Group approximation in Cayley topology & Coarse geometry,

Part II: Fibered coarse embedding

"FCE" (CHEN, WANG, YU)

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Setting: $k \geq 1$ fixed

$\mathcal{G}_m = (G_m, S_m)$: finite k-marked grp

(· $\sim$ = "$|G_m| < \infty$"

· $=$ = "$S_m = (s_1^{(m)}, \dots, s_k^{(m)})$, $\langle S_m \cup S_m^{-1} \rangle = G_m$")

$\rightsquigarrow$ $\mathrm{Cay}\,\mathcal{G}_m$: Cayley diagram (with graph metric)

· vertices $= G_m$

· edges $g \xrightarrow{i} s_i^{(m)} g$

$\rightsquigarrow$ $\boxed{X = \bigsqcup_{m \geq 1} \mathrm{Cay}\,\mathcal{G}_m}$ $\Leftarrow$ ∞ metric space

Our philosophy

Coarse property of $X = \coprod_m \mathrm{Cay}(G_m)$ "$\longleftrightarrow$" group property of $\partial \mathrm{Cay}\{G_m\}_m$

e.g.

$A \iff$ amenable

$\xleftrightarrow[\text{FCE}]{}$ Hilb $\iff$ uniformly a-T-menable

Today

geom. (T) $\iff$ (T)

Main Thm (M.–SAKO)

$\{G_m\}_m \subseteq \mathcal{G}(k)$ finite marked grp

$\mathcal{Y}$ stable under $\begin{cases} \text{- (continuum) } \ell_p\text{-sum} \\ \quad \text{for } \exists p \in [1, \infty) \\ \text{- scaling limit} \end{cases}$

Then,

$\bigsqcup \mathrm{Cay}\, G_m \underset{\text{FCE}}{\hookrightarrow} \mathcal{Y} \iff \partial_{\mathrm{Cay}}\{G_m\}$ is uniformly a-$\mathcal{F}\mathcal{Y}$-menable

$\mathcal{Y}$: class of metric space

($\mathcal{Hilb} = \{\text{Hilb sp}\}$)

Def $X \xhookrightarrow[CE]{} \mathcal{Y}$ (Coarse Embedding)

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists Y \in \mathcal{Y}$, $\exists f : X \to Y$, $\exists \rho : \mathbb{R}_+ \nearrow \mathbb{R}_+$, $(\nearrow \infty)$

s.t. $\forall u, v \in X$

$$\rho(d_X(u,v)) \leq d_Y(f(u), f(v)) \leq d_X(u,v)$$

Fact: $\coprod_m \Gamma_m$ expanders $\begin{cases} \cdot \ \deg \Gamma_m \equiv d \\ \cdot \ |\Gamma_m| \to \infty \\ \cdot \ \lambda_1(\Gamma_m) > \exists \varepsilon > 0 \end{cases}$ (uniformly)

$\Rightarrow \coprod_m \Gamma_m \ \cancel{\xhookrightarrow[CE]{}} \ \mathcal{Hilb}$

ex of expanders

$$(F_2; \text{standard}) = (F_2; \underbrace{\begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}}) \quad F_2 \subset SL_2\mathbb{Z}$$

For p : large prime,

$$(F_2; \text{standard}) \twoheadrightarrow \mathbb{H}_p := (SL_2(\mathbb{Z}/p\mathbb{Z}); \underset{(\mathrm{mod}\ p)}{\sim\sim})$$

Fact: $\displaystyle\bigsqcup_{p:\text{ large prime}} \mathbb{H}_p$ is expanders

one example of $\Box F_2$ ("box space")

FCE "=" weak form of CE

Thm (CHEN-WANG-WANG '13)

$\square$ $G \xrightarrow[\text{FCE}]{} \mathcal{H}ilb \iff G$ a-T-menable

Thm (CHEN-WANG-YU '13)

$X \xrightarrow[\text{FCE}]{} \mathcal{H}ilb \Rightarrow$ the maximal coarse BAUM-CONNES conj true for X!

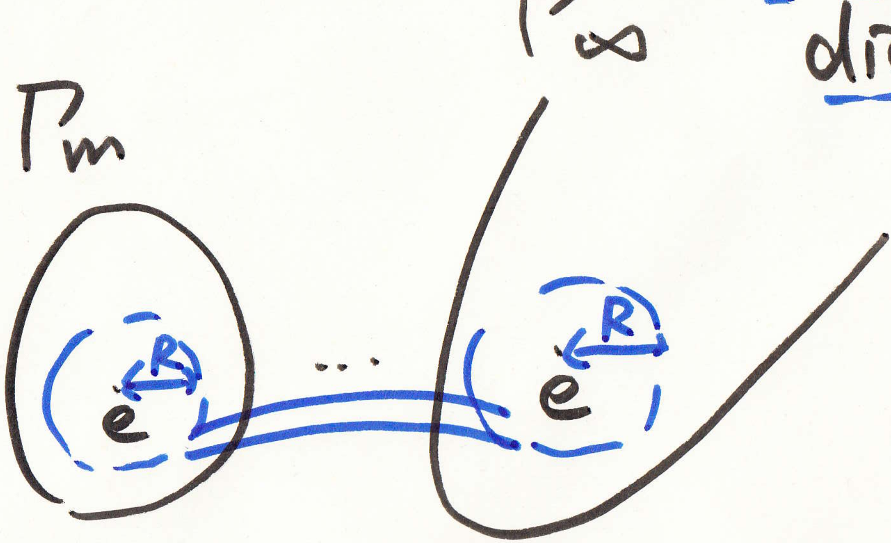
The space of k-marked groups (GRIGORCHUK)

$\boxed{\mathcal{G}(k)} := \{k\text{-mark group}\}$

★ topology?: CAYLEY topology (cpt & metrizable)

$$G_m \xrightarrow[\text{Cay}]{} G_\infty \iff \overbrace{\mathrm{Cay}\, G_m}^{=:\Gamma_m} \longrightarrow \overbrace{\mathrm{Cay}\, G_\infty}^{=:\Gamma_\infty}$$

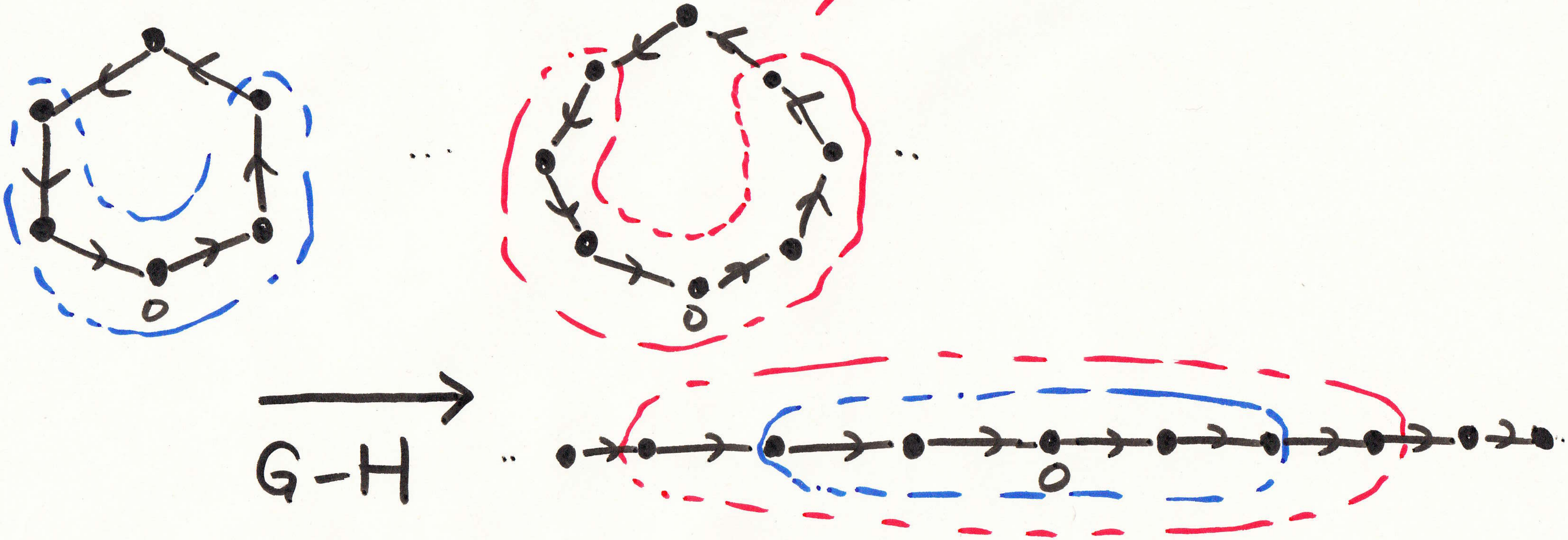
GROMOV-Hausdorff conv. of diagrams i.e.



$\Big[\forall R>0,\ \exists\, m_R \in \mathbb{N}$ st. $\forall m \geq m_R$

$$\begin{pmatrix}R\text{-balls}\\ \text{in}\\ \Gamma_m\end{pmatrix} = \begin{pmatrix}R\text{-balls}\\ \text{in}\\ \Gamma_\infty\end{pmatrix} \text{ as rooted diagrams}$$

Ex • $(\mathbb{Z}/m\mathbb{Z};\ 1) \xrightarrow[\text{Cay}]{} (\mathbb{Z};\ 1)$ in $\mathcal{G}(1)$



• $\mathbb{H}_p \left(= \left(SL_2(\mathbb{Z}/p\mathbb{Z});\ \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix} \ (\mathrm{mod}\ p)\right)\right) \xrightarrow[\text{Cay}]{} \mathbb{H}_\infty$ in $\mathcal{G}(2)$

$\parallel$

$(F_2;\ \text{standard})$

$\parallel$

$\left((F_2;\ \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix})\right)$

[in general, groups in box sp of $\mathbb{G} \xrightarrow[\text{Cay}]{} \mathbb{G}$]

$\{\mathbb{G}_m\}_{m\geq 1} \subseteq \mathcal{G}(k)$ a seq of <u>finite</u> k-marked groups

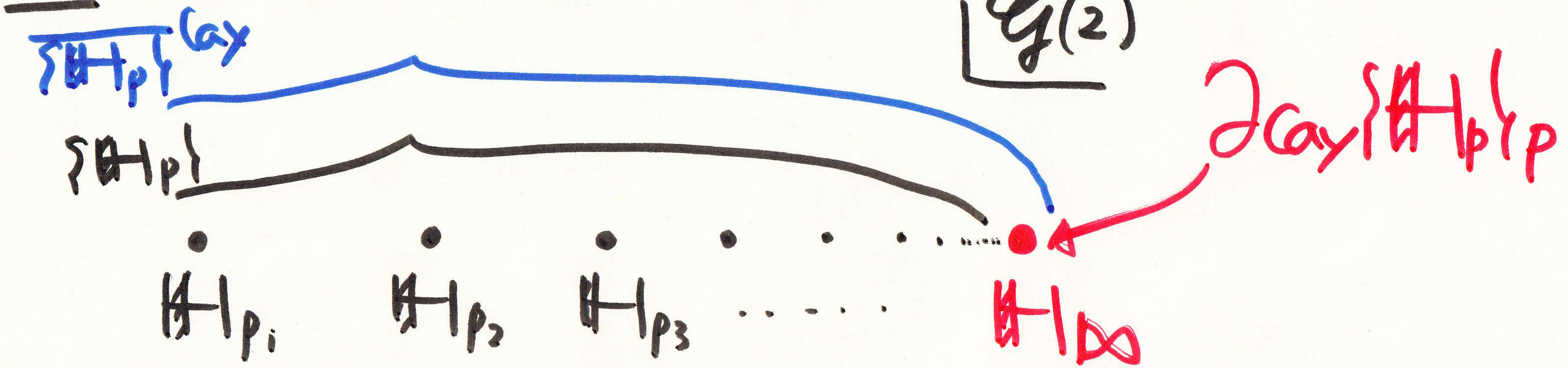
Def

$$\partial_{Cay}\{\mathbb{G}_m\}_m := \overline{\{\mathbb{G}_m\}_m}^{Cay} \setminus \{\mathbb{G}_m\}_m \quad (\text{CPT set})$$

: "CAYLEY boundary"

Ex $\partial_{Cay}\{\mathbb{H}_p\}_p = \{\mathbb{H}_\infty\} \ (= \{(F_2; \text{standard})\})$

$\mathbb{H}_\infty \in \mathcal{G}(2)$



Main Thm (M. – SAKO)

$\{G_m\}_m \subseteq \mathcal{G}(k)$ finite marked grp

$\mathcal{Y}$ stable under
- (continuum) ℓ_p-sum for $\exists p \in [1, \infty)$
- scaling limit

Then,

$\coprod \mathrm{Cay}\, G_m \xhookrightarrow[\text{FCE}]{} \mathcal{Y}$ $\iff$ $\partial_{\mathrm{Cay}}\{G_m\}$ is uniformly a-$\mathcal{F}\mathcal{Y}$-menable

Applications

① MT applies to:

* $\mathcal{Y} = \mathcal{H}ilb$ $(\flat = 2)$
* $\mathcal{Y} = \left\{ \begin{array}{l} L_p\text{-sp over} \\ \text{any measure sp} \end{array} \right\}$ for fixed $p \in [1, \infty)$ $(\flat = p)$
* For fixed $\epsilon \in [0, 1]$, $\mathcal{Y} = \mathcal{CAT}(0)_{\leq \epsilon} = \left\{ \begin{array}{l} CAT(0) \text{ sp's} \\ \text{with } \underline{\underline{\delta}} \leq \epsilon \end{array} \right\}$

IZEKI-NAYATANI inv.

Rem Hilb, trees, Hadamard mfds $\in \mathcal{CAT}(0)_{\leq 0}$

② $\rightsquigarrow$ recovers C-W-W.

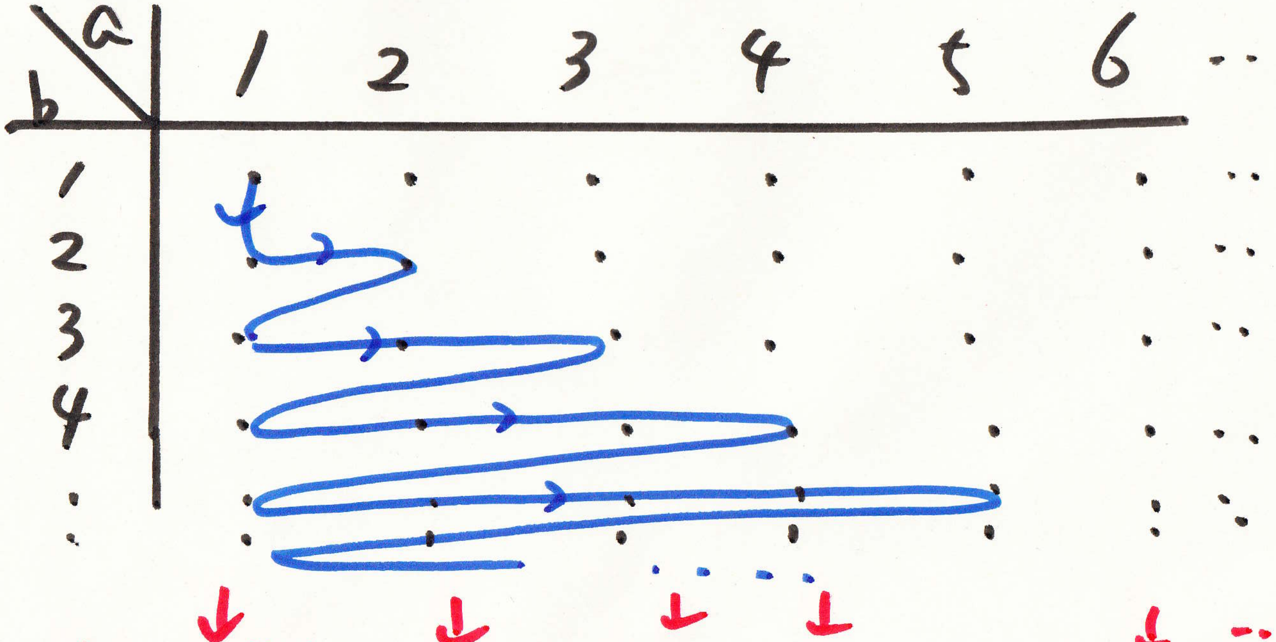
③ Example of expanders s.t. $\cancel{\xrightarrow[\text{T.CE}]{}}$ $\exists A\ \mathcal{T}(0)_{<1}$

(Recall $\mathbb{H}_p := \left(SL_2(\mathbb{Z}/p\mathbb{Z}); \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix} \pmod p\right)_p$ $\in \mathcal{G}(2)$ for p large prime)

$p_1 < p_2 < \cdots\cdots$ large prime.

m	1	2	3	4	5	6	7	8	9	10	11 ⋯
a_m	$\underbrace{1}$	$\underbrace{1 \quad 2}$		$\underbrace{1 \quad 2 \quad 3}$			$\underbrace{1 \quad 2 \quad 3 \quad 4}$				$\underbrace{1}$ ⋯
b_m	$\underbrace{1}$	$\underbrace{2 \quad 2}$		$\underbrace{3 \quad 3 \quad 3}$			$\underbrace{4 \quad 4 \quad 4 \quad 4}$				$\underbrace{5}$ ⋯

Set $\boxed{\mathbb{E}_m := \mathbb{H}_{p_{a_m}} \times \mathbb{H}_{p_{b_m}}}$ $(\in \mathcal{G}(4))$



$\left(\mathbb{H}_\infty = (F_2;\ \text{standard}) \right)$

$\mathbb{H}_{p_1}\times\mathbb{H}_\infty \quad \mathbb{H}_{p_2}\times\mathbb{H}_\infty \quad \cdots\cdots \longrightarrow \mathbb{H}_\infty\times\mathbb{H}_\infty$

$\rightsquigarrow \partial \mathrm{Cay}\{\mathbb{E}_m\} = \{\mathbb{H}_{p_i}\times\mathbb{H}_\infty \mid 1\le i\le\infty \ (p_\infty := \infty)\}$

$\rightsquigarrow$ ☆ By MT, $\bigsqcup_m \mathbb{E}_m \overset{}{\underset{\text{FiCE}}{\not\hookrightarrow}} \mathrm{CAT}(0)_{<1}$

$\left(\because \bigsqcup_p \mathbb{H}_p \right.$ expanders and so $\left. \underset{\text{CE}}{\not\hookrightarrow} \mathrm{CAT}(0)_{<1} \right)$

★ However, $\exists Y$ CAT(0) sp with $\delta = 1$

s.t. $\bigsqcup_m \mathbb{E}_m \xhookrightarrow[\text{bi-Lip}]{} Y$

(∵ $\{\mathbb{H}_p\}_p$ has large girth & [KONDO])

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④ Answer to Q: Is "$\xhookrightarrow[\text{FCE}]{} \mathcal{H}ilb$" stable under finite product?

A. **NO.**

(∵ $\bigsqcup_m \mathbb{E}_m \xhookrightarrow[\text{bi-Lip}]{} \left(\bigsqcup_p \mathbb{H}_p\right) \times \left(\bigsqcup_p \mathbb{H}_p\right)$)

谢谢