HYPER-KÄHLER MANIFOLDS AND LAGRANGIAN FIBRATIONS

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Abstract. Hyper-Kähler manifolds have been studied in the past in many contexts, from an arithmetic, algebraic, geometric point of view, and in applications to physics and dynamics. The compact theory in dimension two, namely K3 surfaces, is well understood. The aim of this note is to give an informal introduction to the theory of compact hyper-Kähler manifolds in higher dimension, from a point of view of their classification; in particular, about the existence of Lagrangian fibrations. We present some results in dimension four, obtained recently in collaboration with Olivier Debarre, Daniel Huybrechts and Claire Voisin.

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Contents

1. Introduction

Compact hyper-Kähler (HK) manifolds, often also called irreducible holomorphic symplectic manifolds, can be thought of either as compact Riemannian manifolds (M, g) with holonomy group $\mathrm{Sp}(n)$ or as compact Kähler manifolds X with a unique holomorphic symplectic form. The passage between the two viewpoints is made possible by Yau's celebrated solution of the Calabi conjecture. Together with complex tori and Calabi–Yau varieties, HK manifolds provide one of the three building blocks of all Ricci-flat Kähler manifolds, a class of geometric objects that occupies a central place in differential and algebraic geometry as well as in mathematical physics.

The history of compact HK manifolds is quite extraordinary. The theory in two complex dimensions, that is of *K3 surfaces*, is one of the gems of algebraic geometry, with its interplay between lattice theory and geometry in Weil's program and in the global Torelli

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theorem [B1]. Naturally, mathematicians became curious about higher dimensions. The first two series examples were discovered by Beauville [B2], *Hilbert schemes of points* on K3 surfaces and *generalized Kummer varieties* associated to abelian surfaces. After that, with the work of Mukai [Mu1], it seemed that moduli spaces of sheaves on K3 surfaces would provide more examples in abundance. However, while proving that they often are indeed hyperkähler, Huybrechts [H1], O'Grady [O'G1], and Yoshioka [Y1] showed that they are topologically all equivalent to one of Beauville's series. Besides the two further sporadic families constructed by O'Grady [O'G2, O'G3], no further examples have been found until this day. At this point, HK manifolds seem, from several perspectives, richer and more interesting than abelian varieties, but not as uncontrollable as Calabi–Yau varieties for which tens of thousands of different topologies are known already in dimension three. The topological classification of HK manifolds is one of the central open questions in the area. In comparison, the topological classification of complex tori is trivial whereas for Calabi–Yau manifolds it seems unreasonable to expect any.

The goal of this note is to present the first steps towards the classification of HK fourfolds. In [DHMV], we show a conjecture by O'Grady: if a HK manifold of dimension 4 has the same cohomology ring as a Hilbert square of a K3 surface, then it is actually in the same deformation class. This can be thought as the 4-dimensional analogue of Kodaira's theorem [Ko] that all K3 surfaces are deformation equivalent.

The proof, based on deep ideas by O'Grady in [O'G4], builds on all the theory developed for HK fourfolds in the past decades: Verbitsky's global Torelli theorem [V, H3, M1], Markman's results on monodromy operators and prime exceptional divisors [M2], Salamon–Guan topological constraints [S, G], results on the base of Lagrangian fibrations by Ou and Huybrechts-Xu [O, HX], and birational geometry results by Fujino–Kawamata [F, K], Fukuda [Fu], and Wierzba–Wisniewsky [W, WW]. It is hard to imagine to generalize this proof either in dimension 6 or above, or to complete the classification of HK fourfolds, where other inputs are needed.

The structure of this note is as follows. In Section 2, we will present K3 surfaces and Kodaira's theorem (Theorem 2.3). In Section 3, we discuss Lagrangian fibrations and the main conjecture (Conjecture 3.4) regarding their existence for HK. Proving such conjecture in dimension 4 is the new ingredient in the proof of O'Grady's conjecture (the main theorem of the paper, Theorem 4.2): the two statements are actually closely related to each other, and we will prove both at the same time. The main theorem is explained in Section 4 and a rough idea of the proof is in Section 5.

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2. K3 surfaces

Let us denote by X a compact complex manifold, namely a compact manifold locally homeomorphic to open subsets of $\mathbf{C}^m$ with holomorphic transition functions. For simplicity, in this note we will consider projective manifolds, namely we will assume that X has a closed embedding in the complex projective space $\mathbf{P}^N$; by Chow's theorem, this means that X is defined by algebraic equations. Moreover, it carries a Kähler metric, by using the induced Fubini–Study metric from $\mathbf{P}^N$.

Definition 2.1 (Calabi, Hitchin). Say that X is *irreducible holomorphic symplectic* if

(i) X is simply connected;
(ii) there exists a unique holomorphic symplectic form η, up to multiplicative constants.

As common use in the literature, we will use the imprecise name *compact hyper-Kähler* (HK); though important in the theory, in this note we will never explicitly use the actual hyper-Kähler metric. Since X is holomorphic symplectic, its complex dimension is even: we will denote it by $2n$, $n \geq 1$. HK manifolds in dimension 2 are called *K3 surfaces*. Basic references for HK manifolds are [H2, D]; the literature on K3 surfaces is quite rich and we mention the books [BB+, H4].

Example 2.2. Let S be a compact surface with a closed embedding in $\mathbf{P}^3$ as a hypersurface of degree 4, a *quartic surface*. Namely, it is given as the 0-locus of a homogeneous polynomial f of degree 4 in 4 variables $x_0, \ldots, x_3$. Then:

(i) by Bott's version of the Lefschetz hyperplane theorem, S is simply connected;
(ii) by the Poincaré residue theorem, the holomorphic 2-form

$$\eta := \operatorname{Res}\left(\frac{\sum_{i=0}^{3} x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_3}{f}\right)$$

is a symplectic form. Hence, every quartic surface in $\mathbf{P}^3$ is a K3 surface.

The first fundamental result in the theory of K3 surfaces is the following (see [Ko, Theorem 13]), showing in particular, by Ehresmann's lemma, that all K3 surfaces are diffeomorphic.

Theorem 2.3 (Kodaira)**.** *Every K3 surface is a smooth complex deformation of a non-singular quartic surface, namely all K3 surfaces are in the same deformation class.*

Explicitly, Kodaira's theorem means that, for all K3 surface S, there exists a smooth proper morphism $\mathscr{S} \to \mathbf{D}$, from a smooth complex manifold $\mathscr{S}$ to a complex disk $\mathbf{D}$, such that $\mathscr{S}|_0 = S$ and $\mathscr{S}|_1$ is a quartic surface as in Example 2.2. For a quartic surface, the family $\mathscr{S}$ simply means to vary the coefficients in the defining polynomial f.

The proof of Kodaira's theorem is based on two important points. The first one is deformation theory: the deformation space of a K3 surface is smooth of the expected dimension (equal to $h^1(X, T_X)$; for K3 surfaces, it is 20-dimensional) and the local Torelli theorem holds. One of the starting point for the theory of HK manifolds is that, by the Bogomolov–Tian–Todorov theorem, this is true for all HK manifolds. The second point in the proof of Kodaira's theorem is to be able to recognize a quartic K3 surface numerically (and they are dense), namely from the second singular cohomology group $H^2(S, \mathbf{Z})$ (together with its Hodge structure). For K3 surfaces this can be done effectively by using the theory of linear systems (see [BB+, Exposé VI]). In higher dimension, this is more complicate, since the theory of linear systems is not yet so precise. O'Grady's approach to show Theorem 4.2 below was indeed to try to recognize numerically HK fourfolds of degree 2. Our approach in [DHMV] was instead to use Lagrangian fibrations as intermediate step; in some sense this was used as well in Kodaira's original proof, by using elliptic K3 surfaces.

The basic example of elliptic K3 surface is the following.

Example 2.4. Let $S \subset \mathbf{P}^3$ be a quartic K3 surface as in Example 2.2. We can deform S, namely vary the coefficients of the defining polynomial f, in such a way that S contains a line ℓ. For example, we can take the polynomial

$$f = x_0^4 - x_1^4 + x_2^4 - x_3^4,$$

the line being $x_0 - x_1 = x_2 - x_3 = 0$.

We can perform the following elementary projective geometry construction. We project S from ℓ to another line ℓ' such that $\ell \cap \ell' = \varnothing$. Explicitly, this means for a point $x \in S \smallsetminus \ell$, we consider the plane generated by ℓ and x and denote by $\pi(x)$ the unique intersection point of this plane with ℓ'. This gives a morphism $\pi\colon S \smallsetminus \ell \to \ell' \simeq \mathbf{P}^1$ and it is easy to see that this extends to an actual morphism

$$\pi\colon S \longrightarrow \mathbf{P}^1.$$

The fiber of π over a point $x' \in \ell'$ is given by the intersection of the plane generated by ℓ and x' with the quartic surface S. This gives a planar quartic curve containing the line ℓ, namely the fiber is nothing but the residual planar cubic curve. For a generic x', this is a smooth cubic curve, namely an elliptic curve, topologically a compact complex 1-dimensional torus. By dimension reasons the symplectic form η on S restricts trivially on all fibers: the morphism π is a Lagrangian fibration.

Every K3 surface can be deformed to an elliptic one as in Example 2.4; moreover, these can be recognized numerically from the second cohomology group (and its Hodge structure). The first question in the theory of HK manifolds is if this generalizes to higher dimensions: can we always deform a HK manifold to a fibration in complex Lagrangian tori? We will discuss this in Section 3, where we will present an explicit version for this question, Conjecture 3.4. The next question is then to understand how this helps in the classification problem for HK: this can be seen in the fourfold case, under certain assumptions. We will sketch this in Section 5.

3. Lagrangian fibrations

Let X be a projective HK manifold, $\dim(X) = 2n$; let us denote by η the holomorphic symplectic form.

Definition 3.1. Let $\pi\colon X \to B$ morphism. We say that π is a

(a) *fibration*, if
   - B normal projective variety
   - π is surjective with connected fibers
   - $0 < \dim(B) < 2n$.

(b) *Lagrangian fibration* if π is a fibration and
   - $\eta|_{\pi^{-1}(b)} \equiv 0$, for all $\pi^{-1}(b)$ smooth fiber
   - $\dim(B) = n$ (or equivalently, $\dim(\pi^{-1}(b)) = n$ for $b \in B$ general point).

Note that, by the Liouville–Arnold theorem, the smooth fibers of a Lagrangian fibration are abelian varieties of dimension n, namely complex compact tori which are projective as well. The basic fact about HK manifolds is that they are quite rigid objects with respect to morphisms, as one can see from the following result [Ma1].

Theorem 3.2 (Matsushita)**.** *Let $\pi\colon X \to B$ be a fibration. Then π is a Lagrangian fibration with equidimensional fibers.*

In the definition of Lagrangian fibration, we allow B to be singular: a general expectation is that B is actually always a smooth projective manifold. The following result gives a quite striking consequence for this expectation [Hw, O, HX].

Theorem 3.3. *Let $\pi\colon X \to B$ be a Lagrangian fibration.*
(a) *(Hwang) If B is smooth, then $B \simeq \mathbf{P}^n$.*
(b) *(Ou, Huybrechts–Xu) If $\dim(X) = 4$, then B is always smooth (and thus $B \simeq \mathbf{P}^2$).*

We can now state the main conjecture about existence of Lagrangian fibrations on HK manifolds. To understand the statement, let us recall that on the singular cohomology group $H^2(X, \mathbf{Z})$ we have the intersection pairing

$$\alpha \in H^2(X, \mathbf{Z}) \mapsto \int_X \alpha^{2n} \in \mathbf{Z}. \tag{1}$$

If $\pi\colon X \to B$ is a fibration, then since B is projective, we have a non-zero class $\alpha_B \in H^2(B, \mathbf{Z})$ representing an ample divisor[1] on B. Let us denote by $\mathsf{l} := \pi^*\alpha_B \in H^2(X, \mathbf{Z})$. Then we have that:

- l is "algebraic", namely there exists a line bundle L on X such that $\mathrm{c}_1(L) = \mathsf{l}$,
- $\int_X \mathsf{l}^{2n} = 0$,
- l is *nef*, namely for all closed complex curve $C \hookrightarrow X$, $\int_C \mathsf{l} \geq 0$.

The main conjecture, giving a "numerical characterization" for the existence of a Lagrangian fibration, says that these conditions are also necessary. Recall that an element in a finite free abelian group is *primitive* if it is non-zero and the quotient by the subgroup generated by that element is also free.

Conjecture 3.4 (SYZ conjecture for HK manifolds, strong version). Let X be a HK manifold, $\dim(X) = 2n$. Let L be a line bundle on X and let $\mathsf{l} := \mathrm{c}_1(L)$. Assume that:

- $\int_X \mathsf{l}^{2n} = 0$,
- l is nef and primitive.

Then there exists a Lagrangian fibration $\pi\colon X \to \mathbf{P}^n$ such that $\mathsf{l} = \pi^* h$, where h denotes the class of a hyperplane.

There are weaker forms for this conjecture, by allowing the base B of the fibration to be singular, as in Definition 3.1, and by simply asking that there is a constant $k \geq 1$ such that $k\mathsf{l} = \pi^*\alpha_B$, for the class α_B of an ample divisor on B. In this weaker form, Conjecture 3.4 is the HK version of Kawamata's famous *abundance conjecture*.

The main evidence for the stronger version of the conjecture is that it holds on all known deformation classes of HK manifolds [Ma2, M3, BM, Y2, MR, MO]:

Theorem 3.5. *Let X be a HK manifold of either $\mathrm{K3}^{[n]}$, generalized Kummer, OG10, or OG6 deformation type. Then Conjecture 3.4 holds for X.*

This is the key example of Theorem 3.5 for $\mathrm{K3}^{[2]}$ deformation type.

Example 3.6 (Fujiki, Beauville, Mukai). Let $S \to \mathbf{P}^2$ be a 2-1 cover ramified over a smooth planar curve of degree 6. Explicitly, given f_6 a polynomial of degree 6 in two variables[2], S is

[1] Namely the class of a hyperplane section.

[2] In this example, we will always assume that the polynomial f_6 is chosen sufficiently general.

given as the smooth minimal compactification of the complex surface in the affine 3-dimensional space $\mathbf{C}^3$ (with coordinates x, y, z) given by the equation:

$$z^2 = f_6(x, y).$$

It is not too difficult to see that S is a K3 surface.

We now consider the *Hilbert square* $\mathrm{Hilb}^2(S)$ of S.[3] Intuitively, this parametrizes pairs of unordered points on S: when they collide, we want to remember further the direction. More precisely, we consider first the product $S \times S$. This is a smooth projective manifold, simply connected, but with two symplectic forms. To fix this, we take the quotient by the involution switching the two factors. The resulting space $\mathrm{Sym}^2(S)$ has now a unique symplectic form, but it is a singular variety. The singular locus is the diagonal, where locally the space looks like $\mathbf{C}^2 \times Q$, where Q is the quadric cone $z^2 = xy$, singular at the vertex 0. We then blow-up the singular locus $\mathbf{C}^2 \times \{0\}$, and obtain a new space $\mathrm{Hilb}^2(S)$. Locally, this corresponds to the following geometric construction: we take the product $\mathbf{C}^2 \times T^*\mathbf{P}^1$ of the cotangent bundle $T^*\mathbf{P}^1$ of $\mathbf{P}^1$ and we contract the 0-section $\mathbf{C}^2 \times \mathbf{P}^1$ to a point

$$\mathbf{C}^2 \times T^*\mathbf{P}^1 \longrightarrow \mathbf{C}^2 \times Q.$$

Intuitively, $\mathrm{Hilb}^2(S)$ has a unique symplectic form, since we did not modify it outside the diagonal and on the diagonal we have replaced it with a cotangent bundle, which always carries the Liouville symplectic form. Moreover this operation does not change the topology. We deduce, $\mathrm{Hilb}^2(S)$ becomes a smooth projective HK manifold.[4]

Given a general point in $\mathrm{Hilb}^2(S)$, namely two general points in S, we can associate to it the unique line passing through their image in $\mathbf{P}^2$ (if we choose the points in S general, their images are distinct). This gives a map

$$\pi_1\colon \mathrm{Hilb}^2(S) \dashrightarrow \check{\mathbf{P}}^2 \simeq \mathbf{P}^2.$$

This map is not defined everywhere: there is a copy of $\mathbf{P}^2$ in $P \subset \mathrm{Hilb}^2(S)$ given by the preimage of a point in $\mathbf{P}^2$ by the 2-1 cover. Locally, $\mathrm{Hilb}^2(S)$ near P looks like $T^*\mathbf{P}^2$. We can cut this locus and replace it with the cotangent $T^*\check{\mathbf{P}}^2$ of the dual projective plane. This operation gives another projective HK manifold X_1, which is a deformation[5] of $\mathrm{Hilb}^2(S)$ with an actual morphism

$$\pi_1\colon X_1 \to \mathbf{P}^2.$$

This is a Lagrangian fibration, by Matsushita's theorem. Its general fibers can be explicitly described: by the Abel–Jacobi theorem, they can be identified naturally with the Jacobian $\mathrm{Jac}(C)$ of the hyperelliptic curve C in S given as preminage of a line in $\mathbf{P}^2$. Hence, X_1 is the (compactified) relative Jacobian over the linear system of curves which are pre-images of lines.

Finally, we can twist this construction, by considering line bundles on C of degree 1; these are parametrized by the abelian surface $\mathrm{Pic}^1(C)$, which is still isomorphic to the Jacobian, but varying the curve it gives globally a different HK manifold X_2, which is a torsor over X_1, still deformation equivalent to the Hilbert square and with a Lagrangian fibration

$$\pi_2\colon X_2 \to \mathbf{P}^2.$$

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[3] This sketch of construction works for any K3 surface and any number of points.

[4] Note that, by Kodaira's theorem, all Hilbert schemes of K3 surfaces are deformation equivalent: this explains why we talk about $\mathrm{K3}^{[n]}$ deformation type.

[5] This can be checked directly in this example; it is a general theorem of Huybrechts [H1] that this is actually always the case for birational maps of HK manifolds.

This last example has an advantage. Namely, the Θ-divisors on the Jacobian do patch together to define a global unique Θ-divisor E on X_2. If we let, as above, $\mathsf{l} := \pi_2^* h$ and $\mathsf{m} := c_1(E \otimes L)$, we have the following relations:

$$\int_X \mathsf{l}^4 = \int_X \mathsf{m}^4 = 0 \qquad \text{and} \qquad \frac{1}{2}\int_X \mathsf{l}^2\mathsf{m}^2 = 1. \tag{2}$$

This example is the starting point of our main result, which we will describe in the next section.

4. O'Grady's conjecture

With the aim at starting classifying HK manifolds in lower dimension, with a view to Kodaira's theorem for K3 surfaces, O'Grady [O'G4] conjectured the following.

Conjecture 4.1 (O'Grady). Let X be a HK fourfold. Assume that X has the same cohomology ring[6] as the Hilbert square of a K3 surface. Then X is deformation equivalent to the Hilbert square of a K3 surface.

In [DHMV] we prove a slightly more general version of O'Grady's conjecture, motivated by the situation in Example 3.6, equation (2).

Theorem 4.2 (Debarre–Huybrechts–Macrì–Voisin)**.** *Let X be a HK fourfold. Assume that there exist two classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that*

$$\int_X \mathsf{l}^4 = \int_X \mathsf{m}^4 = 0 \qquad \textit{and} \qquad \frac{1}{2}\int_X \mathsf{l}^2\mathsf{m}^2 = 1.$$

Then X is deformation equivalent to the Hilbert square of a K3 surface.

Theorem 4.2 gives a cohomological characterization of HK manifolds of $\mathrm{K3}^{[2]}$ deformation type. As mentioned, there are other known examples of HK fourfolds: generalized Kummer fourfolds. In this case, their cohomology still has two isotropic classes l and m such that

$$\frac{1}{2}\int_X \mathsf{l}^2\mathsf{m}^2 = 3.$$

The next question, towards classification of smooth HK fourfolds, is to understand if a similar result as Theorem 4.2 holds in such a situation and then if these cover all possible cases.

5. Ideas from the proof

In this section we give a few ideas from the proof of Theorem 4.2. Let X be a HK fourfold and let $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ satisfying the assumptions in the theorem (equation (2)). The first step is to use deformation theory. We can deform X in such a way that the two classes l and m are both algebraic and they generate the $\mathbf{R}$-vector space of all algebraic classes. We can further assume that l is nef and that m is in the boundary of the positive cone. Let us denote by L and M the corresponding line bundles on X.

The second step is to use results by Salamon and Guan [S, G] on constraints on Betti numbers of HK fourfolds to be able to compute the dimension of the space of sections of tensor products $L^{\otimes a} \otimes M^{\otimes b}$, for $a, b > 0$. The key result, an algebraic consequence of our topological assumptions, is that they do behave as in the Hilbert square case.

[6] With respect to the intersection product (1).

The third step of the proof is to establish Conjecture 3.4, under our assumptions. Note that, by Theorem 3.5, HK manifolds deformation equivalent to Hilbert schemes on K3 surfaces do satisfy the SYZ conjecture. In our approach, the SYZ conjecture is a step towards the main theorem, the precise logical relationships being as follows. A study of the effective cone of X shows that there are two cases. The first case is when L and M are both nef and thus $L \otimes M$ is ample. In this case,

- either, after possibly permuting L and M, the SYZ conjecture holds; this uses results by Fujino–Kawamata [F, K] and Fukuda [Fu]. But this is impossible, since a general argument shows that L and M cannot be both nef if either of the two induces a Lagrangian fibration in the strongest form of Conjecture 3.4.
- or any divisor in the linear system $|L \otimes M|$ is irreducible and the image of the rational map $\varphi_{L\otimes M} \colon X \dashrightarrow \mathbf{P}^5$ is rationally connected. This is the hardest part of the proof, which uses the algebraic structure of the space of sections of the line bundles $L \otimes M$, $L^{\otimes 2} \otimes M$, and $L^{\otimes 3} \otimes M^{\otimes 2}$, which we can compute thanks to Step 2. A fundamental theorem by Voisin [Vo], extending a previous result by O'Grady [O'G4] shows that this situation cannot happen.

So this case in fact does not arise.

The second case is when X admits a divisorial contraction and M is not nef. The existence of the divisorial contraction follows by results of Wierzba–Wiśniewski [WW] and Markman [M2]. We then analyse the divisorial contraction, by using [W]. This allows us to prove first the SYZ conjecture for L and to use this to have an explicit description of the exceptional divisor of the divisorial contraction: by using a result by Mukai [Mu2], this is the same as the global Θ divisor $E = M \otimes L^\vee$ in the last case X_2 treated in Example 3.6. Then, both X and X_2 are birational to the relative Albanese variety, and thus they are isomorphic, concluding the proof of Theorem 4.2.

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