Towards Strong Banach (T) for higher rank Lie groups

Mikael de la Salle

Wuhan, 10/06/2014

Table of contents

Motivation 1: (non) coarse embeddability of expanders

Consider

- S a finite generating subset of $\mathrm{SL}(3, \mathbf{Z})$ (e.g. elementary matrices $\mathrm{Id} + e_{i,j}$, $i \neq j \in \{1, 2, 3\}$).
- X_n graph with vertices $\mathrm{SL}(3, \mathbf{Z}/nZ)$ and an edge between a and b is $a^{-1}b \in S \mod n$.

Then (Kazhdan-Margulis) $(X_n)_{n \geq 0}$ is an *expander*.

Question

What are the Banach spaces X that contain coarsely $(X_n)_{n \geq 1}$?

Motivation 1: (non) coarse embeddability of expanders

Consider

- S a finite generating subset of $\mathrm{SL}(3, \mathbf{Z})$ (e.g. elementary matrices $\mathrm{Id} + e_{i,j}$, $i \neq j \in \{1, 2, 3\}$).
- X_n graph with vertices $\mathrm{SL}(3, \mathbf{Z}/nZ)$ and an edge between a and b is $a^{-1}b \in S \mod n$.

Then (Kazhdan-Margulis) $(X_n)_{n \geq 0}$ is an *expander*.

Question

What are the Banach spaces X that contain coarsely $(X_n)_{n \geq 1}$?

Recall this means there exists $\rho \colon \mathbf{N} \to \mathbf{R}_+$ increasing with $\lim_n \rho(n) = \infty$ and 1-Lipschitz functions $f_n \colon X_n \to X$ such that for all n

$$\rho(d_n(x, y)) \leq \|f_n(x) - f_n(y)\|_X \text{ for all } x, y \in X_n.$$

Question

What are the Banach spaces that contain coarsely $(X_n)_n$?

- ℓ^∞ (obvious) and hence X with trivial cotype (Maurey-Pisier).

Question

What are the Banach spaces that contain coarsely $(X_n)_n$?

- ℓ^∞ (obvious) and hence X with trivial cotype (Maurey-Pisier).
- **Not** Hilbert spaces (Gromov) and **Not** θ-Hilbertian spaces.

Question

What are the Banach spaces that contain coarsely $(X_n)_n$?

- ℓ^∞ (obvious) and hence X with trivial cotype (Maurey-Pisier).
- **Not** Hilbert spaces (Gromov) and **Not** θ-Hilbertian spaces.
- for $\mathrm{SL}(3, \mathbf{Z})$ replaced by a lattice in $\mathrm{SL}(3, \mathbf{Q}_p)$: **Not** in a space with type > 1 (Lafforgue, see also Liao).
  New results (for $\mathrm{SL}(3, \mathbf{Z}/n\mathbf{Z})$):

Question

What are the Banach spaces that contain coarsely $(X_n)_n$?

- ℓ^∞ (obvious) and hence X with trivial cotype (Maurey-Pisier).
- **Not** Hilbert spaces (Gromov) and **Not** θ-Hilbertian spaces.
- for $\mathrm{SL}(3, \mathbf{Z})$ replaced by a lattice in $\mathrm{SL}(3, \mathbf{Q}_p)$: **Not** in a space with type > 1 (Lafforgue, see also Liao).
  New results (for $\mathrm{SL}(3, \mathbf{Z}/n\mathbf{Z})$):
- **Not** a space X_0 for which $\exists \beta < 1/4$, C s.t. $d_n(X_0) \leq Cn^\beta$ where

$$d_n(X_0) = \sup\{d(Y, \ell^2_n), Y \subset X_0 \text{ of dimension } n\}.$$

Question

What are the Banach spaces that contain coarsely $(X_n)_n$?

- ℓ^∞ (obvious) and hence X with trivial cotype (Maurey-Pisier).
- **Not** Hilbert spaces (Gromov) and **Not** θ-Hilbertian spaces.
- for $\mathrm{SL}(3, \mathbf{Z})$ replaced by a lattice in $\mathrm{SL}(3, \mathbf{Q}_p)$: **Not** in a space with type > 1 (Lafforgue, see also Liao).
  New results (for $\mathrm{SL}(3, \mathbf{Z}/n\mathbf{Z})$):
- **Not** a space X_0 for which $\exists \beta < 1/4$, C s.t. $d_n(X_0) \leq Cn^\beta$ where

$$d_n(X_0) = \sup\{d(Y, \ell^2_n), Y \subset X_0 \text{ of dimension } n\}.$$

- **Not** (a subquotient of) $X_\theta = [X_0, X_1]_\theta$ with $\theta < 1$ and X_1 arbitrary.

Table of contents

X Banach space, $Aff(X) = \{$ affine isometries of $X\}$.

Definition

A (locally compact) group G has (F_X) if every (continuous) $\sigma\colon G \to Aff(X)$ has a fixed point ($=x \in X$ s.t. $\sigma(g)x = x\ \forall g \in G$).

X Banach space, $Aff(X) = \{$ affine isometries of $X\}$.

Definition

A (locally compact) group G has (F_X) if every (continuous) $\sigma: G \to Aff(X)$ has a fixed point ($=x \in X$ s.t. $\sigma(g)x = x \; \forall g \in G$).

Example: G has (F_{ℓ^2}) $\Leftrightarrow$ G has (T) $\Leftrightarrow$ G has (F_{L^p}) for some (or all) $1 \leq p \leq 2$ (Delorme-Guichardet+Bader-Furman-Gelander-Monod).

X Banach space, $Aff(X) = \{$ affine isometries of $X\}$.

Definition

A (locally compact) group G has (F_X) if every (continuous) $\sigma\colon G \to Aff(X)$ has a fixed point ($=x \in X$ s.t. $\sigma(g)x = x\ \forall g \in G$).

Example: G has $(\mathrm{F}_{\ell^2}) \Leftrightarrow G$ has (T) $\Leftrightarrow G$ has (F_{L^p}) for some (or all) $1 \leq p \leq 2$ (Delorme-Guichardet+Bader-Furman-Gelander-Monod).

If Γ is a hyperbolic group, $\exists p < \infty$ such that $\Gamma \notin (\mathrm{F}_{\ell^p})$ (Bourdon-Pajot).

X Banach space, $Aff(X) = \{$ affine isometries of $X\}$.

Definition

A (locally compact) group G has (F_X) if every (continuous) $\sigma \colon G \to Aff(X)$ has a fixed point ($=x \in X$ s.t. $\sigma(g)x = x \; \forall g \in G$).

Example: G has $(\mathrm{F}_{\ell^2}) \Leftrightarrow G$ has (T) $\Leftrightarrow G$ has (F_{L^p}) for some (or all) $1 \leq p \leq 2$ (Delorme-Guichardet+Bader-Furman-Gelander-Monod).

If Γ is a hyperbolic group, $\exists p < \infty$ such that $\Gamma \notin (\mathrm{F}_{\ell^p})$ (Bourdon-Pajot).

Conjecture (BFGM)

Higher rank alebraic groups and their lattices have (F_X) for every superreflexive X.

X Banach space, $Aff(X) = \{$ affine isometries of $X\}$.

Definition

A (locally compact) group G has (F_X) if every (continuous) $\sigma\colon G \to Aff(X)$ has a fixed point (=$x \in X$ s.t. $\sigma(g)x = x \ \forall g \in G$).

Example: G has $(\mathrm{F}_{\ell^2}) \Leftrightarrow G$ has (T) $\Leftrightarrow G$ has (F_{L^p}) for some (or all) $1 \leq p \leq 2$ (Delorme-Guichardet+Bader-Furman-Gelander-Monod).

If Γ is a hyperbolic group, $\exists p < \infty$ such that $\Gamma \notin (\mathrm{F}_{\ell^p})$ (Bourdon-Pajot).

Conjecture (BFGM)

Higher rank alebraic groups and their lattices have (F_X) for every superreflexive X.

(Lafforgue, Liao) the conjecture holds for non-archimedean fields (eg $\mathbf{Q}_p$). Main open case: $\mathrm{SL}(3, \mathbf{R})$, $\mathrm{SL}(3, \mathbf{Z})$.

X Banach space, $Aff(X) = \{$ affine isometries of $X\}$.

Definition

A (locally compact) group G has (F_X) if every (continuous) $\sigma\colon G \to Aff(X)$ has a fixed point ($=x \in X$ s.t. $\sigma(g)x = x\ \forall g \in G$).

Example: G has $(\mathrm{F}_{\ell^2}) \Leftrightarrow G$ has (T) $\Leftrightarrow G$ has (F_{L^p}) for some (or all) $1 \leq p \leq 2$ (Delorme-Guichardet+Bader-Furman-Gelander-Monod).

If Γ is a hyperbolic group, $\exists p < \infty$ such that $\Gamma \notin (\mathrm{F}_{\ell^p})$ (Bourdon-Pajot).

Conjecture (BFGM)

Higher rank alebraic groups and their lattices have (F_X) for every superreflexive X.

(Lafforgue, Liao) the conjecture holds for non-archimedean fields (eg $\mathbf{Q}_p$).
New result: for $\mathrm{SL}(3,\mathbf{R})$ and $\mathrm{SL}(3,\mathbf{Z})$, the conjecture holds for the Banach spaces X_θ as previously.

Table of contents

How are these two questions related?

How are these two questions related? Through Strong Banach (T)!

How are these two questions related? Through Strong Banach (T)!

Definition of Banach (T) (Lafforgue)

G has (T_X) if there exists m_n (compactly supported symmetric) probability measures on G such that for every (continuous) linear isometric representation of G on X, $\pi(m_n)$ converges **in the norm topology of** $B(X)$ to a projection on $X^\pi = \{x \in X, \pi(g)x = x \forall g \in G\}$.

(Lafforgue) Γ has ($\mathrm{T}_{\ell^2(\mathbf{N};X)}$) $\Rightarrow$ the expanders coming from Γ do not coarsely embed in X.

How are these two questions related? Through Strong Banach (T)!

Definition of Banach (T) (Lafforgue)

G has (T_X) if there exists m_n (compactly supported symmetric) probability measures on G such that for every (continuous) linear isometric representation of G on X, $\pi(m_n)$ converges **in the norm topology of** $B(X)$ to a projection on $X^\pi = \{x \in X, \pi(g)x = x \forall g \in G\}$.

(Lafforgue) Γ has $(\mathrm{T}_{\ell^2(\mathbf{N};X)})$ $\Rightarrow$ the expanders coming from Γ do not coarsely embed in X.

To define a stronger form, take $\ell\colon G \to \mathbf{N}$ the word-lenfth function associated to a compact generating set in G.

How are these two questions related? Through Strong Banach (T)!

Definition of Banach (T) (Lafforgue)

G has (T_X) if there exists m_n (compactly supported symmetric) probability measures on G such that for every (continuous) linear isometric representation of G on X, $\pi(m_n)$ converges **in the norm topology of** $B(X)$ to a projection on $X^\pi = \{x \in X, \pi(g)x = x \forall g \in G\}$.

(Lafforgue) Γ has $(\mathrm{T}_{\ell^2(\mathbf{N};X)})$ $\Rightarrow$ the expanders coming from Γ do not coarsely embed in X.

To define a stronger form, take $\ell \colon G \to \mathbf{N}$ the word-lenfth function associated to a compact generating set in G.

Definition of Strong Banach (T) (Lafforgue)

G has Strong (T_X) if there exists m_n (c.s. symm.) prob. measures on G and $s > 0$ such that for every (continuous) linear representation of G on X with $\sup_g e^{-s\ell(g)} < \infty$, $\pi(m_n)$ converges **in the norm topology of** $B(X)$ to a projection on X^π.

Properties of (Strong) Banach (T) (due to Lafforgue)

- Γ has $(\mathrm{T}_{\ell^2(X)}) \Rightarrow$ expanders not coarsely embed in X.

Properties of (Strong) Banach (T) (due to Lafforgue)

- Γ has $(\mathrm{T}_{\ell^2(X)}) \Rightarrow$ expanders not coarsely embed in X.
- G has Strong $(\mathrm{T}_{X \oplus \mathbf{C}}) \Rightarrow G$ has (F_X).

Properties of (Strong) Banach (T) (due to Lafforgue)

- Γ has $(\mathrm{T}_{\ell^2(X)}) \Rightarrow$ expanders not coarsely embed in X.
- G has Strong $(\mathrm{T}_{X\oplus\mathbf{C}}) \Rightarrow G$ has (F_X).
- If $\Gamma \subset G$ (cocompact) lattice. G has (Strong) $(\mathrm{T}_{L^2(G/\Gamma;X)}) \Rightarrow \Gamma$ has (Strong) (T_X).

Properties of (Strong) Banach (T) (due to Lafforgue)

- Γ has $(\mathrm{T}_{\ell^2(X)}) \Rightarrow$ expanders not coarsely embed in X.
- G has Strong $(\mathrm{T}_{X\oplus\mathbf{C}}) \Rightarrow G$ has (F_X).
- If $\Gamma \subset G$ (cocompact) lattice. G has (Strong) $(\mathrm{T}_{L^2(G/\Gamma;X)}) \Rightarrow \Gamma$ has (Strong) (T_X).

Examples:

- Hyperbolic groups do not have Strong (T_{ℓ^2}).
- $\mathrm{SL}(3,\mathbf{R})$ has Strong (T_H) for Hilbert spaces H.
- $\mathrm{SL}(3,\mathbf{Q}_p)$ has Strong (T_X) for every X with type > 1.
- (Liao) same result for G higher rank group on a nonarchimedean field.

Main results

Theorem (dlS)

$\mathrm{SL}(3,\mathbf{R})$ has Strong (T_X) for every $X \in \mathcal{E}_4$.

Theorem (de Laat–dlS)

G connected simple Lie group of $\mathrm{rank}_{\mathbf{R}} \geq 2$. $\mathrm{SL}(3,\mathbf{R})$ has Strong (T_X) for every $X \in \mathcal{E}_{10}$.

where for $2 < r < \infty$, $\mathcal{E}_r$ is the smallest set of Banach spaces such that

- $d_n(X_0) = O(n^\beta)$ for some $\beta < 1/r \Rightarrow X_0 \in \mathcal{E}_r$.
- X is isomorphic to (a subquotient) of $[X_0, X_1]_\theta$ with $\theta < 1 \Rightarrow X_0 \in \mathcal{E}_r$.

The family of Banach spaces $\mathcal{E}_r$

What is known about d_n :

- $d_n(X) \leq n^{1/2}$ always, with equality if $X = \ell^1_n$ and hence if X has trivial type.
- (Milman-Wolfson) if X has type > 1, $d_n(X) = o(n^{1/2})$.
- (Koenig–Retherford–Tomczak-Jaegermann) if X has type p and cotype q,

$$d_n(X) \leq Cn^{1/p-1/q}.$$

The family of Banach spaces $\mathcal{E}_r$

What is known about $d_n\mathcal{E}_r$:

- $d_n(X) \leq n^{1/2}$ always, with equality if $X = \ell^1_n$ and hence if X has trivial type.
- (Milman-Wolfson) if X has type > 1, $d_n(X) = o(n^{1/2})$.
- (Koenig–Retherford–Tomczak-Jaegermann) if X has type p and cotype q,
$$d_n(X) \leq Cn^{1/p-1/q}.$$
- every space in $\mathcal{E}_r$ has type > 1.
- (Pisier-Xu) for every $r < \infty$, $\mathcal{E}_r$ contains non-superreflexive spaces.

The family of Banach spaces $\mathcal{E}_r$

What is known about $d_n\mathcal{E}_r$:

- $d_n(X) \leq n^{1/2}$ always, with equality if $X = \ell^1_n$ and hence if X has trivial type.
- (Milman-Wolfson) if X has type > 1, $d_n(X) = o(n^{1/2})$.
- (Koenig–Retherford–Tomczak-Jaegermann) if X has type p and cotype q,

$$d_n(X) \leq Cn^{1/p-1/q}.$$

- every space in $\mathcal{E}_r$ has type > 1.
- (Pisier-Xu) for every $r < \infty$, $\mathcal{E}_r$ contains non-superreflexive spaces.

Open questions:

Does $\mathcal{E}_r$ depend on r?

Does $\cup_{r>2}\mathcal{E}_r$ contain all spaces with type > 1? All superreflexive spaces?

Table of contents

Banach-space representations of $U = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(2) \end{pmatrix} \subset K = \mathrm{SO}(3)$

If $\pi\colon \mathrm{SO}(3) \to \mathrm{GL}(X)$ is an isometric representation of $\mathrm{SO}(3)$.
A *U-biinvariant coefficient of* π is a map

$$c(k) = \langle \pi(k)\xi, \eta \rangle$$

for $\xi \in X$, $\eta \in X^*$ U-invariant unit vectors.

Banach-space representations of $U = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(2) \end{pmatrix} \subset K = \mathrm{SO}(3)$

If $\pi \colon \mathrm{SO}(3) \to \mathrm{GL}(X)$ is an isometric representation of $\mathrm{SO}(3)$.
A *U-biinvariant coefficient of* π is a map

$$c(k) = \langle \pi(k)\xi, \eta \rangle$$

for $\xi \in X$, $\eta \in X^*$ U-invariant unit vectors.
Goal: find $C > 0$, $\alpha > 0$ such that

$$|c(k_0) - c(k_\delta)| \leq C|\delta|^s \text{ where } k_\theta = \begin{pmatrix} R_{\pi/2+\delta} & 0 \\ 0 & 1 \end{pmatrix}. \qquad (1)$$

Banach-space representations of $U = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(2) \end{pmatrix} \subset K = \mathrm{SO}(3)$

If $\pi\colon \mathrm{SO}(3) \to \mathrm{GL}(X)$ is an isometric representation of $\mathrm{SO}(3)$.
A *U-biinvariant coefficient of* π is a map

$$c(k) = \langle \pi(k)\xi, \eta \rangle$$

for $\xi \in X$, $\eta \in X^*$ U-invariant unit vectors.
Goal: find $C > 0$, $\alpha > 0$ such that

$$|c(k_0) - c(k_\delta)| \leq C|\delta|^s \text{ where } k_\theta = \begin{pmatrix} R_{\pi/2+\delta} & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

(1) for every $\pi\colon \mathrm{SO}(3) \to \mathrm{GL}(X)$ implies that $\mathrm{SL}(3, \mathbf{R})$ has Strong (T_X).
Lafforgue proved (1) for $X = \ell^2$ with $s = 1/2$.

Reduction

Enough to prove

$$\|T_0 - T_\delta\|_{L^2(K;X)\to L^2(K;X)}$$

where $T_\delta f(k) = \iint_{U\times U} f(kuk_\delta u')du du'$.
Knowing

$$\|T_0 - T_\delta\|_{L^2(K)\to L^2(K)} \leq C\sqrt{|\delta|}.$$

A general question

Let $A\colon L^2(\Omega) \to L^2(\Omega)$ an operator of small norm ε. Under what condition on X can we say $\|A_X\| := \|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} << 1$?

A general question

Let $A\colon L^2(\Omega) \to L^2(\Omega)$ an operator of small norm ε. Under what condition on X can we say $\|A_X\| := \|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} << 1$?

In general (Kwapien), $\|A_X\| < \infty \Rightarrow X \simeq$ a Hilbert space!

A general question

Let $A\colon L^2(\Omega) \to L^2(\Omega)$ an operator of small norm ε. Under what condition on X can we say $\|A_X\| := \|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} << 1$?

In general (Kwapien), $\|A_X\| < \infty \Rightarrow X \simeq$ a Hilbert space!
$\longrightarrow$ Assume moreover $\|A_{\ell^\infty}\| = 1$ (implies $\|A_X\| \leq 1$ for all X).

A general question

Let $A\colon L^2(\Omega) \to L^2(\Omega)$ an operator of small norm ε. Under what condition on X can we say $\|A_X\| := \|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} << 1$?

In general (Kwapien), $\|A_X\| < \infty \Rightarrow X \simeq$ a Hilbert space!
$\longrightarrow$ Assume moreover $\|A_{\ell^\infty}\| = 1$ (implies $\|A_X\| \leq 1$ for all X).

In general (Pisier) the existence of $\theta > 0$ such that $\|A_X\| \leq C\varepsilon^\theta$ implies X superreflexive (θ-Hilbertian).

A general question

Let $A\colon L^2(\Omega) \to L^2(\Omega)$ an operator of small norm ε. Under what condition on X can we say $\|A_X\| := \|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} << 1$?

In general (Kwapien), $\|A_X\| < \infty \Rightarrow X \simeq$ a Hilbert space!
$\longrightarrow$ Assume moreover $\|A_{\ell^\infty}\| = 1$ (implies $\|A_X\| \leq 1$ for all X).

In general (Pisier) the existence of $\theta > 0$ such that $\|A_X\| \leq C\varepsilon^\theta$ implies X superreflexive (θ-Hilbertian).
Conclusion: we have to find some specific properties of the operators $T_0 - T_\delta$.

A general question

Let $A\colon L^2(\Omega) \to L^2(\Omega)$ an operator of small norm ε. Under what condition on X can we say $\|A_X\| := \|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} << 1$?

In general (Kwapien), $\|A_X\| < \infty \Rightarrow X \simeq$ a Hilbert space!
$\longrightarrow$ Assume moreover $\|A_{\ell^\infty}\| = 1$ (implies $\|A_X\| \leq 1$ for all X).

In general (Pisier) the existence of $\theta > 0$ such that $\|A_X\| \leq C\varepsilon^\theta$ implies X superreflexive (θ-Hilbertian).
Conclusion: we have to find some specific properties of the operators $T_0 - T_\delta$.
This property is p-summability!

Schatten classes and geometry of Banach spaces

The inequality (1) for $X \in \mathcal{E}_4$ follows from two facts :

Lemma (Lafforgue–dlS 2011)

For every $p > 4$ there is a constant C_p such that

$$\|T_0 - T_\delta\|_{S^p(L^2K)} \leq C_p|\delta|^{1/2-2/p}.$$

Proposition (Pietsch, Pisier, ? dlS ?)

If $A \in S^p(L^2\Omega)$ and $d_n(X) \leq Cn^\beta$ for $\beta < 1/p$ then

$$\|A\|_{L^2(\Omega;X)\to L^2(\Omega;X)} \leq C'\|A\|_{S^p}.$$

Table of contents

Questions/ Open problems

- Does Strong (T) pass to non cocompact lattices? What about $\mathrm{SL}(3, \mathbf{Z})$?

Questions/ Open problems

- Does Strong (T) pass to non cocompact lattices? What about $\mathrm{SL}(3, \mathbf{Z})$?
- Does $\mathcal{E}_r$ depend on r?

Questions/ Open problems

- Does Strong (T) pass to non cocompact lattices? What about $\mathrm{SL}(3,\mathbf{Z})$?
- Does $\mathcal{E}_r$ depend on r?
- Is $\cup_{r>2}\mathcal{E}_r$ equal to all spaces with type > 1. Does it contain all superreflexive spaces? ($\Leftrightarrow$ old open problem from the 1970's).

Questions/ Open problems

- Does Strong (T) pass to non cocompact lattices? What about $\mathrm{SL}(3, \mathbf{Z})$?
- Does $\mathcal{E}_r$ depend on r?
- Is $\cup_{r>2}\mathcal{E}_r$ equal to all spaces with type > 1. Does it contain all superreflexive spaces? ($\Leftrightarrow$ old open problem from the 1970's).
- How to prove $\|T_O - T_\delta\|_{L^2(K;X)\to L^2(K;X)} \leq C|\delta|^s$ for every X of type > 1?

Questions/ Open problems

- Does Strong (T) pass to non cocompact lattices? What about $\mathrm{SL}(3,\mathbf{Z})$?
- Does $\mathcal{E}_r$ depend on r?
- Is $\cup_{r>2}\mathcal{E}_r$ equal to all spaces with type >1. Does it contain all superreflexive spaces? ($\Leftrightarrow$ old open problem from the 1970's).
- How to prove $\|T_O - T_\delta\|_{L^2(K;X)\to L^2(K;X)} \leq C|\delta|^s$ for every X of type >1?

Thank you!