SPECIAL SURFACES ON SPECIAL CUBIC FOURFOLDS

/$\mathbb{C}$

work in progress with

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Thm

Notation: $Y \in \mathcal{C}_d$

Y special cubic 4 fold of discriminant

$$d = 6a^2 + 6a + 2, \quad a \geq 1$$

Then $\exists\, \Sigma \subseteq Y$ surface s.t.

family param. by K3 surface of deg d

- $\deg(\Sigma) = 1 + \frac{3}{2}a(a+1)$

$$\Sigma^2 = \frac{d + (\deg)^2}{3}$$

- $\mathcal{I}_\Sigma(a) \in Ku(Y)$

$$H^\bullet(Y, \mathcal{I}_\Sigma(a-i)) = 0 \quad \forall i = 0,1,2$$

<u>Exs</u> (1) $a=1$, $d=14$

Y general $\in \mathcal{C}_{14}$ $\rightsquigarrow$ Σ = smooth quartic scroll

(if Y Pfaffian, then

$$\begin{tikzcd}
\Sigma_p \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[rr, bend left, hook] & \widetilde{\Sigma} \arrow[r, two heads, "4-1"'] \arrow[d] & Y \\
p \arrow[r, phantom, "\in"] & S &
\end{tikzcd}$$

)

(2) $a = 2$, $d = 38$

$Y \in \mathcal{C}_{36}$ gen. $\rightsquigarrow$ Σ = smooth "generalized" Coble surface

Now

$$\mathrm{Bl}_{10\text{ pts}}\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

$$10 \cdot L - 3 \cdot (E_1 + \dots + E_{10}).$$

Open questions :

- Σ integral ?
- Σ smooth ?

$\gamma \in \mathcal{C}_d$ gen.

K3 categories and Brill-Noether loci

Y cubic 4 fold

$$\begin{tikzcd} \mathrm{Ku}(Y) \arrow[r, hook, "i_*"'] & D^b Y \arrow[l, bend right, "i^*"'] \end{tikzcd}$$

Kuznetsov component

$$\mathrm{Ku}(Y) = \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle^{\perp}$$

Facts: • $\mathrm{Ku}(Y)$ K3 category
[Kuz]

• $H(\mathrm{Ku}(Y),\mathbb{Z}) = \langle [\mathcal{O}_Y], [\mathcal{O}_Y(1)], [\mathcal{O}_Y(2)] \rangle^{\perp}$
[AT]

$\subseteq K_{top}(Y).$

lattice $\cong U^4 \oplus E_8(-1)^2$

+ Hodge structure

- $A_2 = \langle \lambda_1, \lambda_2 \rangle \subseteq H_{alg}(Ku(Y), \mathbb{Z})$

$$\lambda_1 = i^*[\mathcal{O}_{line}(1)], \quad \lambda_2 = i^*[\mathcal{O}_{line}(2)].$$

- $Stab(Ku(Y)) \neq \emptyset$ & moduli spaces

[BLMNPS]

$M_\sigma(v)$ as "nice" as semistable sheaves on K3 surfaces

$\exists$ "canonical" $\sigma_0 \in Stab$

$\sigma \in Stab$

$v \in H_{alg}(Ku(Y), \mathbb{Z})$

v prim., σ gen. $\quad M_\sigma(v) \neq \emptyset$ iff $v^2 + 2 \geq 0$ $\quad \|$

- [LPZ] $Y \not\ni$ plane

$$M_{\gamma_0}(\lambda_2 - \lambda_1) \cong \text{LLSvS 8fold}$$

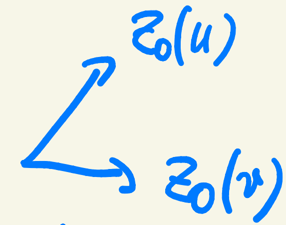
$$\begin{aligned} Y &\hookrightarrow M_{\gamma_0}(\lambda_2-\lambda_1) \quad \text{closed embedding.} \\ y &\longmapsto i^* \pi^!(y) \end{aligned}$$

<u>Idea</u>:

$v := \lambda_2 - \lambda_1$

$u \in H_{alg}(\mathrm{Ku}(X), \mathbb{Z})$ s.t. $(u, v) = -1$

$u^2 + 2 \geq 0$



$F \in M_{\sigma_0}(u)$

$$BN_F := \{ E \in M_{\sigma_0}(v) : \min(\hom(E,F), \mathrm{ext}^1(E,F)) \geq 1 \}$$

Then if BN_F has exp. codim $=$ codim $= 2$

$\rightsquigarrow$ $BN_F \cap Y$ has chance to be surface.

$Y \in \mathcal{C}_d$ gen.

(i) u exists ✓

(ii) BN_F has codim 2

(iii) $BN_F \cap Y$ surface.

if $d = 6a^2 + 6a + 2$

$\rightsquigarrow$ ✓

(i) u exists iff $d \equiv 2 \bmod 6$. $\leftarrow d = 6m+2$

$\forall t > 0$ s.t. $t^2 + t + (1-m) \geq 0$

$\Rightarrow \exists u_t \in H_{alg}(Km(Y), \mathbb{Z})$ s.t. $\begin{cases} u_t^2 + 2 = 2\cdot(t^2+t+(1-m)) \\ u_t \cdot v = -1 \end{cases}$

(ii), (iii) $\quad d = 6a^2 + 6a + 2$

$\rightsquigarrow \quad t = a \quad \rightsquigarrow \quad u \quad$ w/ $\quad u^2 + 2 = 2$

$\rightsquigarrow \quad S = M_{\sigma_0}(u) \qquad$ K3 surface

$\rightsquigarrow \qquad\qquad \mathrm{Ku}(Y) \simeq D^b S$

[AT, BLMNPS]

<u>Lem 1</u> $(a \geq 2)$

- $\forall E \in M_{\gamma_0}(v)$, $E \cong B_\Gamma$,

  $\Gamma \subseteq S$ dim 0, $|\Gamma| = 4$

  $M_{\gamma_0}(v) \cong \mathrm{Hilb}^4 S$

- $\forall F \in M_{\gamma_0}(u)$, $F \cong \mathcal{K}(p)$, $p \in S$.

<u>Lem 2</u> $F \in M_{\gamma_0}(u)$ general

Then

$$i_* F \cong B_{\Sigma_F}(a), \quad \Sigma_F \subseteq Y \text{ \underline{surface}}$$

Morphisms

As before; v, u, $F \in M_{\gamma_0}(u)$

$$\rightsquigarrow \quad j_F : M_{\gamma_0}(v) \dashrightarrow M_{\gamma_0}(v-u)$$

In "optimal situations", j_F well-defined and induces exc. div.

closed embeddings

$$\begin{tikzcd}
\Delta \arrow[d] \arrow[r, phantom, "\subseteq"] & \mathrm{Bl}_{\Sigma_F} Y \arrow[d, "\sigma"] \arrow[r, hook] \arrow[drr, hook, "f"] & \mathrm{Bl}_{BN_F} M_{\gamma_0}(v) \arrow[d] \arrow[dr] & \\
\Sigma_F \arrow[r, hook] & Y \arrow[r, hook] & M_{\gamma_0}(v) \arrow[r, dashed] & M_{\gamma_0}(v-u)
\end{tikzcd}$$

look at (rational) morphisms from here!

Lem $\forall D \in NS(M_{\gamma_0}(v-u)) \cong (v-u)^{\perp} \subseteq H_{alg}(Ku(Y), \mathbb{Z})$

$$f^*D = -\frac{(D+(D,u)u,\ \lambda_1+\lambda_2)}{2}\cdot \sigma^* H + (D,u)\cdot \Delta$$

"optimal situation" $\quad d = 6a^2+6a+2, \quad p \in S$

assum. Σ_p sm.

$$\begin{tikzcd}
\mathrm{Bl}_{\Sigma_p} Y \arrow[r, hook] \arrow[d] \arrow[drr, hook, bend right] & \mathrm{Bl}_{BN_p} \mathrm{Hilb}^4 S \arrow[d] \arrow[dr] & \\
Y \arrow[r, hook] & \mathrm{Hilb}^4 S \arrow[r, dashed] & \mathrm{Hilb}^5 S \\
 & \Gamma \arrow[r, mapsto] & p+\Gamma
\end{tikzcd}$$

Ex $(a=2,\ d=38)$

$$\begin{tikzcd}
\mathrm{Hilb}^5 S \arrow[r, dashed, leftrightarrow, "\sim"] \arrow[dr] & M_1 \arrow[dl] \arrow[dr] \arrow[r, dashed, leftrightarrow] & M_2 \arrow[dl] \\
 & \overline{M}_1 & \overline{M}_2
\end{tikzcd}$$

Mukai flop at $\mathbb{P}^3$-bdle / FY

Fano variety of lines

Mukai flop at $\mathbb{P}^2$-bdle over

$S \times FY$

[Russo-Staglianò]

flop at 3-secant lines to Σ cont. in Y

$3H-\Delta$

$$
\begin{tikzcd}
\mathrm{Hilb}^5 S \cong \mathrm{Bl}_\Sigma Y \arrow[d] \arrow[dr] & & Y_1 \subseteq M_1 \arrow[ll, dashed, "\sim"'] \arrow[ll, dashed, <->] \arrow[dl] \arrow[d] \\
Y & \overline{Y}_1 \subseteq \overline{M}_1 & \overline{Y}_2 \cong \mathbb{P}^4 \subseteq \overline{M}_2
\end{tikzcd}
$$

div. contr. at 5-secant conics

$5H-\Delta$.