Riesz transforms on group von Neumann algebras

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Classical Riesz transform

$f \in L_2(\mathbb{R}^n)$,

$$R = \partial \triangle^{-\frac{1}{2}}.$$

with

$$\partial = (\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_j}, \cdots \frac{\partial}{\partial x_n}), \triangle = -\frac{\partial^2}{\partial x^2} = \sum_j \frac{\partial^2}{\partial x_j^2}$$

$$Rf = (R_1 f, R_2 f, \cdots R_n f)$$

with $R_i = \partial_i \Delta^{-\frac{1}{2}}$ the $i - th$ Riesz transform,

$$\widehat{R_i f}(\xi) = c \frac{\xi_i}{|\xi|} \widehat{f}(\xi), \xi \in \mathbb{R}^n.$$

(Riesz; Stein/Meyer-Bakry/Pisier/Gundy/Varopoulos and many others)

$$\|(\sum_j |R_j f|^2)^{\frac{1}{2}}\|_p \simeq \|f\|_p, 1 < p < \infty,$$

$$\|\partial f\|_p \simeq \|\Delta^{\frac{1}{2}} f\|_p, 1 < p < \infty,$$

Carré du Champ—P. A. Meyer's Gradient form

$\Delta = -\partial^2 x$ on $(\mathbb{R}, dx)$; Chain rule:

$$-\Delta(f_1 f_2) + (\Delta f_1) f_2 + f_1 \Delta f_2 = 2\partial f_1 \partial f_2.$$

Given a generator of a Markov semigroup L (e.g. an elliptic operator), set

$$\begin{aligned} 2\Gamma(f_1, f_2) &= -L(f_1^* f_2) + L(f_1^*) f_2 + f_1^* L(f_2); \\ |R_L(f)|^2 &= \Gamma(L^{-\frac{1}{2}} f, L^{-\frac{1}{2}} f) \end{aligned}$$

Semiclassical Riesz transform —Markov Semigroups of Operators

$(\mathcal{M}, \mu)$: Sigma finite measure space,
$(S_t)_{t\geq 0}$: a semigroup of operators on $L_\infty(\mathcal{M})$
We say $(S_t)_t$ is Markov, if

- S_t are contractions on $L_\infty(\mathcal{M}, \mu)$.
- S_t are symmetric i.e. $\langle S_t f, g\rangle = \langle f, S_t g\rangle$ for $f, g \in L^1(\mathcal{M}) \cap L_\infty(\mathcal{M})$.
- $S_t(1) = 1$
- $S_t(f) \to f$ in the w^* topology for $f \in L_\infty(\mathcal{M})$.

Infinitesimal generator: $L = -\frac{\partial S_t}{\partial t}|_{t=0}; S_t = e^{-tL}$.
More general case: $L_\infty(\mathcal{M})$ replaced by semi finite von Neuman algebras.

Abstract theories by E. Stein, Cowling, Mcintoch...., Junge/Xu, Le Merdy-Junge-Xu.

Examples, P. A. Meyer's Gradient form

- $L = \Delta$: Laplace-Betrami operator
  $\Gamma(f, f) = |\nabla f|^2$.
- (M, dx): complete Riemannian manifold.
  $Lf(x) = \sum_{i,j} a_{ij}(x)\partial_i\partial_j f + \sum_i g_i(x)\partial_i f$.
  $\Gamma(f, f) = \sum_{i,j} a_{i,j}\partial_i f \partial_j f$.
- $L = \Delta^{\frac{1}{2}}$, $S_t = e^{-tL}$ on $\mathbb{R}^n$.
  $\Gamma(f, f) = \int_0^\infty |\nabla S_t f|^2 + |\partial_t S_t f|^2 dt$.
- $G = \mathbb{F}_2$: the free group of two generaters.
  $L : \lambda_g \mapsto |g|\lambda_g$, with $|g|$ the word length of $g \in \mathbb{F}_2$.
  $\Gamma(\lambda_g, \lambda_h) = \frac{|g|+|h|-|g^{-1}h|}{2}\lambda_{g^{-1}h}$.

P. A. Meyer's question When do we have

$$\|\Gamma(f, f)\|^2_{L_{\frac{p}{2}}} \simeq \|L^{\frac{1}{2}} f\|_{L_p}? \qquad (*)$$

Semiclassical Riesz transform—P.A. Meyer's question

Theorem (D.Bakry for diffusion S_t 1986;) Assume L generates a diffusion Markov semigroup (on commutative L_p spaces) satisfying $\Gamma_2 \geq 0$, then

$$\|\Gamma(f,f)\|^2_{L_{\frac{p}{2}}} \simeq \|L^{\frac{1}{2}}f\|_{L_p}$$

holds for any $1 < p < \infty$.

$\Gamma(f,f) \geq 0$ iff $|S_t f|^2 \leq S_t|f|^2$ for $S_t = e^{-tL}$.

$\Gamma_2 \geq 0$ means

$$\Gamma_2(f,f) = -L\Gamma(f,f) + \Gamma(Lf,f) + \Gamma(f,Lf) \geq 0.$$

P. A. Meyer, D. Bakry, M. Emery, X. D. Li, F. Baudoin-N.

Garofalo, etc. ($\Gamma_2 \geq 0 \Leftrightarrow CD(0,\infty)$ criterion
$\Leftrightarrow 2S_t|S_t f|^2 \leq S_{2t}|f|^2 + |S_{2t}f|^2$.)

Fractional power of Δ

P. A. Meyer's inequality (*) Fails for $L = \Delta^{\frac{1}{2}}$ on $\mathbb{R}^n$.

For $p \leq \frac{2n}{n+1}$, and any Schwarz function f on $\mathbb{R}^n$,

$$\|\Gamma(f,f)\|_{L^{\frac{p}{2}}} = \infty$$

while

$$\|L^{\frac{1}{2}} f\|_{L^p} < \infty.$$

Noncommutative extension

Theorem (Junge;Junge/M noncommutative S_t 2010;)
Assume L generates a noncommutative Markov semigroup on a semi finite von Neumann algebra $\mathcal{M}$ satisfying $\Gamma_2 \geq 0$, then

$$\|L^{\frac{1}{2}} f\|_{L_p(\mathcal{M})} \leq c_p \max\{\|\Gamma(f,f)\|^2_{L_{\frac{p}{2}}(\mathcal{M})}, \|\Gamma(f^*,f^*)\|^2_{L_{\frac{p}{2}}(\mathcal{M})}\}$$

for any $2 \leq p < \infty$.

$$\|f\|_{L_p(\mathcal{M})} = (\tau |f|^p)^{\frac{1}{p}}.$$

Application to M. Rieffel's quantumn metric spaces;..

Question: Can we get an equivalence for all $1 < p < \infty$?

Riesz transforms via cocycles

G: discrete (abelian) group
(b, α, H): cocycle of group actions on Hilbert space H,
i.e. $b : G \mapsto H, \alpha : G \mapsto Aut(H), \alpha_g b(g^{-1}h) = b(h) - b(g)$
v_k: orthonormal basis of H.

$L : \lambda_g \mapsto \|b(g)\|^2 \lambda_g$ generates a Markov semigroup on $L_\infty(\widehat{\mathbb{G}})$.

$$\begin{aligned} \Gamma(\lambda_g, \lambda_h) &= \frac{\|b(g)\|^2 + \|b(h)\|^2 - \|b(g^{-1}h)\|^2}{2} \lambda_{g^{-1}h} \\ &= \langle b(g), b(h) \rangle \lambda_{g^{-1}h} \end{aligned}$$

$|R_L(f)|^2 = \Gamma(L^{-\frac{1}{2}} f, L^{-\frac{1}{2}} f) = \sum_k |R_k(f)|^2$ with

$$R_k : \lambda_g \to \frac{\langle b(g), v_k \rangle}{\|b(g)\|} \lambda_g .$$

Semiclassical Riesz transform—Examples

- $G = \mathbb{R}^n$
  $(b, \alpha, H) = (id, id, \mathbb{R}^n)$.
  $L = \Delta$: $e^{i\langle \xi, \cdot \rangle} \to -|\xi|^2 e^{i\langle \xi, \cdot \rangle}$.
  $R_k = \partial_k \Delta^{-\frac{1}{2}}$.
- Let $G = \mathbb{F}_2$: the free group generated by $\{h_1, h_2\}$.
  $\Lambda = \{\delta_g - \delta_{g^-}; g \in G\} \subset \ell_2(G)$
  $H = \ell_2(\Lambda)$. $b : g \to \delta_g - \delta_e \in H$.
  $\|b(g)\|^2 = |g|$, with $|g|$ the word length of $g \in \mathbb{F}_2$.
  S_t: $\lambda_g \to e^{-t|g|}\lambda_g$.
  $|R_L \cdot|^2 = \sum_k |R_k \cdot|^2$ with

$$R_k : \lambda_g \to \frac{\langle b(g), v_k \rangle}{|g|^{\frac{1}{2}}} \lambda_g.$$

  Let $v_1 = \delta_{h_1} - \delta_e, v_2 = \delta_{h_2} - \delta_e, v_3 = \delta_{h_1^{-1}} - \delta_e, v_4 = \delta_{h_2^{-1}} - \delta_e$,

$$R_1 + R_2 + R_3 + R_4 : \lambda_g \to \frac{1}{|g|^{\frac{1}{2}}} \lambda_g, g \neq e$$

P. A. Meyer's question revisited

P. A. Meyer's question G: (discrete) abelian group.

$$\|(\sum |R_k(f)|^2)^{\frac{1}{2}}\|_{L_p(\hat{G})} \simeq \|f\|_{L_p(\hat{G})}?$$

$$\|\sum_k R_k(f)\gamma_k\|_{L_p(\Omega\times\hat{G})} \simeq \|f\|_{L_p(\hat{G})}?$$

Fails for R_k correspondence to $\Delta^{\frac{1}{2}}$.
Recall G acts on $H \simeq L_2(\Omega)$.
Revision of the question

$$\|\sum_k R_k(f) \rtimes \gamma_k\|_{L_p(G\rtimes L_\infty(\Omega))} \simeq \|f\|_{L_p(\hat{G})}?$$

Yes! Junge/M/Parcet 2014

L_p Fourier multipliers on Discrete Groups

G: (nonabelian) discrete group. e.g. $G = \mathbb{Z}, \mathbb{F}_2$,
$\delta_g, g \in G$: the canonical basis of $\ell_2(G)$.

λ_g: left regular representation of G on $\ell_2(G)$,

$$\lambda_g(\delta_h) = \delta_{gh}, \quad \text{for } g, h \in G.$$

$L_\infty(\hat{\mathbb{G}})$: the w^* closure of $Span\{\lambda_g\}$'s in $B(\ell_2(G))$.
Example: $G = \mathbb{Z}, \lambda_k = e^{ik\theta}, L_\infty(\hat{\mathbb{G}}) = L_\infty(\mathbb{T})$.
τ: For $f = \sum_g f_g \lambda_g$,

$$\tau f = f_e.$$

$$||f||_{L^p(\widehat{\mathbb{G}})} = [\tau(|f|^p)]^{\frac{1}{p}}, 1 \le p < \infty.$$

Example: $G = \mathbb{Z}, \tau f = \hat{f}(0) = \int f, L_p(\hat{\mathbb{G}}) = L_p(\mathbb{T})$.

Question Find (nontrivial) $L^p(\widehat{\mathbb{G}})(1 < p < \infty)$ bounded multipliers

$$T_m : \lambda_g \to m(g)\lambda_g.$$

L_p bound of Riesz transforms via cocycles

Theorem (Junge/M/Parcet, 2014) For any discrete group G with a cocycle (b, H, α), we have

$$\|(\sum_k |R_k f|^2)^{\frac{1}{2}}\|_{L_p(\widehat{\mathbb{G}})} + \|(\sum_k |R_k f^*|^2)^{\frac{1}{2}}\|_{L_p(\widehat{\mathbb{G}})} \simeq \|f\|_{L_p(\widehat{\mathbb{G}})},$$

for any $p \geq 2$. And

$$\|f\|_{L_p(\widehat{\mathbb{G}})} \simeq \inf_{R_j f = a_j + b_j} \Big\| \Big(\sum_{j \geq 1} a_j^* a_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\widehat{\mathbb{G}})} + \Big\| \Big(\sum_{j \geq 1} \widetilde{b}_j \widetilde{b}_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\widehat{\mathbb{G}})},$$

for any $1 < p < 2$.

The equivalence constant only depends on p.

Pisier's method+Khintchine inequality for crossed products.
$\widetilde{b}_j$ is a twist of b_j coming from the Khintchine inequality for crossed products.

Classical Hörmander-Mihlin multipliers are averages of Riesz transforms

- Let $G = \mathbb{R}^n$.
  $H = L_2(\mathbb{R}^n, \frac{dx}{|x|^{n+2\varepsilon}})$. $b : \xi \in \mathbb{R}^n \to e^{i\langle \xi, \cdot \rangle} - 1 \in H$.
  $\|b(\xi)\|^2 = |\xi|^{2\varepsilon}$,
  $L = -\Delta^{\varepsilon}$: $e^{i\xi x} \to -|\xi|^{2\varepsilon} e^{i\langle \xi, x \rangle}$.
- For $v \in H$, let

$$R_v : e^{i\langle \xi, x \rangle} \to \frac{\langle b(g), v \rangle}{\|b(\xi)\|} e^{i\langle \xi, x \rangle}.$$

Given $T_m : \int \hat{f}(\xi) e^{i\langle \xi, x \rangle} \to \int \hat{f}(\xi) m(\xi) e^{i\langle \xi, x \rangle} d\xi$, then

$$T_m = R_v$$

with

$$v(x) = |x|^{n+2\varepsilon} \widehat{m(\cdot)|\cdot|^{\varepsilon}}.$$

Hörmander-Mihlin multipliers on a branch of free groups

Given a branch B of $G = \mathbb{F}_\infty$, let

$$L_p(\widehat{\mathbf{B}}) = \{f = \sum_{g \in B} a_g \lambda_g ; \|f\|_{L^p(\widehat{\mathbb{G}})} = (\tau |f|^p)^{\frac{1}{p}} < \infty\}.$$

For $m : \mathbb{Z}_+ \to \mathbb{C}$, let

$$T_m f = \sum_{g \in B} m(g) a_g \lambda_g .$$

Theorem (Junge/M/Parcet, 2014)
Suppose $m : \mathbb{Z}_+ \to \mathbb{C}$ satisfies

$$\sup_{j \geq 1} |m(j)| + j|m(j) - m(j-1)| < c$$

then

$$\|T_m f\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} c\|f\|_{L^p(\widehat{\mathbb{G}})}$$

for any $f \in L_p(\widehat{\mathbf{B}})$.

Littlewood-Paley estimates.

Theorem (Junge/M /Parcet, 2014) Consider a standard Littlewood-Paley partition of unity $(\varphi_j)_{j\geq 1}$ in $\mathbb{R}_+$. Let $\Lambda_j : \lambda(g) \mapsto \sqrt{\varphi_j(|g|)}\lambda(g)$ denote the corresponding radial multipliers in $\mathcal{L}(\mathbb{F}_\infty)$. Then, the following estimates hold for $f \in L_p(\widehat{\mathbf{B}})$ and $1 < p < 2$

$$\inf_{\Lambda_j f = a_j + b_j} \Big\| \Big(\sum_{j\geq 1} a_j^* a_j + \widetilde{b}_j \widetilde{b}_j^* \Big)^{\frac{1}{2}} \Big\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} \|f\|_{L^p(\widehat{\mathbb{G}})},$$

$$\|f\|_{L^p(\widehat{\mathbb{G}})} \lesssim_{c(p)} \inf_{\Lambda_j f = a_j + b_j} \Big\| \Big(\sum_{j\geq 1} a_j^* a_j + b_j b_j^* \Big)^{\frac{1}{2}} \Big\|_{L^p(\widehat{\mathbb{G}})}.$$