequence of spaces

$$(T_X)_n := X(\mathbb{C} \oplus \mathbb{C}[n])$$

and the equivalences $\Omega(T_X)_{n+1} \simeq (T_X)_n$ into a spectrum. Here $\mathbb{C}\oplus\mathbb{C}[n]$ denotes the trivial square-zero extension of $\mathbb{C}$ by a class in homological degree n.

Exercise 9.5. Describe the tangent spectrum for (pro-)representable formal moduli problems using the tangent fibre formalism.

Remark 9.6. The Lurie–Pridham theorem holds over any field of characteristic zero.

9.4. **Deformation theory in characteristic** p**.** Many key players in number theory, arithmetic geometry, and representation theory are not defined over $\mathbb{C}$, but over $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$, and it is vital to understand how they deform. Unfortunately, differential graded Lie algebras are ill-behaved in this context, and no longer classify formal moduli problems. In fact, they do not even admit a model structure satisfying the most basic desiderata:

Exercise 9.7. Show that if k is a field of characteristic p, the category $\mathbf{dgla}_k$ of differential graded Lie algebras over k *does not* admit a model structure whose weak equivalences are given by quasi-isomorphisms and whose fibrations are given by levelwise surjections.

To resolve this problem, we must use *partition Lie algebras*, which we introduced in the preceeding lecture. They can be defined in two equivalent ways:

(1) Abstractly via ∞-categories, cf. Section 8.4 from last lecture, [BM19, Construction 1.9]. The (contravariant) tangent fibre functor $A \mapsto \cot(A)^\vee = (k \otimes_A L_{A/k})^\vee$ from augmented simplicial commutative k-algebras to chain complexes over k is part of an adjunction. The associated monad T on Mod_k behaves badly, but we can use Goodwillie's functor calculus to approximate T by a monad Lie^π_k which preserves filtered colimits and geometric realisations.

If $V^\bullet$ is a cosimplicial k-module with associated chain complex $\mathrm{Tot}(V^\bullet)$, then

$$\mathrm{Lie}^\pi_k(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Here $\Sigma|\Pi_n|^\diamond$ is a simplicial Σ_n-complex known as the n^{th} (doubly suspended) *partition complex*. For $d > 0$, the nondegenerate d-simplices of $\Sigma|\Pi_n|^\diamond$ correspond to chains of increasingly coarse partitions $[\hat{0} = x_0 < x_1 < \ldots < x_t = \hat{1}]$ of $\{1, \ldots n\}$.

Partition Lie algebras are then defined as algebras over the monad Lie^π_k.

(2) Concretely via model categories, cf. [BCN21, Construction 5.34].

We can construct an ordinary category whose objects are cosimplicial-simplicial k-modules with n-ary operations indexed by *nested chains of partitions* of $\{1,\ldots,n\}$, that is, by pairs

$$(\sigma, S)$$

where $\sigma = [\hat{0} = x_0 < x_1 < \ldots < x_t = \hat{1}]$ is a chain of increasingly coarse partitions of $\{1,\ldots,n\}$ and $S = (S_0 \subseteq \ldots \subseteq S_d)$ is a chain of increasing subsets of $S_d = \{0,\ldots,t\}$. There are also "divided power operations", and all these satisfy various conditions.

We can then equip this ordinary category with a natural model structure, and after inverting weak equivalences, we obtain the ∞-category of partition Lie algebras.

Partition Lie algebras satisfy the following gold standard property (cf. [BM19, Theorem 1.11]) which singles them out as the correct analogues of differential graded Lie algebras in characteristic p:

Theorem 9.8. Given a field k of arbitrary characteristic, the functor

$$\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k}} \simeq \mathrm{Moduli}_k, \ \mathfrak{g} \mapsto \mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(-), \mathfrak{g})$$

defines an equivalence between the ∞-categories of partition Lie algebras and formal moduli problems over k.

Hence partition Lie algebras completely classify derived infinitesimal deformation functors, and thereby provide a helpful tool in deformation theory.

The above generalisation of the Lurie–Pridham theorem relies on the following affine statement, which asserts that for suitably nice $R \in \mathrm{SCR}^{\mathrm{aug}}_k$, the complex $\cot(R)^\vee$ with its partition Lie algebra structure remembers the structure of R:

Theorem 9.9 ([BM19])**.** The functor $\mathfrak{D} : \mathrm{SCR}^{\mathrm{aug}}_k \longrightarrow \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}^{\pi}_{k}}$, $R \mapsto \cot(R)^\vee$ defined in Section 8.3 of last lecture restricts to an equivalence

$$\mathrm{SCR}^{\mathrm{cN}}_k \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{\mathrm{op}},$$

between

(1) the full subcategory

$$\mathrm{SCR}^{\mathrm{cN}}_k \subset \mathrm{SCR}^{\mathrm{aug}}_k$$

spanned by all R for with $\pi_0(R)$ is a complete local Noetherian ring and $\pi_i(R)$ is a finitely generated $\pi_0(R)$-module for all i.

(2) the (opposite of the) full subcategory

$$\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}) \subset \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k}}$$

spanned by all partition Lie algebras $\mathfrak{g}$ whose underlying chain complex is coconnective and satisfies $\dim(\pi_i(\mathfrak{g})) < \infty$ for all i.

References

[BCN21] Lukas Brantner, Ricardo Campos, and Joost Nuiten, *PD operads and explicit partition lie algebras*, arXiv preprint arXiv:2104.03870 (2021).

[BM19] Lukas Brantner and Akhil Mathew, *Deformation theory and partition lie algebras*, arXiv preprint arXiv:1904.07352 (2019).

[Dri] V. Drinfeld, *A letter from Kharkov to Moscow*, EMS Surv. Math. Sci. **1**, no. 2, 241–248, Translated from Russian by Keith Conrad. MR 3285856

[GM88] William M. Goldman and John J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96. MR 972343

[Hin01] Vladimir Hinich, *DG coalgebras as formal stacks*, J. Pure Appl. Algebra **162** (2001), no. 2-3, 209–250. MR 1843805

[KS58] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. I, II*, Ann. of Math. (2) **67** (1958), 328–466. MR 0112154

[KS02] Maxim Kontsevich and Yan Soibelman, *Deformation theory*, Livre en préparation (2002).

[Lur10] Jacob Lurie, *Moduli problems for ring spectra*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 1099–1125. MR 2827833

[Lur16] ______, *Spectral algebraic geometry*, Preprint available from the author's web page (2016).

[Man09] Marco Manetti, *Differential graded Lie algebras and formal deformation theory*, Algebraic geometry—Seattle 2005. Part 2, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 785–810. MR 2483955

[Pri10] Jon P Pridham, *Unifying derived deformation theories*, Advances in Mathematics **224** (2010), no. 3, 772–826.

[Toë14] Bertrand Toën, *Derived algebraic geometry*, EMS Surv. Math. Sci. **1** (2014), no. 2, 153–240. MR 3285853