ANTISYMPLECTIC INVOLUTIONS ON PROJECTIVE HYPERKÄHLER MANIFOLDS

work in progress with

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1. Main Thm

(X, λ) $\quad$ X: smooth proj HK manifold

$\dim X = 2g \ , \quad g \geq 1$

$K3^{[g]}$-type

wrt BBF form

λ: prim. polariz. , $\lambda^2 = 2$

(Recall: $\mathrm{div}(\lambda) := 1, 2$ gen. of the ideal

$\lambda . H^2(X, \mathbb{Z}) \subseteq \mathbb{Z}$)

$$\rho_\lambda(x) = -x + (\lambda . x)\lambda \qquad \text{on } H^2(X,\mathbb{Z})$$

Torelli Thm
+
Monodromy Thm $\Longrightarrow$ $\exists\, \tau \in \mathrm{Aut}(X)$, $\tau^2 = \mathrm{id}$, $\tau_* = \rho_\lambda$

antisympl.

Main Thm (1) The number of conn. components of $\mathrm{Fix}(\tau)$ is equal to $\mathrm{div}(\lambda)$.

(2) If $\mathrm{div}(\lambda) = 2$, then one conn. comp. is Fano manifold of dim g and index 3.

2. Questions / Conjectures

div(λ)=1 : $\Sigma := \mathrm{Fix}(\tau)$ irred.

- Σ gen. type

- $m \cdot \Sigma$ covering family of Lagrangian cycles (O'Grady) of X.

  $m > 0$

Ex $g=2$
(O'Grady, Ferretti)

$\underline{\mathrm{div}(\lambda)=2}$: $\mathrm{Fix}(\tau) = Y \amalg \tilde{Y}$

- Y Fano ✓, index 3 ✓
- $\rho(Y)=1$? ← OK, if $q=4,8$
- $h^{3,1}(Y)=1$?
- $\tilde{Y}$ gen. type ?
- $m\tilde{Y}$ cov. family of Lagr. cycles $m>0$.

Conj $D^bY = \langle \mathcal{D}^{[h]},$ $\qquad h = g/4$

K3 category

$\mathcal{O}_Y, \mathcal{D}, \dots, \mathcal{D}^{[h-1]},$ $\qquad \mathcal{D}^{[h-1]} \rightsquigarrow \mathcal{D}^{[n]}$

analogue of der. cat. of $\mathrm{Hilb}^n K3$

$\mathcal{O}_Y(1), \mathcal{D}(1), \dots, \mathcal{D}^{[h-1]}(1),$

$\mathcal{O}_Y(2), \mathcal{D}(2), \dots, \mathcal{D}^{[h-1]}(2) \rangle$

- $\exists\, \sigma \in \mathrm{Stab}(\mathcal{D})$ st. $(X, \lambda) \cong (M_\sigma(\underline{v}), \ell_\sigma)$?

Ex 1

Y cubic 4 fold.

$X =$ LLSvS HK8 $\overset{72:1}{\dashrightarrow} Gr(4,6)$

$\lambda =$ Plücker pol. $\lambda^2 = 2$, $\mathrm{div}(\lambda) = 2$.

$\tau =$ inv. on twisted cubics

$$\mathrm{Fix}(\tau) = \underset{\nearrow}{Y} \sqcup \tilde{Y}$$

cubic 4 fold

$$D^bY = \langle \underset{\substack{\uparrow \\ \text{K3 categ.}}}{\mathcal{D}}, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

- $\exists \sigma \in \mathrm{Stab}(\mathcal{D})$ [BLMS]
- $(X, \lambda) \cong (M_\sigma(\underline{v}), \ell_\sigma)$ [Li-Pertusi-Zhao].

Motivation :

$$\begin{tikzcd} (F, h) \arrow[r, squiggly] & (X, \lambda) \arrow[r, squiggly] & Y \text{ Fano} \arrow[ll, dashed, bend right, "?"'] \end{tikzcd}$$

F: HK 4 fold K3-type

h: $h^2 = 2n$, $\operatorname{div}(h) = 2$

λ: HK dim $2n+2$, $\lambda^2 = 2$, $\operatorname{div}(\lambda) = 2$

$$H^2(F,\mathbb{Z})_h \cong H^2(X,\mathbb{Z})_\lambda$$

Ex 2

V_6 v.sp. of dim 6

$A \subseteq \Lambda^3 V_6$ Lagr. subsp.

$\uparrow$ dim 10

$$Y_A := \{ [v] \in \mathbb{P}V_6 : \begin{array}{c} A \xrightarrow{\varphi_v} \Lambda^4 V_6 \\ \alpha \longmapsto v \wedge \alpha \end{array} \text{ has non-zero kernel} \}$$

$\subseteq \mathbb{P}V_6$ sextic hypersurface

$\mathrm{sing}(Y_A) =: \Sigma_A$ irred sm. surface of gen. type.

$\exists\ X_A \xrightarrow[f]{2-1} Y_A$ ramif. at Σ_A

X_A HK4, K3-type

$\lambda_A := f^* \mathcal{O}_{\mathbb{P}V_6}(1)$ $\lambda^2 = 2$, $\mathrm{div}(\lambda) = 1$.

$\rightsquigarrow (X_A, \lambda_A)$ $(g=2)$

$\mathrm{Fix}(\tau_A) = \Sigma_A$ $/\ 2\Sigma_A$ cov. fam. of Lagr.

Recall: $\underset{\text{dim } 2g}{\overset{\nearrow}{X}}, \underset{\lambda^2=2}{\overset{\nwarrow}{\lambda}}) \rightsquigarrow \mathrm{Fix}(\tau)$

$$(\underset{\underset{\text{dim } 2g}{\nearrow}}{X}, \underset{\underset{\lambda^2=2}{\nwarrow}}{\lambda}) \rightsquigarrow \mathrm{Fix}(\tau)$$

Ex Y cubic $\rightsquigarrow$ specialize to Y **nodal**

[C. Lehn] (S, h) S v.gen. K3 surface

$h^2 = 6$

$\underline{v} = (0, h, -3)$

$$\begin{tikzcd}
M = M_h(\underline{v}) \arrow[loop left, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop } s(\check{\mathbb{P}}^4)"'] \arrow[d, "\pi"] & M_0 \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop at } \mathbb{P}^3\text{-bdle over } S\times S"'] & M_1 \arrow[loop above, "\tau"] \arrow[d, "\varphi"] \\
\check{\mathbb{P}}^4 \arrow[u, bend left, "s"] & & \boxed{\overline{M} \circlearrowleft \tau}
\end{tikzcd}$$

div. contr.

specializ. of LLSvS 8fold.

$\mathrm{Fix}(\tau) = s(\mathbb{P}^4) \sqcup \tilde{Y}'$

L_C on $C \in |h|$ s.t. $L_C^2 \cong \mathcal{O}_C$

e.g. $s(\check{\mathbb{P}}^4)$:

$$\begin{tikzcd}
\check{\mathbb{P}}^4 \arrow[r, dashed, leftrightarrow] & \mathbb{P}^4 & \mathrm{Bl}_S\,\mathbb{P}^4 \arrow[l] \arrow[r] & \overline{Y} & \\
& \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\in$} & \text{nodal cubic} \arrow[ul] \\
& S & Q \arrow[r, mapsto] & p &
\end{tikzcd}$$

In general: (S, h) S K3 surface

$h^2 = 2n$ $g = n+1$

$\underline{v} = (0, h, -n)$

$(\mathrm{div}(\lambda) = 2 \rightsquigarrow 4 \mid g)$

$$\begin{tikzcd}
M = M_h(\underline{v}) \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop at } s(\check{\mathbb{P}}^g)"'] \arrow[d, "\pi"] & M_0 \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop at } \mathbb{P}^a\text{-bdle over } S\times S"] & M_1 \arrow[loop below, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop at } \mathbb{P}^a\text{-bdle over } S^{[2]}\times S^{[2]}"'] & M_2 \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow] & \cdots \arrow[r, dashed, leftrightarrow] & M_{g/4} \arrow[loop above, "\tau"] \arrow[d, "\varphi \ \text{div. contr.}"] \\
\check{\mathbb{P}}^g \arrow[u, bend left, "s"] & & & & & \boxed{\bar{M} \circlearrowleft \tau}
\end{tikzcd}$$

For Fano component:

$$\begin{tikzcd}
\check{\mathbb{P}}^g \arrow[r, dashed, leftrightarrow] & \mathbb{P}^g \arrow[d, phantom, "\cup"] & Bl_S \mathbb{P}^g \arrow[l] \arrow[r, phantom, "="] & Y_0 \arrow[r, dashed, leftrightarrow, "\sim", "\text{anti flip at } S^{[2]}"'] & Y_1 \arrow[r, dashed, leftrightarrow, "\sim"] & Y_2 \arrow[r, dashed, leftrightarrow] & Y_{g/4} \arrow[d, "\varphi"] \\
 & S & & & & & \bar{Y}
\end{tikzcd}$$

specialization of Fano component.

g = 8 :

$$\begin{tikzcd}
\check{\mathbb{P}}^8 \arrow[r, dashed] & \mathbb{P}^8 & \mathrm{Bl}_S \mathbb{P}^8 \arrow[l] \arrow[r, dashed, leftrightarrow, "\text{stand. antiflip at } S^{[2]}"'] & Y_1 \arrow[dr] & \supseteq & E \arrow[dr, "\mathrm{LGr}(2,4)"] & \\
& & & & \bar{Y} & \supseteq & Z \\
& & & & & & \text{cubic 4 fold of discriminant 14} \arrow[u]
\end{tikzcd}$$

Thm $g=8$

Hodge numbers of Y are:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&&& 1 &&&& \quad h^{1,1} \\ && 1 && 21 && 1 && \\ && 1 && 23 && 1 && \\ 1 && 22 && 253 && 22 && 1 \end{array}$$

$$D^bY = \langle \underset{\mathcal{D}^{[2]}}{D^b(S^{[2]})},\ \underset{\mathcal{D}}{D^bS},\ \underset{\mathcal{D}}{D^bS},\ \underset{\mathcal{D}}{D^bS},\ \mathcal{O}_Y,\ \mathcal{O}_Y(1),\ \mathcal{O}_Y(2) \rangle$$

If $\sigma \in \mathrm{Stab}(\mathcal{D})$

$$(F, h) \cong (M_\sigma(\lambda), \ell_\sigma)$$

$$(X, \lambda) \cong (M_\sigma(\mu), \ell_\sigma)$$