Lagrangian fibrations on HK 4 folds

[w/ O. Debarre, D. Huybrechts, C. Voisin]

Def X compact Kähler manifold, $\dim X = 2n$

Say X hyper-Kähler if
- X simply conn.
- $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \eta$

Conj 1 Any HK manifold can be deformed into a HK manifold w/ Lagrangian fibration

$f: X \to B$ proper
w/ conn. fibers
B normal, proj
$b \in B$ gen $\rightsquigarrow f^{-1}(b) \subseteq X$ abelian variety of dim n

Rmk
B proj $\rightsquigarrow L := f^* \mathcal{O}_B(1)$ nef on X w/ $\int_X c_1(L)^{2n} = 0$

Conj 2 (SYZ Conj for HK, abundance Conj for HK, TBHTHS)

X HK, L *nef* line bdle st. $\int_X c_1(L)^{2n} = 0$

$\Rightarrow L$ *semiample*

∎

[Matsushita] Conj 2 $\Rightarrow$ $|mL|$, $m \gg 0$ gives Lagr. fibr.

$\rightsquigarrow$ Conj 2 $\underset{b_2(X)\geq 5}{\Longrightarrow}$ Conj 1.

[Hwang] X proj., B smooth $\Rightarrow B \cong \mathbb{P}^n$

[Huybrechts-Xu] X 4 fold $\Rightarrow B \cong \mathbb{P}^2$

__Ex__ (S,H) very gen. K3 surface of genus g

$$\textcircled{H} \subseteq X = M(0,H,0) \ni i_{C*}\mathcal{L}, \quad C \in |H| \text{ smooth}$$

$\mathcal{L}$ line bdle of degr. $g-1$

$$\begin{array}{ccc} & \textcircled{H} & \\ S \swarrow & & X \\ S \subseteq \bar{X} \swarrow & & \downarrow \\ \uparrow & & \mathbb{P}^g \\ \underline{\dim 4} & & \end{array}$$

$L := \Theta((0,0,-1))$ __nef__

$M := \Theta((1,0,0))$

$$\int_X c_1(L)^{2g} = 0$$

$$\int_X c_1(M)^{2g} = 0$$

$$\int_X c_1(L)^g \cdot c_1(M)^g = g!$$

__Rmk__ $\textcircled{H} = \Theta((1,0,1))$

$\textcircled{H} \cong M \otimes L^\vee \rightsquigarrow M$ __not__ __nef__

$\triangle$

The Main Thm

$D = \ell + m \quad q_X(D) = 2$

X HK 4 fold.

Assume: $\exists\ \ell, m \in H^2(X,\mathbb{Z})$ st.

- $\int_X \ell^4 = 0$
- $\int_X \ell^2 . m^2 = 2$

$\Rightarrow \int_X m^4 = 0$

Thm Either X is of $K3^{[2]}$-type

(or $\exists$ defo of X w/ "special geometry")

work in progress by Voisin, this does NOT happen!

Idea of pf

St. 1 We can deform X st.

$$\ell = c_1(L) \qquad L, M \text{ line bdles on } X$$

$$m = c_1(M)$$

$$\rightsquigarrow \int_X m^4 = 0 \qquad NS(X) = \mathbb{Z}\cdot L \oplus \mathbb{Z}\cdot M$$

X very general

Thm A The Hirzebruch-Riemann-Roch poly of X is

$$P_X(2k) = \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k+1)) = \frac{(k+2)(k+3)}{2}$$

$$:= \chi(X, L^a \otimes M^b), \quad k = ab.$$

Moreover, the Fujiki constant and the Hodge and Chern numbers of X are those of $K3^{[2]}$.

$\rightsquigarrow$ know how to compute $\chi(X, L^a \otimes M^b)$.

St. 2 : SYZ Conj.

Assume further that L nef

Thm B $\exists$ Lagr. fibr. $f: X \to \mathbb{P}^2$ w/ $f^*\mathcal{O}_{\mathbb{P}^2}(1) \cong L$

(OR X "~~special~~ geometry")

$\square$

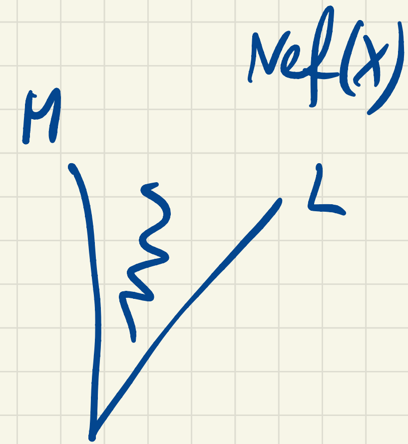
Explanations :

Thm B vs Main Thm

$\Leftarrow$

Conversely, L, M , L nef

$M \in \mathrm{Pseff}(X)$



There are 2 possib. :

(I) both L and M are nef

$\rightsquigarrow$ $D = L \otimes M$ ample $\quad \varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$

Then : (I.1) $h^0(L), h^0(M) \neq 0$

Prop $h^0(L) + h^0(M) \geq 3$

$\Rightarrow f = \varphi_{|L|} : X \to \mathbb{P}^2$, $f^*\mathcal{O}(1) \cong L$

$\rightsquigarrow$ f_*M <u>line bdle</u> on $\mathbb{P}^2$

$M \overset{\text{nef}}{\cong}$ L

$\overset{\rightsquigarrow}{\text{Thm A}}$ $f_*M = \mathcal{O}(1)$ $M \otimes L^\vee$

$\rightsquigarrow$ $H^0(X, M \otimes L^\vee) = H^0(\mathbb{P}^2, \mathcal{O}) = \mathbb{C}$

$\rightsquigarrow$ M not <u>nef</u> $\overset{\text{\{}}{\underline{\Downarrow}}$

Voisin in progress $\overset{\text{\{}}{\underline{\Downarrow}}$ (I.2) Or <u>Prop</u> $h^0(L) = 0$

~~"special geometry"~~ $\rightsquigarrow$ Then $\varphi_{|L \otimes M|} : X \dashrightarrow \mathbb{P}^5$

its image is <u>rationally</u> <u>connected</u>

(II) M not nef

[Markman] $\exists$ divisorial contr.

$\varphi : X \to \overline{X}$ st.

- $E = \mathrm{exc}(\varphi)$ integral div. w/ $[E] = m \cdot \ell$

not ample

- φ induced by multiple of $L \otimes M$

[Wierzba + Prop.] the morph. $\varphi|_E : E \to \varphi(E) \subseteq \overline{X}$ is <u>$\mathbb{P}^1$-bdle</u> over $S = \varphi(E)$ <u>smooth</u>

K3 surface very gen. of genus 2.

Can prove a weaker version of Thm B

$\exists\ f: X \to \mathbb{P}^2,\ f^! \mathcal{O}(1) \cong L^k$, some $k>0$.

$\rightsquigarrow$ look at

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ E \arrow[ur, hook] \arrow[rr, "f_E"] \arrow[d, "\varphi_E"', "\mathbb{P}^1"] & & \mathbb{P}^2 \\ S & & \end{tikzcd}$$

$\rightsquigarrow$ this diagram is isom. to

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\mathbb{P}^1"'] & \mathbb{P}^2 \\ S & \end{tikzcd}$$

$\rightsquigarrow X \cong M(0, H, 0)$.