Sur la classification des sous-algèbres co-idéales des groupoïdes quantiques finis

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$$M^G \subset M \subset M \rtimes G \subset (M \rtimes G) \rtimes \hat{G} \subset ....$$

Outer action : $(M^G)' \cap M = \mathbb{C}1 \Longrightarrow$ All vN algebras are factors and $[M : M^G] = [M \rtimes G : M] = ... = |G|$

Galois correspondence : $M^G \subset K \subset M \iff$ subgroups of G.

II_1-factors : (M, τ), where τ is a faithful normal finite trace : $\tau(ab) = \tau(ba)$.

Example of II_1 factor : $M = \mathcal{L}(H)$, H is a discrete ICC group, $\tau(m) =< m\delta_e, \delta_e >$

Fundamental construction

$$M_0 \subset (M_1, \tau) \longrightarrow e : L^2(M_1, \tau) \to \overline{M_0}, \quad M_2 = \{M_1, e\} \text{ - a factor}$$

$$M_0 \subset M_1 \subset M_2 \subset \ldots \quad \text{Jones tower}$$

$$[M_1, M_0] := \tau(e)^{-1} \in \{4\cos^2\frac{\pi}{n} \;\; | n \geq 3\} \cup [4, \infty]$$

Derived tower :

$$M_0' \cap M_1 \subset M_0' \cap M_2 \subset M_0' \cap M_3 \subset \ldots \quad \text{Finite dim}$$

Depth k -fundamental construction from step k

Example : $M^G \subset M \subset M \rtimes G \subset (M \rtimes G) \rtimes \hat{G} \subset \ldots.$ - depth 2

Theorem (R. Longo, W. Szymanski, **M.-C. David**)

$M_0 \subset M_1$: type II_1 subfactors of finite index and depth 2,

$M_0' \cap M_1 = \mathbb{C}1$, $M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$....... Jones tower

- $A = M_0' \cap M_2$, $B = M_1' \cap M_3$: Kac algebras in duality
- B acts on M_2 and $M_1 = M_2^B$, $\quad M_3 = \theta(M_2 \rtimes B)$
- $[M_k : M_{k-1}] = \dim A = \dim B \quad (k = 1, 2, ..)$

Further motivation : $M_0' \cap M_1 \neq \mathbb{C}1$, non integer index,

Depth > 2

Theorem (D. Nikshych-LV)

Let $M_0 \subset M_1$ be an inclusion of type II_1 factors of finite index and finite depth with the corresponding Jones tower

$$M_0 \subset M_1 \subset M_2 \subset M_3 \subset .... \qquad Then:$$

1. $A = M_0' \cap M_{2n-1}$, $B = M_{n-1}' \cap M_{3n-2}$ are C^*-weak Hopf algebras in duality.

2. B acts on M_{2n-1} and $M_{n-1} = M_{2n-1}^B$, $\quad M_{3n-2} \cong (M_{2n-1} \rtimes B)$.

3. The lattices of intermediate vN subalgebras and of right coideal C^*-subalgebras of B are isomorphic (Galois correspondence).

Weak Hopf Algebras (Finite Quantum Groupoids)

Definition

$\mathfrak{G} = (B, \Delta, \varepsilon, S)$, where (B, Δ, ε) is a finite dimensional bialgebra (but $\Delta(1) \neq 1 \otimes 1$ and $\varepsilon(ab) \neq \varepsilon(a)\varepsilon(b)$, in general !) s. t.

(i) $(\Delta \otimes \mathrm{id})\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))$,

(ii) $\varepsilon(abc) = \varepsilon(ab_{(1)})\varepsilon(b_{(2)}c) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c)$, $\forall a, b, c \in B$,

(here $\Delta(b) = b_{(1)} \otimes b_{(2)}$ - Sweedler notation)

(iii) Antipode $S : B \longrightarrow B$ is a bialgebra anti-isomorphism s. t.

$$m(\mathrm{id} \otimes S)\Delta(a) = \varepsilon(1_{(1)}a)1_{(2)},$$

$$m(S \otimes \mathrm{id})\Delta(a) = 1_{(1)}\varepsilon(a1_{(2)}),$$

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a).$$

A C^*-WHA : B is a C^*-algebra, $\Delta \circ * = (* \otimes *)$ and $(S \circ *)^2 = id_B$.

Some properties of WHA's

1. The dual $\hat{B}$ has a structure of a weak C^*-Hopf algebra $\hat{\mathfrak{G}}$

2. $\mathfrak{G}$ is a Hopf C^*-algebra if and only if either

$$\Delta(1) = 1 \otimes 1 \quad \text{or} \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

3. Target and source counital maps and counital C^*-subalgebras :

$$\varepsilon_t(a) = m(\mathrm{id} \otimes S)\Delta(a), \quad \varepsilon_s(a) = m(S \otimes \mathrm{id})\Delta(a), \ \forall a \in B$$

C^*-subalgebras $B_t := Im(\varepsilon_t)$ and $B_s := Im(\varepsilon_s)$, $[B_s, B_t] = 0$.

$$\Delta(1) \in B_s \otimes B_t$$

4. $\mathfrak{G}$ (resp., $\widehat{\mathfrak{G}}$) is commutative iff $B \cong$ to)the algebra of functions on (resp., to the groupoid algebra of) a usual groupoid.

Tensor C^*-categories

A C^*-category is a category whose morphism spaces $Hom(X, Y)$ are Banach spaces such that $||S \circ T|| \leq ||S||||T||$ ($\circ$ is a composition of morphisms) admitting antilinear contravariant involutive conjugation $* : Hom(X, Y) \to Hom(Y, X) : \ T \mapsto T^*$ s.t. $||T^*T|| = ||T||^2$. So any $End(X)$ is a unital C^*-algebra and there are orthogonal projections $p \in End(X) : \ p = p^* = p^2$, unitaries $u : X \to Y : u^*u = id_X, \ uu^* = id_Y$ and partial isometries $w : X \to Y : w^*w = p_X, \ ww^* = q_Y$.

A tensor C^*-category is a C^*-category which is a tensor category in the usual sense with tensor product $\otimes$ an unit object $\mathbf{1}$.

The category $\mathbf{Rep}(\mathfrak{G})$ of Unitary Representations

Objects : f.d. Hilbert spaces which are also left B-modules :

$$< b \cdot \xi, \eta > = < \xi, b^* \cdot \eta >, \quad \forall b \in B, \xi, \eta \in H$$

Morphisms : B-linear maps

Tensor product :

- of objects : $\Delta(1_B) \cdot (H \otimes K)$
- of morphisms : restriction of the usual tensor product of B-module morphisms

Unit : 1 $= B_t$ via $b \cdot z = \varepsilon_t(bz)$, $\forall b \in B, z \in B_t$

Dual object : $\overline{H}$ with action $< b \cdot \overline{v}, v > := < \overline{v}, S(b) \cdot v >$

If $B_t \cap Z(B) = \mathbb{C}$, $\mathbf{Rep}(\mathfrak{G})$ is a finite tensor rigid C^*-category.

The category $\mathbf{Corr}(\mathbf{R})$ of correspondences

Let R be a unital f.d. C^*-algebra

Objects - Hilbert spaces H with a unital $*$-homomorphism $\alpha : R \to B(H)$ and a unital $*$-anti-homomorphism $\beta : R \to B(H)$ whose images commute in $B(H)$ (so H is an R-bimodule via $a \cdot v \cdot b = \alpha(a)\beta(b)v, \ \forall a, b \in R, v \in H$).

Morphisms - bounded linear maps intertwining $\alpha(R)$ and $\beta(R)$.

The conjugate morphism is the adjoint linear map.

Tensor product. $H_1 \otimes_R H_2$.

$\mathbf{Corr}(\mathbf{R})$ is a semisimple multi-tensor C^*-category with unit $\mathbf{1} = R$ (the scalar product on R via the normalized trace).

The Hayashi's reconstruction theorem

Let $\mathcal{C}$ be a finite rigid tensor C^*-category. Define a functor $\mathcal{H}$:

$\mathcal{C} \to Corr_f(R)$, where $R = \bigoplus_{x \in \Omega} \mathbb{C} p_x$ is a commutative C^*-algebra :

$$\mathcal{H}(x) = H^x = \bigoplus_{y,z \in \Omega} Hom(z, y \otimes x), \quad \forall\, x \in \Omega = Irr(\mathcal{C}).$$

H^x is R-bimodule : $p_y \cdot H^x \cdot p_z = Hom(z, y \otimes x)$, $\forall x, y, z \in \Omega$

The tensor structure on $\mathcal{H}$ is $\mathcal{H}_{x,y} : H^x \underset{R}{\otimes} H^y \to H^{x \otimes y}$

$$\text{defined by} \quad \mathcal{H}_{x,y}(v \otimes w) = a_{z,x,y} \circ (v \otimes id_y) \circ w,$$

for all $v \in p_z \cdot H^x \cdot p_t, w \in p_t \cdot H^y \cdot p_s, z, s, t \in \Omega$.

The scalar product on $Hom(z, y \otimes x) : < f, g >_x = g^* \circ f \in End(z) = \mathbb{C}$, the subspaces $Hom(z, y \otimes x)$ are declared to be orthogonal, so $H^x \in Corr_f(R)$. Then $\mathcal{H}$ is a C^*-tensor functor.

Theorem

The vector space

$$B = \bigoplus_{x \in \Omega} (H^x \otimes \overline{H}^x),$$

carries a regular biconnected WHA structure $\mathfrak{G}$ s. t. $Rep(\hat{\mathfrak{G}}) \cong \mathcal{C}$.

If $v, w \in H^x$, $g, h \in H^y$, $\{v_j^x\}$ is an orthonormal basis in H^x, then :

$$(w \otimes \overline{v})_x \cdot (g \otimes \overline{h})_y = (\mathcal{H}_{x,y}(w \otimes g) \otimes \overline{\mathcal{H}_{x,y}(v \otimes h)})_{x \otimes y} \in H^{(x \otimes y)} \otimes \overline{H^{(x \otimes y)}}$$

$$\Delta(w \otimes \overline{v}) = \bigoplus_j (w \otimes \overline{v_j^x})_x \otimes (v_j^x \otimes \overline{v})_x \quad (\forall x, y \in \Omega)$$

Definition

A $\mathfrak{G}$-C^*-algebra is a unital C^*-algebra A with

a $*$-homomorphism $\mathfrak{a} : A \to A \otimes B$ (a right coaction) s.t.

- $(\mathfrak{a} \otimes id_B)\mathfrak{a} = (id_A \otimes \Delta)\mathfrak{a}$
- $(id_A \otimes \varepsilon)\mathfrak{a} = id_A$
- $\mathfrak{a}(1_A) \in A \otimes B_t$

Example : a weak right coideal $*$-subalgebra $A \subset B$ (i.e., $\Delta(A) \subset A \otimes B$, $\Delta(1_A) \in A \otimes B_t$) with the coaction $\Delta_{|A}$

One has necessarily $1_A \in B_t$. If $1_A = 1_B$, A is called a coideal $*$-subalgebra (in what follows, just a coideal).

A $\mathfrak{G}$-C^*-algebra is called indecomposable if it is not a direct sum of two nontrivial $\mathfrak{G}$-C^*-algebras.

Spectral subspaces (isotypical components)

Theorem ([VV], 2017)

Let $(A, \mathfrak{a})$ be a unital $\mathfrak{G}$-C^*-algebra A. For any $U \in \mathbf{Rep}(\hat{\mathfrak{G}})$ there exists a vector subspace A_U of A such that the family $\{A_U | U \in \mathbf{Rep}(\hat{\mathfrak{G}})\}$ satisfies the following properties :

(i) A_U are closed and we have $A = \bigoplus_{x \in \Omega} A_{U^x}$, where $\{U^x\}$ are irreducible representations of $\hat{\mathfrak{G}}$.

(ii) $A_{U^x} A_{U^y} \subset \bigoplus_z A_{U^z}$, where z runs over the set of irreducibles contained in $U^x \otimes U^y$.

(iii) $\mathfrak{a}(A_U) \subset \mathfrak{a}(1_A)(A_U \otimes B_U)$, where B_U is the subspace of B generated by the matrix units of $U \in \mathbf{Rep}(\hat{\mathfrak{G}})$.

(iv) A_ε is a unital C^*-subalgebra of A

Module categories

Definition

A C^*-category $\mathcal{M}$ is called a left module C^*-category over a C^*-tensor category $\mathcal{C}$ if there is a bilinear $*$-functor $\boxtimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ with natural unitary transformations $(X \otimes Y) \boxtimes M \to X \boxtimes (Y \boxtimes M)$ $(X, Y \in \mathcal{C}, M \in \mathcal{M})$ satisfying the module associativity conditions.

An object $M \in \mathcal{M}$ generates $\mathcal{M}$ if any object of $\mathcal{M}$ is isomorphic to a subobject of $X \boxtimes M$ for some $X \in \mathcal{C}$.

A morphism between two $\mathcal{C}$-module C^*-categories is a morphism of underlying C^*-categories respecting $\boxtimes$.

Reconstruction theorem for $\mathfrak{G}$-C^*-algebras

Theorem ([VV], 2017)

Let $\mathfrak{G}$ be a regular coconnected WHA (i.e., $B_t \cap B_s = \mathbb{C}1$).

Then the following two categories are equivalent :

(i) The category of unital $\mathfrak{G}$-C^*-algebras $(A, \mathfrak{a})$ with unital $\mathfrak{G}$-equivariant $*$-homomorphisms as morphisms.

(ii) The category of pairs $(\mathcal{M}, M)$, where $\mathcal{M}$ is a left $\mathbf{Rep}(\hat{\mathfrak{G}})$-module C^*-category with trivial module associativities and M is a generator in $\mathcal{M}$, with equivalence classes of unitary $\mathbf{Rep}(\hat{\mathfrak{G}})$-module functors respecting the prescribed generators as morphisms.

Remark For compact quantum groups, similar result was obtained by K. De Commer, M. Yamashita (2013) and S. Neshveyev (2014).

From a pair $(\mathcal{M}, M)$ to a $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$

$$A = \bigoplus_{x \in \Omega} A_{U^x} := \bigoplus_{x \in \Omega} (F(U^x) \otimes \overline{H^x}), \quad \text{where}$$

$F(U) = Hom_{\mathcal{M}}(M, U \boxtimes M)$ $(\forall\ U \in \mathbf{Rep}(\hat{\mathfrak{G}}))$, defines a functor

$F : \mathbf{Rep}(\hat{\mathfrak{G}}) \to Corr(End(M))$ having a weak tensor structure [Nes]

$J_{X,Y}(X \otimes Y) = (id \otimes Y)X$ $(X \in F(U), Y \in F(V), U, V \in \mathbf{Rep}(\hat{\mathfrak{G}})$.

The unit is $1_{\tilde{A}} = id_M \otimes \overline{1_B}$ and the product in A is

$$(X \otimes \overline{\xi})(Y \otimes \overline{\eta}) = J_{X,Y}(X \otimes Y) \otimes (\overline{\xi} \otimes_{B_s} \overline{\eta}),\ \forall (X \otimes \overline{\xi}) \in A_U, (Y \otimes \overline{\eta}) \in A_V.$$

A right coaction of $\mathfrak{G}$ on A is given by

$$\mathfrak{a}(X \otimes \overline{v_i^x}) = X \otimes \Sigma_j (\overline{v_j^x} \otimes U_{j,i}^x)$$

$(\{v_i^x\} \in H^x$ is an orthonormal basis, $U_{i,j}^x$ are the matrix elements).

$TY(G,\chi,\beta) = Vec_G \oplus \{m\}$ (G is abelian finite group, $\chi : G \times G \to \mathbb{T}$ is a non degenerate symmetric bicharacter, $\beta = \pm(\sqrt{|G|})^{-1}$)

$\Omega = Irr(TY(G,\chi,\beta)) = G \sqcup \{m\}$, the fusion rule is

$$g \otimes h = g + h, \quad g^* = -g, \quad \mathbf{1} = 0 \in G,$$

$$g \otimes m = m \otimes g = m = m^*, \quad m \otimes m = \bigoplus_{g \in G} g,$$

and the only nontrivial associativities are

- $a_{g,m,h} = \chi(g,h)\mathrm{id}_m$, for all $g, h \in G$,
- $a_{m,g,m} = \bigoplus_{h \in G} \chi(g,h)\mathrm{id}_h$, for all $g \in G$,
- $a_{m,m,m} = \left(\beta\chi(g,h)^{-1}\mathrm{id}_m\right)_{g,h}$,

WHA associated with $TY(G, \chi, \beta)$

$\mathfrak{G}_{TY}$ is a WHA s. t. $\mathbf{Rep}(\mathfrak{G}_{\mathbf{TY}}) \cong TY(G, \chi, \beta)$ (C. Mével (2010)).

The structure of the Hayashi's functor gives

$$H^g = \underset{x \in \Omega}{\oplus} \mathbb{C} v_x^g, \quad H^m = [\underset{h \in G}{\oplus} \mathbb{C} v_h^m] \oplus [\underset{\overline{h} \in \overline{G}}{\oplus} \mathbb{C} v_{\overline{h}}^m].$$

Indecomp. $\mathfrak{G}_{TY}$-C^*-algebras $\cong$ indecomp. $\mathbf{Rep}(\mathfrak{G}_{\mathbf{TY}})$-$C^*$-module categories $\mathcal{M}$ with trivial associativities and fixed generators M.

Notation : $H^\perp := \{g \in G | \chi(g, h) = 1$, for all $h \in H < G\}$.

One can deduce from the results of E. Meir, E. Musicantov (2012) :

Proposition. There are two cases :

(i) $\mathcal{M}$ is indecomposable over Vec_G iff there is $H < G$

s.t. $H = H^\perp$, $\Lambda = Irr(\mathcal{M}) = G/H$ and the fusion rule of $\mathcal{M}$ is :

$$U^g \boxtimes M_\lambda = M_{g+\lambda}, \quad U^m \boxtimes M_\lambda = \bigoplus_{\mu \in G/H} M_\mu \quad (\forall g \in G, \lambda \in G/H)$$

(ii) $\mathcal{M}$ is decomposable over Vec_G iff there is $H < G$,

$\Lambda = Irr(\mathcal{M}) = G/H \sqcup G/H^\perp$ and the fusion rule of $\mathcal{M}$ is :

$$U^g \boxtimes M_\lambda = M_{g+\lambda} \ (g \in G,\ \lambda \in \Lambda), \quad \text{and}$$

$$U^m \boxtimes M_\rho = \bigoplus_{\mu \in G/H^\perp} M_\mu, \qquad U^m \boxtimes M_\mu = \bigoplus_{\rho \in G/H} M_\rho.$$

Remark Any object $M = \bigoplus_{\lambda \in \Lambda} m_\lambda M_\lambda$ is a generator of $\mathcal{M}$.

Theorem 1. There are 2 types of indecomp. $\mathfrak{G}_{TY}$-C^*-algebras :

(i) Labeled by tuples $(H = H^\perp < G, \{m_\lambda | \lambda \in G/H\})$, where $m_\lambda \in \mathbb{Z}_+$, $\sum_{\lambda \in \Lambda} m_\lambda > 0$. Tuples $(H, \{m_\lambda\})$ and $(H', \{m'_\lambda\})$ give isomorphic algebras iff $H = H'$ and $m'_\lambda = m_{\lambda+k}$ $(\exists k \in G/H)$

(ii) Labeled by tuples $(H < G, \{m_\lambda | \lambda \in G/H\}, \{m_\rho | \rho \in G/H^\perp\})$, $m_\lambda, m_\rho \in \mathbb{Z}_+$, at least one is > 0.

Tuples $(H, \{m_\lambda\}, \{m_\rho\})$ and $(H', \{m'_\lambda\}, \{m'_\rho\})$ give isomorphic algebras iff $H = H'$ and $m'_\lambda = m_{\lambda+k_1}$, $m'_\mu = m_{\mu+k_2}$ $(\exists (k_1, k_2) \in G/H \times G/H^\perp)$. If $H = H^\perp$, also $m'_\lambda = m_{\sigma(\mu)+k_1}$ and $m'_\mu = m_{\sigma(\lambda)+k_2}$, where σ permutes 2 copies of G/H.

Distinguishing weak coideals

Lemma

If A is a weak coideal, we have $m_\lambda \in \{0,1\}$ for all $\lambda \in \Lambda$, so we can identify $M = \bigoplus_{\lambda \in \Lambda} m_\lambda M_\lambda$ with $Z \subset \Lambda$. This gives

$$dim(F(U^g)) = |Z \cap (g + Z)|, \ \forall g \in G.$$

In case (ii) we identify $M \cong (Z, Z_1)$, where $Z \subset G/H$, $Z_1 \subset G/H^\perp$, so $dim(F(U^m)) = 2|Z||Z_1|$.

Theorem 2

1. There is no weak coideals of type (i).

2. Indecomposable weak coideals A s.t. $A^m = \{0\}$ are labeled by couples (H, Z), where $H < G$ and Z is a nonempty subset either of G/H or of $G/H^\perp$. A is ($\cong$ to) a coideal iff $|Z| = 1$. Couples (H, Z) and (H', Z') give isomorphic weak coideals iff $H = H'$ and :

a) when $H \neq H^\perp$ and $Z, Z' \subset G/H$ (resp., $Z, Z' \subset G/H^\perp$), there is $p \in G/H$ (resp., $p \in G/H^\perp$) s.t. that $Z' = p + Z$;

b) when $H = H^\perp$, there is $p \in G/H$ s. t. either $Z' = p + Z$ or $Z' = p + \sigma(Z)$, where σ interchanges two copies of G/H.

Classification of weak coideals : case $A^m \neq \{0\}$

Theorem 3

1. There is no weak coideals of type (i).

2. Indecomposable weak coideals A s.t. $A^m \neq \{0\}$ are labeled by triples (H, Z, Z_1), where $H < G$ and $Z \subset G/H, Z_1 \subset G/H^\perp$, $min\{|Z|, |Z_1|\} = 1$. A is isomorphic to a coideal iff either $Z = G/H$ or $Z_1 = G/H^\perp$. Two triples, (H, Z, Z_1) and (H', Z', Z_1'), give rise to isomorphic weak coideals iff $H = H'$ and :

a) when $H \neq H^\perp$, there exists $(p_1, p_2) \in G/H \times G/H^\perp$ such that $Z' = p_1 + Z$, $Z_1' = p_2 + Z_1$;

b) when $H = H^\perp$, there exists $(p_1, p_2) \in G/H \times G/H$ such that either $Z' = p_1 + Z$, $Z_1' = p_2 + Z_1$ or $Z' = p_1 + \sigma(Z_1)$, $Z_1' = p_2 + \sigma(Z)$, where σ interchanges two copies of G/H.

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