

ANTISYMPLECTIC INVOLUTIONS

ON PROJECTIVE HYPERKÄHLER MANIFOLDS

work in progress with

Laure Flapan

Kieran O'Grady

Giulia Saccà

1. Main Thm

(X, λ)

X : smooth proj HK manifold

$$\dim X = 2g, \quad g \geq 1$$

$K_3^{[g]}$ -type

with
BBF form

λ : prim. polariz., $\lambda^2 = 2$

(Recall: $\text{div}(\lambda) := 1, 2$ gen. of the ideal
 $\lambda, H^2(X, \mathbb{Z}) \subseteq \mathbb{Z}$)

$$f_\lambda(n) = -n + (\lambda \cdot n) \lambda \quad \text{on } H^2(X, \mathbb{Z})$$

Torelli Thm
+
Monodromy Thm $\Rightarrow \exists \tau \in \text{Aut}(X)$, $\tau^2 = \text{id}$, $\tau_* = f_\lambda$
antisympl.

Main Thm (1) The number of conn. components of $\text{Fix}(\tau)$ is equal to $\text{div}(\lambda)$.

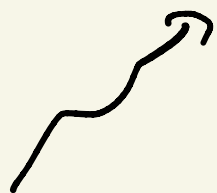
(2) If $\text{div}(\lambda) = 2$, then one conn. comp. is Fano manifold of $\dim g$ and index 3.

2. Questions / Conjectures

$\text{div}(\lambda) = 1$: $\Sigma := \text{Fix}(\tau)$ irred.

- Σ gen. type

- $m \cdot \Sigma$ covering family of Lagrangian cycles (O'Grady) of X .



Ex $g = 2$
(O'Grady, Fuortti)

$m > 0$

div(λ)=2 : $\text{Fix}(\tau) = Y \sqcup \tilde{Y}$

- Y Fano^v, index 3^v
- $g(Y)=1$? \leftarrow OK, if $g=4, 8$
- $h^{3,1}(Y)=1$?
- \tilde{Y} gen. type ?
- $m \tilde{Y}$ cov. family of Lagr. cycles
 $m > 0$.

Conj $D^b Y = \langle \mathcal{D}^{[h]} \rangle$ $h = g/4$

K_3 category $\mathcal{Q}_Y, \mathcal{D}, \dots, \mathcal{D}^{[h-1]}$ $\mathcal{D}^{[r]}$
analogue
of der.
cat. of
 $\text{Hilb}^r_{k_3}$

$\mathcal{Q}_Y(1), \mathcal{D}(1), \dots, \mathcal{D}^{[h-1]}(1)$

$\mathcal{Q}_Y(2), \mathcal{D}(2), \dots, \mathcal{D}^{[h-1]}(2) \rangle$

• $\exists \sigma \in \text{Stab}(\mathcal{D})$ st. $(X, \lambda) \cong (\text{Mod}(\underline{\sigma}), \ell_\sigma)$?

Ex 1

γ cubic 4 fold.

$X = \text{LLSvS} \quad \text{HK8} \quad \overset{\mathbb{Z}_2 \cdot 1}{\dots}, \text{Gr}(4,6)$

$\lambda = \text{Plücker pol.} \quad \lambda^2 = 2, \text{div}(\lambda) = 2.$

$\tau = \text{inv. on twisted cubics}$

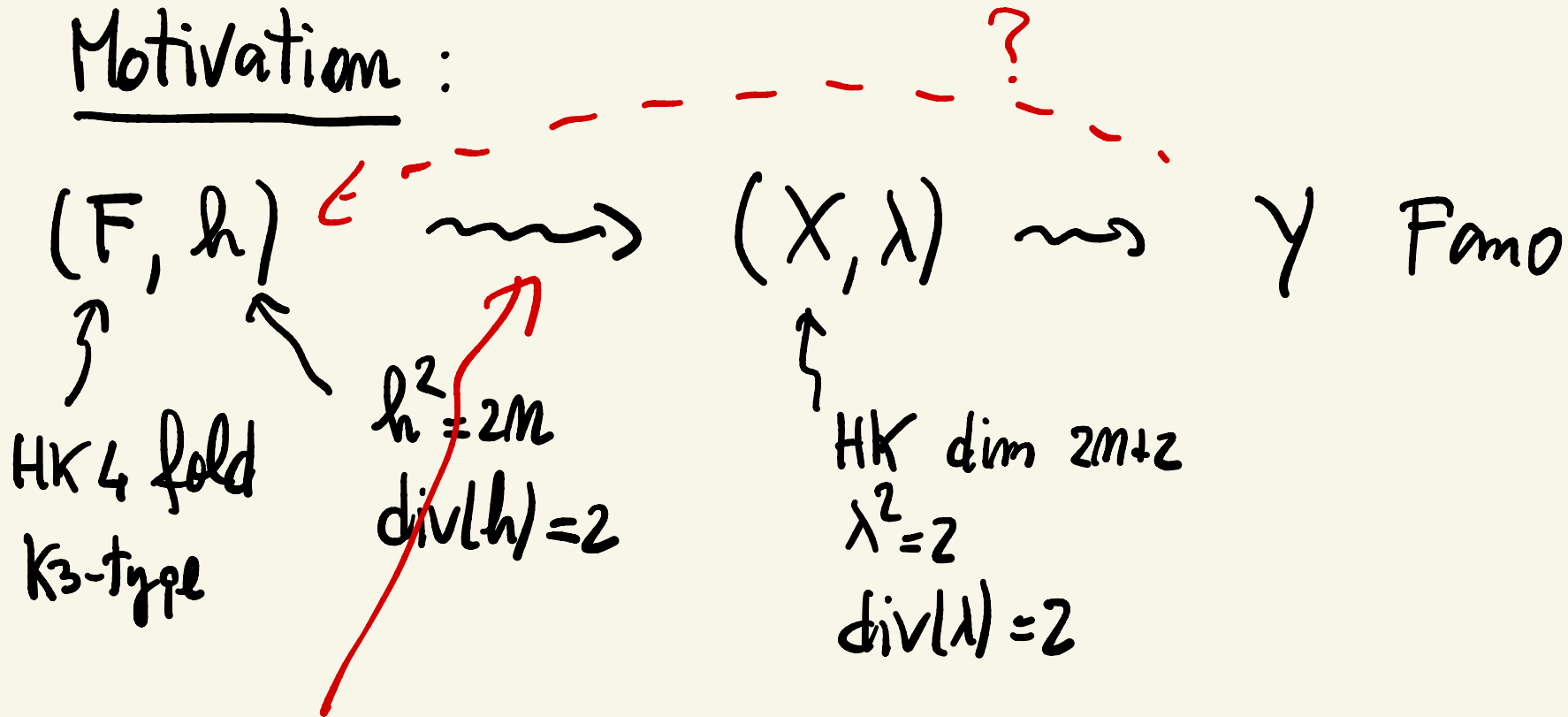
$\text{Fix}(\tau) = \gamma \perp \tilde{\gamma}$
 \nearrow
cubic 4 fold

$$D^b Y = \langle \mathcal{D}, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

\uparrow
K3 categ.

- $\exists \sigma \in \text{Stab}(\mathcal{D})$ [BLMS]
- $(X, \lambda) \cong (M_g(\mathbb{P}^2), \rho)$ [Li-Petrusci-Zhao].

Motivation:



$$H^2(F, \mathbb{Z})_h \cong H^2(X, \mathbb{Z})_\lambda$$

Ex 2

V_6 v.s.p. of dim 6

$A \in \Lambda^3 V_6$ Lagr. subsp.

\nwarrow
dim 10

$Y_A := \{ [v] \in \mathbb{P}V_6 : A \xrightarrow{\varphi_r} \Lambda^4 V_6 \text{ has non-zero kernel} \}$
 $d \mapsto \sigma \wedge d$

$\subseteq \mathbb{P}V_6$

sextic hypersurface

$\text{sing}(Y_A) =: \Sigma_A$ irred sm. surface
of gen. type.

$\exists X_A \xrightarrow[\not{f}]{2-1} Y_A$ ramif. at Σ_A

X_A HK4, K3-type

$\lambda_A := f^* \mathcal{O}_{\mathbb{P}V_6}(1)$ $\lambda^2 = 2$, $\text{div}(\lambda) = 1$.

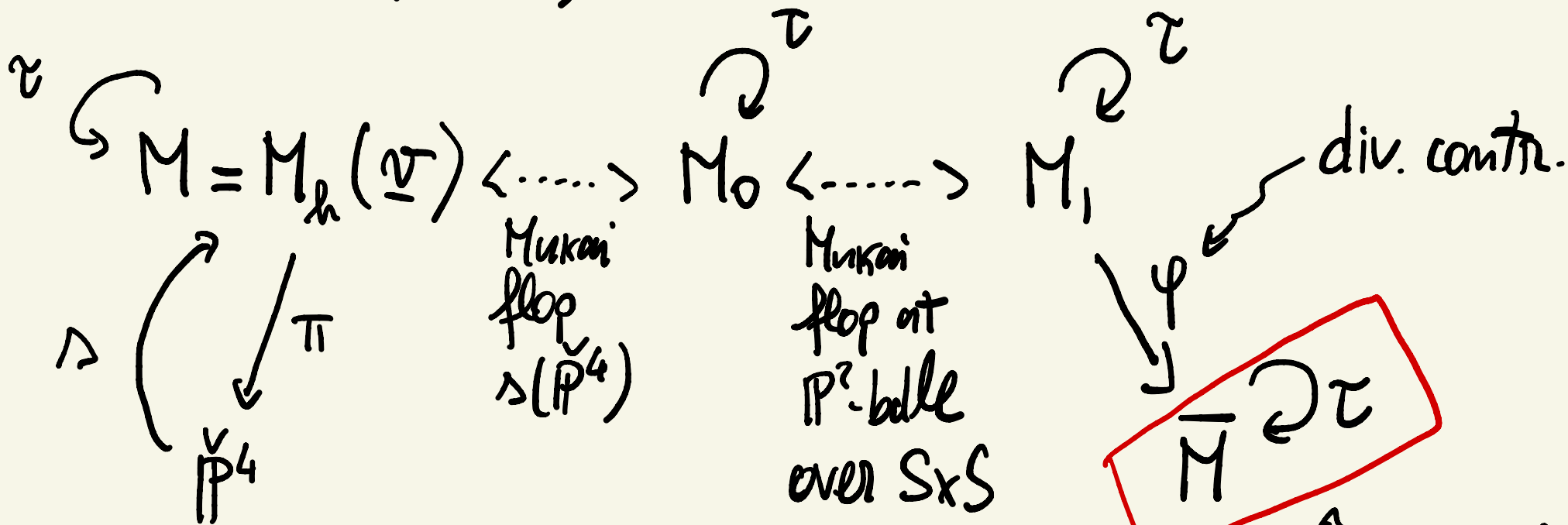
$\leadsto (X_A, \lambda_A)$ ($g=2$)
 $\text{Fix}(\mathcal{C}_A) = \Sigma_A / 2\Sigma_A$ cov. fam.
of Lagr.

Recall: $(X, \lambda) \rightsquigarrow \text{Fix}(\tau)$
 $\nearrow \dim 2g$ $\nwarrow \lambda^2 = 2$

Ex Y cubic \rightsquigarrow specialize to Y modul

[C. Lehn] (S, h) S v.gen. K3 surface
 $h^2 = 6$

$$\underline{v} = (0, h, -3)$$

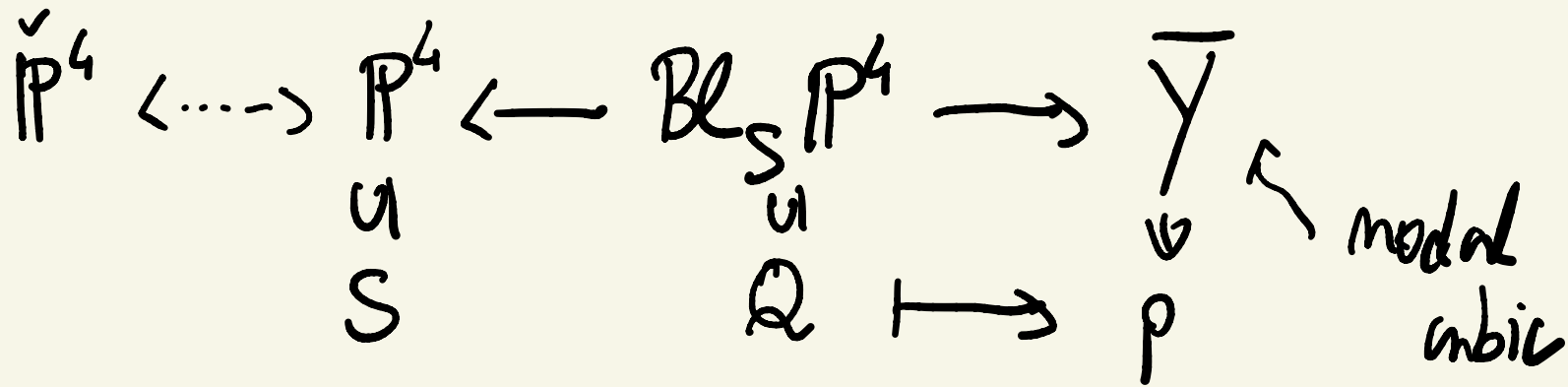


$$\text{Fix}(\tau) = \Delta(\mathbb{P}^4) \amalg \tilde{Y} \xrightarrow{\sim} \tilde{Y}'$$

L_C on $C \in |h|$
 st. $L_C^2 \cong \mathcal{O}_C$

specializ. of LLSvS 8 fold.

e.g. $\Delta(\check{\mathbb{P}}^4)$:



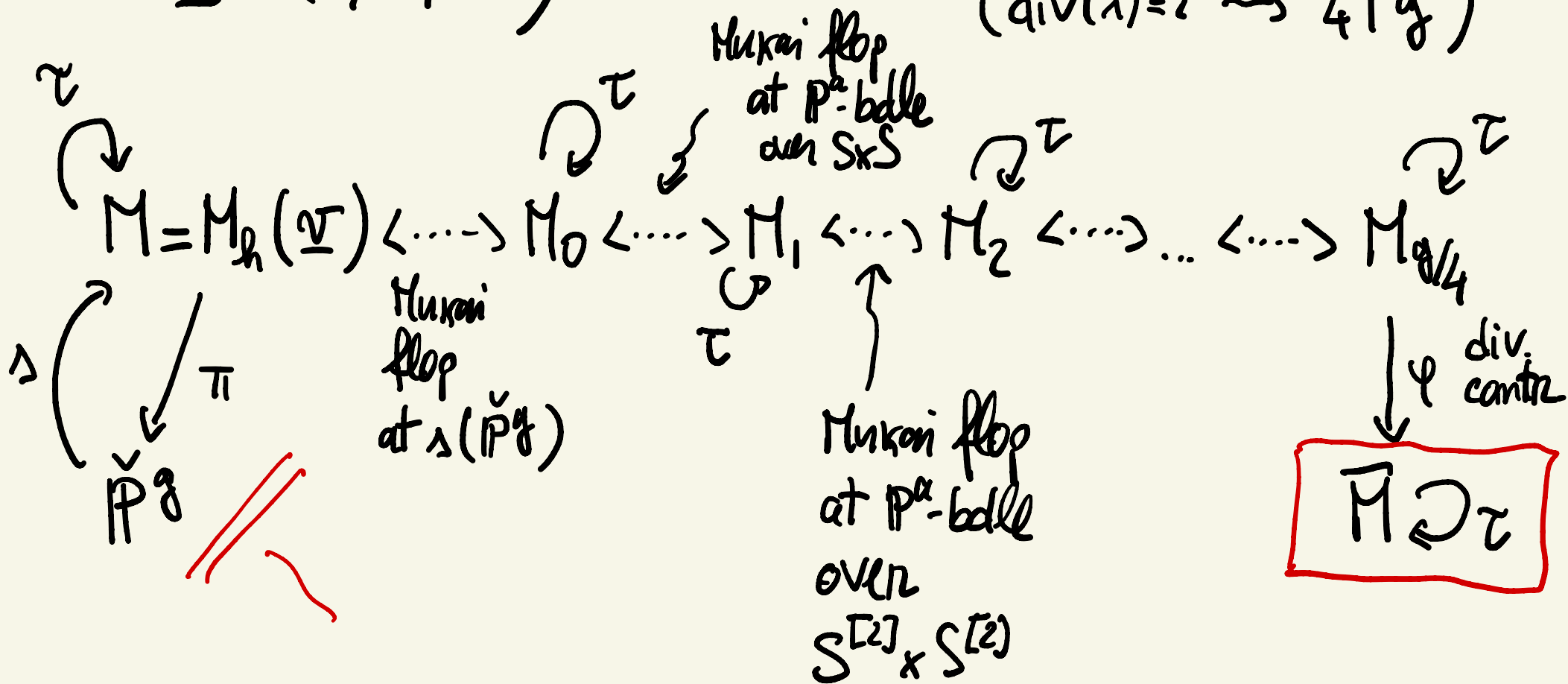
In general: (S, h)

$$\underline{v} = (0, h, -m)$$

S K3 surface

$$h^2 = 2m \quad g = m+1$$

$$(\text{div}(\lambda) = 2 \rightarrow 4 \mid g)$$

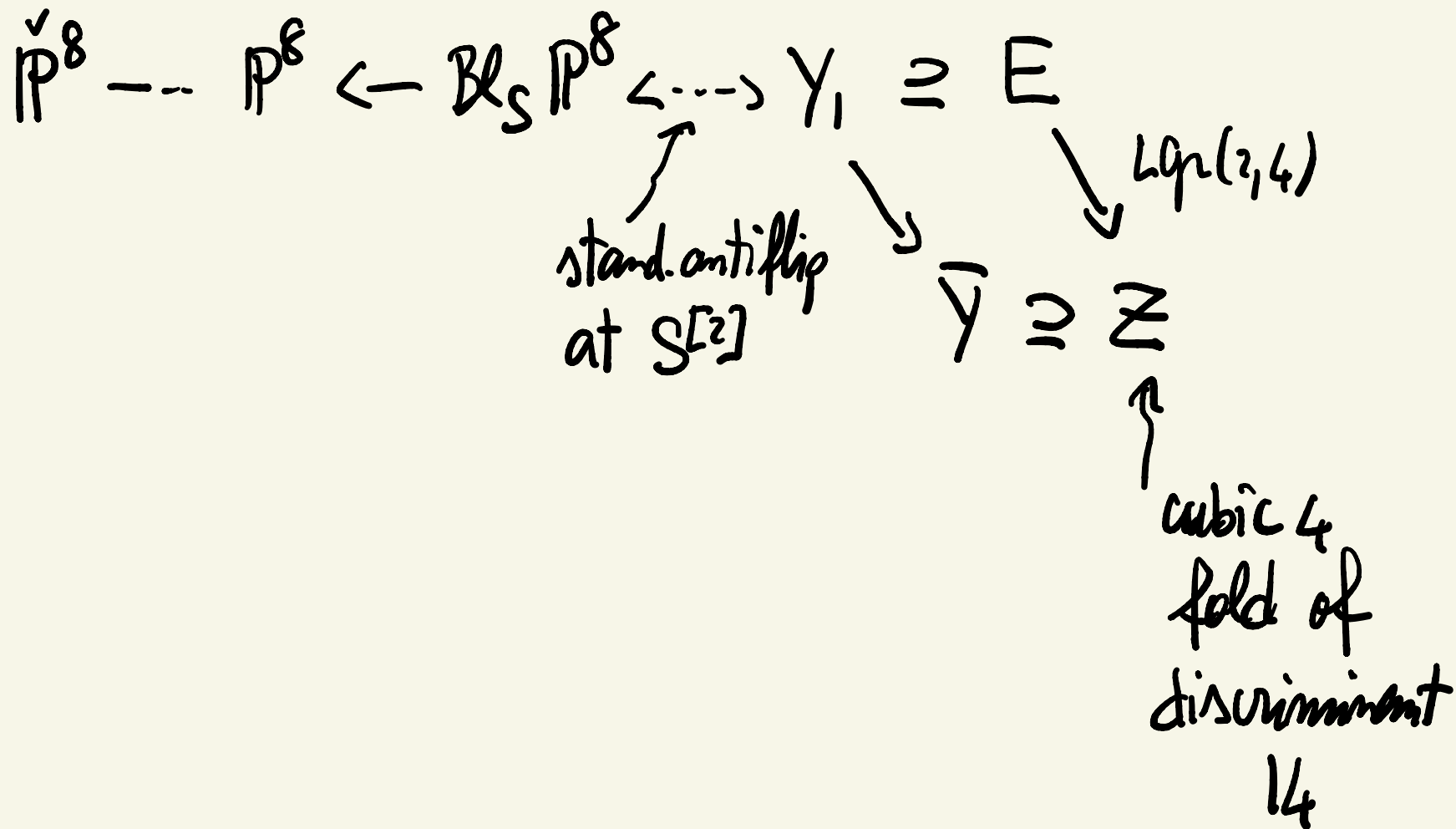


For Fano component:

$$\begin{array}{ccccccc}
 \check{\mathbb{P}}^2 & \langle \dots \rangle & \mathbb{P}^2 & \leftarrow \text{Bl}_S \mathbb{P}^2 = Y_0 & \langle \overset{\sim}{\dots} \rangle & Y_1 & \langle \overset{\sim}{\dots} \rangle & Y_2 & \langle \dots \rangle & Y_{3/4} \\
 & & \downarrow \text{v1} & & \downarrow \text{anti flip} & & & & & \downarrow \psi \\
 & & S & & \text{at } S^{(2)} & & & & & \check{Y}
 \end{array}$$

specialization of
Fano component.

$g=8$:



If $\sigma \in \text{Stab}(\mathcal{Q})$

$$(F, h) \cong (M_{\sigma}(\lambda), l_{\sigma})$$

$$(X, \lambda) \cong (M_{\sigma}(\mu), l_{\sigma})$$