K-HOMOLOGY AND K-THEORY OF PURE BRAID GROUPS

SARA AZZALI, SARAH L. BROWNE, MARIA P. GOMEZ APARICIO,
LAUREN C. RUTH, HANG WANG

Abstract. We produce an explicit description of the K-theory and K-homology of the pure braid group on n strands. We describe the Baum–Connes correspondence between the generators of the left- and right-hand sides for $n = 4$. Using functoriality of the assembly map and direct computations, we recover Oyono-Oyono's result on the Baum–Connes conjecture for pure braid groups [OO01a]. We also discuss the case of the full braid group B_3.

1. Introduction

Given a locally compact group G, the Baum–Connes conjecture predicts a way of computing the K-theory of the reduced group C^*-algebra of G in terms of the equivariant K-homology of $\underline{E}G$, the classifying space for proper actions of G. More precisely, let $K_i^G(\underline{E}G)$ denote the G-equivariant K-homology of the space $\underline{E}G$ of order i and $K_i(C_r^*(G))$ is the K-theory of the reduced C^*-algebra $C_r^*(G)$ of order i; the conjecture, as formulated by Baum, Connes and Higson in [BCH94], states that the assembly map

$$\mu_i \colon K_i^G(\underline{E}G) \to K_i(C_r^*(G))$$

for $i = 0, 1$, is a group isomorphism for all locally compact groups.

The Baum–Connes conjecture has been proven for large classes of groups, including all semi-simple Lie groups and all groups satisfying Haagerup's property ([Laf02], [HK01]). Many of the proofs are based on methods that use heavy machinery, such as the Dirac-dual Dirac method, introduced by Kasparov in the case of connected Lie groups and further developed by Higson and Kasparov in [HK01] to prove the conjecture for groups having Haagerup's property.

In the case of semi-simple Lie groups, a first proof was established by Wassermann ([Was87]) following the work of Penington–Plymen ([PP83]) and Valette ([Val84, Val85]). This proof was based on the idea of giving a complete description of both sides of the assembly map and then proving explicitly that the correspondence was an isomorphism of groups. Indeed, the description of the K-theory of the reduced C^*-algebra of a semi-simple group can be made using the exhaustive work of Harish-Chandra on the classification of their reduced representations. For discrete groups, as no such classification exists, other approaches were needed and led to the development of very powerful techniques. For an account of the history of the conjecture and

Date: October 5, 2021.

the recent developments, we refer to the survey [GAJV19] and the references therein, as well as to the books [Val02, MV03].

In this paper, we study the Baum–Connes correspondence for the pure braid group on n strands. The conjecture for those groups is known to be true by the work of Oyono-Oyono [OO01a].

Our paper fits into the context of the work of Isely [Isl11] followed by the works of Flores, Pooya and Valette [FPV17, PV18, Poo19], in which explicit computations of the Baum–Connes correspondence are given for certain discrete groups. We believe that these explicit computations contribute to a deeper understanding of the Baum–Connes correspondence.

It is important to mention that the conjecture also holds for full braid groups by the work of Schick ([Sch07]) using permanence properties of the conjecture shown by Chabert–Echterhoff in [CE01] and the result of Oyono-Oyono for pure braid groups. The conjecture holds in its strong form, with coefficients, *i.e.* considering the action of the group on a C^*-algebra. Moreover, full braid groups have property RD (see for example [Cha17]). Explicit computations for full braid groups are more difficult, though, and other methods have to be used.

Therefore, the aim of this work is to compute the K-theory and K-homology arising in the Baum–Connes assembly map explicitly for the pure braid group on n strands and then to understand the correspondence of the generators under this map. The case when $n = 4$ is worked out explicitly as a typical example. In this case, the classifying space BP_4 can be given a model of the form $S^1 \times X$, where X is a 2-dimensional CW-complex. We can then apply a lemma from [MV03] that relates K-homology of X to its integer singular homology, leading us to the following result:

Theorem 1.1. *For the pure braid group P_4 the P_4-equivariant K-homology of $\underline{E}P_4$ is*

$$K_0^{P_4}(\underline{E}P_4) \simeq \mathbb{Z}^{12} \quad \textit{and} \quad K_1^{P_4}(\underline{E}P_4) \simeq \mathbb{Z}^{12}.$$

Matthey proved that the K-homology of a CW-complex of dimension ≤ 3 is isomorphic to its integral homology [Mat02]; however, for higher number of strands ($n \leq 5$), the classifying space of P_n admits a model of dimension $n-1$, so that one cannot apply Matthey's results to BP_n when $n \geq 5$.

In the general case, we proceed as follows. First we deduce the K-homology group up to torsion by means of existing results on the group homology of P_n. After that, we use an Atiyah–Hirzebruch spectral sequence to remove the torsion. We are then able to extend our first result to pure braid groups on n strands:

Theorem 1.2. *For the pure braid group P_n we have*

$$K_0^{P_n}(\underline{E}P_n) \simeq \mathbb{Z}^{\frac{n!}{2}} \quad \textit{and} \quad K_1^{P_n}(\underline{E}P_n) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

For the right-hand side of the Baum–Connes correspondence, we use the Pimsner–Voiculescu six-term exact sequence in [MP80] and [PV82] to show the following:

Theorem 1.3. *For the pure braid group P_n we have*

$$K_0(C_r^*(P_n)) \simeq \mathbb{Z}^{\frac{n!}{2}} \qquad \text{and} \quad K_1(C_r^*(P_n)) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

Next, using functoriality of the Baum–Connes assembly map, together with explicit computations, we recover Oyono-Oyono's results for pure braid groups:

Theorem 1.4. *The Baum–Connes assembly map $\mu : K_i(BP_n) \to K_i(C_r^*(P_n))$ for the pure braid group P_n is an isomorphism.*

We explicitly describe the assembly map on each of the generators in the case of P_4.

All our computations can be carried out explicitly, thanks to the iterated semidirect product structure of pure braid groups:

$$P_n = F_{n-1} \rtimes F_{n-1} \rtimes \cdots \rtimes F_1.$$

This also indicates that the rank of the K-groups grows as n increases.

The techniques we use for pure braid groups do not apply to full braid groups. Although there is an extension

$$1 \to P_n \to B_n \to S_n \to 1$$

where we denote by S_n the symmetric group over the set of n-elments, that implies that the braid group B_n contains the pure braid group P_n as a normal subgroup of finite index, the K-groups of B_n have fewer generators than the K-groups for P_n. In fact, using an existing result on the group homology of B_n(see [Arn69], [ACC03] and section 3.3), one knows that, up to torsion, both the even and odd K-homology groups for BB_n are $\mathbb{Z}$. Then the Baum–Connes conjecture says that, up to torsion, the K-theory of the reduced C^*-algebra of B_n is $\mathbb{Z}$ as well. When $n = 3$, B_3 has the special structure of a free amalgamated product, which allows us to perform a direct calculation:

$$K_0(C_r^*(B_3)) = K_1(C_r^*(B_3)) \simeq \mathbb{Z}.$$

For $n = 4$ the K-theory of $C_r^*(B_4)$ is explicitly computed in the recent paper by Li, Omland, and Spielberg ([LOS21]). To our knowledge, the problem of directly computing K-theory for the full braid group C^*-algebra remains open.

The paper is organized as follows. In Section 2.1 we recall the structure and properties of braid and pure braid groups (in the appendix we give some of the corresponding diagrams that illustrate the structure of this groups). In Section 3 we describe the classifying space for P_4 explicitly, compute its K-homology and generalize to the case of P_n. In Section 4 we apply the Pimsner–Voiculescu six-term exact sequence to calculate the K-theory for the reduced group C^*-algebras for P_n with $n = 4$ as a typical example. In Section 5 we describe the Baum–Connes

assembly map on each generator for P_4 and show that the map is an isomorphism for all n. In Section 6 we compute the example for B_3 on both sides of the assembly map and show that the map is an isomorphism.

Acknowledgments. We thank Alain Valette for the suggestion to examine K-theory and K-homology of pure braid groups. We thank the organisers of the Women in Operator Algebras Conference that took place at BIRS where this project started. HW acknowledges the support from Shanghai Rising-Star Program 19QA1403200 and NSFC-11801178. MGA was partially supported by ANR project Singstar.

2. Braid and pure braid groups

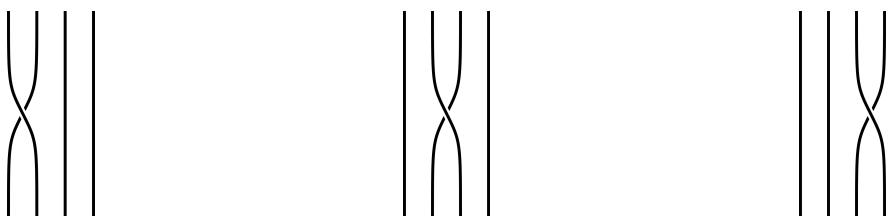
2.1. **Structure of braid and pure braid groups.** Throughout the paper we will denote by $F_n(x_1 \dots x_n)$ the free group generated by $x_1, \dots, x_n$. Let us recall the definition and some properties of braid groups. We refer to [Bir74].

The Artin Braid Group on n letters, denoted by B_n, is a finitely-generated group with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ that satisfy the following relations:

$$\sigma_j \sigma_i = \sigma_i \sigma_j \qquad |i-j| > 1, \quad i,j \in \{1, \dots, n-1\}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad i \in \{1, \dots, n-2\}$$

It can also be described as the group of equivalence classes of all braids on n strands. The generators are illustrated here for B_4.



σ_1 σ_2 σ_3

In this framework, composition of two elements is visualized as the concatenation of the corresponding braid pictures. The identity is represented visually by four straight lines.

As every n-braid determines a permutation of the set of n elements in an obvious way, it is easy to see that there is a surjective map from B_n to S_n, the symmetric group consisting of all permutations of n elements

$$p : B_n \to S_n.$$

This map is compatible with the structures of the two groups so that it is a morphism of groups. Notice that the image of the element σ_i is the permutation exchanging i and $i+1$, hence $p(\sigma_i) = (i, i+1)$, a transposition.

By definition, the *pure braid group on n-strands* is the kernel of p (hence a subgroup of B_n of index $n!$). It is usually denoted by P_n and it is easy to see that in the strand framework it corresponds to the elements of B_n for which all strands start and end at the same point. Notice that as $(i, i+1)$ is a transposition of S_n, the element σ_i^2 belongs to P_n for all $i \in \{1, ..., n\}$.

Starting from P_n we can construct a surjective morphism

$$f : P_n \to P_{n-1}$$

by forgetting the nth strand whose kernel is known to be isomorphic to the free group on $n-1$ generators; this is easy to understand when viewing braids as configuration spaces. In that context, the kernel of f correspond to the fundamental group of the space obtained by removing $n-1$ points to the plane $\mathbb{C}$, which is isomorphic to the free group on $n-1$ generators, F_{n-1}. We have therefore a short exact sequence

$$1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$$

that is split because it is always possible to add a strand to a braid in P_{n-1} to obtain a braid in P_n. Hence P_n is isomorphic to a semi-direct product $F_{n-1} \rtimes P_{n-1}$, and hence isomorphic to an iterated semi-direct product as follows :

$$P_n \simeq F_{n-1} \rtimes P_{n-1} \simeq F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1.$$

Throughout this paper we will us the following presentation of P_n which is due to Artin (see [Bir74] Lemma 1.8.2.). Notice that we are conjugating in the reverse order of [Bir74], for the sake of compatibility with the diagrams in the appendix, so our presentations appears slightly different than the presentation in [Bir74].

The generators of P_n are given by the following formula :

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1} \quad \text{for} \quad 1 \leq i < j \leq n$$

where the σ_i, for $i = 1, \ldots, n$ are the generators of B_n given above; they are subject to the following relations

$$A_{rs}A_{ij}A_{rs}^{-1} = \begin{cases} A_{ij} & r < s < i < j, \quad i < r < s < j \\ A_{sj}^{-1}A_{ij}A_{sj} & r = i \\ (A_{rj}A_{ij})^{-1}A_{ij}(A_{rj}A_{ij}) & s = i \\ (A_{rj}A_{sj})^{-1}(A_{sj}A_{rj})A_{ij}(A_{sj}A_{rj})^{-1}(A_{rj}^{-1}A_{sj}) & r < i < s < j \end{cases}$$

Let $\alpha_i = A_{n-i,n}$ for $i = 1, \ldots, n-1$. Then the subgroup of P_n isomorphic to F_{n-1} appearing in the decomposition $P_n = F_{n-1} \rtimes P_{n-1}$ is generated by the elements α_i, $i = 1, \ldots, n-1$. The

semidirect product decomposition can be written as

$$P_n \simeq F_{n-1} \rtimes_\varphi P_{n-1}$$

where the action φ of P_{n-1} on F_{n-1} is given by the map

$$\varphi : P_{n-1} \to \mathrm{Aut}(F_{n-1})$$

defined by

$$\varphi(A_{rs})(A_{in}) = \begin{cases} A_{in} & r < s < i < n, \quad i < r < s < n \\ A_{sn}^{-1} A_{in} A_{sn} & r = i \\ (A_{rn}A_{in})^{-1} A_{in} (A_{rn}A_{in}) & s = i \\ (A_{rn}A_{sn})^{-1}(A_{sn}A_{rn})A_{in}(A_{sn}A_{rn})^{-1}(A_{rn}^{-1}A_{sn}) & r < i < s < n \end{cases}$$

Following this notation, we have that $P_n = F_{n-1}(\alpha_1, \alpha_2, \dots \alpha_{n-1}) \rtimes_\varphi P_{n-1}$.

The center of B_n is generated by the element

$$(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$$

which can be expressed in terms of elements of P_n by

$$(A_{12})(A_{13}A_{23}) \cdots (A_{1n}A_{2n} \cdots A_{(n-1)n}).$$

(This is illustrated in the appendix in the case $n = 4$.)

For $n = 3$, the generators of P_3 are

$$A_{13} = \sigma_1\sigma_2^2\sigma_1^{-1}, \quad A_{23} = \sigma_2^2, \quad A_{12} = \sigma_1^2$$

Letting $\alpha_2 = A_{13}$, $\alpha_1 = A_{23}$ and $\sigma_1^2 = A_{12}$, we get that P_3 has the following presentation

$$P_3 = \left\langle \alpha_1, \alpha_2, \sigma_1^2 \quad | \quad \sigma_1^{-2}\alpha_1\sigma_1^2 = (\alpha_2\alpha_1)\alpha_1(\alpha_2\alpha_1)^{-1}, \quad \sigma_1^{-2}\alpha_2\sigma_1^2 = \alpha_1\alpha_2\alpha_1^{-1} \right\rangle,$$

whence

$$P_3 \simeq F_2(\alpha_1, \alpha_2) \rtimes \langle \sigma_1^2 \rangle,$$

that is P_3 is isomorphic to the semi-direct product of the free group generated by α_1 and α_2 and the group generated by σ_1^2, where the action of σ_1^2 on $F(\alpha_1, \alpha_2)$ is given by conjugation.

Denoting by c the element $\sigma_1^2\alpha_1\alpha_2$, we can check that $\alpha_1 c = c\alpha_1$ and $\alpha_2 c = c\alpha_2$ so that

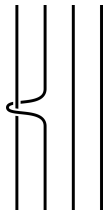
$$P_3 = F(\alpha_1, \alpha_2) \times \langle c \rangle.$$

For $n = 4$, to simplify notation in the rest of the paper, we will denote the generators of P_4 as follows.

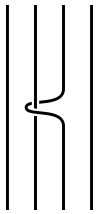
$$\begin{aligned} \sigma_1^2 &= A_{12} & \beta_1 &= A_{34} = \sigma_3^2 \\ \alpha_1 &= A_{23} = \sigma_2^2 & \beta_2 &= A_{24} = \sigma_3\sigma_2^2\sigma_3^{-1} \\ \alpha_2 &= A_{13} = \sigma_2\sigma_1^2\sigma_2^{-1} & \beta_3 &= A_{14} = \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1} \end{aligned}$$

From the relations, and from the diagrams in the appendix, we have

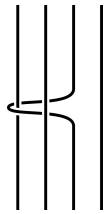
$$P_4 \simeq F_3(\beta_1, \beta_2, \beta_3) \rtimes \big(F_2(\alpha_1, \alpha_2) \rtimes F(\sigma_1^2)\big).$$



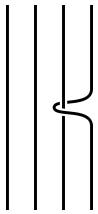
σ_1^2



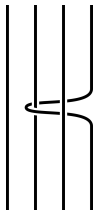
$\alpha_1 = \sigma_2^2$



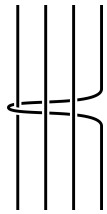
$\alpha_2 = \sigma_2\sigma_1^2\sigma_2^{-1}$



$\beta_1 = \sigma_3^2$



$\beta_2 = \sigma_3\sigma_2^2\sigma_3^{-1}$



$\beta_3 = \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}$

where the actions are given by the following relations :

(1) $\alpha_1\beta_1\alpha_1^{-1} = (\beta_2\beta_1)^{-1}\beta_1(\beta_2\beta_1)$
(2) $\alpha_1\beta_2\alpha_1^{-1} = \beta_1^{-1}\beta_2\beta_1$
(3) $\alpha_1\beta_3\alpha_1^{-1} = \beta_3$
(4) $\alpha_2\beta_1\alpha_2^{-1} = (\beta_3\beta_1)^{-1}\beta_1(\beta_3\beta_1)$
(5) $\alpha_2\beta_2\alpha_2^{-1} = (\beta_3\beta_1)^{-1}\beta_1\beta_3\beta_2(\beta_1\beta_3)^{-1}(\beta_3\beta_1)$
(6) $\alpha_2\beta_3\alpha_2^{-1} = \beta_1^{-1}\beta_3\beta_1$
(7) $\sigma_1^2\beta_1\sigma_1^{-2} = \beta_1$
(8) $\sigma_1^2\beta_2\sigma_1^{-2} = (\beta_3\beta_2)^{-1}\beta_2(\beta_3\beta_2)$
(9) $\sigma_1^2\beta_3\sigma_1^{-2} = \beta_2^{-1}\beta_3\beta_2$
(10) $\sigma_1^2\alpha_1\sigma_2^{-2} = (\alpha_2\alpha_1)^{-1}\alpha_1(\alpha_2\alpha_1)$
(11) $\sigma_1^2\alpha_2\sigma_1^{-2} = \alpha_1^{-1}\alpha_2\alpha_1$

Remark 2.1. In this paper, we will use a splitting off the center of P_4 in order to realize P_4 as the direct product of its center and a semidirect product of free groups. The center of P_4 is generated by $c = (\sigma_1\sigma_2\sigma_3)^4 = \sigma_1^2\alpha_1\alpha_2\beta_1\beta_2\beta_3$, as illustrated in the appendix (see 7.4), and we have

$$P_4 \simeq (F(\beta_1, \beta_2, \beta_3) \rtimes F(\alpha_1, \alpha_2)) \times \langle c \rangle.$$

2.2. **The Baum–Connes conjecture for P_n and K-amenability.** A property of P_n that we will use in order to give explicit computations of its K-theory groups is its K-amenability. This property implies that for every C^*-algebra A endowed with an action of P_n, the K-theory of the maximal crossed product $A \rtimes P_n$ is isomorphic to the K-theory of the reduced crossed product $A \rtimes_r P_n$ (see [JV84] for the definition of K-amenability). In particular,

$$K_*(C^*(P_n)) \simeq K_*(C_r^*(P_n)).$$

The K-amenability of P_n can be proven using the following result of Pimsner combined with the following proposition that is an adaptation of a result appearing in the proof of the Baum–Connes conjecture for P_n given by Oyono-Oyono (see Proposition 7.3 in [OO01a]) :

Theorem 2.2 ([Pim86])**.** *A locally compact group acting on an oriented tree such that the stabilizer group of any vertices is K-amenable is K-amenable.*

Proposition 2.3. *Let $D_0, \ldots, D_n$ be a finite sequence of groups such that $D_0 = \{e\}$ and for $1 \leq k \leq n$ there exists n_k in $\mathbb{N}$ such that $D_k = F_{n_k} \rtimes D_{k-1}$. Then D_n is K-amenable.*

Proof. The proof is the same as the proof of Proposition 7.3 in [OO01a] and it is held by induction on n : if $0 \leq k \leq n-1$, let D'_k be the kernel of the morphism mapping D_{k+1} to $D_1 = F_{n_1}$. Then, $D'_0 = \{e\}$ and, if $1 \leq k \leq n-1$ then $D'_k = F_{n_{k-1}} \rtimes D'_{k-1}$. Hence, by induction, D'_{n-1} is K-amenable. But the group D_n acts on the Cayley graph of F_{n_1} (which is a tree) through the morphism mapping D_n to $D_1 = F_{n_1}$ and the stabilizer group of the vertex corresponding to the neutral element of F_{n_1} is exactly D'_{n-1}; as the action is transitive, the stabilizer group of all vertices is K-amenable and hence, by Pimsner's theorem, D_n is K-amenable. □

Corollary 2.4. *The pure braid group P_n is K-amenable.*

In [OO01a], Oyono-Oyono proved that a countable discrete group acting on an oriented tree satisfies the Baum–Connes conjecture with coefficients[1] if and only if the groups stabilizing the vertices of the tree satisfy Baum–Connes with coefficients. This result allows him to prove the stability of the conjecture under free and amalgamated products and HNN extensions. It also allows him to prove the analogous of Proposition 2.3 in the context of Baum–Connes and hence to give his first proof of the Baum–Connes conjecture for P_n. He then proved in [OO01b] the Baum–Connes conjecture for groups which are extensions of a group satisfying the Haagerup property by a group satisfying Baum–Connes, which allowed him to give a second proof of Baum-Connes for P_n (as the free group is known to have the Haagerup property).

3. Classifying space and K-homology for P_n

In this section, we deal with the compactly supported Γ-equivariant K-homology of $\underline{E}\Gamma$, the space classifying Γ-proper actions for $\Gamma = P_n$.

As P_n is a discrete torsion-free group, $\underline{E}P_n$ coincides with EP_n, the universal cover of the classifying space BP_n, and the P_n-equivariant K-homology of EP_n is the K-homology of the space BP_n, that is $K_*^{P_n}(EP_n) \simeq K_*(BP_n)$.

We start with the case $n = 4$. We will give a model for BP_4 and compute its K-homology explicitly.

[1]The Baum-Connes conjecture with coefficients is a stronger version of the Baum–Connes that considers actions of the group on a C^*-algebra.

3.1. **A model for BP_4.** Let us give a model for BP_4. Recall that

$$P_4 \simeq (F(\beta_1, \beta_2, \beta_3) \rtimes F(\alpha_1, \alpha_2)) \times \langle c \rangle,$$

the center of P_4 is generated by $c = (\sigma_1\sigma_2\sigma_3)^4$, and the generators $\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2$ are subject to 6 relations:

$R1.$ $\alpha_1\beta_1\alpha_1^{-1} = (\beta_2\beta_1)^{-1}\beta_1(\beta_2\beta_1)$

$R2.$ $\alpha_1\beta_2\alpha_1^{-1} = \beta_1^{-1}\beta_2\beta_1$

$R3.$ $\alpha_1\beta_3\alpha_1^{-1} = \beta_3$

$R4.$ $\alpha_2\beta_1\alpha_2^{-1} = (\beta_3\beta_1)^{-1}\beta_1(\beta_3\beta_1)$

$R5.$ $\alpha_2\beta_2\alpha_2^{-1} = (\beta_3\beta_1)^{-1}(\beta_1\beta_3)\beta_2(\beta_1\beta_3)^{-1}(\beta_3\beta_1)$

$R6.$ $\alpha_2\beta_3\alpha_2^{-1} = \beta_1^{-1}\beta_3\beta_1$

Conjugating the relation R5 by the last relation R6 we obtain:

$$R5'.\ \alpha_2(\beta_3\beta_2\beta_3^{-1})\alpha_2^{-1} = \beta_3\beta_2\beta_3^{-1}.$$

We may replace relation R5 by R5' without changing the presentation of the group. Notice that the pairs of relations R1 & R4, R2 & R6, R3 & R5' are of the same type.

Let X be the 2-CW complex associated to the group $F_3 \rtimes F_2$. That is, X consists of 1 0-cell p, 5 1-cells attached as loops on p, and 6 2-cells whose boundaries are given by the relations above. Then $\pi_1(X) = F_3 \rtimes F_2$. Denote by $\widetilde{X}$ the universal cover of X. We want to show

Proposition 3.1. *The 2-CW complex X is a model for $B(F_3 \rtimes F_2)$.*

Proof. We first construct $\widetilde{X}$ and then show it is contractible (Lemma 3.3).

Step 1: We start from the 1-skeleton of X, denoted by $X^{(1)}$. This is a bouquet of 5 circles, and its universal covering $\widetilde{X^{(1)}}$ is a tree, the Cayley graph of the free product $F_2(\alpha_1, \alpha_2) * F_3(\beta_1, \beta_2, \beta_3)$. The group $F_3 * F_2$ acts freely on $\widetilde{X^{(1)}}$.

As the group $F_3 \rtimes F_2$ is the quotient of $F_2 * F_3$ by the six relations stated above,

$$F_3 \rtimes F_2 \simeq (F_3 * F_2)/\langle R1, R2, R3, R4, R5', R6\rangle,$$

we shall modify $\widetilde{X^{(1)}}$ so that the relations act trivially. This is done in Step 2 by gluing 2-cells and by identification of branches.

Step 2: Note that every point in the Cayley graph $\widetilde{X^{(1)}}$ is generic. Choose an arbitrary vertex $P \in \widetilde{X^{(1)}}$ and define $Q := \alpha_1(P)$ and $R := \alpha_2(P)$ in $\widetilde{X^{(1)}}$. The six relations R1, R2, R3, R4, R5', R6 require that the pair of points on both sides of the following equations have to be identified:

a) $\alpha_1(\beta_2 P) = \beta_1^{-1}\beta_2\beta_1 Q$

b) $\alpha_1(\beta_1 Q) = \beta_1^{-1}\beta_2^{-1}\beta_1\beta_2\beta_1 Q$

c) $\alpha_1(\beta_3 P) = \beta_3 Q$

d) $\alpha_2(\beta_1 P) = \beta_1^{-1}\beta_3^{-1}\beta_1\beta_3\beta_1 R$

e) $\alpha_2(\beta_3\beta_2\beta_3^{-1}P) = \beta_3\beta_2\beta_3^{-1}R$

f) $\alpha_2(\beta_3 P) = \beta_1^{-1}\beta_3\beta_1 R$

Attach 2-cells given by the six relations and identify the branches at the vertices being glued in the 2-cells. Performing this process for each vertex in $\widetilde{X^{(1)}}$, we then obtain a 2-dimensional CW-complex, we denoted by $\widetilde{X}_0$.

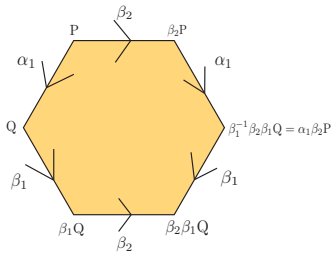


FIGURE 1. 2 cell associated to R1

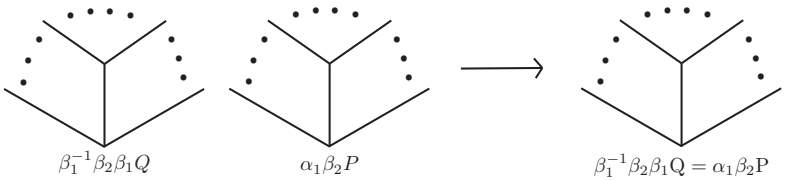


FIGURE 2. Identify branches over two identified points

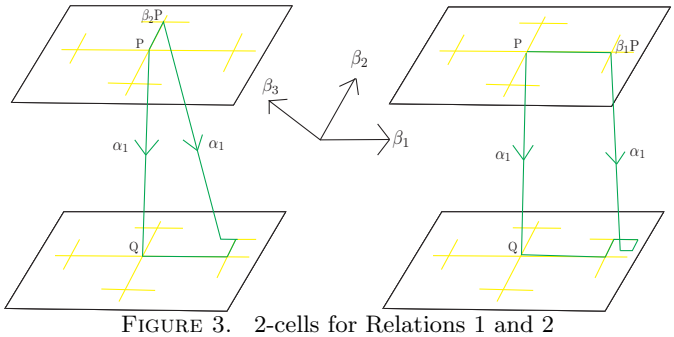


FIGURE 3. 2-cells for Relations 1 and 2

Lemma 3.2. *The space $\widetilde{X}_0$ constructed in Step 1 and Step 2 is the universal cover $\widetilde{X}$ of X.*

Proof. Let us go through the process for R1, as an example. Starting from P and following the expression $\alpha_1(\beta_2 P)$, we obtain the vertices P, $\beta_2 P$, and $\alpha_1(\beta_2 P)$; and starting from $Q = \alpha_1(P)$ and following the expression $\beta_1^{-1}\beta_2\beta_1 Q$, we obtain vertices Q, $\beta_1 Q$, $\beta_2\beta_1 Q$ and $\beta_1^{-1}\beta_2\beta_1 Q$. So, by tracing out each letter in R1, one obtains a hexagon with 6 vertices. Fill in the interior of the hexagon with a 2-cell. See Figure 1.

Because the two points $\alpha_1(\beta_2 P)$ and $\beta_1^{-1}\beta_2\beta_1 Q$ are identified in the hexagon, the branches rooted over the two points will be identified under the group action. See Figure 2 for an illustration.

In the left hand side of the Figure 3, the hexagon associated to R1 is the green hexagon relative to other relevant vertices in the space. See Figures 3, 4 and 5 for the typical 2-cell associated to each of the relations.

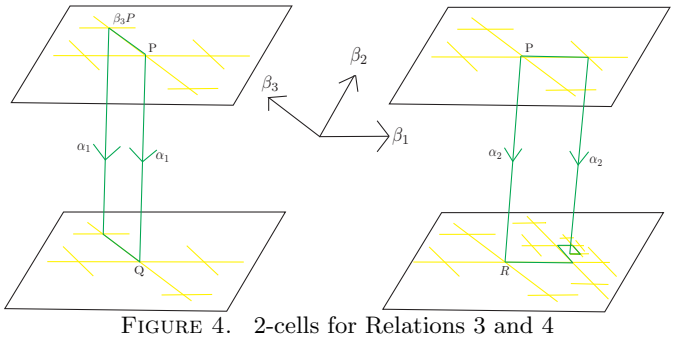


FIGURE 4. 2-cells for Relations 3 and 4

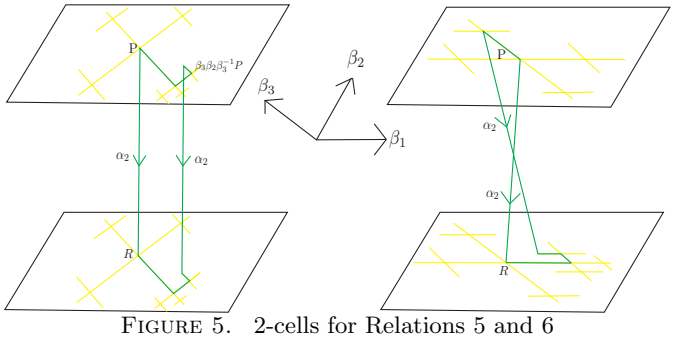


FIGURE 5. 2-cells for Relations 5 and 6

After the gluing of 2-cells on $\widetilde{X^{(1)}}$ and identifications of branches, one can check that $F_3 \rtimes F_2$ acts freely on $\widetilde{X}_0$ with quotient X. In fact, let $g \in F_3 \rtimes F_2$ and suppose $gx = x$ for some vertex x in $\widetilde{X}_0$. Assume $g \neq e$. Then there exists $\tilde{g} \in F_3 * F_2$ such that $\pi(\tilde{g}) = g$, where π is the morphism $\pi : F_3 * F_2 \to F_3 \rtimes F_2$. Let $\tilde{x} \in \widetilde{X^{(1)}}$ such that $\pi(\tilde{x}) = x$. Because $\tilde{g}$ is not the identity in $F_3 * F_2$ and $F_3 * F_2$ acts freely on $\widetilde{X^{(1)}}$, we have $\tilde{g}\tilde{x} \neq \tilde{x}$. By definition, $\tilde{g}\tilde{x}$ and $\tilde{x}$ are identified with the same point x in $\tilde{X}_0$. So $\tilde{g}\tilde{x}$ and $\tilde{x}$ can be connected by relations R1, R2, R3, R4, R5', R6. So $\tilde{g}$ can be expressed as $R_{i_1} \cdots R_{i_k}$, and then $g = e$ in $F_3 \rtimes F_2$, which is a contradiction. This shows that $F_3 \rtimes F_2$ acts freely on $\widetilde{X}_0$.

By construction, the quotient of $\widetilde{X}_0$ by $F_3 \rtimes F_2$ is X. Therefore the lemma is proved. □

Proposition 3.1 then follows from the lemma below.

Lemma 3.3. *The universal cover $\widetilde{X}$ is contractible.*

Proof. We shall define a uniform deformation of every 2-cell in $\widetilde{X}$ so that the resulting deformed 2-complex $\widetilde{X}'$ is contractible.

Let $P_0 = P$ be a 0-cell in $\widetilde{X}$, and set $P_i = \beta_1^i(P)$ and $Q_i = \alpha_1(P_i)$. Here $\beta_1^i(P)$ means applying β_1 to P i times. Let c_i be the 2-cell containing P_i and Q_i associated to Relation 1. Define a homotopy of c_i by moving Q_i continuously to $\beta_2\beta_1(Q_i)$ through the path $Q_i \to \beta_1(Q_i) \to \beta_2\beta_1(Q_i)$. See Figure 6. Then c_i is continuously deformed to a rectangle c_i', where

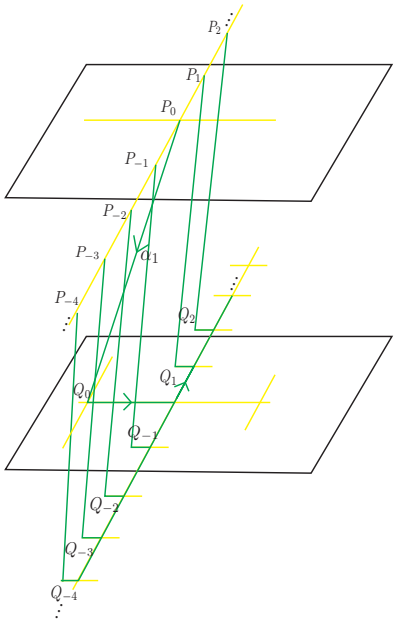


FIGURE 6. Homotopy for cells c_i

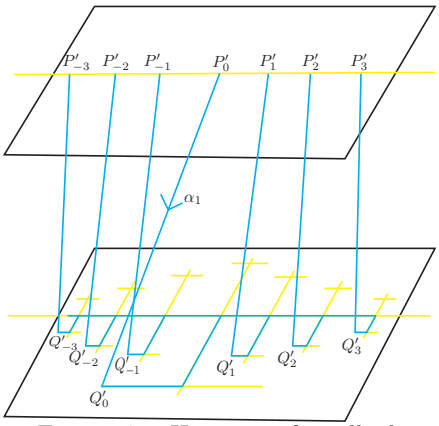


FIGURE 7. Homotopy for cells d_i

$\cup_i c'_i \simeq \mathbb{R}\times[0,1]$. This homotopy is uniform with respect to i. Similarly, one can define a homotopy for the 2-cells associated to Relation 6. (In this case, $\alpha_2(P_i)$ should be moved to $\beta_3\beta_1(\alpha_2(P_i))$).

Let $P'_0 = P$, $P'_i = \beta_2^i(P)$, and $Q'_i = \alpha_1(P_i)$. Let d_i be the 2-cell containing P'_i and Q'_i associated to Relation 2. Define a homotopy of d_i by moving each Q'_i continuously to $\beta_2\beta_1 Q'_i$ though the path $Q'_i \to \beta_1 Q'_i \to \beta_2\beta_1 Q'_i$; see Figure 7. Then d_i deforms continuously to a rectangle d'_i, uniformly with respect to i, with $\cup_{i\in\mathbb{Z}} d'_i \simeq \mathbb{R}\times[0,1]$.

Similarly, one can define a homotopy for the 2-cells associated to Relation 4. (In this case, $\alpha_2(P'_i)$ should be moved to $\beta_3\beta_1(\alpha_2(P_i))$).

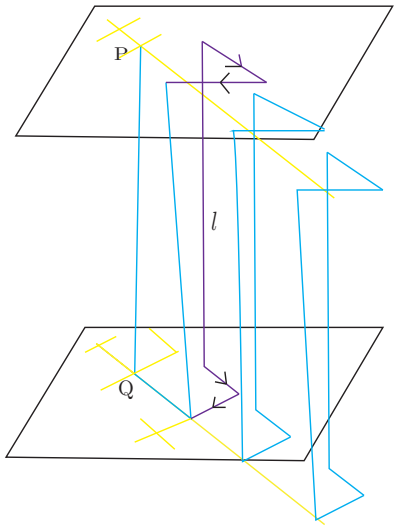


FIGURE 8. Homotopy for cells e

Relation 3 gives rise to 2-cells in the shape of a rectangle. Let a be a rectangle containing P and $\alpha_1 Q$. Deform a by a homotopy carrying the edge l with vertices Q and $\beta_3 Q$ to $\beta_2\beta_1 l$ through the path $l \to \beta_1 l \to \beta_2\beta_1 l$.

For Relation 5, let $Q = \alpha_2(P)$. In the cell e containing P and Q and corresponding to Relation 5, move the edge l with vertices $\beta_3\beta_2\beta_3^{-1}(P)$ and $\beta_3\beta_2\beta_3^{-1}(Q)$ through the path $l \to \beta_3^{-1} l \to \beta_2^{-1}\beta_3^{-1} l$; see Figure 8. Then e is deformed to a 2-cell e' in the shape of a rectangle. Deform e' again by a homotopy carrying the edge l with vertices Q and $\beta_3^{-1} Q$ to $\beta_3\beta_1 l$ through the path $l \to \beta_1 l \to \beta_3\beta_1 l$.

After this process, we obtain a 2-CW complex $\widetilde{X}'$. All the 2-cells in $\widetilde{X}$ are turned into rectangular-shaped 2-cells in $\widetilde{X}'$. Algebraically, this process corresponds to the abelianization of all relations. Note that here we are using the special structure of the pure braid group: Indeed, the deformations can be done uniformly, because all relations in P_4 have the form

$$\alpha_i \beta_j \alpha_i^{-1} = C \beta_j C^{-1} \tag{3.1}$$

where C is a word depending on i, j, having finite letters chosen from $\beta_1, \beta_2, \beta_3$. The deformation from $\widetilde{X}$ to $\widetilde{X}'$ corresponds to replacing (3.1) by $\alpha_i \beta_j \alpha_i^{-1} = \beta_j$. Indeed, the relations determine the group

$$F_2 \times F_3 = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 | \alpha_i \beta_j = \beta_j \alpha_i, \forall i, j \rangle.$$

Therefore we have shown that $\widetilde{X}$ is homotopic to

$$\widetilde{X}' = E(F_2 \times F_3) = EF_2 \times EF_3$$

which is a contractible space. The lemma is then proved. □

This completes the proof of the proposition. □

3.2. **K-homology of BP_4.** We are ready to compute the K-homology of P_4.

Theorem 3.4. *We have that*

$$K_0(BP_4) \simeq K_1(BP_4) \simeq \mathbb{Z}^{12} \ .$$

Hence, as P_4 is torsion-free, $K_0^{P_4}(\underline{E}P_4) \simeq K_1^{P_4}(\underline{E}P_4) \simeq \mathbb{Z}^{12}$.

To prove this theorem, we first observe the following two easy well-known facts:

Lemma 3.5. *For any finite CW complex X, we have*

$$K_i(X \times S^1) \simeq K_0(X) \oplus K_1(X) \qquad \textit{for} \quad i = 0, 1.$$

Proof. Let i be 0 or 1. Note that $K_i(X \times S^1) \simeq K^i(C(X \times S^1)) \simeq K^i(C(X) \otimes C(S^1))$. As $C(S^1) \simeq C_0(\mathbb{R}) \oplus \mathbb{C}$, we have that

$$K^i(C(X) \otimes C(S^1)) \simeq K^i(C(X) \otimes C_0(\mathbb{R})) \oplus K^i(C(X)).$$

Noting that $K^i(C(X) \otimes C_0(\mathbb{R})) = K^{i+1}(C(X))$, the lemma is proved. □

Lemma 3.6. *We have*

$$BP_4 \simeq B(F_3 \rtimes F_2) \times S^1.$$

Proof. We use the isomorphism $P_4 \simeq F_3 \rtimes F_2 \rtimes F_1$. By changing the representative of the generator of $F_1 \simeq \mathbb{Z}$ in $P_4 \simeq F_3 \rtimes F_2 \rtimes F_1$, we can obtain the trivial action of F_1; see Remark 2.1. Hence $P_4 \simeq (F_3 \rtimes F_2) \times \mathbb{Z}$. Thus

$$BP_4 \simeq B(F_3 \rtimes F_2) \times B\mathbb{Z} \simeq B(F_3 \rtimes F_2) \times S^1;$$

which proves the lemma. □

Recall that a model of $B(F_3 \rtimes F_2)$ is given (Proposition 3.1) by the 2-dimensional CW complex X constructed in Lemma 3.2 (and associated to the group presentation of $F_3 \rtimes F_2$).

Lemma 3.7. *We have $K_0(X) \simeq \mathbb{Z}^7$ and $K_1(X) \simeq \mathbb{Z}^5$.*

Proof. By Lemma 4.1 in [MV03], because X is a 2-dimensional CW complex, we have

$$K_0(X) \simeq H_0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z}) \qquad K_1(X) \simeq H_1(X, \mathbb{Z}) \ .$$

Note that $H_0(X, \mathbb{Z}) = \mathbb{Z}$, since X is connected; and

$$H_1(X, \mathbb{Z}) \simeq (F_3 \rtimes F_2)/[F_3 \rtimes F_2, F_3 \rtimes F_2] \simeq \mathbb{Z}^5.$$

To calculate $H_2(X, \mathbb{Z})$, one notes that all 6 relations are cycles (nontrivial and distinct), and there are at most 6 2-cells, so $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^6$. The lemma is thus proved. □

Proof of Theorem 3.4. Making use of the Lemmas, we have for $i = 0$ or 1:

$$K_i^{P_4}(\underline{E}P_4) \simeq K_i(BP_4) \simeq K_0(B(F_3 \rtimes F_2)) \oplus K_1(B(F_3 \rtimes F_2)),$$

where the first isomorphism is due to P_4 being torsion-free, and the second isomorphism follows from Lemma 3.5. Then by Proposition 3.1 and Lemma 3.7, we have

$$K_i(B(F_3 \rtimes F_2)) \simeq K_i(X) \simeq \begin{cases} \mathbb{Z}^7 & i = 0 \\ \mathbb{Z}^5 & i = 1 \end{cases}.$$

The theorem then follows. □

3.3. **K-homology of BP_n.** In this section, we show that for $i = 0$ or 1,

$$K_i(BP_n) \simeq K_i(Y) \simeq \mathbb{Z}^{\frac{n!}{2}}, \tag{3.2}$$

where $Y = BF_{n-1} \times BF_{n-2} \times \cdots \times BF_1$.

Recall that rationally, the Chern character on K-homology for any finite CW complex X is an isomorphism:

$$K_0(X) \otimes \mathbb{Q} \simeq \bigoplus_i H_{2i}(X, \mathbb{Q}) \qquad K_1(X) \otimes \mathbb{Q} \simeq \bigoplus_i H_{2i-1}(X, \mathbb{Q}). \tag{3.3}$$

Thus the second isomorphism in (3.2) holds up to torsion.

Lemma 3.8. *Let $Y = BF_{n-1} \times BF_{n-2} \times \cdots \times BF_1$. Then modulo torsion, we have*

$$K_0(Y) \simeq \mathbb{Z}^{\frac{n!}{2}} \qquad \text{and} \quad K_1(Y) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

Proof. We need to show that $\sum_i \dim H_{2i}(Y) = \frac{n!}{2} = \sum_i \dim H_{2i-1}(Y)$. Observe that $H_i(Y, \mathbb{Q})$ is generated by i-cells in Y. Denote by a_k the number of k-cells in the CW-complex Y. It is enough to show that

$$\sum_i a_{2i} = \sum_i a_{2i-1} = \frac{n!}{2}.$$

Because BF_k has one 0-cell and k 1-cells, we have that for $Y = BF_{n-1} \times BF_{n-2} \times \cdots \times BF_1$,

$$a_0 = 1, \qquad a_1 = \sum_{i=1}^{n-1} i, \qquad a_2 = \sum_{1 \le i < j \le n-1} ij, \qquad \ldots, \qquad a_{n-1} = 1 \cdot 2 \cdots (n-1).$$

That is, a_m is equal to the coefficient of t^m in the series $(1+t)(1+2t)\cdots(1+[n-1]t)$. Therefore we have

$$\sum_{k=1}^{n-1} a_k = \prod_{l=1}^{n-1} (1+l) = \prod_{l=1}^{n} l = n!$$

and

$$\sum_{k=1}^{n-1} (-1)^k a_k = \prod_{l=1}^{n-1} (1-l) = 0.$$

The lemma is then proved. □

The first isomorphism in (3.2) also holds up to torsion. In other words, the K-homology for BP_n can be computed rationally.

Proposition 3.9. *Up to torsion,*

$$K_0(BP_n) \simeq K_1(BP_n) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

Proof. The integer cohomology of BP_n of degree m is torsion-free, with the power over $\mathbb{Z}$ being equal to the coefficient of t^m in the series $(1+t)(1+2t)\cdots(1+[n-1]t)$. This was first computed by Arnold [Arn69]; see also Example 2.3 of [ACC03]. Applying the proof of Lemma 3.8, we find that up to torsion,

$$K^0(BP_n) \simeq H^{even}(BP_n, \mathbb{Z}) \simeq \mathbb{Z}^{\frac{n!}{2}} \quad K^1(BP_n) \simeq H^{odd}(BP_n, \mathbb{Z}) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

Together with (3.3) and Poincaré duality, this gives the result. Note that BP_n is a Poincaré complex in the sense of Wall [Wal67] and hence satisfies the Poincaré duality theorem. To see that BP_n is a Poincaré complex, notice that BP_n is composed of iterated extensions of the BF_is, and it is easy to verify that the BF_is are Poincaré complexes. □

Finally, the torsion in Proposition 3.9 can be removed using Atiyah-Hirzebruch spectral sequences, analogous to Arnold's calculation of $H_*(BP_n)$ using Serre spectral sequences. Let us review some key steps and properties of pure braid groups in computing $H_*(BP_n)$. For more details, see [Wil].

It is well known that the ordered configuration space of n points in $\mathbb{C}$ is a model for the classifying space of P_n:

$$BP_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ if } i \neq j\}.$$

Note that for $n = 4$, this model is homotopic to the model of BP_4 that we constructed explicitly earlier. Consider the map

$$\rho_n : BP_n \to BP_{n-1} \qquad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}).$$

This is a fibration whose fiber is homeomorphic to the $(n-1)$-times punctured plane, which is homotopic to a wedge of $n-1$ circles, so that the filtration can be written as $BF_n \to BP_n \to BP_{n-1}$. This can also be induced by the short exact sequence $1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$. The map

$$i_n : BP_{n-1} \to BP_n \qquad (z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{n-1}, \max|z_i| + 1)$$

gives rise to a splitting of the fibration ρ_n. Associated to a fibration, there is a monodromy action of $\pi_1(B)$ on the (co)homology of the fiber F. In our setting, it can be checked that $P_{n-1} = \pi_1(BP_{n-1})$ acts on the homology of BF_{n-1} trivially. Then the E_2 page of the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q}(BP_n) = H^p(BP_{n-1}, H^q(BF_{n-1}))$$

is a cohomology with untwisted coefficients in $H^q(BF_{n-1})$ and is calculated by

$$E_2^{p,q}(BP_n) = \begin{cases} H^p(BP_{n-1}) & q = 0 \\ H^p(BP_{n-1}) \otimes \mathbb{Z}^{n-1} & q = 1 \\ 0 & q > 1. \end{cases}$$

One can then show that the spectral sequences collapse on the E_2 page, and we obtain

$$H^k(BP_n) = \bigoplus_{p+q=k} E_2^{p,q}(BP_n) = E_2^{k,0}(BP_n) \oplus E_2^{k-1,1}(BP_n).$$

The group can then be calculated using induction.

Theorem 3.10. *We have*

$$K_0(BP_n) \simeq K_1(BP_n) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

Proof. From [MV03], there is an Atiyah–Hirzebruch spectral sequence

$$E^2_{p,q}(BP_n) = H_p(BP_{n-1}, K_q(BF_{n-1})) \Rightarrow K_{p+q}(BP_n).$$

Because the spectral sequence collapses on the E^2 page for similar reasons as above, one has

$$K_0(BP_n) \simeq \bigoplus_{p+q\equiv 0(2)} E^2_{p,q}(BP_n) \qquad K_1(BP_n) \simeq \bigoplus_{p+q\equiv 1(2)} E^2_{p,q}(BP_n).$$

Because the monodromy action is trivial, the homology

$$E^2_{p,q}(BP_n) = H_p(BP_{n-1}, K_q(BF_{n-1}))$$

has no twisted coefficients and can thus be calculated by

$$H_p(BP_{n-1}, K_q(BF_{n-1})) = \begin{cases} H_p(BP_{n-1}) & q = 0 \\ H_p(BP_{n-1}) \otimes \mathbb{Z}^{n-1} & q = 1. \end{cases}$$

Therefore

$$K_0(BP_n) \simeq H_{even}(BP_{n-1}) \oplus (H_{odd}(BP_{n-1}) \otimes \mathbb{Z}^{n-1}) \tag{3.4}$$

$$K_1(BP_n) \simeq H_{odd}(BP_{n-1}) \oplus (H_{even}(BP_{n-1}) \otimes \mathbb{Z}^{n-1}). \tag{3.5}$$

Because $H_*(BP_n)$ is torsion-free for all n, from the above we obtain that $K_*(BP_n)$ is torsion-free as well. Thus,

$$K_0(BP_n) \simeq K_0(BP_{n-1}) \oplus (K_1(BP_{n-1}) \otimes \mathbb{Z}^{n-1})$$
$$K_1(BP_n) \simeq K_1(BP_{n-1}) \oplus (K_0(BP_{n-1}) \otimes \mathbb{Z}^{n-1}).$$

The theorem then follows by induction. □

Note that the calculation of the K-homology of $Y = BF_{n-1} \times \cdots \times BF_1$ is not used in the proof of the theorem, but the comparison to the K-homology of BP_n provides intuition for the cycles of BP_n that form a set of generators for $K_*(BP_n)$ in light of (3.4) and (3.5). By the fibration structure $BF_{n-1} \to BP_n \to BP_{n-1}$, and by induction, BP_n has a model of dimension $n-1$, and every k-simplex of BP_n is labelled by $A_{j_1,i_1}, \ldots, A_{j_k,i_k}$, where $1 \le i_1 < \cdots < i_r \le n$,

$1 \leq j_l < i_l$, and $F_l(A_{1,l+1}, \ldots, A_{l,l+1})$ is the free subgroup of P_n. They are cycles because of the pure braid group relations, and they correspond bijectively to cycles of Y. There is a canonical map $BP_n \to Y$ inducing an isomorphism on K-homology; see Section 5.4 for more details.

4. K-theory of the reduced group C^*-algebra of P_n

In this section, we compute the right-hand side of the Baum–Connes morphism for P_n. We shall use the Pimsner–Voiculescu six-term exact sequence, as it allows us to compute the K-theory of the reduced crossed product of a C^*-algebra with a free group [PV82]. Let A be a C^*-algebra endowed with an action of the free group on n generators $F_n = F_n(x_1, \ldots x_n)$ by automorphisms $\phi : F_n \to \mathrm{Aut}(A)$. Following Pimsner and Voiculescu (see [PV82, Theorem 3.1, Theorem 3.5]), we have two six-term exact sequences

$$\begin{tikzcd} K_0(A) \arrow[r, "w_n"] & K_0(A \rtimes_{\phi',r} F_{n-1}) \arrow[r, "k_*"] & K_0(A \rtimes_{\phi,r} F_n) \arrow[d] \\ K_1(A \rtimes_{\phi,r} F_n) \arrow[u] & K_1(A \rtimes_{\phi',r} F_{n-1}) \arrow[l, "k_*"'] & K_1(A) \arrow[l, "w_n"'] \end{tikzcd} \tag{4.1}$$

where ϕ' is the restriction of the action to F_{n-1}, $k : A \rtimes_{\phi',r} F_{n-1} \to A \rtimes_{\phi,r} F_n$ is the natural inclusion, $i : A \to A \rtimes_{\phi',r} F_{n-1}$ is the canonical inclusion, $w_n = i_* \circ (id_* - \phi(x_n^{-1})_*)$, and the vertical arrows correspond to the connecting homomorphisms of a sequence induced by a Toeplitz extension; and

$$\begin{tikzcd} (K_0(A))^n \arrow[r, "\theta"] & K_0(A) \arrow[r, "\pi_*"] & K_0(A \rtimes_r F_n) \arrow[d] \\ K_1(A \rtimes_r F_n) \arrow[u] & K_1(A) \arrow[l, "\pi_*"] & (K_1(A))^n \arrow[l, "\theta"'] \end{tikzcd} \tag{4.2}$$

where θ is the map

$$\theta(\gamma_1 \oplus \cdots \oplus \gamma_n) = \sum_{i=1}^{n} (\gamma_i - \phi(x_i^{-1})_*(\gamma_j)) \ .$$

and the vertical arrows come from connecting homomorphisms of a Toeplitz extension.

Moreover, we recall that in [PV82], Pimsner and Voiculescu used these six-term exact sequences to give the following computation of the K-theory of the C^*-algebra of a free group:

$$K_0(C_r^*(F_n(\alpha_1, \ldots, \alpha_n))) \simeq \mathbb{Z}[1] \tag{4.3}$$

and

$$K_1(C_r^*(F_n(\alpha_1, \ldots, \alpha_n))) \simeq \mathbb{Z}^n[u_{\alpha_1}, \ldots, u_{\alpha_n}] \tag{4.4}$$

where for $g \in F_n(\alpha_1, \ldots, \alpha_n)$, we denote by u_g the element in $C_r^*(F_n(\alpha_1, \ldots, \alpha_n))$ such that $u_g(h) = 1$ if $h = g$, and $u_g(h) = 0$ if $h \neq g$.

As the group F_n is K-amenable (see [Cun83] and [Cun82]),

$$K_0(C^*(F_n(\alpha_1,\dots,\alpha_n))) \simeq \mathbb{Z}[1] \tag{4.5}$$

and

$$K_1(C^*(F_n(\alpha_1,\dots,\alpha_n))) \simeq \mathbb{Z}^n[u_{\alpha_1},\dots,u_{\alpha_n}]\ . \tag{4.6}$$

4.1. **K-theory of $C_r^*(P_4)$.** We will start with the case of $n=4$. To compute the K-theory of $C_r^*(P_4)$, we will use the decomposition $P_4 = (F_3 \rtimes_\varphi F_2)\times\mathbb{Z}$, where the action φ is given by the relations in Remark 2.1. Its reduced C^*-algebra $C_r^*(P_4)$ is then isomorphic to $C_r^*(F_3\rtimes F_2)\otimes C_r^*(\mathbb{Z})$, and by the Künneth formula we have the following decompositions:

$$K_0(C_r^*(P_4)) \simeq K_0(C_r^*(F_3\rtimes F_2))\otimes_{\mathbb{Z}} K_0(C_r^*(\mathbb{Z})) \oplus K_1(C_r^*(F_3\rtimes F_2))\otimes_{\mathbb{Z}} K_1(C_r^*(\mathbb{Z})),$$

$$K_1(C_r^*(P_4)) \simeq K_0(C_r^*(F_3\rtimes F_2))\otimes_{\mathbb{Z}} K_1(C_r^*(\mathbb{Z})) \oplus K_1(C_r^*(F_3\rtimes F_2))\otimes_{\mathbb{Z}} K_0(C_r^*(\mathbb{Z}))\ .$$

We then compute the K-theory of $C_r^*(F_3\rtimes F_2)$. By the following well-known lemma, the former C^*-algebra is the reduced crossed product $C_r^*(F_3)\rtimes_r F_2$ (see [CELY17, Example 2.3.6], [Wil07, §3.3]).

Lemma 4.1. *Let H and N be discrete groups, and let $\phi : H \to \mathrm{Aut}(N)$ be an action by group automorphisms. Then ϕ gives actions of H on the full and reduced group C^*-algebras $C^*(N)$ and $C_r^*(N)$ that are given by the formula $\phi_h(f)(n) = f(h^{-1}nh)$, for $f\in C_c(N)$, $h\in H$ and $n\in N$, and one has*

$$C_r^*(N\rtimes_\phi H) \simeq C_r^*(N)\rtimes_{\phi,r} H,$$
$$C^*(N\rtimes_\phi H) \simeq C^*(N)\rtimes_\phi H.$$

Applying Pimsner–Voiculescu's first sequence to compute the K-theory of $C_r^*(F_3)\rtimes_r F_2$, where $F_3 = F_3(\beta_1,\beta_2,\beta_3)$ and $F_2 = F_2(\alpha_1,\alpha_2)$, we get the following result.

Proposition 4.2. *The K-theory of $C_r^*(F_3\rtimes F_2)$ is as follows:*

$$K_0(C_r^*(F_3\rtimes F_2)) \simeq \mathbb{Z}^7, \qquad K_1(C_r^*(F_3\rtimes F_2)) \simeq \mathbb{Z}^5.$$

Proof. Denote by $\varphi : F_2 \to \mathrm{Aut}(F_2)$ the action of F_3 on F_2 given by the relations which determine the structure of P_4. Set $B := C_r^*(F_3)\rtimes_r F_2 = C_r^*(F_3(\beta_1,\beta_2,\beta_3))\rtimes_r F_2(\alpha_1,\alpha_2)$. From (4.1), we have the exact sequence:

(4.7)

$$\begin{tikzcd}
K_0(C_r^*(F_3(\beta_1,\beta_2,\beta_3))) \arrow[r, "w_2"] & K_0(C_r^*(F_3(\beta_1,\beta_2,\beta_3))\rtimes_r F(\alpha_1)) \arrow[rr, "k_*"] & & K_0(B) \arrow[d] \\
K_1(B) \arrow[u] & K_1(C_r^*(F_3(\beta_1,\beta_2,\beta_3))\rtimes_r F(\alpha_1)) \arrow[l, "k_*"] & & K_1(C_r^*(F_3(\beta_1,\beta_2,\beta_3))) \arrow[ll, "w_2"]
\end{tikzcd}$$

We claim that the maps $w_2 = i_*\circ(id_* - \varphi(\alpha_2^{-1})_*)$ are equal to zero. On K_0, we have $\varphi(\alpha_2^{-1})_*([1]) = [1] = id_*([1])$, so $w_2([1]) = 0$; and on K_1, for every $j\in\{1,2,3\}$ we have

$\varphi(\alpha_2^{-1})_*([u_{\beta_j}]) = [u_{\varphi(\alpha_2^{-1})(\beta_j)}] = [u_{\alpha_2^{-1}\beta_j\alpha_2}]$. By the braid group relations 4., 5. and 6. in Remark 2.1, we have $\alpha_2^{-1}\beta_j\alpha_2 = f_\beta\beta_j f_\beta^{-1}$, where f_β is a product of elements in $\{\beta_1, \beta_2, \beta_3\}$. Hence

$$[u_{\alpha_2^{-1}\beta_j\alpha_2}] = [u_{f_\beta\beta_j f_\beta^{-1}}] = [u_{\beta_j}] \text{ in } K_1(C^*(F(\beta_1, \beta_2, \beta_3)) \rtimes_r F(\alpha_1)),$$

so $w_2([u_{\beta_j}]) = 0$ for all j, and the claim is proved.

From (4.7), we get two short exact sequences:

$$0 \to K_0(C_r^*(F_3(\beta_1, \beta_2, \beta_3)) \rtimes_r F(\alpha_1)) \to K_0(B) \to K_1(C_r^*(F_3(\beta_1, \beta_2, \beta_3)) \to 0 \tag{4.8}$$

$$0 \to K_1(C_r^*(F_3(\beta_1, \beta_2, \beta_3)) \rtimes_r F(\alpha_1)) \to K_1(B) \to K_0(C_r^*(F_3(\beta_1, \beta_2, \beta_3)) \to 0 \tag{4.9}$$

We now use another PV sequence to compute $K_i(C_r^*(F_3(\beta_1, \beta_2, \beta_3)) \rtimes_r F(\alpha_1))$ for $i = 0, 1$. Set $B_1 := C_r^*(F_3(\beta_1, \beta_2, \beta_3)) \rtimes_r F(\alpha_1)$. We have from (4.2)that

$$\begin{tikzcd} K_0(C_r^*(F_3(\beta_1, \beta_2, \beta_3))) \arrow[r, "\theta"] & K_0(C_r^*(F_3(\beta_1, \beta_2, \beta_3))) \arrow[r] & K_0(B_1) \arrow[d] \\ K_1(B_1) \arrow[u] & K_1(C_r^*(F_3(\beta_1, \beta_2, \beta_3))) \arrow[l] & K_1(C_r^*(F_3(\beta_1, \beta_2, \beta_3))) \arrow[l, "\theta"'] \end{tikzcd} \tag{4.10}$$

where $\theta = id_* - \varphi(\alpha_1^{-1})_*$ is again the trivial map in K-theory, because by relations 1., 2. and 3. in Remark 2.1, we have that $\alpha_1^{-1}\beta_j\alpha_= f_\beta\beta_j f_\beta^{-1}$, where f_β is a product of elements in $\{\beta_1, \beta_2, \beta_3\}$, as above.

We get the short exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(B_1) \longrightarrow \mathbb{Z}^3 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow K_1(B_1) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

from which we deduce

$$K_0(B_1) \simeq K_1(B_1) \simeq \mathbb{Z}^4 \ .$$

By inserting this in (4.8), we get

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow K_0(B) \longrightarrow \mathbb{Z}^3 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow K_1(B) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

so we deduce

$$K_0(B) \simeq \mathbb{Z}^7, \qquad K_1(B) \simeq \mathbb{Z}^5, \tag{4.11}$$

which proves the proposition. □

By applying the Künneth formula to $P_4 = B \times \mathbb{Z}$, we obtain the following corollary.

Corollary 4.3. *The K-theory of $C_r^*(P_4)$ is given by*

$$K_0(C_r^*(P_4)) \simeq \mathbb{Z}^{12}, \qquad K_1(C_r^*(P_4)) \simeq \mathbb{Z}^{12}.$$

4.2. **Going from P_{n-1} to P_n.** Before computing the K-theory of $C_r^*(P_n)$ for all n, let us explain how to compute the K-theory of $C_r^*(P_5)$. Recall that $P_n = F_{n-1} \rtimes P_{n-1}$, so that $P_5 = F_4 \rtimes P_4 = F_4 \rtimes F_3 \rtimes F_2 \times \mathbb{Z}$. We first compute the K-theory of $C_r^*(F_4 \rtimes F_3 \rtimes F_2)$. By Lemma 4.1, we have

$$C_r^*(F_4 \rtimes F_3 \rtimes F_2) \simeq (C_r^*(F_4 \rtimes F_3)) \rtimes_r F_2,$$

so the K-theory of $C_r^*(P_5)$ can be computed via a Pimsner–Voiculescu sequence once we know the K-theory of $C_r^*(F_4 \rtimes F_3)$, which is isomorphic to $C_r^*(F_4) \rtimes F_3$. Notice that, as F_n is K-amenable for all n, we have $K_i(C_r^*(F_4) \rtimes F_3) \simeq K_i(C_r^*(F_4) \rtimes_r F_3)$, for $i = 0, 1$.

We compute the K-theory groups $K_i(C_r^*(F_4) \rtimes_r F_3)$ via Pimsner–Voiculescu sequences. Letting $n = 3$ in (4.2), and using (4.6) and (4.5), we get the sequence below.

$$\begin{tikzcd}
\mathbb{Z}^3 \arrow[r, "\theta"] & \mathbb{Z} \arrow[r] & K_0(C_r^*(F_4) \rtimes_r F_3) \arrow[d] \\
K_1(C_r^*(F_4) \rtimes_r F_3) \arrow[u] & \mathbb{Z}^4 \arrow[l] & (\mathbb{Z}^4)^3 \arrow[l, "\theta"']
\end{tikzcd}$$

Since again $\theta = 0$ on both K_0 and K_1, we have

$$K_0(C_r^*(F_4) \rtimes_r F_3) \simeq \mathbb{Z}^{13}, \qquad K_1(C_r^*(F_4) \rtimes_r F_3) \simeq \mathbb{Z}^7.$$

To compute the K-theory of $(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2$, we apply (4.2)

$$\begin{tikzcd}
(\mathbb{Z}^{13})^2 \arrow[r, "\theta"] & \mathbb{Z}^{13} \arrow[r] & K_0(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2) \arrow[d] \\
K_1(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2 \arrow[u] & \mathbb{Z}^7 \arrow[l] & (\mathbb{Z}^7)^2 \arrow[l, "\theta"']
\end{tikzcd}$$

and we get

$$K_0(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2) \simeq \mathbb{Z}^{33}, \qquad K_1(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2) \simeq \mathbb{Z}^{27}\ .$$

By the Künneth formula, we finally obtain

$$K_0(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2 \times \mathbb{Z}) \simeq K_1(C_r^*(F_4) \rtimes_r F_3) \rtimes_r F_2 \times \mathbb{Z}) \simeq \mathbb{Z}^{33} \oplus \mathbb{Z}^{27} \simeq \mathbb{Z}^{60}\ .$$

Let us now generalise this procedure to compute the K-theory of $C_r^*(F_n) \rtimes_r F_{n-1}$ in the following lemma.

Lemma 4.4. *The K-theory of $C_r^*(F_n) \rtimes_r F_{n-1}$ is given by*

$$K_0(C_r^*(F_n) \rtimes_r F_{n-1}) \simeq \mathbb{Z}^{1+n(n-1)}, \qquad K_1(C_r^*(F_n) \rtimes_r F_{n-1}) \simeq \mathbb{Z}^{2n-1}.$$

Proof. We apply (4.2)

$$\begin{tikzcd} (K_0(C_r^*(F_n)))^{n-1} \arrow[r, "\theta"] & K_0(C_r^*(F_n)) \arrow[r] & K_0(C_r^*(F_n) \rtimes_r F_{n-1})) \arrow[d] \\ K_1(C_r^*(F_n) \rtimes_r F_{n-1})) \arrow[u] & K_1(C_r^* F_n) \arrow[l] & (K_1(C_r^* F_n))^{n-1} \arrow[l, "\theta"'] \end{tikzcd} \tag{4.12}$$

the maps θ are trivial in K-theory, so for $i = 0, 1$ this gives

$$K_0(C_r^*(F_n) \rtimes_r F_{n-1}) \simeq K_0(C_r^*(F_n)) \oplus (K_1(C_r^*(F_n)))^{n-1}, \tag{4.13}$$

$$K_1(C_r^*(F_n) \rtimes_r F_{n-1}) \simeq K_1(C_r^*(F_n)) \oplus (K_0(C_r^*(F_n)))^{n-1} \ . \tag{4.14}$$

By the relations (4.6), we obtain the result. □

4.3. **K-theory of $C_r^*(P_n)$.** We are now ready to compute the K-theory of $C_r^*(P_n)$ using Pimsner–Voiculescu sequences. We use the following remark.

Remark 4.5. Let A be a C^*-algebra whose K-theory is torsion-free and finitely generated. This means that there exist two integers a_0 and a_1 such that $K_0(A) \simeq \mathbb{Z}^{a_0}$ and $K_1(A) \simeq \mathbb{Z}^{a_1}$. If $\phi : F_k \to \mathrm{Aut}(A)$ is an action of the free group F_k by automorphisms on A such that the induced map θ in the PV sequence is zero, then we obtain

$$K_0(A \rtimes_r F_k) \simeq \mathbb{Z}^{a_0 + k a_1}, \qquad K_1(A \rtimes_r F_k) \simeq \mathbb{Z}^{a_1 + k a_0}.$$

In particular, the K-theory groups of $A \rtimes_r F_k$ are also torsion-free.

Proposition 4.6. *The K-theory of $C_r^*(P_n)$ is given by*

$$K_0(C_r^*(P_n)) \simeq K_1(C_r^*(P_n)) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

Proof. We set $Q_j := F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_{n-j}$, for $j = 1, \ldots, n-2$.

We first show inductively that all the groups $K_*(C_r^*(Q_j))$ are torsion free. For $j = 1$, this holds true; at each subsequent step, the K-theory of $C_r^*(Q_j) \simeq C_r^*(F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_{n-j+1}) \rtimes_r F_{n-j}$ is computed as in Remark 4.5 via a Pimsner-Voiculescu sequence involving the K-theory of the reduced C^*-algebra $C_r^*(F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_{n-j+1})$. We repeatedly apply Pimsner–Voiculescu sequences (4.2) and we have

$$K_0(C_r^*(Q_i)) \simeq K_0(C_r^*(F_{n-i})) \oplus (K_1(C_r^*(Q_{i-1})))^{n-i} \tag{4.15}$$

$$K_1(C_r^*(Q_i)) \simeq K_1(C_r^*(F_{n-i})) \oplus (K_0(C_r^*(Q_{i-1})))^{n-i} \ . \tag{4.16}$$

To determine the rank, set

$$x_i := \operatorname{rank} K_0(C_r^*(Q_i)), \qquad y_i := \operatorname{rank} K_1(C_r^*(Q_i)).$$

From (4.15), we have for $i = 2, \ldots, n-2$

$$x_i = \operatorname{rank} K_0(C_r^*(Q_{i-1}) \rtimes_r F_{n-i}) = x_{i-1} + y_{i-1}(n-i),$$

$$y_i = \operatorname{rank} K_1(C_r^*(Q_{i-1}) \rtimes_r F_{n-i}) = y_{i-1} + x_{i-1} \cdot (n-i).$$

with $x_1 = 1$ and $y_1 = n-1$. This implies

$$x_i + y_i = x_{i-1} + y_{i-1} + (x_{i-1} + y_{i-1})(n-i) \ .$$

Note that $P_n = Q_{n-2} \times \mathbb{Z}$; and by the Künneth formula, we have rank $K_*(C_r^*(P_n)) = x_{n-2} + y_{n-2}$. The sum $s_i := x_i + y_i$ satisfies

$$s_i = s_{i-1}(n-i+1)$$

hence we deduce

$$\begin{aligned} s_{n-2} =& s_{n-3}(n-n+2+1) \\ &= s_2 \cdot (n-2)(n-3) \cdot \ldots 4 \cdot 3 \\ &= n(n-1) \cdot \ldots 4 \cdot 3 \\ &= \frac{n!}{2}. \end{aligned}$$

□

As the group P_n is K-amenable (see section 2.3), the K-theory of the maximal C^*-algebra of P_n coincides with the reduced one, see for instance, [JV84, Corollary 3.6]: If G is K-amenable, then for every C^*-dynamical system (A, α, G) one has

$$K_i(A \rtimes_{\alpha,r} G) \simeq K_i(A \rtimes_\alpha G), \qquad i = 0, 1 \ . \tag{4.17}$$

Since free groups are K-amenable, by Lemma 4.1 one has immediately:

(1) $K_i(C^*(P_n)) \simeq K(C_r^*(P_n))$
(2) For the iterated semidirect products

$$\begin{aligned} Q_1 &:= F_{n-1} \\ Q_2 &:= F_{n-1} \rtimes F_{n-2} \\ &\vdots \\ Q_{n-2} &:= F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \ . \end{aligned}$$

the K-theory of the maximal C^*-algebra is the same as the reduced one.

Therefore

$$K_0(C^*(P_n)) \simeq K_1(C^*(P_n)) \simeq \mathbb{Z}^{\frac{n!}{2}}.$$

5. Isomorphism of the Baum–Connes assembly map

Let us denote by $\Gamma = F(\beta_1, \beta_2, \beta_3) \rtimes F(\alpha_1, \alpha_2)$ the group generated by $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ satisfying the 6 relations described in section 3.1. In this section, we use the structure of P_4 given by $P_4 \simeq \Gamma \times \mathbb{Z}$, where $\Gamma = F_3 \rtimes F_2$, and the Künneth theorem in K-theory, to reduce the proof of the Baum–Connes isomorphism for P_4 to that of Γ. We will use the explicit computations given in [Val02] of the Baum–Connes assembly map in small homological degree to write down

the explicit image for Γ under the Baum–Connes assembly map. We will prove the following theorem.

Theorem 5.1. *The assembly map*

$$\mu_r : K_i(B\Gamma) \to K_i(C_r^*(\Gamma))$$

is an isomorphism for $i = 0$ (even-degree case) and $i = 1$ (odd-degree case).

5.1. **Odd-degree Baum–Connes isomorphism for Γ.** We will start with the case $i = 1$. Recall that $H_1(\Gamma, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ so that $H_1(\Gamma, \mathbb{Z})$ is the abelian group generated by $\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2$. There is a classical isomorphism $H_1(\Gamma, \mathbb{Z}) \simeq H_1(B\Gamma)$, where every generator of $H_1(\Gamma, \mathbb{Z})$ corresponds to a unique 1-cycle coming from the 1-skeleton of the space $B\Gamma$. The correspondence is determined by the fact that $\Gamma = \pi_1(B\Gamma)$: Every element $\gamma \in \Gamma$ can be viewed as a pointed continuous map $\gamma : S^1 \to B\Gamma$, thus inducing a map in K-homology $\gamma_* : K_1(S^1) \to K_1(B\Gamma)$. Let D be the Dirac operator on S^1 and π the representation of $C(S^1)$ on $L^2(S^1)$ given by pointwise multiplication. Then the class of the cycle (π, D) is the generator of $K_1(S^1) \simeq \mathbb{Z}$; we denote it by $[D]$. Every element $\gamma \in \Gamma$ can then be mapped to the class of the cycle $\gamma_*(\pi, D) = \gamma_*([D])$ in $K_1(B\Gamma)$ (see [Val02, Chapter 7]). Moreover, every element $\gamma \in \Gamma$ can be mapped to the invertible element $[\gamma] \in K_1(C_r^*(\Gamma))$, which is determined by the class of the Dirac element δ_γ in $C_c(\Gamma)$.

We are going to prove the following theorem.

Theorem 5.2. *Let Γ be the group $F(\beta_1, \beta_2, \beta_3) \rtimes F(\alpha_1, \alpha_2)$ generated by the elements α_1, α_2, β_1, β_2, β_3 that satisfy the 6 relations described in Section 3.1. The Baum-Connes assembly map*

$$\mu_r : K_1(B\Gamma) \to K_1(C_r^*(\Gamma))$$

is an isomorphism with

$$\mu_r((\alpha_i)_*[D]) = [\alpha_i], \qquad i = 1, 2;$$
$$\mu_r((\beta_i)_*[D]) = [\beta_i], \qquad i = 1, 2, 3,$$

where $[D]$ is the K-homology cycle given by the Dirac operator on the unit circle S^1.

Following [Val02] (see page 58), define a morphism $\tilde{\beta}_a : \Gamma \to K_1(C_r^*(\Gamma))$ by sending $\gamma \in \Gamma$ to the invertible element $[\gamma]$ in $K_1(C_r^*(\Gamma))$. The map $\tilde{\beta}_a$ gives rise to a well-defined morphism

$$\beta_a : H_1(\Gamma, \mathbb{Z}) \to K_1(C_r^*(\Gamma)), \qquad \gamma[\Gamma, \Gamma] \mapsto [\gamma],$$

because $K_1(C_r^*(\Gamma))$ is an abelian group. The Dirac operator D on S^1 gives rise to a class which generates $K_1(S^1)$. For every group element $\gamma \in \Gamma$, denote by $\gamma : S^1 \to B\Gamma$ a 1-cycle representative for $[\gamma] \in H_1(B\Gamma)$.

Define a morphism $\tilde{\beta}_t : \Gamma \to K_1(B\Gamma)$ by sending γ to $\gamma_*[D]$. By [Val02, Proposition 7.1], $\tilde{\beta}_t$ descends to

$$\beta_t : H_1(\Gamma, \mathbb{Z}) \to K_1(B\Gamma), \qquad [\gamma] \mapsto \gamma_*[D],$$

and we have the following lemma due to Natsume, which we will use to prove Theorem 5.2.

Lemma 5.3 ([Nat88], [Val02] Proposition 7.2)**.** *The following diagram commutes.*

$$\begin{tikzcd} & H_1(\Gamma,\mathbb{Z}) \arrow[dl, "\beta_t"'] \arrow[dr, "\beta_a"] & \\ K_1(B\Gamma) \arrow[rr, "\mu_r"] & & K_1(C_r^*(\Gamma)) \end{tikzcd}$$

That is, $\beta_a = \mu_r \circ \beta_t$.

Proof of Theorem 5.2. We are going to use the fact that there is an assembly map

$$\mu : K_i(B\Gamma) \to K(C^*(\Gamma))$$

such that $\mu_r = \lambda_{\Gamma,*} \circ \mu$ where the morphism $\lambda_{\Gamma,*} : K_i(C^*(\Gamma)) \to K(C_r^*(\Gamma))$ is induced by the regular representation λ_Γ of Γ. We will then use the K-amenability of P_n which implies that $\lambda_{\Gamma,*}$ is an isomorphism. The advantage of μ with respect to μ_r is that μ is functorial in Γ and we will make us of its functoriality. Consider the group homomorphism $\psi : \Gamma \to \Gamma_{ab}$, where $\Gamma_{ab} = \Gamma/[\Gamma,\Gamma] \simeq \mathbb{Z}^5$ is generated by the cosets of $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$. This map induces a continuous map $\psi : B\Gamma \to B\mathbb{Z}^5$ and a morphism of group C^*-algebras $\psi : C^*(\Gamma) \to C^*(\mathbb{Z}^5)$. By the functoriality of the Baum–Connes assembly map at the level of the maximal C^*-algebra, we have the following commutative diagram.

$$\begin{tikzcd} K_1(B\Gamma) \arrow[r, "\mu"] \arrow[d, "\psi_*"'] & K_1(C^*(\Gamma)) \arrow[d, "\psi_*"'] \\ K_1(B\mathbb{Z}^5) \arrow[r, "\mu'"] & K_1(C^*(\mathbb{Z}^5)) \end{tikzcd}$$

where μ and μ' are the Baum-Connes assembly map defined at the level of the full C^*-algebra for Γ and $\mathbb{Z}^5$. From our calculation of $K_1(B\Gamma)$ (see Lemma 3.7), we see that $\psi_* : K_1(B\Gamma) \to K_1(B\mathbb{Z}^5)$ is an isomorphism. In fact, the following diagram is commutative by definition

$$\begin{tikzcd} H_1(\Gamma,\mathbb{Z}) \arrow[r] \arrow[d, "\simeq"'] & K_1(B\Gamma) \arrow[d, "\psi_*"'] \\ H_1(\mathbb{Z}^5,\mathbb{Z}) \arrow[r, "\simeq"] & K_1(B\mathbb{Z}^5) \end{tikzcd}$$

and the left and bottom arrows are isomorphisms. This shows that ψ_* is a surjective morphism. But ψ_* is a surjective morphism from $\mathbb{Z}^5$ to itself. So ψ_* is an isomorphism on K-homology.

As the Baum–Connes conjecture is known to be true for abelian groups, the map μ' is an isomorphism; so using the commutativity of the diagram, we get that the map $\psi_* : K_1(C^*(\Gamma)) \to K_1(C^*(\mathbb{Z}^5))$ is surjective. On the other hand, we have computed that

$$K_1(C^*(\Gamma)) \simeq K_1(C^*(\mathbb{Z}^5)) \simeq \mathbb{Z}^5;$$

therefore, as a surjective homomorphism from $\mathbb{Z}^5$ to itself is an isomorphism, the map ψ_* is an isomorphism on K-theory as well, and hence μ is an isomorphism.

By commutativity of the diagram in Lemma 5.3, the map β_a is surjective; and, being a surjective morphism from $\mathbb{Z}^5$ to itself, it is an isomorphism. Thus, β_a is an isomorphism, mapping $\alpha_i[\Gamma,\Gamma]$ to $[\alpha_i]$ and $\beta_j[\Gamma,\Gamma]$ to $[\beta_j]$. Because μ is an isomorphism, we have that β_t is an isomorphism, mapping $[\alpha_i : S^1 \to B\Gamma] \in H_1(B\Gamma) \simeq H_1(\Gamma,\mathbb{Z})$ to $(\alpha_i)_*[D] \in K_1(B\Gamma)$. Therefore we know that the generators of $K_1(B\Gamma)$ are of the form $(\alpha_i)_*[D]$, $i = 1,2$; or $(\beta_j)_*[D]$, $j = 1,2,3$. The commutativity of the diagram of Lemma 5.3 then implies that

$$\mu((\alpha_i)_*[D]) = [\alpha_i], \qquad i = 1,2;$$
$$\mu((\beta_i)_*[D]) = [\beta_i], \qquad i = 1,2,3.$$

The theorem is then proved by noting the K-amenability of Γ and applying $\lambda_{\Gamma,r}$ to get the elements of $C_r^*(\Gamma)$. □

5.2. **Even-degree Baum–Connes isomorphism for Γ.** Recall that Γ is the group $F(\beta_1,\beta_2,\beta_3)\rtimes F(\alpha_1,\alpha_2)$ whose generators $\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3$ satisfy the 6 relations of Section 3.1. Each relation R_i corresponds to a surface Σ_i whose fundamental group is canonically related to R_i as follows.

$$\begin{aligned}
\pi_1(\Sigma_1) &= \langle a_1,a_2,a_3,a_4 \mid a_1a_2a_1^{-1} = (a_3a_4)^{-1}a_4(a_3a_2)\rangle,\\
\pi_1(\Sigma_2) &= \langle b_1,b_2,b_3,b_4 \mid b_1b_2b_1^{-1} = b_3^{-1}b_4b_3b_4^{-1}b_2\rangle,\\
\pi_1(\Sigma_3) &= \langle c_1,c_2 \mid c_1c_2c_1^{-1} = c_2\rangle,\\
\pi_1(\Sigma_4) &= \langle d_1,d_2,d_3,d_4 \mid d_1d_2d_1^{-1} = (d_3d_4)^{-1}d_4(d_3d_2)\rangle,\\
\pi_1(\Sigma_5) &= \langle e_1,e_2,e_3,e_4,e_5,e_6 \mid e_1e_2e_1^{-1} = (e_6e_5)^{-1}e_5e_6e_2(e_3e_4)^{-1}(e_4e_3)\rangle,\\
\pi_1(\Sigma_6) &= \langle f_1,f_2,f_3,f_4 \mid f_1f_2f_1^{-1} = f_3^{-1}f_4f_3f_4^{-1}f_2\rangle.
\end{aligned}$$

Let $\Gamma_i = \pi_1(\Sigma_i)$, and set $\tilde{\Gamma} := \Gamma_1 * \Gamma_2 * \cdots * \Gamma_6$, the free product of the Γ_i. By the van Kampen theorem, the group $\tilde{\Gamma}$ is the fundamental group of

$$\Sigma := \Sigma_1 \vee \Sigma_2 \vee \cdots \vee \Sigma_6$$

obtained by joining together a base point from each of the Σ_i. Then the mapping defined by

$$\begin{aligned}
a_1,b_1,c_1 &\mapsto \alpha_1 & a_3,b_2,b_4,e_2 &\mapsto \beta_2\\
d_1,e_1,f_1 &\mapsto \alpha_2 & c_2,d_3,e_4,e_6,f_2,f_4 &\mapsto \beta_3\\
a_2,a_4,b_3,d_2,d_4,e_3,e_5,f_3 &\mapsto \beta_1
\end{aligned}$$

sends relations of $\tilde{\Gamma}$ to relations of Γ, and it determines a surjective morphism

$$f : \tilde{\Gamma} \to \Gamma. \tag{5.1}$$

The map f also leads to the morphism of full group C^*-algebras below.

$$f : C^*(\tilde{\Gamma}) \to C^*(\Gamma).$$

Let $X = B\Gamma$, and denote by $X_1, \ldots, X_6$ the 2-simplices of X corresponding to each R_i. The union of the X_i is the 2-skeleton $X^{(2)}$; and since X is a space of dimension 2, we have $X^{(2)} = X$. The map f in (5.1) induces a map at the level of the classifying spaces $f : B\tilde{\Gamma} \to B\Gamma = X$. Taking the 2-skeleton, we obtain a continuous map

$$f : \Sigma \to X \tag{5.2}$$

such that $f(\Sigma_i) = X_i$. Applying the fundamental group functor for (5.2) recovers

$$f : \pi_1(\Sigma) \to \pi_1(X)$$

in (5.1). By the functoriality of the Baum–Connes assembly map, we have the following commutative diagram.

$$\begin{tikzcd} K_0(\Sigma) \arrow[r, "\mu"] \arrow[d, "f_*"'] & K_0(C^*(\tilde{\Gamma})) \arrow[d, "f_*"'] \\ K_0(B\Gamma) \arrow[r, "\mu"] & K_0(C^*(\Gamma)). \end{tikzcd}$$

Note that the map in (5.2) gives rise to the two isomorphisms

$$H_0(\Sigma) \simeq H_0(B\Gamma) \simeq \mathbb{Z}$$

and

$$f_* : H_2(\Sigma) \to H_2(B\Gamma), \qquad [\Sigma_i] \mapsto [X_i].$$

The existence of an inverse Chern character map

$$H_{even}(\Sigma) = H_0(\Sigma) \oplus H_2(\Sigma) \to K_0(\Sigma)$$

sending the generator $[\Sigma_i] \in H_2(\Sigma)$ to $[D_{\Sigma_i}] \in K_0(\Sigma)$, where D_{Σ_i} is the Dirac operator on Σ_i, which induces an isomorphism at the level of K-homology classes, allows one to construct a morphism

$$\beta_t : H_{even}(B\Gamma) \simeq H_{even}(\Sigma) \to K_0(\Sigma) \xrightarrow{f_*} K_0(B\Gamma),$$

taking the composition with f_* on K-homology,

By construction,

$$\beta_t([X_i]) = f_*[D_{\Sigma_i}]. \tag{5.3}$$

The map β_t is part of the lower-left of diagram

$$\begin{tikzcd} H_{even}(B\Gamma) \arrow[dr, dotted, "\beta_t"'] \arrow[drr, dotted, bend right, "\beta_a"'] & H_{even}(\Sigma) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & K_0(\Sigma) \arrow[r, "\mu"] \arrow[d, "f_*"'] & K_0(C^*(\tilde{\Gamma})) \arrow[d, "f_*"'] \\ & & K_0(B\Gamma) \arrow[r, "\mu_r"] & K_0(C_r^*(\Gamma)). \end{tikzcd}$$

The upper-right of this diagram is defined to be β_a; by definition, (see [Val02])

$$\beta_a([X_i]) = f_*(\mu([D_{\Sigma_i}])).$$

The commutativity of this diagram is implied by the following lemma.

Lemma 5.4 ([Val02] Prop. 7.3)**.** *The diagram commutes with β_t rationally injective:*

$$\begin{tikzcd} & H_{even}(\Gamma,\mathbb{Z}) \arrow[dl, "\beta_t"'] \arrow[dr, "\beta_a"] & \\ K_0(B\Gamma) \arrow[rr, "\mu_r"] & & K_0(C_r^*(\Gamma)) \end{tikzcd}$$

That is, $\beta_a = \mu_r \circ \beta_t$.

Now, from our calculations (see Lemma 3.7), we have,

$$H_{even}(B\Gamma) \simeq K_0(B\Gamma) \simeq \mathbb{Z}^7$$

Knowing that β_t maps generators to generators from (5.3), and in view of the fact that β_t is rationally injective (see [Val02, Proposition 7.3], β_t is an isomorphism. The commutativity $\beta_a = \mu_r \circ \beta_t$ hence implies that β_a is an isomorphism if and only if μ_r is an isomorphism.

Since $\beta_a = \mu_r \circ \beta_t$, we can also describe the map μ_r explicitly.

$$\begin{aligned} \mu_r : K_0(B\Gamma) &\to K_0(C_r^*(\Gamma)) \\ f_*[D_{\Sigma_i}] &\mapsto f_*\mu([D_{\Sigma_i}]) \end{aligned}$$

We are ready to prove the following theorem.

Theorem 5.5. *The assembly map*

$$\mu_r : K_0(B\Gamma) \to K_0(C_r^*(\Gamma))$$

is an isomorphism, with

$$\begin{aligned} \mu_r(f_*[D_{\Sigma_i}]) &= f_*(\mu[D_{\Sigma_i}]), \qquad 1 \le i \le 6 \\ \mu_r(1) &= [1]. \end{aligned}$$

Proof. Consider the trivial homomorphism $1\colon \Gamma \to \{e\}$. It induces a map $B\Gamma \to \{\mathrm{pt}\}$ and a map $C^*(\Gamma) \to \mathbb{C}$ such that the K-homology and K-theory functor lead to two morphisms $1_* : K_0(B\Gamma) \to K_0(\mathrm{pt})$ and $1_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$. The first morphism in K-homology is a surjective map capturing the 0-simplex of $B\Gamma$. The functoriality of the Baum–Connes assembly map gives rise to the commutative diagram

$$\begin{tikzcd} K_0(B\Gamma) \arrow[r, "\mu"] \arrow[d, "1_*"'] & K_0(C^*(\Gamma)) \arrow[d, "1_*"'] \\ K_0(\mathrm{pt}) \arrow[r, "\mu_0"] & K_0(\mathbb{C}) \end{tikzcd}$$

where μ_0 is the identity map from $\mathbb{Z}$ to itself. Let $i = 1$ or 2, and let $j = 1, 2$, or 3. Denote by ϕ_{ij} the surjective morphism given by

$$\phi_{ij} : \Gamma \to \mathbb{Z}^2 \qquad \alpha_p \mapsto \delta_{pi}\alpha_i, \quad \beta_p \mapsto \delta_{pj}\beta_j,$$

where δ_{ij} is the Kronecker delta. As above, it induces two maps

$$\phi_{ij} : B\Gamma \to B\mathbb{Z}^2 \simeq T^2, \qquad \phi_{ij} : C^*(\Gamma) \to C^*(\mathbb{Z}^2),$$

where we denote by T the torus. Note that the map on the classifying space is given by collapsing all 1-cells which do not represent α_i or β_j to a point, and collapsing all 2-cells that do not represent the group relation involving $\alpha_i\beta_j\alpha_i^{-1}$ (denoted R_{ij}) to a point. Thus, the induced map on K-homology $\phi_{ij,*} : K_0(B\Gamma) \to K_0(T^2)$ is a surjective map that maps the 2-cell represented by R_{ij} to the Bott generator of $K_0(T^2)$. As above, we also have the induced map on K-theory $\phi_{ij,*} : K_0(C^*(\Gamma)) \to K_0(C^*(\mathbb{Z}^2))$ the commutative diagram

$$\begin{tikzcd} K_0(B\Gamma) \arrow[r, "\mu"] \arrow[d, "\phi_{ij,*}"'] & K_0(C^*(\Gamma)) \arrow[d, "\phi_{ij,*}"'] \\ K_0(T^2) \arrow[r, "\mu_{ij}"] & K_0(C^*(\mathbb{Z}^2)) \end{tikzcd}$$

Here μ_{ij} is the assembly map for $\mathbb{Z}^2$. Putting 7 diagrams (involving 1_* and $\phi_{ij,*}$ where $i = 1, 2$ and $j = 1, 2, 3$) together, we have a commutative diagram

$$\begin{tikzcd} K_0(B\Gamma) \arrow[r, "\mu"] \arrow[d, "\phi_*"'] & K_0(C^*(\Gamma)) \arrow[d, "\phi_*"'] \\ K_0(\mathrm{pt}) \bigoplus \left[\bigoplus_{i,j} \tilde{K}_0(T^2)\right] \arrow[r, "\mu'", "\simeq"'] & K_0(\mathbb{C}) \bigoplus \left[\bigoplus_{i,j} \tilde{K}_0(C^*(\mathbb{Z}^2))\right]. \end{tikzcd}$$

Here, $\tilde{K}_0(T^2)$ is the reduced K-homology, excluding elements generated by the trivial cycle from $K_0(T^2)$, and $\tilde{K}_0(C^*(\mathbb{Z}^2))$ is the reduced K-theory, eliminating elements generated by the trivial projection from $K_0(C^*(\mathbb{Z}^2))$. By construction, ϕ_* on K-homology (the left arrow) is an isomorphism. It is well known that μ' is an isomorphism for abelian groups $\mathbb{Z}^2$ and for the trivial group $\{e\}$. Together with the commutativity of the diagram, the map ϕ_* on K-theory (the right arrow) is surjective. Because ϕ_* is a surjective group homomorphism from $\mathbb{Z}^7$ to itself, we conclude that ϕ_* on K-theory is an isomorphism. Therefore the commutativity of the diagram implies that μ for Γ is an isomorphism. As the group Γ is K-amenable, we have $K_*(C^*(\Gamma)) \simeq K_*(C_r^*(\Gamma))$, hence we find that the assembly map μ_r is an isomorphism. □

5.3. **Isomorphism for P_4.** Let us now recover Oyono-Oyono's theorem for P_4 using the Künneth formula.

Theorem 5.6. *The assembly map*

$$\mu_r : K_i(BP_4) \to K_i(C_r^*(P_4))$$

is an isomorphism for $i = 0$ or 1.

Proof. Note that the isomorphism $P_4 \simeq \Gamma \times \mathbb{Z}$ implies that

$$K_i(BP_4) \simeq K_0(B\Gamma) \otimes K_i(B\mathbb{Z}) \oplus K_1(B\Gamma) \otimes K_{i+1}(B\mathbb{Z}),$$
$$K_i(C_r^*(P_4)) \simeq K_0(C_r^*(\Gamma)) \otimes K_i(C_r^*(\mathbb{Z})) \oplus K_1(C_r^*(\Gamma)) \otimes K_{i+1}(C_r^*(\mathbb{Z})).$$

Following the definition of the assembly map in [Val02] by twisting Mishchenko line bundles, we have $\mu_r^{P_4}(x \otimes y) = \mu_r^{\Gamma}(x) \otimes \mu_r^{\mathbb{Z}}(y)$ for $x \in K_i(B\Gamma)$ and $y \in K_j(B\mathbb{Z})$, which are represented by Dirac-type operators. Then the assembly map for P_4 is an isomorphism if $\mu_r : K_i(B\Gamma) \to K_i(C_r^*(\Gamma))$ for $i = 0{,}1$ is an isomorphism. The theorem thus follows from Theorems 5.2 and 5.5. □

5.4. **Baum–Connes isomorphism for P_n.** Let us now prove the following theorem, which is originally due to Oyono-Oyono.

Theorem 5.7 ([OO01a] Proposition 7.2)**.** *The Baum–Connes assembly map for the pure braid group P_n*

$$\mu_r : K_i(BP_n) \to K_i(C_r^*(P_n))$$

is an isomorphism.

For $k \in \{1, \dots, n-1\}$, let

$$F_k := F_k(A_{1,k+1}, A_{2,k+1}, \dots, A_{k,k+1})$$

be the free subgroup in P_n (see section 2.1). There is a canonical homomorphism

$$\rho : P_n \to F_1 \times F_2 \times \cdots \times F_{n-1}, \qquad A_{s,t} \mapsto (e, \dots, e, A_{s,t}, e, \dots, e)$$

where e is the identity element; here, $A_{s,t} \in F_{t-1}$. In particular, all relations in the presentation for P_n reduce to the form

$$\rho(A_{r,s})\rho(A_{i,j}) = \rho(A_{i,j})\rho(A_{r,s}), \qquad i < j, \;\; r < s, \;\; s < j$$

in the image. The map ρ induces maps between the classifying spaces and the C^*-algebras:

$$B\rho : BP_n \to BF_1 \times \cdots \times BF_{n-1}, \qquad \rho : C^*(P_n) \to C^*(F_1) \otimes \cdots \otimes C^*(F_{n-1}).$$

Consider the induced maps on K-homology and K-theory. By the functoriality of the Baum–Connes assembly map at the level of the maximal C^*-algebra, one has the following commutative diagram.

$$\begin{tikzcd} K_i(BP_n) \arrow[r, "\mu"] \arrow[d, "B\rho_*"'] & K_i(C^*(P_n)) \arrow[d, "\rho_*"'] \\ K_i(BF_1 \times \cdots \times BF_{n-1}) \arrow[r, "\mu'", "\simeq"'] & K_i(C^*(F_1) \otimes \cdots \otimes C^*(F_{n-1})) \end{tikzcd} \tag{5.4}$$

where μ' is an isomorphism because the Baum–Connes conjecture is stable under direct products (see [OO01b]) and it is an isomorphism for the free groups F_k because they have Haagerup's

property. Let us describe the map $B\rho$ as a morphism between CW-complexes. For $0 \leq r \leq n$, choose pairs of numbers (i_k, j_k), where $k \in \{1, 2, \ldots, r\}$, satisfying

$$1 \leq i_1 < i_2 < \cdots < i_r \leq n, \qquad 1 \leq j_k < i_k. \tag{5.5}$$

It can be checked that every r-simplex of BP_n depends uniquely on the pairs (i_k, j_k), where $1 \leq k \leq r$. Denote the r-simplex by $[A_{j_1,i_1}, \ldots, A_{j_r,i_r}]$. Note that for a fixed r, the number of distinct r-simplices in BP_n is equal to

$$\sum_{1 \leq i_1 < \cdots < i_r \leq n} (i_1 - 1) \cdots (i_r - 1) = a_r,$$

which is the rank of the free part of $H^r(BP_n)$. Recall that $a_0 + \cdots a_n = n!$, so that BP_n has $n!$ simplices in total.

Example 5.8. *The CW complex BP_4 has 1 0-simplex; 6 1-simplices $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$; 11 2-simplices $R1, R2, R3, R4, R5, R6, c \times \alpha_1, c \times \alpha_2, c \times \beta_1, c \times \beta_2, c \times \beta_3$; and 6 3-simplices $R_i \times c$ for $i \in \{1, \cdots, 6\}$.*

The map $B\rho : BP_n \to BF_1 \times \cdots \times BF_{n-1}$ is defined by sending the r-simplex $[A_{j_1,i_1}, \ldots, A_{j_r,i_r}]$ in BP_n to the r-simplex

$$([A_{j_1,i_1}], \ldots, [A_{j_r,i_r}]) \in BF_{i_1-1} \times \cdots \times BF_{i_r-1} \subset BF_1 \times \cdots \times BF_{n-1}.$$

Observe that $B\rho$ gives rise to an isomorphism

$$B\rho_* : H_i(BP_n) \to H_i(BF_1 \times \cdots \times BF_{n-1}).$$

Because the Chern character maps

$$\begin{aligned} &\mathrm{Ch} : K_{0/1}(BP_n) \to H_{even/odd}(BP_n) \\ &\mathrm{Ch} : K_{0/1}(BF_{i_r-1} \subset BF_1 \times \cdots BF_{n-1}) \to H_{even/odd}(BF_1 \times \cdots \times BF_{n-1}) \end{aligned}$$

are isomorphisms, and by the functoriality of the Chern character, we obtain an isomorphism on K-homology

$$B\rho_* : K_i(BP_n) \to K_i(BF_1 \times \cdots \times BF_{n-1}), \qquad i = 0, 1.$$

Because the Baum–Connes conjecture holds for free groups, we obtain that μ' in (5.4) is an isomorphism. By the commutativity of (5.4), the map on K-theory

$$\rho_* : K_i(C^*(P_n)) \to K_i(C^*(F_1) \otimes \cdots \otimes C^*(F_{n-1}))$$

is surjective. It is an easy exercise to compute that $K_i(C^*(F_1) \otimes \cdots \otimes C^*(F_{n-1})) \simeq \mathbb{Z}^{\frac{n!}{2}}$. Thus ρ_* is a surjective morphism from $\mathbb{Z}^{\frac{n!}{2}}$ to itself. So ρ_* is in fact an isomorphism. Therefore μ is an isomorphism, by the commutativity of the diagram (5.4). As P_n is K-amenable (see section 2.3), Theorem 5.7 is then proved.

6. The full braid group on three stands B_3

In this section we consider full braid groups. The Baum–Connes correspondence for B_n is known to be an isomorphisms by the work of Schick ([Sch07]). We provide the explicit description in the case $n = 3$, modulo torsion. Note that in the paper [BBV99], the authors had computed the K-theory of $C_r^*(B_3)$.

6.1. **K-homology of BB_n.** Modulo torsion, the K-homology of BB_n is easier to compute using the rational isomorphism of the Chern character:

$$\mathrm{Ch} : K_0(BB_n) \to \bigoplus_i H_{2i}(B_n, \mathbb{Z}), \qquad \mathrm{Ch} : K_1(BB_n) \to \bigoplus_i H_{2i-1}(B_n, \mathbb{Z}).$$

Arnold computed the integral cohomology ring of the braid groups:

$$H^0(B_n, \mathbb{Z}) \simeq \mathbb{Z}, \qquad H^1(B_n, \mathbb{Z}) \simeq \mathbb{Z},$$

and $H^i(B_n, \mathbb{Z})$ is finite for $i > 1$; see [Arn69] and [Ver98]. By Poincaré duality, we obtain the following result.

Proposition 6.1. *Up to torsion,*

$$K_0(BB_n) \simeq K_1(BB_n) \simeq \mathbb{Z}.$$

Remark 6.2. Calculating $K_*(BB_n)$ is a challenging task, since $K_*(BB_n)$ may contain torsion. For example, in Example 5.10 in [LOS21], the K-theory of the reduced group C^*-algebra of B_4 is computed to be

$$K_0(C_r^*(B_4)) \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), \qquad K_1(C_r^*(B_4)) \simeq \mathbb{Z}.$$

By the Baum-Connes isomorphism for the braid group, one knows that $K_0(BB_4)$ has torsion.

6.2. **K-theory of $C_r^*(B_3)$.** Let $B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be the braid group on three strands. The center of this group is generated by $(\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$. Let $x = \sigma_1\sigma_2\sigma_1$ and $y = \sigma_1\sigma_2$. Then B_3 can be presented alternatively as

$$B_3 \simeq \langle x, y \mid x^2 = y^3 \rangle,$$

where $\langle x^2 \rangle = \langle y^3 \rangle = Z(B_3)$. Setting $G = \langle x \rangle, H = \langle y \rangle$ and $K = \langle x^2 \rangle = \langle y^3 \rangle$, then

$$B_3 = \langle x \rangle *_{Z(B_3)} \langle y \rangle = G *_K H.$$

For an amalgamated free product, one has the following six-term exact sequence (See [Nat85] Theorem A1).

$$\begin{tikzcd}
K_0(C_r^*(K)) \arrow[r, "a"] & K_0(C_r^*(G)) \oplus K_0(C_r^*(H)) \arrow[r, "d"] & K_0(C^*(B_3)) \arrow[d] \\
K_1(C_r^*(B_3)) \arrow[u] & K_1(C_r^*(G)) \oplus K_1(C_r^*(H)) \arrow[l, "c"'] & K_1(C_r^*(K)) \arrow[l, "b"']
\end{tikzcd}$$

Note that $K_i(C_r^*(K)) = K_i(C_r^*(G)) = K_i(C_r^*(H)) \simeq \mathbb{Z}$. By definition,

$$a : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}, \qquad a(x) = (x, x);$$
$$b : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}, \qquad b(x) = (2x, 3x).$$

Thus a and b are injective, and then c and d are surjective. Therefore we have

$$K_1(C_r^*(B_3)) \simeq \mathbb{Z} \oplus \mathbb{Z}/\mathrm{im}(b) \simeq \mathbb{Z},$$

where the last isomorphism is due to the linear transformation

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}, \qquad (x, y) \mapsto (3x - 2y, -x + y)$$

in $SL(2, \mathbb{Z})$. Similarly,

$$K_0(C_r^*(B_3)) \simeq \mathbb{Z} \oplus \mathbb{Z}/\mathrm{im}(a) \simeq \mathbb{Z}.$$

Thus the following proposition is proved.

Proposition 6.3. *We have that*

(1) $K_0(C_r^*(B_3)) \simeq \mathbb{Z}$ is generated by the unit of $C_r^*(B_3)$, and
(2) $K_1(C_r^*(B_3)) \simeq \mathbb{Z}$ is generated by $[\sigma_1] = [\sigma_2]$.

Remark 6.4. The K-theory of $C_r^*(B_4)$ is computed in [LOS21], Example 5.10. At present, we are not aware of any direct method of computing $K_*(C_r^*(B_n))$ when $n \geq 5$.

The proof of the Baum–Connes isomorphism (rationally) for B_3 can be carried out analogous to Theorem 5.2 by considering the trivial morphism $B_3 \to \{e\}$, the quotient morphism $B_3 \to B_3/[B_3, B_3] \simeq \mathbb{Z}$, and these commutative diagrams:

$$\begin{tikzcd} K_0(BB_3) \arrow[r, "\mu"] \arrow[d] & K_0(C_r^*(B_3)) \arrow[d] \\ K_0(B\{e\}) \arrow[r, "\mu'"] & K_0(C_r^*(\{e\})) \end{tikzcd} \qquad \begin{tikzcd} K_1(BB_3) \arrow[r, "\mu"] \arrow[d] & K_1(C_r^*(\Gamma)) \arrow[d] \\ K_1(B\mathbb{Z}) \arrow[r, "\mu'"] & K_1(C_r^*(\mathbb{Z})) \end{tikzcd}$$

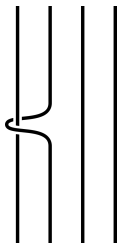
Theorem 6.5. *The Baum–Connes assembly map*

$$K_i(BB_3) \to K_i(C_r^*(B_3)), \qquad i = 0, 1$$

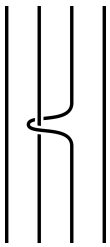
is an isomorphism rationally.

7. Appendix

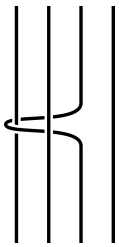
In this appendix, we give some of diagrams that illustrate the structure of pure braid groups.



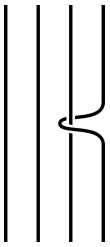
σ_1^2



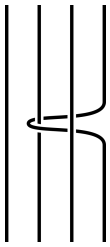
$\alpha_1 = \sigma_2^2$



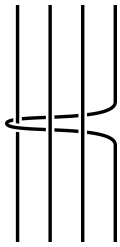
$\alpha_2 = \sigma_2\sigma_1^2\sigma_2^{-1}$



$\beta_1 = \sigma_3^2$

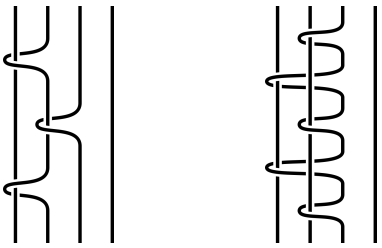


$\beta_2 = \sigma_3\sigma_2^2\sigma_3^{-1}$

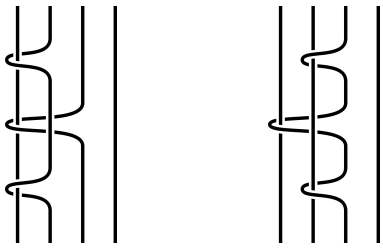


$\beta_3 = \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}$

7.1. Generators of P_4.

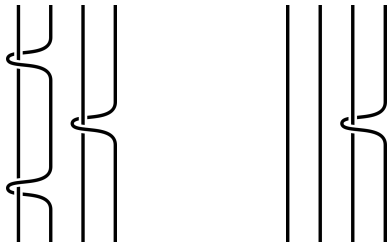


$\sigma_1^2\alpha_1\sigma_1^{-2} = (\alpha_2\alpha_1)^{-1}\alpha_1(\alpha_2\alpha_1)$

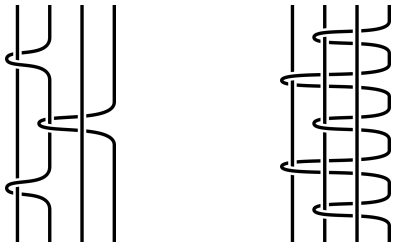


$\sigma_1^2\alpha_2\sigma_1^{-2} = \alpha_1^{-1}\alpha_2\alpha_1$

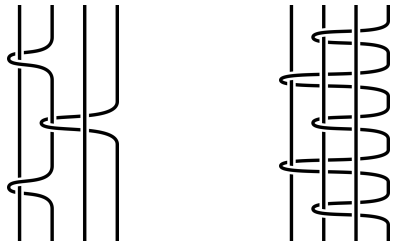
7.2. **Relations for $F(\alpha_1, \alpha_2) \rtimes F(\sigma_1^2)$.**



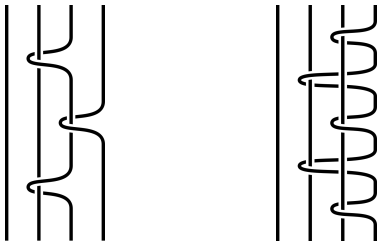
$\sigma_1^2\beta_1\sigma_1^{-2} = \beta_1$



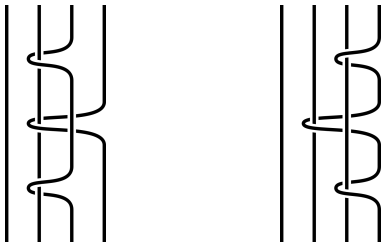
$\sigma_1^2\beta_2\sigma_1^{-2} = (\beta_3\beta_2)^{-1}\beta_2(\beta_3\beta_2)$



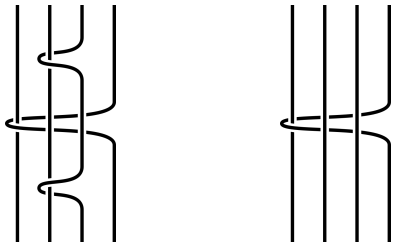
$\sigma_1^2\beta_2\sigma_1^{-2} = (\beta_3\beta_2)^{-1}\beta_2(\beta_3\beta_2)$



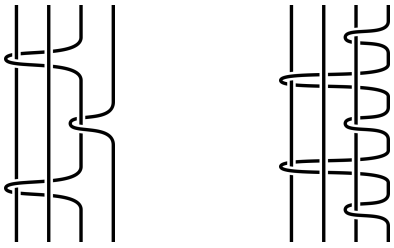
$\alpha_1\beta_1\alpha_1^{-1} = (\beta_2\beta_1)^{-1}\beta_1(\beta_2\beta_1)$



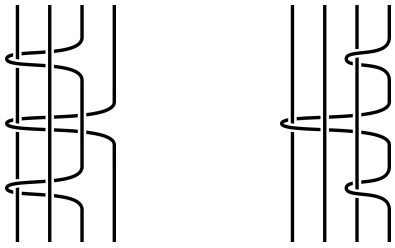
$\alpha_1\beta_2\alpha_1^{-1} = \beta_1^{-1}\beta_2\beta_1$



$\alpha_1\beta_3\alpha_1^{-1} = \beta_3$



$\alpha_2\beta_1\alpha_2^{-1} = (\beta_3\beta_1)^{-1}\beta_1(\beta_3\beta_1)$



$\alpha_2\beta_3\alpha_2^{-1} = \beta_1^{-1}\beta_3\beta_1$

$\alpha_2\beta_2\alpha_2^{-1} = (\beta_3\beta_1)^{-1}(\beta_1\beta_3)\beta_2(\beta_1\beta_3)^{-1}(\beta_3\beta_1)$

7.3. **Relations for $F(\beta_1,\beta_2,\beta_3) \rtimes \big(F(\alpha_1,\alpha_2) \rtimes F(\sigma_1^2)\big)$.**

$(\sigma_1\sigma_2\sigma_3)^4 = \sigma_1^2\alpha_2\alpha_1\beta_3\beta_2\beta_1$

7.4. **The center of P_4.** These diagrams show that the center splits off and gives us the direct product decomposition:

$$F(\beta_1,\beta_2,\beta_3) \rtimes \big(F(\alpha_1,\alpha_2) \rtimes \langle\sigma_1^2\rangle\big) = (F(\beta_1,\beta_2,\beta_3) \rtimes F(\alpha_1,\alpha_2)) \times \langle\sigma_1^2\alpha_2\alpha_1\beta_3\beta_2\beta_1\rangle.$$

References

[ACC03] A. Adem, D. Cohen, and F.R. Cohen. On representations and K-theory of the braid groups. *Math. Ann.*, 326(3):515–542, 2003.

[Arn69] V. I. Arnol'd. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[BBV99] Cédric Béguin, Hela Bettaieb, and Alain Valette. K-theory for C^*-algebras of one-relator groups. *K-Theory*, 16(3):277–298, 1999.

[BCH94] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[Bir74] J. S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[CE01] J. Chabert and S. Echterhoff. Permanence properties of the Baum-Connes conjecture. *Doc. Math.*, 6:127–183, 2001.

[CELY17] J. Cuntz, S. Echterhoff, X. Li, and G. Yu. *K-theory for group C^*-algebras and semigroup C^*-algebras*, volume 47 of *Oberwolfach Seminars*. Birkhäuser/Springer, Cham, 2017.

[Cha17] I. Chatterji. Introduction to the rapid decay property. In *Around Langlands correspondences*, volume 691 of *Contemp. Math.*, pages 53–72. Amer. Math. Soc., Providence, RI, 2017.

[Cun82] J. Cuntz. The K-groups for free products of C^*-algebras. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 81–84. Amer. Math. Soc., Providence, R.I., 1982.

[Cun83] J. Cuntz. K-theoretic amenability for discrete groups. *J. Reine Angew. Math.*, 344:180–195, 1983.

[FPV17] R. Flores, S. Pooya, and A. Valette. K-homology and K-theory for the lamplighter groups of finite groups. *Proc. Lond. Math. Soc. (3)*, 115(6):1207–1226, 2017.

[GAJV19] M. P. Gomez-Aparicio, P. Julg, and A. Valette. *The Baum-Connes conjecture: an extended survey*, pages 127–244. Springer International Publishing, Cham, 2019.

[HK01] N. Higson and G. Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001.

[Isl11] O. Isley. K-theory and k-homology for semi-direct products of $\mathbb{Z}_2$ by $\mathbb{Z}$. Thèse de doctorat, Université de Neuchâtel, 2011.

[JV84] P. Julg and A. Valette. K-theoretic amenability for $\mathrm{SL}_2(\mathbf{Q}_p)$, and the action on the associated tree. *J. Funct. Anal.*, 58(2):194–215, 1984.

[Laf02] V. Lafforgue. K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149(1):1–95, 2002.

[LOS21] X. Li, T. Omland, and J. Spielberg. C*-algebras of right LCM one-relator monoids and Artin-Tits monoids of finite type. *Comm. Math. Phys.*, 381(3):1263–1308, 2021.

[Mat02] M. Matthey. Mapping the homology of a group to the K-theory of its C^*-algebra. *Illinois J. Math.*, 46(3):953–977, 2002.

[MP80] D. Voiculescu M. Pimsner. Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras. *J. Operator Theory*, 4(1):93–118, 1980.

[MV03] G. Mislin and A. Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[Nat85] Toshikazu Natsume. On $K_*(C^*(\mathrm{SL}_2(\mathbf{Z})))$. Appendix to "$K$-theory for certain group C^*-algebras" [Acta Math. **151** (1983), no. 3-4, 209–230; MR0723010 (86f:46076)] by E. C. Lance. *J. Operator Theory*, 13(1):103–118, 1985.

[Nat88] Toshikazu Natsume. The Baum-Connes conjecture, the commutator theorem, and Rieffel projections. *C. R. Math. Rep. Acad. Sci. Canada*, 10(1):13–18, 1988.

[OO01a] H. Oyono-Oyono. Baum-Connes conjecture and group actions on trees. *K-Theory*, 24(2):115–134, 2001.

[OO01b] Hervé Oyono-Oyono. Baum-Connes conjecture and extensions. *J. Reine Angew. Math.*, 532:133–149, 2001.

[Pim86] Mihai V. Pimsner. KK-groups of crossed products by groups acting on trees. *Invent. Math.*, 86(3):603–634, 1986.

[Poo19] S. Pooya. K-theory and K-homology of finite wreath products with free groups. *Illinois J. Math.*, 63(2):317–334, 2019.

[PP83] M. G. Penington and R. J. Plymen. The Dirac operator and the principal series for complex semisimple Lie groups. *J. Funct. Anal.*, 53(3):269–286, 1983.

[PV82] M. Pimsner and D. Voiculescu. K-groups of reduced crossed products by free groups. *J. Operator Theory*, 8(1):131–156, 1982.

[PV18] S. Pooya and A. Valette. K-theory for the C^*-algebras of the solvable Baumslag-Solitar groups. *Glasg. Math. J.*, 60(2), 2018.

[Sch07] T. Schick. Finite group extensions and the Baum-Connes conjecture. *Geom. Topol.*, 11:1767–1775, 2007.

[Val84] Alain Valette. K-theory for the reduced C^*-algebra of a semisimple Lie group with real rank 1 and finite centre. *Quart. J. Math. Oxford Ser. (2)*, 35(139):341–359, 1984.

[Val85] Alain Valette. Dirac induction for semisimple Lie groups having one conjugacy class of Cartan subgroups. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 526–555. Springer, Berlin, 1985.

[Val02] A. Valette. *Introduction to the Baum-Connes conjecture*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.

[Ver98] V. V. Vershinin. Homology of braid groups and their generalizations. In *Knot theory (Warsaw, 1995)*, volume 42 of *Banach Center Publ.*, pages 421–446. Polish Acad. Sci. Inst. Math., Warsaw, 1998.

[Wal67] C. T. C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.

[Was87] A. Wassermann. Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(18):559–562, 1987.

[Wil] J. Wilson. The geometry and topology of braid groups, available at, http://www.math.lsa.umich.edu/ jchw/RTG-Braids.pdf.

[Wil07] D. P. Williams. *Crossed Products of C^*-Algebras*. Mathematical Surveys and Monographs. AMS, 2007.

Sara Azzali, Institut für Mathematik und Informatik, Universität Greifswald, Walther-Rathenau-Str. 47, 17489 Greifswald Germany, sara.azzali@uni-greifswald.de

Sarah L. Browne, The University of Kansas, Department of Mathematics, 1460 Jayhawk Blvd, Lawrence, KS, 66045. slbrowne@ku.edu

Maria Paula Gomez Aparicio, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France,
maria-paula.gomez-aparicio@universite-paris-saclay.fr

Lauren C. Ruth, Mercy College, 555 Broadway, Dobbs Ferry, NY 10522, USA LRuth@mercy.edu

Hang Wang, School of Mathematical Sciences, East China Normal University, Shanghai, China 200241 wanghang@math.ecnu.edu.cn