On an L^2-Lefschetz fixed point formula

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Outline

- L^2-Lefschetz numbers.
- Applications to orbifolds with quotient singularities and to representation theory.

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Reference

- L^2-index formula for proper cocompact group actions, arXiv:1106.4542.
- (with Bai-Ling Wang) Localized index and L^2-Lefschetz fixed formula for orbifolds. arXiv:1307.2088.

Lefschetz number

M: a compact triangulable topological space
$f : M \to M$: a continuous map.

The Lefschetz number of f is given by

$$L(f) = \sum_{i \geq 0} (-1)^i \mathrm{Tr}\left[f_{*,i} : H_i(M, \mathbb{R}) \to H_i(M, \mathbb{R}) \right].$$

Example

If $f = id$, then $L(f) = \chi(M)$.

Theorem (Lefschetz)

If $L(f) \neq 0$, then f has a fixed point.

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$$L(g) = \mathrm{Tr}(g : \ker D^+ \to \ker D^+) - \mathrm{Tr}(g : \ker D^- \to \ker D^-).$$

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Definition

For a Dirac type operator D on M. The Lefschetz number is given by the equivariant index

$$L(g, D) = \mathrm{Tr}(gP_{\ker D^+}) - \mathrm{Tr}(gP_{\ker D^-}).$$

Lefschetz fixed point formula

Theorem (Atiyah-Segal-Singer)

Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be a Dirac type operator on M (dimension n). Then

$$L(g,D) = \mathrm{Tr}(gP_{\ker D^+}) - \mathrm{Tr}(gP_{\ker D^-})$$
$$= \sum_{m\le n} \int_{M^g_{(m)}} \frac{(2\pi)^{\dim n/2}}{(2\pi i)^{m/2}} T_M\left(\frac{\hat{A}(M^g)\mathrm{ch}_G(g,E/S)}{\det^{1/2}(1-g^N\exp(-R^N))}\right)|dx_0|$$

where $M^g_{(m)}$ is the component of M^g of dimension m , S is the spinor bundle (locally exists).

Remark

When $g=e$, the Lefschetz number is the Fredholm index:

$$L(g,D) = \mathrm{ind}D.$$

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  commutes with action of G.
- A cut-off function exists for proper cocompact actions. A non-negative function $c \in C_c^\infty(X)$ is a cut-off function if

$$\int_G c(g^{-1}x)\mathrm{d}g = 1 \quad \forall x \in X.$$

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Definition (L^2-index)

$$\begin{aligned} \mathrm{L}^2\text{-ind} F :=& \dim_G \ker F^+ - \dim_G \ker F^- \\ =& \mathrm{tr}_G(P_{\ker F^+}) - \mathrm{tr}_G(P_{\ker F^-}) \in \mathbb{R}. \end{aligned}$$

L^2-index formula

Theorem (HW)

F: G-invariant elliptic operator. Then

$$L^2\text{-ind}F = \int_{TX} c(x)\mathrm{ch}(\sigma_F)\pi^*\mathrm{Td}(TX \otimes \mathbb{C}).$$

Here $\mathrm{ch}(\sigma_F)$ *and* $\mathrm{Td}(TX \otimes \mathbb{C})$ *are as defined in the Atiyah-Singer index theorem (G-equivariant).*

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  - *de Rham operator* $d + d^*$: L^2*-harmonic forms.*
  - $\bar{\partial}$*-operator: Modular forms.*
- *When G is a semi-simple real Lie group,* L^2-ind$D$ *is related to the multiplicity of discrete series of G.*

L^2-Lefschetz number

From index to L^2-index:

$$\mathrm{Tr}(P_{\ker D^{\pm}}) \quad \text{to} \quad \mathrm{Tr}_G(P_{\ker D^{\pm}});$$

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does not make sense.

Assume in addition that D is a Dirac type operator and G is discrete. The L^2-Lefschetz number is given by

$$L(g,D)_{L^2} = \mathrm{Tr}_G\left(\sum_{h\in(g)} hP_{\ker D^+}\right) - \mathrm{Tr}_G\left(\sum_{h\in(g)} hP_{\ker D^-}\right).$$

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- When X compact and G finite, we have

$$L(g, D)_{L^2} = \frac{1}{|Z_G(g)|} L(g, D)$$

- When X is not compact, $L(g, D)_{L^2}$ is a replacement for the Lefschetz number/equivariant index.

L^2-Lefschetz fixed point formula

Theorem (BW-HW)

The L^2-Lefschetz number of a G-invariant Dirac type operator D on X is given by

$$L(g,D)_{L^2} = \sum_{m\le n}\int_{X^g_{(m)}} c^{(g)}(x)\frac{\hat{A}(X^g)\mathrm{ch}_g(V)}{\det(1-ge^{R_{\mathcal{N}^g}/2\pi i})^{\frac{1}{2}}},$$

where $c^{(g)}(x)$ is a cut-off function on X^g with respect to $Z_G(g)$ action.

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Remark

$$L(e, D)_{L^2} = L^2\text{-ind}D.$$

K-theoretic interpretation

The localised (g)-trace on $L^1(G)$ is given by

$$\tau^{(g)} : L^1(G) \to \mathbb{C} : \sum_{h \in G} \alpha_h h \mapsto \sum_{h \in (g)} \alpha_h .$$

$$\tau_*^{(g)} : K_0(L^1(G)) \to \mathbb{R}.$$

Theorem (BW-HW)

Let $\mu : K_G^*(C_0(X)) \to K_*(L^1(G))$ *where* $\mathrm{Ind} D = \mu[D]$ *be the higher index. Then*

$$L(g, D)_{L^2} = \tau_*^{(g)}(\mathrm{Ind} D).$$

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$$G \times_{G_i} U := G \times U \big/ \{(gh, u) \sim (g, h^{-1}u)\}$$

where G_i is a finite subgroup and U_i is an open G_i-module.

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Inertia orbifold of $\mathfrak{X}$ (extended quotient of X by G):

$$\begin{aligned} I\mathfrak{X} =& \{(h, x) \in G \times X : hx = x\}/(ghg^{-1}, gx) \sim (h, x) \\ =& \bigsqcup_{(g) \subset G} \mathfrak{X}_{(g)}. \end{aligned}$$

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$$\mathfrak{X}_{(e)} = \mathfrak{X} = G \backslash X \qquad \mathfrak{X}_{(g)} = Z_G(g) \backslash X^g.$$

L^2-Lefschetz numbers and orbifold index

Discrete $G \curvearrowright X$ properly, cocompactly and isometrically.

D (resp. $\overline{D}$): elliptic operator on X (resp. $\mathfrak{X} = [G \backslash X]$);

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The L^2-Lefschetz numbers $L(g, D)_{L^2}$:

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$$\mathrm{ind}\overline{D} = \sum_{(g) \in C_G} L(g, D)_{L^2};$$

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- Detect the fixed points of g on X.
- If G acts freely,

$$\mathrm{ind}\overline{D} = L(e, D)_{L^2} = L^2\text{-ind}D \quad \text{(Atiyah's } L^2\text{-index).}$$

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$$[\sigma] \in K_0(C^*_r(G)) \xrightarrow{R_*} K_0(\mathcal{K}(V_\sigma)) \ni m_\sigma.$$

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S : spin K-module; $E := S \otimes W$; $\mathcal{E} = G \times_K E$ homogeneous vector bundle; Construct Dirac operators associated to σ:

$$D : L^2(M, \mathcal{E}_+) \to L^2(M, \mathcal{E}_-);$$
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Then

$$d_\sigma = L^2\text{-ind}_G D \qquad m_\sigma = \text{ind}_\Gamma \overline{D}.$$

A special case

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\downarrow \mathrm{Ind}_G & & \downarrow \mathrm{Ind}_\Gamma \\
[\sigma] \in K_0(C^*_r(G)) & \longrightarrow & K_0(C^*_r(\Gamma)) \\
\downarrow \tau^{(e)}_* & & \downarrow \rho_* \\
d_\sigma = L^2\text{-}\mathrm{ind}_G D \in \mathbb{R} & \xrightarrow{\mathrm{vol}(\Gamma\backslash G)} & \mathbb{Z} \ni \mathrm{ind}_\Gamma \overline{D} = m_\sigma
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Observation:

$$m_\sigma = \mathrm{ind}_\Gamma \overline{D} = \mathrm{vol}(\Gamma\backslash G)[L^2\text{-}\mathrm{ind}_G D] = \mathrm{vol}(\Gamma\backslash G) d_\sigma.$$

A remark on representation theory

When Γ is a discrete cocompact subgroup of G, the multiplicity $m_\sigma = \mathrm{ind}_\Gamma \overline{D}$ and formal degree $d_\sigma = L^2\text{-}\mathrm{ind}_G D$ are related as

$$\begin{aligned}\mathrm{ind}_\Gamma \overline{D} = & \sum_{(\gamma)\in C_\Gamma} L(\gamma, D)_{L^2} \\ = & \mathrm{vol}(\Gamma\backslash G)[L^2\text{-}\mathrm{ind}_G D] + \text{correction terms.}\end{aligned}$$

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- The "correction terms" are obtained from finite quotient singularities and recognised by L^2-Lefschetz numbers $L(\gamma, D)_{L^2}$, where $e \neq g \in \Gamma$.

A remark on representation theory

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- The "correction terms" are obtained from finite quotient singularities and recognised by L^2-Lefschetz numbers $L(\gamma, D)_{L^2}$, where $e \neq g \in \Gamma$.
- There is a 1-to-1 correspondence between $L(\gamma, D)_{L^2}$ and the orbital integrals on the geometric side of Selberg trace formula for the regular representation of G on $L^2(\Gamma\backslash G)$ and for some test function.