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Critical recurrence in real quadratic and rational dynamics

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Critical recurrence in real quadratic and rational dynamics

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Lund University

Doctoral thesis
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Abstract

In this thesis we study the dynamics of real quadratic functions on the interval, and rational functions on the Riemann sphere. The problems we are considering are concerned with the recurrent nature of the critical orbit(s). In Paper I we investigate the real quadratic family and prove a theorem regarding the rate of recurrence of the critical point to itself, extending a previous result by Avila and Moreira. In Paper II and Paper III we consider rational functions. Here we do not study the rate of recurrence, rather we assume that the critical points approach each other at a slow rate, and investigate some of the consequences. Assuming this slow recurrence condition, we prove in Paper II that certain Collet–Eckmann rational functions can in a strong sense be approximated by hyperbolic ones. In Paper III we observe that within the family of slowly recurrent rational maps, the well-known Collet–Eckmann conditions are all equivalent.

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Populärvetenskaplig sammanfattning

Studiet av dynamiska system grundar sig i att förstå det långsiktiga beteendet hos ett system som fortskrider i tiden, enligt vissa för systemet specifika regler. Dynamiska system uppkommer naturligt inom olika vetenskapliga discipliner, exempelvis då man vill studera planeternas rörelse, ta fram väderprognoser, eller förstå hur ett virus sprider sig i samhället.

För att studera dessa naturliga system behöver man matematiska modeller. Dessa modeller är naturligt parametriserade och det är därför av intresse att inte enbart studera ett specifikt dynamiskt system, utan en parametriserad familj av dynamiska system. En viktig fråga man kan ställa är hur robusta dessa system är, eller med andra ord, hur dynamiken förändras vid små störningar av parametrarna. Fastän modellerna man tar fram ofta är förenklade, och parameterberoendet väldigt explicit, uppkommer teoretiskt intressanta och mycket icke-triviala problem. Av betydande intresse är interaktionen mellan tamt beteende och kaotiskt beteende. I parameterrummet är dessa två skilda företeelser ofta komplext sammanvävda.

I denna avhandling studeras små störningar av kaotiska system. Dessa system kommer att beskrivas av funktioner på intervallet och på Riemannsfären. Systemen vi studerar har kritiska punkter, det vill säga punkter där funktionens derivata är lika med noll. Hur dynamiken för dessa specifika punkter ter sig visar sig ha stor betydelse för den globala dynamiken. En viktig aspekt är rekurrent beteende: med vilken hastighet återkommer de kritiska punkterna till varandra under iteration? Avhandlingen bygger vidare på tidigare väl etablerade resultat, och det centrala temat är just dessa frågeställningar angående rekurrens och dess konsekvenser.

List of papers

This thesis is based on the following three papers.

Paper I **Critical recurrence in the real quadratic family**

M. Bylund
To appear in Ergodic Theory and Dynamical Systems
Preprint: arXiv:2103.17200

Paper II **Slowly recurrent Collet–Eckmann maps with non-empty Fatou set**

M. Aspenberg, M. Bylund, W. Cui
Submitted
Preprint: arXiv:2207.14046

Paper III **Equivalence of Collet–Eckmann conditions for slowly recurrent rational maps**
M. Bylund
Submitted
Preprint: arXiv:2209.05237

Paper II is the result of an equal contribution from the authors regarding all aspects of the work.

The papers as they appear in this thesis might differ slightly from the corresponding preprint versions.

Contents

Part I

Introduction and summary

Chapter 1

Introduction

This introductory chapter gives a brief overview of the theory and results on which the scientific papers of this thesis are based upon. It is divided into five sections as follows. We begin by introducing what a dynamical system is, and some of the most fundamental notions. The second section is devoted to the real quadratic family, which is the system studied in Paper I, and which is one of the most well studied families of dynamical systems. In Paper II and Paper III we study the dynamics of rational functions on the Riemann sphere, and this topic is briefly introduced in the third section. In the fourth section we discuss the Collet–Eckmann condition and some of its variants. These are conditions of non-hyperbolicity and they play a central role in the thesis. In the final section we give a schematic outline of the Benedicks–Carleson techniques, which are the foundational tools used in Paper I and Paper II.

These sections below are by no means complete in terms of their scope, and many important results and notions are left out. Rather, the goal is to give the minimal information needed to motivate the problems studied in Paper I–III. Relevant references will be given throughout the text, but for the more general theory of interval dynamics and complex (rational) dynamics, we refer to [dMvS93, Dev92] and [CG93, Mil06, Bea91], respectively.

1 Some notions in dynamical systems

In this thesis we are concerned with the study of *discrete dynamical systems*. At its core this constitutes a set X of points and a mapping $f : X \to X$. The set X is usually referred to as the *state space* (or *phase space*), with each $x \in X$ representing a specific state of the system.

The mapping f is the *evolution mapping* which determines the future of the system, taking state x to its future state $f(x)$. One of the main objectives when studying a dynamical system is to understand its long term behaviour: given a state $x \in X$, how does its *orbit*

$$x, f(x), f(f(x)), \dots, f^n(x), \dots$$

distribute in state space? Here and elsewhere, f^n always denotes the nth iterate of f. That is, $f^0 = \mathrm{id}$ and $f^n = f \circ f^{n-1}$, with $n \geq 1$ an integer.

More generally, given one or more parameters λ belonging to some *parameter space*, one can consider a *family* of dynamical systems $f_\lambda : X \to X$. In this setting it is of interest to understand how certain behaviours of the system are affected by small changes of the parameter value.

The above questions are of course too general to answer if no structure on X nor regularity on f are imposed. In this thesis we study the so-called real quadratic family

$$x \mapsto x^2 + a = Q_a(x),$$

acting on the real line, and more general rational functions

$$z \mapsto \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0} = R(z),$$

acting on the Riemann sphere. The quadratic family can be seen as a 'toy model' for the more general study of rational maps, but has also been used in, for instance, biological modelling [May76]. Nevertheless, already in this analytically simple family of dynamical systems one finds very rich dynamics.

To understand the dynamics of a function f such as above, acting on some appropriate space, it is important to look for points which are left invariant under the action of f, and to study the local behaviour of f near these points. Such points are called *fixed points*, and per definition they solve the equation $f(x) = x$. More generally, one can look for so-called *periodic points*. A point x is a periodic point of f if there exists an integer $k > 0$ such that

$$x \mapsto f(x) \mapsto f^2(x) \mapsto \cdots \mapsto f^k(x) = x.$$

Such an above orbit is usually referred to as a *cycle*, and if $k > 0$ is the least integer such that the above holds, then k is called the length of the cycle. A cycle of length k is classified as

- *attracting* if $|(f^k)'(x)| < 1$,
- *repelling* if $|(f^k)'(x)| > 1$,
- *neutral* if $|(f^k)'(x)| = 1$.

These names are very suggestive: nearby points get closer to the cycle under iteration if the cycle is attracting, get further away if the cycle is repelling, and in the neutral case both instances may occur.

Another important notion is that of *critical points*. A point x is a critical point of f if the derivative of f at x vanishes, i.e. critical points are the solutions to the equation $f'(x) = 0$. From now on we denote the set of critical points of f by $\mathrm{Crit}(f)$. It turns out that the behaviour of the critical orbit(s) is of great importance to the global dynamics, and we give some motivation to this claim in the following sections.

The results of this thesis are in one way or another concerned with the notion of *critical recurrence*. In Paper I we investigate the real quadratic family and prove a theorem regarding the rate of recurrence of the critical point to itself. This extends a previous result, and completes the picture of so-called *polynomial recurrence*. In Paper II and Paper III we consider rational functions. Here we do not prove any results regarding the rate of recurrence, rather we investigate some of the consequences when the critical points are allowed to approach each other only at a slow rate.

2 The real quadratic family

A quadratic polynomial acting on the real line is from an analytic point of view the simplest non-linear dynamical system one can study. Let $x \mapsto Ax^2 + Bx + C = p(x)$ be a quadratic polynomial with real coefficients $A \neq 0$, B, and C. Conjugating this polynomial with $x \mapsto Ax$ we get the monic quadratic polynomial $x \mapsto x^2+Bx+AC$, and further conjugating with $x \mapsto x+B/2$, i.e. translating the critical point to the origin, we end up with the so-called *real quadratic family*

$$x \mapsto x^2 + a = Q_a(x),$$

with $a = B/2 - B^2/4 + AC$ being the *parameter*. Given a real parameter a, going the other way around does not determine a unique quadratic polynomial. Rather, each a corresponds to a *conjugacy class*. In this thesis we are concerned with the recurrent behaviour of the critical orbit. To motivate this study, and also settle some notation, let us first briefly mention some of the major results regarding this family of dynamical systems.

To understand the dynamics of Q_a for different values of a, understanding the behaviour of the critical orbit is of interest, as can be understood from the following result.

Proposition 2.1. *For each parameter a there can exist at most one (finite) attracting cycle for the corresponding quadratic map Q_a. Moreover, if an attracting cycle exists, the orbit of the critical point $x = 0$ will accumulate along this cycle.*

As a first step towards understanding the behaviour of the iterations of the critical point, we allow ourselves to restrict the parameter interval.

Proposition 2.2. *If a does not belong to the interval $[-2, 1/4]$, then $Q_a^n(0)$ tends to infinity as n tends to infinity. On the other hand, if a belongs to $[-2, 1/4]$ then there exists an interval $I_a \subset [-2, 2]$, containing the critical point, such that $Q_a(I_a) \subset I_a$.*

To begin the study of the qualitative behaviour of the real quadratic family, the following proposition can be checked by hand.

Proposition 2.3. *For the quadratic family Q_a:*

(1) For $a = 1/4$, there is a single fixed point that is neutral.

(2) For $-3/4 < a < 1/4$, there is an attracting fixed point.

(3) For $a = -3/4$, the attracting fixed point given in (2) becomes neutral.

(4) For $-5/4 < a < -3/4$, there is an attracting cycle of length two.

Hence, for parameter values in the interval $(-5/4, 1/4]$, the dynamics is rather trivial. In fact, for such a parameter, almost every point of I_a (with respect to Lebesgue measure) will tend to the attracting fixed point, or 2-cycle, under iteration. To calculate attracting cycles by hand soon becomes impractical, and one must rely on more qualitative and sophisticated techniques. The transition from an attracting fixed point to an attracting 2-cycle is an example of a so-called *period-doubling bifurcation*. By plotting the iterations of the critical point for different values of a, this period-doubling bifurcation can be illustrated as in Figure 1.1. Here one sees, going from right to left, the transition from an attracting fixed point to an attracting 2-cycle, from an attracting 2-cycle to an attracting 4-cycle, and so on. At the parameter value $a = -1.401\ldots$ (the so-called Feigenbaum point), we see a sudden change in the behaviour of the orbit of the critical point. Namely, the orbit does not seem to be attracted to any cycle. This motivates the following definition.

Definition 2.4. A parameter $c \in [-2, 1/4]$ is called a *regular parameter* if $x \mapsto x^2 + c$ has an attracting cycle, and otherwise it is called a *nonregular parameter*. The set of regular parameters is denoted $\mathcal{R}$, and the set of nonregular parameters is denoted $\mathcal{NR}$.

It is customary to call the corresponding function Q_a regular (or nonregular) if the parameter a is regular (or nonregular). Looking at the bifurcation diagram of Figure 1.1, the 'white windows' correspond to regular parameters, while the 'black lines' correspond to nonregular parameters. To understand these two sets of parameters, and how they are intertwined, has been a central topic of study during the last couple of decades.

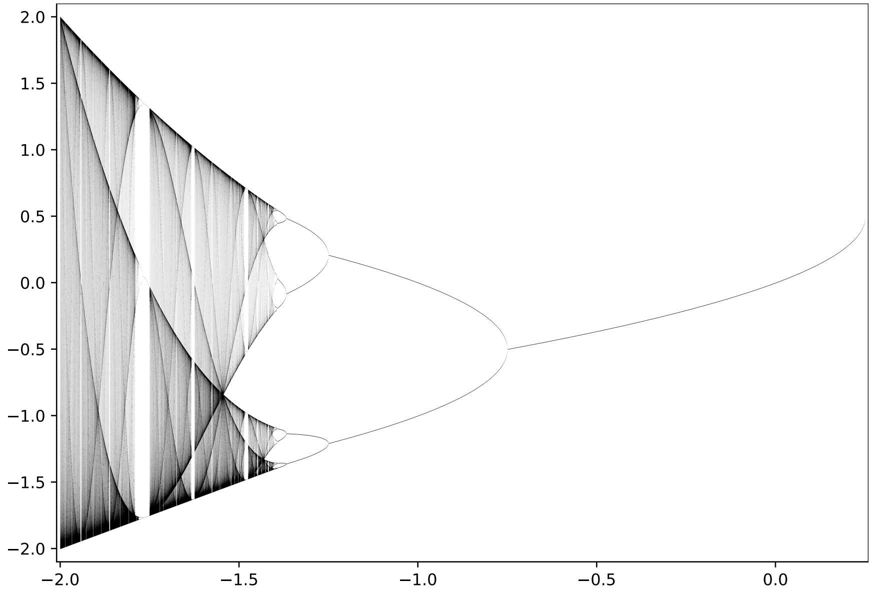

Figure 1.1: Bifurcation diagram for $x \mapsto x^2 + a$, $a \in [-2, 1/4]$.

When studying a parameterised family of dynamical systems, one is often interested in whether some specific property holds on a positive measure set of parameters. In the case of the quadratic family, the natural measure on the parameter interval is the Lebesgue measure (which we from now on denote by Leb). For instance, it is obvious that the set of regular parameters has positive measure since the interval $(-3/4, 1/4)$ is contained in $\mathcal{R}$. Moreover, it is not difficult to show that the set of parameters having neutral cycles constitute only a set of measure zero. More difficult is the question about the measure of the set of nonregular parameters. In 1981, M. Jakobson [Jak81] initiated the study of nonregular parameters by proving that there exists a set Δ_J of positive measure such that for each $a \in \Delta_J$ there exists an absolutely continuous (with respect to Lebesgue) invariant probability measure (acip) for the corresponding quadratic function Q_a. This in turn implies that the Lebesgue measure of $\mathcal{NR}$ is positive, since for a regular parameter any finite invariant measure is necessarily singular with respect to Lebesgue measure, being the sum of point measures along the attracting cycle. We make the following definition for this subset of the nonregular parameters.

Definition 2.5. A parameter $a \in [-2, 1/4]$ is called a *stochastic parameter* if $x \mapsto x^2 + a$ has an absolutely continuous (with respect to Lebesgue) invariant probability measure. The set of stochastic parameters is denoted $\mathcal{S}$.

We recall that the measure μ is acip with respect to the function f if it is a probability measure and if, for every measurable set A, $\mu(f^{-1}(A)) = \mu(A)$ and

$$\mu(A) = \int_A \frac{d\mu}{d\,\mathrm{Leb}} d\,\mathrm{Leb},$$

with $d\mu/d\,\mathrm{Leb}$ denoting the so-called Radon-Nikodým derivative.

Having an acip is one characterisation of being nonregular. Other characterisations can be formulated in terms of the derivative along the critical orbit. Indeed, since for a regular map the critical orbit accumulates on the attracting cycle, the condition

$$\liminf_{n\to\infty} |(Q_a^n)'(a)| > 0$$

clearly implies a being nonregular. However this condition is not necessary: in [Bru94] examples of parameters a are provided such that $x \mapsto x^2 + a$ has no attracting or neutral cycles, but $\liminf_{n\to\infty} |(Q_a^n)'(a)| = 0$. Instead, let us denote by $\chi_-(a)$ the so-called *lower Lyapunov exponent*

$$\chi_-(a) = \liminf_{n\to\infty} \frac{\log |(Q_a^n)'(a)|}{n}.$$

It turns out that the condition $\chi_-(a) \geq 0$ is the correct one to consider, since it is not only sufficient for a to be nonregular, but also necessary [NS98, LPS16].

Focusing on a similar condition as the above, M. Benedicks and L. Carleson [BC85] proved in the early 1980s that there exists a set Δ_{BC} of positive measure such that, for each $a \in \Delta_{BC}$, the derivative along the critical orbit grows at least subexponentially:

$$\liminf_{n\to\infty} \frac{\log |(Q_a^n)'(a)|}{\sqrt{n}} > 0.$$

Moreover, for each $a \in \Delta_{BC}$, the corresponding quadratic map has an acip. In the subsequent paper [BC91], working with the so-called Hénon family, Benedicks and Carleson improved this growth condition and showed that it is in fact exponential. This condition of having exponential growth of the derivative along the critical orbit is called the *Collet–Eckmann condition*, and it was first introduced by P. Collet and J. P. Eckmann [CE83,CE80] where they used this condition to prove the abundance of functions with chaotic dynamics within certain families of dynamical systems. The Collet–Eckmann condition, and some of its variants, are further discussed in Section 4 below. For the quadratic family, we make the following definition.

Definition 2.6. A parameter $a \in [-2, 1/4]$ is called a *Collet–Eckmann parameter* if the corresponding quadratic map satisfies the *Collet–Eckmann condition*

$$\liminf_{n\to\infty} \frac{\log |(Q_a^n)'(a)|}{n} > 0.$$

The set of Collet–Eckmann parameters is denoted $\mathcal{CE}$.

The techniques developed in [BC85,BC91] are of great importance in the field of dynamical systems, and are also central to this thesis. We come back to these in Section 5.

It turns out that both the property of being stochastic, and that of being Collet–Eckmann, are typical within the real quadratic family, namely

$$\operatorname{Leb} \mathcal{NR} = \operatorname{Leb} \mathcal{S} = \operatorname{Leb} \mathcal{CE}.$$

That the stochastic parameters are typical within nonregular parameters was proved by M. Lyubich [Lyu02], following the work in [Lyu00, MN00]. That the Collet–Eckmann parameters are typical within nonregular parameters was proved by A. Avila and C. G. Moreira [AM05]. For this reason, one can consider both of these conditions as good characterisations of being nonregular.

Considering the set of regular parameters, one can with an application of the inverse function theorem show that this set is open, i.e. small changes in the parameter value of a regular map do not alter the existence of an attracting cycle. A much deeper result is that these parameters form a dense set in $[-2, 1/4]$. This result, known as the *real Fatou conjecture*, was proved by J. Graczyk and G. Świątek [GS97, GS98b], and independently by Lyubich [Lyu97]. This genericity result was later extended to the class of real polynomials of arbitrary fixed degree, by O. Kozlovski, W. Shen, and S. van Strien [KSvS07]. With the characterisation of nonregular maps, and the density of regular maps, one can say that from a qualitative point of view, the real quadratic family is well-understood.

Considering the orbit of the critical point, we know from Proposition 2.1 that if a is a regular parameter, then its orbit accumulates on the attracting cycle. If a on the other hand is a nonregular parameter then, by definition, there can be no accumulation on an attracting cycle, and we are left with two possible cases:

$$\text{either} \quad \liminf_{n\to\infty} |Q_a^n(0)| > 0 \quad \text{or} \quad \liminf_{n\to\infty} |Q_a^n(0)| = 0.$$

The first case is known as the *Misiurewicz case*, and it implies that there exists $\delta = \delta(a) > 0$ such that $|Q_a^n(0)| > \delta$ for all $n \geq 1$. It was conjectured by M. Misiurewicz in the early 1980s that these parameters constitute only a set of measure zero, and this conjecture was proved to be true by D. Sands [San98]. Hence for a typical nonregular parameter the second case holds, and we simply call this the *recurrent case*. In this recurrent case, it is natural to ask at what rate the critical point returns to itself or, more precisely, what are the correct conditions on δ_n that guarantee

$$|Q_a^n(0)| < \delta_n \quad \text{for infinitely many } n. \tag{1.1}$$

It was conjectured by Y. Sinai that in the recurrent case, the critical point typically returns with exponent 1. This can be formulated as, for almost every nonregular parameter a,

$$\limsup_{n\to\infty} \frac{-\log|Q_a^n(0)|}{\log n} = 1.$$

This conjecture was indeed proved to be true by Avila and Moreira [AM05]. Another way to phrase this result is as follows: for almost every nonregular parameter a, the set of n such that $|Q_a^n(0)| < 1/n^\theta$ is finite if $\theta > 1$, and infinite if $\theta < 1$. This result motivated Paper I, namely to study the case of the critical exponent $\theta = 1$.

3 Rational dynamics

The study of iterations of rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ was first initiated by P. Fatou [Fat19, Fat20a, Fat20b] and G. Julia [Jul18] around the 1920s. With the emergence of computers with better power of computation, this theory got more popular in the 1980s, much due to the many beautiful pictures. Let us briefly introduce the fundamental notions of rational dynamics.

We consider rational functions of one complex variable z belonging to the *Riemann sphere* $\widehat{\mathbb{C}}$. The Riemann sphere is the complex plane together with the abstract 'point at infinity'. Through stereographic projection, $\widehat{\mathbb{C}}$ is identified with the usual euclidean sphere in $\mathbb{R}^3$, and by pulling back the euclidean metric $|\cdot|$ this provides us with the so-called *chordal metric* σ. For points z and w in the plane the distance between them with respect to the chordal metric is

$$\sigma(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},$$

and if $w = \infty$ then

$$\sigma(z,\infty) = \lim_{w\to\infty} \sigma(z,w) = \frac{2}{\sqrt{1+|z|^2}}.$$

Instead of the chordal metric one can also consider the equivalent so-called *spherical metric* σ_0, which is defined as

$$\sigma_0(z,w) = \inf_\gamma \int_\gamma \frac{|dt|}{1+|t|^2},$$

where the infimum is taken over all continuous curves γ joining z and w.

Each rational function can be represented as the quotient of two polynomials

$$z \mapsto \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0} = \frac{P(z)}{Q(z)} = R(z),$$

with a_i and b_i belonging to $\mathbb{C}$. We always assume that the P and Q do not share any common factors, and if not both a_d and b_d are equal to 0, we say that the *degree of* R $\deg(R)$ is equal to d. Thus, a rational map of degree d is a d-to-1 covering of the Riemann sphere onto itself. The *spherical derivative* of $z \mapsto R(z)$ is defined as

$$DR(z) = R'(z) \frac{1 + |z|^2}{1 + |R(z)|^2},$$

and we notice that it satisfies the chain rule.

The parameter space of rational maps (of a fixed degree d) is more complicated than that of the interval. We can assume that either $a_d = 1$ or $b_d = 1$, thus the parameter space of rational maps of degree d is a $(2d + 1)$-dimensional complex manifold, and also a subspace of the projective space $\mathbb{CP}^{2d+1}$. On each of the two charts corresponding to $a_d = 1$ and $b_d = 1$, respectively, the Lebesgue measures are mutually absolutely continuous. The Lebesgue measures on each chart are also mutually absolutely continuous to the induced Fubini-Study measure on $\mathbb{CP}^{2d+1}$. In Paper II we use a special normalisation of rational functions of degree d, due to G. Levin [Lev14]. We identify two rational functions of degree d as being equal if they are conjugated by a Möbius transformation. Up to equivalence, we then consider the space of rational functions (of degree d) with exactly p' different critical points $c_1, c_2, \ldots, c_{p'}$, with corresponding multiplicities $\overline{p'} = (m_1, m_2, \ldots, m_{p'})$. Within this space, which we denote by $\Lambda_{d,\overline{p'}}$, critical points move analytically with respect to the parameter. In particular, if all critical points are simple, i.e. $\overline{p'} = (1, 1, \ldots, 1)$, then $\Lambda_{d,\overline{p'}}$ is locally equal to the entire parameter space.

An early and important step in the theory of rational dynamics was made by Fatou and Julia when they described a decomposition of the Riemann sphere into two invariant sets with respect to a rational function, namely the *Fatou set* and its complement, the *Julia set*. The Fatou set of a rational map R is denoted $\mathcal{F}(R)$ and is by definition the *domain of normality*: for each $z \in \mathcal{F}(R)$ there exists a neighbourhood U, containing z, such that the set of consecutive iterates of R restricted to U forms a normal family. That is to say, there exists an increasing sequence n_k such that $f^{n_k}|_U$ converges locally uniformly on compact subsets of U, with respect to the spherical metric. Intuitively, nearby points belonging to the Fatou set share similar limiting behaviour, and for this reason the dynamics on the

Fatou set is considered stable. From the definition, it follows that the Fatou set is open, hence the Julia set is compact. We denote the Julia set by $\mathcal{J}(R)$. Using Montel's theorem, one can prove that the Julia set is equal to the closure of the repelling cycles. Hence, nearby points belonging to the Julia set will repel each other, and one speaks of chaotic dynamics. On the Julia set, it therefore makes sense to talk about Lyapunov exponents, invariant measures, and so on.

The dynamics in the Fatou set for a rational function is well understood, and for completeness we state the following classification result. A component U of the Fatou set $\mathcal{F}(R)$ is called *fixed* if $R(U) = U$, *periodic* if $R^k(U) = U$ for some $k > 0$, and *pre-periodic* if $R^l(U)$ is periodic for some $l > 0$. That these are the only possibilities was proved by D. Sullivan [Sul85]: a component U of the Fatou set of a rational map is either fixed, periodic, or pre-periodic. This result by Sullivan, which is often called *Sullivan's no-wandering-domain theorem*, is a milestone in rational dynamics, and introduced the new idea of using quasiconformal mappings in dynamics.

Assuming U to be a fixed component, the dynamics can be classified as follows.

Proposition 3.1. *Let U be a fixed component of the Fatou set of a rational function. Then one of the following alternatives is true.*

(1) U contains an attracting fixed point for which all points in U converge to under iteration,

(2) ∂U contains a neutral fixed point for which all point in U converge to under iteration,

(3) U is either conformally equivalent to the disk or an annulus, and the dynamics is conjugated to a euclidean rotation.

In case (2) above, the neutral fixed point, say $z = R(z)$, is in fact a so-called *parabolic fixed point*. By definition this means that $DR(z) = e^{ip/q}$, with p and q being integers. If U is of type (3), it is called a *Siegel disk* if it is conformally equivalent to the disk, and a *Herman ring* if it is conformally equivalent to an annulus. The rotation angle is, in either case, irrational. Proposition 3.1 can be naturally generalised to periodic components by considering a suitable iterate of the rational map.

Let us now begin to consider the dynamics on the Julia set. It is illustrative to consider the most simple function, namely a complex quadratic one

$$z \mapsto z^2 + a = P_a(z),$$

with $a \in \mathbb{C}$. The following result tells us that the behaviour of the critical point has direct consequences for the geometry of the Julia set.

Proposition 3.2. *If $P_a^n(0)$ tends to infinity as n tends to infinity, then the Julia set $\mathcal{J}(P_a)$ is totally disconnected. Otherwise it is connected.*

The above result motivates the definition of the so-called *connectedness locus*, which is the set consisting of those parameters a for which $\mathcal{J}(P_a)$ is connected. In the case of the (complex) quadratic family this set is usually called the *Mandelbrot set*, after B. Mandelbrot [Man80] who was the first to obtain high quality pictures of it (see also [BM81]). We denote the Mandelbrot set by $\mathcal{M}$, and from Proposition 2.2 we know that $\mathcal{M}$ intersects the real line in $[-2, 1/4]$. Moreover we have the following result.

Proposition 3.3. *$\mathcal{M}$ is a closed simply connected subset of the disk $\{|a| \leq 2\}$, and consists of precisely those a such that $P_a^n(0) \leq 2$ for all $n \geq 0$.*

Figure 1.2 provides a picture of the Mandelbrot set, and we notice the close connection with the bifurcation diagram of Figure 1.1. Indeed, the parameter values for which period doubling bifurcation occurs are precisely those parameters in the Mandelbrot set lying on the real axis connecting the components.

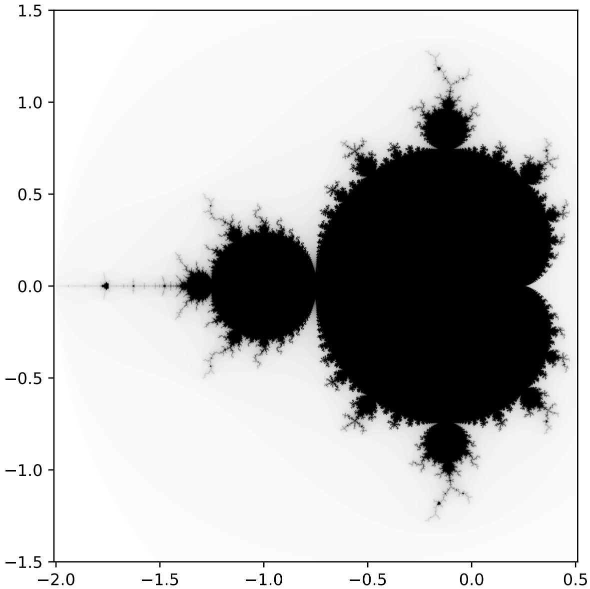

Figure 1.2: Connectedness locus for $z \mapsto z^2 + c$.

In the rational setting there is no analogue of Proposition 3.2, however the behaviour of the critical orbits are equally important for the global dynamics. The following result resembles that of Proposition 2.1.

Proposition 3.4. *For each attracting cycle of a rational function of degree $d \geq 2$ there is at least one critical point whose orbit accumulates on this cycle. The number of critical points (counting multiplicity) is at most $2d - 2$, hence there are at most $2d - 2$ attracting cycles.*

In order to understand the dynamics on the Julia set, the following definition is of central importance. We notice the resemblance with Definition 2.5.

Definition 3.5. A rational function $z \mapsto R(z)$ is called *hyperbolic* if every critical point belongs to the Fatou set $\mathcal{F}(R)$ and is attracted to an attracting cycle. Otherwise it is called *non-hyperbolic*.

Being hyperbolic is equivalent to the existence of a metric, smoothly equivalent to the spherical metric in a neighbourhood of the Julia set, for which R is *expanding*. If we assume that $\infty \notin \mathcal{J}(R)$, then this is equivalent to the existence of $C > 0$ and $\gamma > 0$ such that

$$|(R^n)'(z)| \geq Ce^{\gamma n}$$

for all $z \in \mathcal{J}(R)$ and $n \geq 1$. (This latter notion of expanding on the Julia set is in fact the usual definition of being hyperbolic, and our definition can be proved to be equivalent.)

One of the great open conjectures in the field of rational dynamics is the so-called *Hyperbolicity conjecture*: the set of hyperbolic rational maps form an (open) dense set in parameter space. Even in the case of the quadratic family $z \mapsto z^2 + a$ it is not yet known whether the set of (complex) parameters a forms an open dense set (this is the so-called *Fatou conjecture*).

4 The Collet–Eckmann conditions

As mentioned earlier, the Collet–Eckmann condition was first introduced by Collet and Eckmann [CE83, CE80] in their study of certain real families of dynamical systems, and was used to prove the abundance of acip's.

The Collet–Eckmann condition has proven to be very fruitful to consider also in the rational setting, although things naturally become more complex. We give the following definition.

Definition 4.1. A rational function R without parabolic cycles is said to satisfy the *Collet–Eckmann condition* (CE) if there exist $C > 0$ and $\gamma > 0$ such that, for each critical point c in the Julia set of R,

$$|DR^n(R(c))| \geq Ce^{\gamma n},$$

for all $n \geq 0$.

The requirement of no parabolic cycles is a technical one since, for instance, one usually wants uniform expansion outside a neighbourhood of the critical points in the Julia set. From now on we denote by $\mathrm{Crit}'(R)$ the set of critical points in the Julia set of R, i.e. $\mathrm{Crit}'(R) = \mathrm{Crit}(R) \cap \mathcal{J}(R)$.

The study of rational Collet–Eckmann maps was initiated by F. Przytycki [Prz96, Prz98]. For instance, in [Prz96] it is proved that if $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, then $\mathrm{Leb}\, \mathcal{J}(R) = 0$, i.e. for a rational Collet–Eckmann map, either the Julia set is the entire sphere, or it has measure zero. Moreover, by assuming an extra condition by M. Tsujii, namely that the average distance of $R^n(\mathrm{Crit}')$ to Crit' is not too small, it was also proved that the Hausdorff dimension of $\mathcal{J}(R)$ is strictly less than 2 (provided $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, of course). Later, Graczyk and Smirnov [GS98a] proved, among other things, that rational Collet–Eckmann maps can have no rotation domains, and the Fatou components are Hölder domains. (Using a result by P. Jones and N. Makarov [JM95], this latter property implies that, for a rational Collet–Eckmann map with at least one fully invariant Fatou component, the Hausdorff dimension of its Julia set is strictly less than 2.)

That rational Collet–Eckmann maps are interesting from a measure point of view was established by M. Aspenberg [Asp04, Asp13] in his doctoral thesis: the set of Collet–Eckmann maps has positive (Lebesgue) measure in the parameter space of rational functions of any fixed degree $d \geq 2$. Moreover, using the results of Przytycki [Prz96], and Graczyk and Smirnov [GS98a], these maps described by Aspenberg also support acip's. The existence of a positive measure set of rational maps having acip's was first proved by M. Rees [Ree86].

Considering the recurrent nature of rational functions, Aspenberg [Asp09] furthermore proved that the set of rational Misiurewicz functions of any fixed degree $d \geq 2$ constitutes only a set of measure zero in the parameter space. Therefore, analogous to the case of real quadratic functions, the critical points belonging to the Julia set of a typical non-hyperbolic rational function are recurrent. Results regarding the rate of recurrence of the critical points for non-hyperbolic rational functions are more sparse than in the real quadratic setting. In the quadratic (and even unicritical) setting $z \mapsto z^2 + a = P_a(z)$, the Collet–Eckmann parameters are known to constitute only a set of measure zero [ALS11]. However, Graczyk and Świątek [GS00] proved that for a typical parameter with respect to *harmonic measure* on the boundary of the Mandelbrot set, the Collet–Eckmann condition is satisfied (see also [Smi00]). Moreover, they proved in [GS15] that the *Lyapunov exponent* $\chi(a)$ exists: for a typical parameter $a \in \partial\mathcal{M}$ with respect to harmonic measure,

$$\chi(a) = \lim_{n\to\infty} \frac{\log|(P_a)'(a)|}{n} = \log 2.$$

This in turn immediately gives us a recurrence result: for every $\alpha > 0$ there exists a constant

$C = C(\alpha) > 0$ such that
$$|P_a^n(0)| \geq Ce^{-\alpha n},$$
for all $n \geq 1$. For a rational function, we make the following definition.

Definition 4.2. A rational function R of degree $d \geq 2$ is said to satisfy the *slow recurrence condition* (SR) if for every $\alpha > 0$ there exists $C = C(\alpha) > 0$ such that, for every critical point $c \in \mathrm{Crit}'(R)$,
$$\mathrm{dist}(R^n(c), \mathrm{Crit}') \geq Ce^{-\alpha n},$$
for all $n \geq 1$.

Not much is known about the measure of rational functions satisfying the slow recurrence condition, however it is conjectured to be satisfied for almost every rational Collet–Eckmann map. We should also mention that, to the author's knowledge, no results exist regarding the typical rate of recurrence in the rational setting, i.e. for what δ_n do we have, given $c \in \mathrm{Crit}'$,
$$\mathrm{dist}(R^n(c), \mathrm{Crit}') < \delta_n$$
for infinitely many n? We do believe, however, that the techniques of Paper I can be carried over to the rational setting.

Focusing on this slow recurrence condition, Aspenberg [Asp21] recently proved the following consequence. Let R be a rational Collet–Eckmann map of degree $d \geq 2$, satisfying the slow recurrence condition, and such that $\mathcal{J}(R) = \widehat{\mathbb{C}}$. Then R is a Lebesgue density point of rational Collet–Eckmann maps of degree d within the space $\Lambda_{d,\overline{p}'}$. In particular, this generalises the results in [Asp04, Asp13]. Motivated by this result, together with Aspenberg and W. Cui, in Paper II we consider functions as above but with $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, and prove that these are density points of hyperbolic maps. In particular, assuming that almost every rational Collet–Eckmann map satisfies the slow recurrence condition, then almost every Collet–Eckmann map has its Julia set equal to the Riemann sphere.

Let us finish this section with discussing some other closely related conditions of non-hyperbolicity. Already in [CE83, CE80], a condition now known as the *second (or backward) Collet–Eckmann condition* was considered. The definition in the rational setting is as follows.

Definition 4.3. A rational map R of degree $d \geq 2$ is said to satisfy the *second Collet–Eckmann condition* (CE2) if there exist constants $C_2 > 1$ and $\gamma_2 > 0$ such that, for every $n \geq 1$ and every $w \in R^{-n}(c)$, for $c \in \mathrm{Crit}'(R)$ not in the forward orbit of other critical points,
$$|DR^n(w)| \geq C_2 e^{\gamma_2 n}.$$

Graczyk and Smirnov [GS98a] proved that CE and CE2 are equivalent in the unicritical setting $z \mapsto z^d + a$. In Paper I and Paper II, this condition is utilised to prove strong expansion results outside a neighbourhood of the critical point(s).

In their study of the geometry of Collet–Eckmann Julia sets, Przytycki and S. Rohde [PR98] formulated the following condition.

Definition 4.4. A rational map R of degree $d \geq 2$ is said to satisfy the *topological Collet–Eckmann condition* (TCE) if there exist $M \geq 0$, $P \geq 0$ and $r > 0$ such that for every $z \in \mathcal{J}(R)$ there exists a strictly increasing sequence of positive integers n_j, $j = 1, 2, \ldots$, such that $n_j \leq Pj$ and, for each j,

$$\#\left\{k : 0 \leq k < n_j, \operatorname{Comp}_{R^k(z)} R^{-(n_j-k)}\left(B(R^{n_j}(z), r)\right) \cap \operatorname{Crit} \neq \emptyset\right\} \leq M.$$

Here in the above definition, Comp_w denotes the connected component containing w. Since the above condition is formulated in topological terms, it is invariant under topological conjugacy. One of the more useful properties of the topological Collet–Eckmann condition is its many equivalent formulations [PRLS03, PRL07, RL10]. In particular, CE and CE2 independently imply TCE.

Much work has been done to understand the relationships between these three characterisations of non-hyperbolicity. Przytycki, Smirnov, and J. Rivera-Letelier [PRLS03] made an extensive study and proved, among other things, that these conditions are equivalent within the family of unicritical functions $z \mapsto z^d + a$. In Paper III, we observe yet another consequence of the slow recurrence condition, namely that within the family of slowly recurrent rational maps of degree $d \geq 2$, all of these conditions are equivalent. Since there are known examples where CE does not imply CE2, CE2 does not imply CE, and TCE does not imply CE or CE2, this shows that the slow recurrence condition is in some sense essential for equivalence to hold.

5 The Benedicks–Carleson techniques

In their seminal papers, Benedicks and Carleson [BC85, BC91] developed techniques to prove the abundance of Collet–Eckmann real quadratic functions, and the existence of acip's. However, this machinery of theirs is far reaching, as can be realised by the many papers utilising it. In fact, it is the foundational tool used in Paper I and Paper II of this thesis. In this section we try to provide a schematic outline of these *parameter exclusion techniques*.

At its core, these techniques constitute a technical induction argument, with the Collet–Eckmann condition being the driving force. For the sake of explanation, let $f = f_0$ be the so-called *unperturbed map*, acting on some space X. We ask of this map to satisfy the Collet–Eckmann condition: there exist constants $C > 0$ and $\gamma > 0$ such that, for all critical points c of f belonging to $\mathcal{J}(f)$,

$$|(f^n)'(f(c))| \geq Ce^{\gamma n},$$

for all $n \geq 0$.

For a in some subset $\Delta = \Delta_0$ of the parameter space, we let f_a denote a perturbation of f. The goal is to show that for a large (or small) set of parameters, the corresponding perturbations f_a share similar properties as the unperturbed map.

To this end, suppose that f only has one critical point $c = c(0)$, and that the corresponding perturbation f_a only has one critical point $c(a)$. In fact, let us assume a normalisation so that $c(a) = 0$ for all $a \in \Delta$. We will iterate the critical point simultaneously for different parameters, and we let $\xi_n : \Delta \mapsto X$ denote the function $a \mapsto \xi_n(a) = f_a^n(0)$.

If Δ is chosen sufficiently small then, up to some large time N, the Collet–Eckmann condition is inherited by all perturbations. In particular, as long as the derivatives of f_a^N and f_b^N, evaluated at their corresponding critical values, are comparable, the Collet–Eckmann condition gives *expansion* of the image. This property of having comparable derivatives is called *distortion*. At some time $m_1 \geq N$, the image of Δ will come very close to, and might even cover, the critical point. At this stage one makes a *partition*: $\Delta = \bigcup_k \Delta_{1,k}$. This partition is made so that on each partition element $\Delta_{1,k}$, we have good distortion control. Each of the partition elements will then be iterated individually until the same situation occurs. That is to say, the partition element $\Delta_1 = \Delta_{1,k}$, for instance, will be iterated until at some time $m_2 \geq m_1$ its image $\xi_{m_2}(\Delta_1)$ gets close to the critical point. At this stage we once again make a partition $\Delta_1 = \bigcup_k \Delta_{2,k}$, and the procedure continuous indefinitely.

At each stage of partitioning, one might have to discard parameters that belong to partition elements that come too close to the critical point. The reason for this is to make sure that not too much derivative is lost, hence ensuring a Collet–Eckmann condition for future iterates. This approach rate condition is usually referred to as the *basic assumption*: for all $a \in \Delta$ we ask that

$$\operatorname{dist}\left(f_a^n(0), 0\right) \geq \delta_n,$$

for all $n \geq 1$, and for some suitable sequence δ_n.

Even though some derivative is lost when returning close to the critical point, much (but not all) of what was lost will be recovered during the so called *bound period*. Indeed, the Collet–Eckmann condition is a standing induction assumption, and for some time after the partition, the future iterates will stay close to the past iterates. Using this fact, one can show that during this bound period, derivative from the past iterates will be inherited by the future iterates.

In order to estimate what is left in parameter space after each partition stage, one needs to be able to compare the parameter derivative of ξ_n with the phase derivative of f_a^{n-1}. This kind of comparison is called *transversality*. Assuming good distortion estimates, and good transversality estimates, the measure of what is left in parameter space after infinitely long time is essentially determined by whether the sequence δ_n in the basic assumption is summable or not.

Chapter 2

Summary of results

Paper I

In this paper we study the real quadratic family

$$x \mapsto x^2 + a = Q_a(x),$$

acting on $X = [-2, 2]$, and with parameter $a \in [-2, 1/4]$. Our goal is to investigate the *typical recurrence rate* of the critical point $x = 0$ to itself, when a is a nonregular parameter, i.e. when a is such that $x \mapsto x^2 + a$ has no attracting cycle. With typical recurrence rate we mean a sequence δ_n such that, for almost every nonregular parameter a,

$$|Q_a^n(0)| < \delta_n$$

holds true for infinitely many n. Without loss of generality we may assume $a \in [-2, -1]$, and for such parameters we instead study the equivalent family

$$x \mapsto 1 - ax^2 = F(x; a),$$

acting on $X = [-1, 1]$, and with parameter $a \in [1, 2]$.

A. Avila and C. G. Moreira [AM05] proved two important results regarding the real quadratic family. The first result states that almost every nonregular parameter satisfies the Collet–Eckmann condition. The second result concerns recurrence, and states that for almost every nonregular parameter a

$$\limsup_{n \to \infty} \frac{-\log |F^n(0; a)|}{\log n} = 1.$$

Introducing the set $\Lambda(\delta_n) = \{a \in \mathcal{NR} : |F^n(0;a)| < \delta_n \text{ for infinitely many } n\}$ the above equality can be rephrased as

$$\operatorname{Leb}\Lambda(n^{-\theta}) = \begin{cases} \operatorname{Leb}\mathcal{NR} & \text{if } \theta < 1, \\ 0 & \text{if } \theta > 1. \end{cases}$$

The above lim sup-result is strong and gives us both a typical recurrence rate, namely $\delta_n = n^{-(1-\epsilon)}$ for any $\epsilon > 0$, but also a *typical approach rate*: for almost every nonregular parameter a and $\epsilon > 0$ there exists a constant $C = C(a, \epsilon)$ such that

$$|F^n(0;a)| \geq \frac{C}{n^{1+\epsilon}} \quad \text{for all} \quad n \geq 1.$$

What the lim sup cannot see, though, is the sharpness of the exponent, i.e. the case of $\epsilon = 0$, and to investigate this is the main concern of Paper I.

Let us call a sequence δ_n *admissible* if there exists a constant $K > 0$ and an exponent $\sigma \geq 0$ such that

$$\delta_n \geq \frac{K}{n^\sigma} \quad \text{for all} \quad n \geq 1.$$

In Paper I we prove the following result. There exists $\tau \in (0,1)$ such that if δ_n is admissible and

$$\sum \frac{\delta_n}{\log n} \tau^{(\log^* n)^3} = \infty,$$

then $\operatorname{Leb}\Lambda(\delta_n) = \operatorname{Leb}\mathcal{NR}$. Here $\log^*$ is the so-called iterated logarithm, and it is defined as

$$\log^* x = \begin{cases} 1 & \text{if } x \leq 1, \\ 1 + \log^* \log x & \text{if } x > 1. \end{cases}$$

In particular, $\log^*$ grows slower than any $\log_j = \log \circ \log_{j-1}$, $j \geq 0$. Therefore as a direct corollary we find that

$$\operatorname{Leb}\Lambda(n^{-1}) = \operatorname{Leb}\mathcal{NR},$$

thus covering the missing case of $\theta = 1$.

The proof utilises the Benedicks–Carleson techniques [BC85, BC91], together with more recent developments [Asp21, Lev14]. The main innovation of this paper is the introduction of unbounded distortion estimates.

Paper II

We consider slowly recurrent rational functions of a fixed degree and whose Julia set is not equal to the entire sphere. By assuming that the critical points approach each other only at a slow rate, i.e. by assuming the so-called slow recurrence condition, we prove that these functions can be approximated in a strong sense by hyperbolic functions.

Let us call two rational functions equivalent if they are conjugated by a Möbius transformation. In the parameter space of rational functions of a fixed degree $d \geq 2$, let $\Lambda_{d,\overline{p'}}$ denote the subspace of rational functions, up to equivalence, with exactly p' critical points $c_1, c_2, \ldots, c_{p'}$, and with corresponding multiplicities $\overline{p'} = (m_1, m_2, \ldots, m_{p'})$. Within this subspace, critical points do not split, and move analytically with the parameter. In this paper, we look at small perturbation of $R = R_0 \in \Lambda_{d,\overline{p'}}$, where R satisfies the Collet–Eckmann condition, and $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$. Moreover, R also satisfies the slow recurrence condition: for any $\alpha > 0$ there exists $C > 0$ such that, for every $c \in \mathrm{Crit}'$,

$$\operatorname{dist}\left(R^n(c), \mathrm{Crit}'\right) \geq Ce^{-\alpha n},$$

for all $n \geq 1$. In Paper II we prove that such a rational function is a Lebesgue density point of hyperbolic functions (within $\Lambda_{d,\overline{p'}}$). Moreover, if all critical points are simple, then such a function is a Lebesgue density point of hyperbolic functions in the entire space of rational functions of degree d.

To prove the above result, we utilise the parameter exclusion techniques developed by Benedicks and Carleson [BC85, BC91], together with its evolvement in the rational setting by Aspenberg [Asp04, Asp13, Asp09, Asp21], and strong transversality results by Levin [Lev14]. In fact, Aspenberg [Asp21] recently proved a contrasting result. Namely, if $R \in \Lambda_{d,\overline{p'}}$ satisfies the Collet–Eckmann condition, if $\mathcal{J}(R) = \widehat{\mathbb{C}}$, and if R satisfies the slow recurrence condition, then it is a Lebesgue density point of Collet–Eckmann functions (within $\Lambda_{d,\overline{p'}}$).

The techniques used in Paper II are similar to those in [Asp21]. We begin with a small parameter square centred at R, and our goal is for this square to reach to so-called *large scale*. Since $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, the measure of the Julia set $\mathcal{J}(R)$ is equal to zero [Prz96]. Therefore, upon reaching the large scale, a large portion of our square will correspond to parameters whose critical points lie in the Fatou set. We show that the large scale is reached under bounded transversality, and bounded distortion, and the conclusion is that in parameter space, most parameters correspond to hyperbolic maps, hence our density result.

Paper III

In Paper III we consider rational functions acting on the Riemann sphere $\widehat{\mathbb{C}}$, and the relationships between the Collet–Eckmann condition (CE), the second Collet–Eckmann condition (CE2), and the topological Collet–Eckmann condition (TCE). Much work has been made investigating these conditions. In particular it is known that CE or CE2 implies TCE, whereas to any other possible implication there are known counterexamples. In the unicritical, on the other hand, all of these conditions are equivalent. (See [PRLS03] and references therein.)

In this paper we observe that within the family of slowly recurrent rational functions, all of the above conditions are equivalent. Moreover these conditions are invariant under topological conjugation. The proofs in this paper are short, even though the results on which they are based upon require technical machinery. Indeed, the techniques are those of *shrinking neighbourhoods* as developed by Przytycki [Prz98], and used by Graczyk and Smirnov [GS98a].

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Part II

Scientific papers

Paper I

Critical recurrence in the real quadratic family

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Abstract

We study recurrence in the real quadratic family and give a sufficient condition on the recurrence rate (δ_n) of the critical orbit such that, for almost every nonregular parameter a, the set of n such that $|F^n(0;a)| < \delta_n$ is infinite. In particular, when $\delta_n = n^{-1}$, this extends an earlier result by Avila and Moreira.

1 Introduction

1.1 Regular and nonregular parameters

Given a real parameter a, we let $x \mapsto 1 - ax^2 = F(x;a)$ denote the corresponding real quadratic map. We will study the recurrent behaviour of the critical point $x = 0$ when the parameter belongs to the interval $[0,2]$. For such a choice of parameter there exists an invariant interval $I_a \subset [-1,1]$, i.e.

$$F(I_a;a) \subset I_a,$$

containing the critical point $x = 0$. The parameter interval is naturally divided into a *regular* ($\mathcal{R}$) and *nonregular* ($\mathcal{NR}$) part

$$[0,2] = \mathcal{R} \cup \mathcal{NR},$$

with $a \in \mathcal{R}$ being such that $x \mapsto 1-ax^2$ has an attractive cycle, and $\mathcal{NR} = [0,2] \setminus \mathcal{R}$. These two sets turn out to be intertwined in an intricate manner, and this has led to an extensive study of the real quadratic family. We briefly mention some of the more fundamental results, and refer to [Lyu00b] for an overview.

The regular maps are from a dynamic point of view well behaved, with almost every point, including the critical point, tending to the attractive cycle. This set of parameters, which with an application of the inverse function theorem is seen to be open, constitutes a large portion of $[0, 2]$. The celebrated genericity result, known as the real Fatou conjecture, was settled independently by Graczyk–Świątek [GS97] and Lyubich [Lyu97]: *$\mathcal{R}$ is (open and) dense*. This has later been extended to real polynomials of arbitrary degree by Kozlovski–Shen–van Strien [KSvS07], solving the second part of the eleventh problem of Smale [Sma98]. The corresponding result for complex quadratic maps, the Fatou conjecture, is still to this day open.

The nonregular maps, in contrast to the regular ones, exhibit chaotic behaviour. In [Jak81] Jakobson showed the abundance of *stochastic* maps, proving that the set of parameters $a \in \mathcal{S}$ for which the corresponding quadratic map has an absolutely continuous (with respect to Lebesgue) invariant measure (a.c.i.m), is of positive Lebesgue measure. This showed that, from a probabilistic point of view, nonregular maps are not negligible: for a regular map, any (finite) a.c.i.m is necessarily singular with respect to Lebesgue measure.

Chaotic dynamics is often associated with the notion of sensitive dependence on initial conditions. A compelling way to capture this property was introduced by Collet and Eckmann in [CE80] where they studied certain maps of the interval having expansion along the critical orbit, proving abundance of chaotic behaviour. This condition is now known as the *Collet–Eckmann condition*, and for a real quadratic map it states that

$$\liminf_{n \to \infty} \frac{\log |\partial_x F^n(1; a)|}{n} > 0. \tag{1}$$

Focusing on this condition, Benedicks and Carleson gave in their seminal papers [BC85, BC91] another proof of Jakobson's theorem by proving the stronger result that the set $\mathcal{CE}$ of Collet–Eckmann parameters is of positive measure. As a matter of fact, subexponential increase of the derivative along the critical orbit is enough to imply the existence of an a.c.i.m, but the stronger Collet-Eckmann condition implies, and is sometimes equivalent with, ergodic properties such as exponential decay of correlations [KN92, You92, NS98], and stochastic stability [BV96]. For a survey on the role of the Collet–Eckmann condition in one-dimensional dynamics, we refer to [Ś01].

Further investigating the stochastic behaviour of nonregular maps, supported by the results in [Lyu00a, MN00], Lyubich [Lyu02] established the following famous dichotomy: *almost all real quadratic maps are either regular or stochastic*. Thus it turned out that the stochastic behaviour described by Jakobson is in fact typical for a nonregular map. In [AM05] Avila and Moreira later proved the strong result that expansion along the critical orbit is no exception either: *almost all nonregular maps are Collet–Eckmann*. Thus a typical nonregular map have excellent ergodic properties.

1.2 Recurrence and Theorem A

In this paper we will study recurrence of the critical orbit to the critical point, for a typical nonregular (stochastic, Collet–Eckmann) real quadratic map. For this reason we introduce the following set.

Definition 1.1 (Recurrence Set)**.** Given a sequence $(\delta_n)_{n=1}^{\infty}$ of real numbers, we define the *recurrence set* as

$$\Lambda(\delta_n) = \{a \in \mathcal{NR} : |F^n(0;a)| < \delta_n \text{ for finitely many } n\}.$$

In [AM05] Avila and Moreira also established the following recurrence result, proving a conjecture of Sinai: *for almost every nonregular parameter a*

$$\limsup_{n\to\infty} \frac{-\log|F^n(0;a)|}{\log n} = 1.$$

Another way to state this result is as follows: for almost every nonregular parameter a, the set of n such that $|F^n(0;a)| < n^{-\theta}$ is finite if $\theta > 1$ and infinite if $\theta < 1$. In terms of the above defined recurrence set, this result translates to

$$\operatorname{Leb}\Lambda(n^{-\theta}) = \begin{cases} \operatorname{Leb}\mathcal{NR} & \text{if } \theta > 1, \\ 0 & \text{if } \theta < 1. \end{cases}$$

In [GS14], as a special case, a new proof of the positive measure case in the above stated result was obtained, together with a new proof that almost every nonregular map is Collet–Eckmann. In this paper we will give a new proof of the measure zero case, and in particular we will fill in the missing case of $\theta = 1$, thus completing the picture of polynomial recurrence. Our result will be restricted to the following class of recurrence rates.

Definition 1.2. A nonincreasing sequence (δ_n) of positive real numbers is called *admissible* if there exists a constant $0 \le \bar{e} < \infty$, and an integer $N \ge 1$, such that

$$\delta_n \ge \frac{1}{n^{\bar{e}}} \qquad (n \ge N).$$

The following is the main result of this paper.

Theorem A. *There exists* $\tau \in (0,1)$ *such that if* (δ_n) *is admissible and*

$$\sum \frac{\delta_n}{\log n}\tau^{(\log^* n)^3} = \infty,$$

then $\operatorname{Leb}(\Lambda(\delta_n) \cap \mathcal{CE}) = 0$.

Here $\log^*$ denotes to so-called *iterated logarithm*, which is defined recursively as

$$\log^* x = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 + \log^* \log x & \text{if } x > 1. \end{cases}$$

That is, $\log^* x$ is the number of times one has to iteratively apply the logarithm to x in order for the result to be less than or equal to 1. In particular, $\log^*$ grows slower than $\log_j = \log \circ \log_{j-1}$, for any $j \geq 1$.

Theorem A, together with the fact that almost every nonregular real quadratic map is Collet–Eckmann, clearly implies

Corollary 1.3. $\operatorname{Leb} \Lambda(n^{-1}) = 0$.

Remark 1.4. In fact, one can conclude the stronger statement

$$\operatorname{Leb} \Lambda(1/(n \log \log n)) = 0.$$

At this moment we do not get any result for when $\delta_n = 1/(n \log n)$, and this would be interesting to investigate further.

One of the key points in the proof of Theorem A is the introduction of unbounded distortion estimates; this differs from the classical Benedicks–Carleson techniques.

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2 Reduction and outline of proof

2.1 Some definitions and Theorem B

We reduce the proof of Theorem A to that of Theorem B, stated below. For this we begin with some suitable definitions.

It will be convenient to explicitly express the constant in the Collet–Eckmann condition (1), and for this reason we agree on the following definition.

Definition 2.1. Given $\gamma, C > 0$ we call a parameter a (γ, C)-Collet–Eckmann if

$$|\partial_x F^n(1; a)| \geq Ce^{\gamma n} \qquad (n \geq 0).$$

The set of all (γ, C)-Collet–Eckmann parameters is denoted $\mathcal{CE}(\gamma, C)$.

Our parameter exclusion will be carried out on intervals centred at Collet–Eckmann parameters satisfying the following recurrence assumption.

Definition 2.2. A Collet–Eckmann parameter a is said to have *polynomial recurrence* (PR) if there exist constants $K = K(a) > 0$ and $\sigma = \sigma(a) \geq 0$ such that

$$|F^n(0; a)| \geq \frac{K}{n^\sigma} \qquad (n \geq 1).$$

The set of all PR-parameters is denoted $\mathcal{PR}$.

Finally, we consider parameters for which the corresponding quadratic maps satisfy the reversed recurrence condition after some fixed time $N \geq 1$:

$$\Lambda_N(\delta_n) = \{a \in \mathcal{NR} : |F^n(0; a)| \geq \delta_n \text{ for all } n \geq N\}.$$

Clearly we have that

$$\Lambda(\delta_n) = \bigcup_{N \geq 1} \Lambda_N(\delta_n).$$

Theorem A will be deduced from

Theorem B. *There exists $\tau \in (0, 1)$ such that if (δ_n) is admissible and*

$$\sum \frac{\delta_n}{\log n} \tau^{(\log^* n)^3} = \infty,$$

then for all $N \geq 1$, $\gamma > 0$, $C > 0$, and for all $a \in \mathcal{PR}$, there exists an interval ω_a centred at a such that

$$\mathrm{Leb}(\Lambda_N(\delta_n) \cap \mathcal{CE}(\gamma, C) \cap \omega_a) = 0.$$

2.2 Proof of Theorem A

Using Theorem B, Theorem A is proved by a standard covering argument. Since ω_a is centred at a, so is the smaller interval $\omega_a' = \omega_a/5$. By Vitali covering lemma there exists a countable collection (a_j) of PR-parameters such that

$$\mathcal{PR} \subset \bigcup_{a \in \mathcal{PR}} \omega_a' \subset \bigcup_{j=1}^{\infty} \omega_{a_j}.$$

It now follows directly that

$$\operatorname{Leb}(\Lambda_N(\delta_n) \cap \mathcal{CE}(\gamma, C) \cap \mathcal{PR}) \le \sum_{j=1}^{\infty} \operatorname{Leb}(\Lambda_N(\delta_n) \cap \mathcal{CE}(\gamma, C) \cap \omega_{a_j}) = 0,$$

and therefore

$$\begin{aligned}\operatorname{Leb}(\Lambda(\delta_n) \cap \mathcal{CE} \cap \mathcal{PR}) &\le \sum_{N,k,l \ge 1} \operatorname{Leb}(\Lambda_N(\delta_n) \cap \mathcal{CE}(k^{-1}\log 2, l^{-1}) \cap \mathcal{PR}) \\ &= 0.\end{aligned}$$

Finally, we notice that $\Lambda(\delta_n) \cap \mathcal{CE} \subset \mathcal{PR}$; indeed this is clearly the case since (δ_n) is assumed to be admissible.

Remark 2.3. With the introduction of the set $\mathcal{PR}$ we are avoiding the use of previous recurrence results (e.g. Avila–Moreira) in order to prove Theorem A, by (a priori) allowing $\mathcal{PR}$ to be a set of measure zero. In either case, the statement of Theorem A is true.

2.3 Outline of proof of Theorem B

The proof of Theorem B will rely on the classical parameter exclusion techniques developed by Benedicks and Carleson [BC85, BC91], complemented with more recent results. In particular we allow for perturbation around a parameter in more general position than $a = 2$. In contrast to the usual application of these techniques, our goal here is the show that what remains after excluding parameters is a set of zero Lebesgue measure. One of the key points in our approach is the introduction of unbounded distortion estimates.

We will carefully study the returns of the critical orbit, simultaneously for maps corresponding to parameters in a suitable interval $\omega \subset [0, 2]$, to a small and fixed interval $(-\delta, \delta) = (-e^{-\Delta}, e^{-\Delta})$. (In fact, we will assume that $\omega \subset [1, 2]$ since $[0, 1] \setminus \{3/4\} \subset \mathcal{R}$, with $a = 3/4$ being a parabolic parameter.) These returns to $(-\delta, \delta)$ will be classified as either *inessential*, *essential*, *escape*, or *complete*. Per definition of a complete return, we return close enough to $x = 0$ to be able to remove a large portion of $(-\delta_n, \delta_n)$ in phase space. To estimate what is removed in parameter space, we need distortion estimates. This will be achieved by i) enforcing a (γ, C)-Collet–Eckmann condition, and ii) continuously making suitable partitions in phase space: $(-\delta, \delta)$ is subdivided into partition elements $I_r = (e^{-r-1}, e^{-r})$ for $r > 0$, and $I_r = -I_{-r}$ for $r < 0$. Furthermore, each I_r is subdivided into r^2 smaller intervals $I_{rl} \subset I_r$, of equal length $|I_r|/r^2$. After partitioning, we consider iterations of each partition element individually, and the proof of Theorem B will be one by induction.

We make a few comments on the summability condition appearing in the statement of Theorem A and Theorem B. In order to prove our result we need to estimate how much is removed at a complete return, but also how long time it takes from one complete return to the next. The factor $\tau^{(\log^* n)^3}$ is connected to the estimate of what is removed at complete returns, and more specifically it is connected to distortion; as will be seen, our distortion estimates are unbounded. The factor $(\log n)^{-1}$ is directly connected to the time between two complete returns: if n is the index of a complete return, it will take $\lesssim \log n$ iterations until we reach the next complete return.

In the next section we prove a couple of preliminary lemmas, and confirm the existence of a suitable start-up interval ω_a centred at $a \in \mathcal{PR}$, for which the parameter exclusion will be carried out. After that, the induction step will be proved, and an estimate for the measure of $\Lambda_N(\delta_n) \cap \mathcal{CE}(\gamma, C) \cap \omega_a$ will be given.

3 Preliminary Lemmas

In this section we establish three important lemmas that will be used in the induction step. These are derived from Lemma 2.6, Lemma 2.10, and Lemma 3.1 in [Asp21], respectively, where they are proved in the more general setting of a complex rational map.

3.1 Outside Expansion Lemma

The first result we will need is the following version of the classical Mañé Hyperbolicity Theorem (see [dMvS93], for instance).

Lemma 3.1 (Outside Expansion). *Given a Collet–Eckmann parameter a_0 there exist constants $\gamma_M, C_M > 0$ such that, for all $\delta > 0$ sufficiently small, there is a constant $\epsilon_M = \epsilon_M(\delta) > 0$ such that, for all $a \in (a_0 - \epsilon_M, a_0 + \epsilon_M)$, if*

$$x, F(x;a), F^2(x;a), \dots, F^{n-1}(x;a) \notin (-\delta, \delta),$$

then

$$|\partial_x F^n(x;a)| \geq \delta C_M e^{\gamma_M n}.$$

Furthermore, if we also have that $F^n(x;a) \in (-2\delta, 2\delta)$, then

$$|\partial_x F^n(x;a)| \geq C_M e^{\gamma_M n}.$$

A similar lemma for the quadratic family can be found in [BBS15] and [Tsu93], for instance. The version stated here allows for δ-independence at a more shallow return to the interval $(-2\delta, 2\delta)$. To get this kind of annular result constitutes a minor modification of Lemma 4.1 in [Tsu93]. We refer to Lemma 2.6 in [Asp21] and the proof therein, however, for a proof of the above result. This proof is based on Przytycki's *telescope lemma* (see [Prz90] and also [PRLS03]). In contrast to the techniques in [Tsu93], in the case of the quadratic family, no recurrence assumption is needed.

3.2 Phase-parameter distortion

If $t \mapsto F(x; a + t)$ is a family of (analytic) perturbations of $(x; a) \mapsto F(x; a)$ at a, we may expand each such perturbation as

$$F(x; a + t) = F(x; a) + t\partial_a F(x; a) + \text{higher order terms},$$

and it is easy to verify that

$$\frac{\partial_a F^n(x; a)}{\partial_x F^{n-1}(F(x; a); a)} = \frac{\partial_a F^{n-1}(x; a)}{\partial_x F^{n-2}(F(x; a); a)} + \frac{\partial_a F(F^{n-1}(x; a); a)}{\partial_x F^{n-1}(F(x; a); a)}.$$

Our concern is with the quadratic family $x \mapsto 1-ax^2 = F(x; a)$, with a being the parameter value. In particular we are interested in the critical orbit of each such member, and to this end we introduce the functions $a \mapsto \xi_j(a) = F^j(0; a)$, for $j \geq 0$. In view of our notation and the above relationship, we see that

$$\frac{\partial_a F^n(0; a)}{\partial_x F^{n-1}(1; a)} = \sum_{k=0}^{n-1} \frac{\partial_a F(\xi_k(a); a)}{\partial_x F^k(1; a)}.$$

Throughout the proof of Theorem B it will be of importance to be able to compare phase and parameter derivatives. Under the assumption of exponential increase of the phase derivative along the critical orbit, this can be done, as is formulated in the following lemma. The proof is that of Lemma 2.10 in [Asp21].

Lemma 3.2 (Phase-Parameter Distortion). *Let a_0 be (γ_0, C_0)-Collet–Eckmann, $\gamma_T \in (0, \gamma_0)$, $C_T \in (0, C_0)$, and $A \in (0, 1)$. There exist $T, N_T, \epsilon_T > 0$ such that if $a \in (a_0 - \epsilon_T, a_0 + \epsilon_T)$ satisfies*

$$|\partial_x F^j(1; a)| \geq C_T e^{\gamma_T j} \qquad (j = 1, 2, \ldots, N_T, \ldots n - 1),$$

for some $n - 1 \geq N_T$, then

$$(1 - A)T \leq \left| \frac{\partial_a F^n(0; a)}{\partial_x F^{n-1}(1; a)} \right| \leq (1 + A)T.$$

Proof. According to Theorem 3 in [Tsu00] (see also Theorem 1 in [Lev14])

$$\lim_{j\to\infty} \frac{\partial_a F^j(0;a_0)}{\partial_x F^{j-1}(1;a_0)} = \sum_{k=0}^{\infty} \frac{\partial_a F(\xi_k(a_0);a_0)}{\partial_x F^k(1;a_0)} = T \in \mathbb{R}_{>0}.$$

Let $N_T > 0$ be large enough so that

$$\left| \sum_{k=N_T}^{\infty} \frac{\partial_a F(\xi_k(a_0);a_0)}{\partial_x F^k(1;a_0)} \right| \leq \sum_{k=N_T}^{\infty} \frac{1}{C_0 e^{\gamma_0 k}} \leq \sum_{k=N_T}^{\infty} \frac{1}{C_T e^{\gamma_T k}} \leq \frac{1}{3} AT.$$

Since $a \mapsto \partial_a F(\xi_k(a);a)/\partial_x F^k(1;a)$ is continuous there exists $\epsilon_T > 0$ such that given $a \in (a_0 - \epsilon_T, a_0 + \epsilon_T)$

$$\left| \sum_{k=0}^{N_T-1} \frac{\partial_a F(\xi_k(a);a)}{\partial_x F(1;a)} - T \right| \leq \frac{1}{2} AT.$$

Assuming $x \mapsto 1 - ax^2$ to be (γ_T, C_T)-Collet–Eckmann up to time $n > N_T$, the result now follows since

$$\left| \sum_{k=0}^{n} \frac{\partial_a F(\xi_k(a);a)}{\partial_x F^k(1;a)} - T \right| \leq AT.$$

□

Remark 3.3. The quotient $(1 + A)/(1 - A) = D_A$ can be chosen arbitrarily close to 1 by increasing N_T and decreasing ϵ_T.

3.3 Start-up Lemma

With the above two lemmas we now prove the existence of a suitable interval in parameter space on which the parameter exclusion will be carried out.

Given an admissible sequence (δ_n), let N_A be the integer in Definition 1.2. Fix $N_B \geq 1$, $\gamma_B > 0$, and $C_B > 0$, and let a_0 be a PR-parameter satisfying a (γ_0, C_0)-Collet–Eckmann condition. In Lemma 3.2 we make the choice

$$\gamma_T = \min(\gamma_B, \gamma_0, \gamma_M)/20 \quad \text{and} \quad C_T = \min(C_B, C_0)/3.$$

Furthermore let

$$\gamma = \min(\gamma_B, \gamma_0, \gamma_M)/2 \quad \text{and} \quad C = \min(C_B, C_0)/2,$$

and let $m_{-1} = \max(N_A, N_B, N_T)$.

Lemma 3.4 (Start-up Lemma). *There exist an interval $\omega_0 = (a_0 - \epsilon, a_0 + \epsilon)$, an integer $m_0 \geq m_{-1}$, and a constant $S = \epsilon_1 \delta$ such that*

(i) $\xi_{m_0} : \omega_0 \to [-1, 1]$ *is injective, and*

$$|\xi_{m_0}(\omega_0)| \geq \begin{cases} e^{-r}/r^2 & \text{if } \xi_{m_0}(\omega_0) \cap I_r \neq \emptyset, \\ S & \text{if } \xi_{m_0} \cap (-\delta, \delta) = \emptyset. \end{cases}$$

(ii) Each $a \in \omega_0$ *is* (γ, C)*-Collet–Eckmann up to time* m_0*:*

$$|\partial_x F^j(1; a)| \geq Ce^{\gamma j} \qquad (j = 0, 1, \dots, m_0 - 1).$$

(iii) Each $a \in \omega_0$ *enjoys polynomial recurrence up to time* m_0*: there exist absolute constants* $K > 0$ *and* $\sigma \geq 0$ *such that for* $a \in \omega_0$

$$|\xi_j(a)| \geq \frac{K}{j^\sigma} \qquad (j = 1, 2, \dots, m_0 - 1).$$

Proof. Given $x, y \in \xi_j(\omega_0)$, $j \geq 1$, consider the following distance condition

$$|x - y| \leq \begin{cases} e^{-r}/r^2 & \text{if } \xi_j(\omega_0) \cap I_r \neq \emptyset, \\ S = \epsilon_1 \delta & \text{if } \xi_j(\omega_0) \cap (-\delta, \delta) = \emptyset. \end{cases} \tag{2}$$

By making ϵ smaller, we may assume that (2) is satisfied up to time m_{-1}. Moreover, we make sure that ϵ is small enough to comply with Lemma 3.2. Whenever (2) is satisfied, phase derivatives are comparable as follows

$$\frac{1}{C_1} \leq \left| \frac{\partial_x F(x; a)}{\partial_x F(y; b)} \right| \leq C_1, \tag{3}$$

with $C_1 > 1$ a constant. This can be seen through the following estimate

$$\left| \frac{\partial_x F(x; a)}{\partial_x F(y; b)} \right| = \left| \frac{-2ax}{-2by} \right| \leq \frac{a_0 + \epsilon}{a_0 - \epsilon} \left(\left| \frac{x - y}{y} \right| + 1 \right).$$

If we are outside $(-\delta, \delta)$ then

$$\left| \frac{x - y}{y} \right| \leq \frac{S}{\delta} = \epsilon_1,$$

and if we are hitting I_r with largest possible r,

$$\left| \frac{x - y}{y} \right| \leq \frac{e^{-r}}{r^2} \frac{1}{e^{-(r+1)}} = \frac{e}{r^2} \leq \frac{e}{\Delta^2}.$$

By making sure that ϵ, ϵ_1, and δ are small enough, C_1 can be made as close to 1 as we want. In particular, we make C_1 close enough to 1 so that

$$C_1^{-j} C_0 e^{\gamma_0 j} \geq C e^{\gamma j} \qquad (j \geq 0). \tag{4}$$

As long as the distance condition (2) is satisfied, we will have good expansion along the critical orbits. Indeed by (3) and (4) it follows that, given $a \in \omega_0$,

$$\begin{aligned} |\partial_x F^j(1; a)| &\geq C_1^{-j} |\partial_x F^j(1; a_0)| \\ &\geq C_1^{-j} C_0 e^{\gamma_0 j} \\ &\geq C e^{\gamma j} \qquad (j \geq 0 \text{ such that (2) is satisfied}). \end{aligned}$$

This tells us that, during the time for which (2) is satisfied, every $a \in \omega_0$ is such that the corresponding map is (γ, C)-Collet–Eckmann. In particular, since $\gamma > \gamma_T$ and $C > C_T$, we can apply Lemma 3.2, and together with the mean value theorem we have that

$$\begin{aligned} |\xi_j(\omega_0)| &= |\partial_a F^j(0; a')||\omega_0| \\ &\geq (1 - A)T|\partial_x F^{j-1}(1; a')||\omega_0| \\ &\geq (1 - A)TCe^{\gamma(j-1)}|\omega_0|. \end{aligned}$$

Our interval is thus expanding, and we let $m_0 = j$, with $j \geq m_{-1}$ the smallest integer for which (2) is no longer satisfied. This proves statements (i) and (ii).

To prove statement (iii), let $K_0 > 0$ and $\sigma_0 \geq 0$ be the constants associated to a_0 for which

$$|\xi_j(a_0)| \geq \frac{K_0}{j^{\sigma_0}} \qquad (j \geq 1).$$

In view of (2), when we hit $(-\delta, \delta)$ at some time $j < m_0$,

$$|\xi_j(a)| \geq |\xi_j(a_0)| - |\xi_j(\omega_0)| \geq |\xi_j(a_0)| - \frac{e^{-r}}{r^2}.$$

Here, r is such that

$$e^{-r-1} \leq |\xi_j(a_0)|,$$

and therefore, given δ small enough,

$$|\xi_j(a)| \geq |\xi_j(a_0)| \left(1 - \frac{e}{\Delta^2}\right) \geq \frac{K_0/2}{j^{\sigma_0}} \qquad (j = 1, 2, \dots, m_0 - 1).$$

$\square$

Remark 3.5. By making δ small enough so that $1/\Delta^2 < \epsilon_1$, S will be larger than any partition element $I_{rl} \subset (-\delta, \delta)$. This S is usually referred to as the *large scale*.

Since $\Lambda_{N_B}(\delta_n) \subset \Lambda_{m_0}(\delta_n)$, Theorem B follows if

$$\mathrm{Leb}\left(\Lambda_{m_0}(\delta_n) \cap \mathcal{CE}(\gamma_B, C_B) \cap \omega_0\right) = 0.$$

4 Induction Step

4.1 Initial iterates

Let $\omega_0 = \Delta_0$ be the start-up interval obtained in Lemma 3.4. Iterating this interval under ξ and successively excluding parameters that do not satisfy the recurrence condition, or the Collet–Eckmann condition, we will inductively define a nested sequence $\Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_k \supset \cdots$ of sets of parameters satisfying

$$\Lambda_{m_0}(\delta_n) \cap \mathcal{CE}(\gamma_B, C_B) \cap \omega_0 \subset \Delta_\infty = \bigcap_{k=0}^{\infty} \Delta_k,$$

and our goal is to estimate the Lebesgue measure of Δ_∞. This will require a careful analysis of the so-called *returns* to $(-\delta, \delta)$, and we will distinguish between four types of returns: *inessential*, *essential*, *escape*, and *complete*. At the k^{th} complete return, we will be in the position of excluding parameters and form the partition that will make up the set Δ_k. Below we will describe the iterations from the k^{th} complete return to the $(k+1)^{\text{th}}$ complete return, hence the forming of Δ_{k+1}. Before indicating the partition, and giving a definition of the different returns, we begin with considering the first initial iterates of $\xi_{m_0}(\omega_0)$.

If $\xi_{m_0}(\omega_0) \cap (-\delta, \delta) \neq \emptyset$, then we have reached a return and we proceed accordingly as is described below. If this is not the case, then we are in the situation

$$\xi_{m_0}(\omega_0) \cap (-\delta, \delta) = \emptyset \quad \text{and} \quad |\xi_{m_0}(\omega_0)| \geq S,$$

with S larger than any partition element $I_{rl} \subset (-\delta, \delta)$ (see Remark 3.5). Since the length of the image is bounded from below, there is an integer $n^* = n^*(S)$ such that for some smallest $n \leq n^*$ we have

$$\xi_{m_0+n}(\omega_0) \cap (-\delta, \delta) \neq \emptyset.$$

In this case, $m_0 + n$ is the index of the first return. We claim that, if m_0 is large enough, we can assume good derivative up to time $m_0 + n$. To realise this, consider for $j < n$ the distortion quotient

$$\left|\frac{\partial_x F^{m_0+j}(1;a)}{\partial_x F^{m_0+j}(1;b)}\right| = \left|\frac{\partial_x F^{m_0-1}(1;a)}{\partial_x F^{m_0-1}(1;b)}\right| \left|\frac{\partial_x F^{j+1}(\xi_{m_0}(a);a)}{\partial_x F^{j+1}(\xi_{m_0}(b);b)}\right|.$$

Since the distance conditions (2) are satisfied up to time $m_0 - 1$, the first factor in the above right hand side is bounded from above by the constant $C_1^{m_0-1}$, with $C_1 > 1$ being very close to 1 (see (3)). Furthermore, since $j < n < n^*(S)$, and since we by assumption are iterating outside $(-\delta, \delta)$, the second factor in the above right hand side is bounded from above by some positive constant $C_{S,\delta}$ dependent on S and δ.

If there is no parameter $a' \in \omega_0$ such that $|\partial_x F^{m_0+j}(1;a')| \geq C_B e^{\gamma_B(m_0+j)}$ then we have already reached our desired result. If on the other hand there is such a parameter a' then for all $a \in \omega_0$ it follows from the above distortion estimate and our choice of γ that

$$|\partial_x F^{m_0+j}(1;a)| \geq \frac{C_B e^{\gamma_B(m_0+j)}}{C_1^{m_0-1} C_{S,\delta}} \geq C e^{\gamma(m_0+j)},$$

provided m_0 is large enough. We conclude that

$$|\partial_x F^j(1;a)| \geq C e^{\gamma j} \qquad (a \in \omega_0,\ j = 0, 1, \ldots, m_0 + n - 1). \tag{5}$$

In the case we have to iterate $\xi_{m_0}(\omega_0)$ further to hit $(-\delta, \delta)$ we still let m_0 denote the index of the first return.

4.2 The partition

At the $(k+1)^{\text{th}}$ step in our process of excluding parameters, Δ_k consists of disjoint intervals ω_k^{rl}, and for each such interval there is an associated time m_k^{rl} for which either $\xi_{m_k^{rl}}(\omega_k^{rl}) = I_{rl} \subset (-4\delta, 4\delta)$, or $\xi_{m_k^{rl}}(\omega_k^{rl})$ is mapped onto $\pm(\delta, x)$, with $|x - \delta| \geq 3\delta$. We iterate each such interval individually, and let m_{k+1}^{rl} be the time for which $\xi_{m_{k+1}^{rl}}(\omega_k^{rl})$ hits deep enough for us to be able to remove a significant portion of $(-\delta_{m_{k+1}^{rl}}, \delta_{m_{k+1}^{rl}})$ in phase space, and let E_k^{rl} denote the corresponding set that is removed in parameter space. We now form the set $\hat{\omega}_k^{rl} \subset \Delta_{k+1}$ and make the partition

$$\hat{\omega}_k^{rl} = \omega_k^{rl} \setminus E_k^{rl} = \left(\bigcup_{r',l'} \omega_{k+1}^{r'l'}\right) \cup T_{k+1} = N_{k+1} \cup T_{k+1}.$$

Here, each $\omega_{k+1}^{r'l'} \subset N_{k+1}$ is such that $\xi_{m_{k+1}^{rl}}(\omega_{k+1}^{r'l'}) = I_{r'l'} \subset (-4\delta, 4\delta)$, and T_{k+1} consists of (at most) two intervals whose image under $\xi_{m_{k+1}^{rl}}$ is $\pm(\delta, x)$, with $|x - \delta| \geq 3\delta$.

Remark 4.1. At most four intervals $\omega_{k+1}^{r'l'} \subset N_{k+1}$ will be mapped onto an interval slightly larger than $I_{r'l'}$, i.e.

$$I_{r'l'} \subset \xi_{m_{k+1}^{rl}}(\omega_{k+1}^{r'l'}) \subset I_{r'l'} \cup I_{r''l''},$$

with $I_{r'l'}$ and $I_{r''l''}$ adjacent partition elements.

Remark 4.2. At *essential returns* and *escape returns* we will, if possible, make a partial partition. To these partitioned parameter intervals we associate a *complete return time* even though nothing is removed at these times. This is described in more detail in sections 4.8 and 4.9.

Remark 4.3. Notice that our way of partitioning differs slightly from the original one considered in [BC85], since here we do not continue to iterate what is mapped outside of $(-\delta, \delta)$, but instead stop and make a partition.

4.3 The different returns to $(-\delta, \delta)$

At time m_{k+1}^{rl} we say that ω_k^{rl} has reached the $(k+1)^{\text{th}}$ *complete return* to $(-\delta, \delta)$. In between the two complete returns of index m_k^{rl} and m_{k+1}^{rl} we might have returns which are not complete. Given a return at time $n > m_k^{rl}$, we classify it as follows.

i) If $\xi_n(\omega_k^{rl}) \subset I_{r'l'} \cup I_{r''l''}$, with $I_{r'l'}$ and $I_{r''l''}$ adjacent partition elements ($r' \geq r''$), and if $|\xi_n(\omega_k^{rl})| < |I_{r'l'}|$, we call this an *inessential return*. The interval $I_{r'l'} \cup I_{r''l''}$ is called the *host interval*.

ii) If the return is not inessential, it is called an *essential return*. The outer most partition element I_r contained in the image is called the *essential interval*.

iii) If $\xi_n(\omega_k^{rl}) \cap (-\delta, \delta) \neq \emptyset$ and $|\xi_n(\omega_k^{rl}) \setminus (-\delta, \delta)| \geq 3\delta$, we call this an *escape return*. The interval $\xi_n(\omega_k^{rl}) \setminus (-\delta, \delta)$ is called the *escape interval*.

iv) Finally, if a return satisfies $\xi_n(\omega_k^{rl}) \cap (-\delta_n/3, \delta_n/3) \neq \emptyset$, it is called a *complete return*.

We use these terms exclusively, that is, an inessential return is not essential, an essential return is not an escape, and an escape return is not complete.

Given $\omega_k^{rl} \subset \Delta_k$ we want to find an upper bound for the index of the next complete return. In the worst case scenario we encounter all of the above kind of returns, in the order

$$\text{complete} \to \text{inessential} \to \text{essential} \to \text{escape} \to \text{complete}.$$

Given such behaviour, we show below that there is an absolute constant $\kappa > 0$ such that the index of the $(k+1)^{\text{th}}$ complete return satisfies $m_{k+1}^{rl} \leq m_k^{rl} + \kappa \log m_k^{rl}$.

4.4 Induction assumptions

Up until the start time m_0 we do not want to assume anything regarding recurrence with respect to our recurrence rate (δ_n). Since the perturbation is made around a PR-parameter a_0, we do however have the following *polynomial recurrence* to rely on (Lemma 3.4):

(PR) $|F^j(0;a)| \geq K/j^\sigma$ for all $a \in \omega_k^{rl}$ and $j = 1, 2, \dots, m_0 - 1$.

After m_0 we start excluding parameters according to the following *basic assumption*:

(BA) $|F^j(0;a)| \geq \delta_j/3$ for all $a \in \omega_k^{rl}$ and $j = m_0, m_0 + 1, \dots, m_k^{rl}$.

Since our sequence δ_j is assumed to be admissible, we will frequently use the fact that $\delta_j/3 \geq 1/(3j^{\bar{e}})$.

From (5) we know that every $a \in \omega_k^{rl}$ is (γ, C)-Collet–Eckmann up to time m_0, and this condition is strong enough to ensure phase-parameter distortion (Lemma 3.2). We will continue to assume this condition at complete returns, but in between two complete returns we will allow the exponent to drop slightly due to the loss of derivative when returning close to the critical point $x = 0$. We define the *basic exponent* conditions as follows:

(BE)(1) $|\partial_x F^{m_k^{rl}-1}(1;a)| \geq Ce^{\gamma(m_k^{rl}-1)}$ for all $a \in \omega_k^{rl}$.

(BE)(2) $|\partial_x F^j(1;a)| \geq Ce^{(\gamma/3)j}$ for all $a \in \omega_k^{rl}$ and $j = 0, 1, \dots, m_k^{rl} - 1$.

Assuming (BA) and (BE)(1,2) for $a \in \omega_k^{rl} \subset \Delta_k$, we will prove it for $a' \in \omega_{k+1}^{r'l'} \subset \Delta_{k+1} \subset \Delta_k$. Before considering the iteration of ω_k^{rl}, we define the *bound period* and the *free period*, and prove some useful lemmas connected to them. For technical reasons these lemmas will be proved using the following weaker assumption on the derivative. Given a time $n \geq m_k^{rl}$ we consider the following condition:

(BE)(3) $|\partial_x F^j(1;a)| \geq Ce^{(\gamma/9)j}$ for all $a \in \omega_k^{rl}$ and $j = 0, 1, \dots, n - 1$.

Notice that $\gamma/9 > \gamma_T$, hence we will be able to apply Lemma 3.2 at all times.

To rid ourselves of cumbersome notation we drop the indices from this point on and write $\omega = \omega_k^{rl}$, and $m = m_k^{rl}$.

4.5 The bound and free periods

Assuming we are in the situation of a return for which $\xi_n(\omega) \subset I_{r+1} \cup I_r \cup I_{r-1} \subset (-4\delta, 4\delta)$, we are relatively close to the critical point, and therefore the next iterates $\xi_{n+j}(\omega)$ will closely resemble those of $\xi_j(\omega)$. We quantify this and define the *bound period* associated to this return as the maximal p such that

(BC) $|\xi_\nu(a) - F^\nu(\eta; a)| \leq |\xi_\nu(a)|/(10\nu^2)$ for $\nu = 1, 2, \dots, p$

holds for all $a \in \omega$, and all $\eta \in (0, e^{-|r-1|})$. We refer to (BC) as the *binding condition*.

Remark 4.4. In the proof of Lemma 4.12 we will refer to *pointwise binding*, meaning that for a given parameter a we associate a bound period $p = p(a)$ according to when (BC) breaks for this specific parameter. We notice that the conclusions of Lemma 4.5 and Lemma 4.6 below are still true if we only consider iterations of one specific parameter.

The bound period is of central importance, and we establish some results connected to it (compare with [BC85]). An important fact is that during this period the derivatives are comparable in the following sense.

Lemma 4.5 (Bound distortion). *Let n be the index of a return for which $\xi_n(\omega) \subset I_{r+1} \cup I_r \cup I_{r-1}$, and let p be the bound period. Then, for all $a \in \omega$ and $\eta \in (0, e^{-|r-1|})$,*

$$\frac{1}{2} \leq \left| \frac{\partial_x F^j(1 - a\eta^2; a)}{\partial_x F^j(1; a)} \right| \leq 2 \qquad (j = 1, 2, \dots, p).$$

Proof. It is enough to prove that

$$\left| \frac{\partial_x F^j(1 - a\eta^2; a)}{\partial_x F^j(1; a)} - 1 \right| \leq \frac{1}{2}. \tag{6}$$

The quotient can be expressed as

$$\frac{\partial_x F^j(1 - a\eta^2; a)}{\partial_x F^j(1; a)} = \prod_{\nu=1}^{j} \left(\frac{F^\nu(\eta; a) - \xi_\nu(a)}{\xi_\nu(a)} + 1 \right),$$

and applying the elementary inequality

$$\left| \prod_{\nu=1}^{j} (u_n + 1) - 1 \right| \leq \exp\left(\sum_{\nu=1}^{j} |u_n| \right) - 1,$$

valid for complex u_n, (6) now follows since

$$\sum_{\nu=1}^{j} \frac{|F^\nu(\eta;a) - \xi_\nu(a)|}{|\xi_\nu(a)|} \le \frac{1}{10}\sum_{\nu=1}^{j}\frac{1}{\nu^2} \le \log\frac{3}{2}.$$

□

The next result gives us an estimate of the length of the bound period. As will be seen, if (BA) and (BE)(3) are assumed up to time $n \ge m = m_k^{rl}$, the bound period is never longer than n, and we are therefore allowed to use the induction assumptions during this period. In particular, in view of the above distortion result and (BE)(3), we inherit expansion along the critical orbit during the bound period; making sure m_0 is large enough, and using (BA) together with the assumption that (δ_n) is admissible, we have

$$\begin{aligned}|\partial_x F^{n+j}(1;a)| &= 2a|\xi_n(a)||\partial_x F^{n-1}(1;a)||\partial_x F^j(1-a\xi_n(a)^2;a)| \\ &\ge \frac{2}{3n^{\bar{e}}}C^2 e^{(\gamma/9)(n+j-1)} \\ &= \frac{2}{3}C^2 e^{-\gamma/9}\exp\left\{\left(\frac{\gamma}{9} - \frac{\bar{e}\log n}{n+j}\right)(n+j)\right\} \\ &\ge C_T e^{\gamma_T(n+j)} \qquad (j = 0, 1, \dots, p). \end{aligned} \tag{7}$$

This above estimate is an a priori one, and will allow us to use Lemma 3.2 in the proof of Lemma 4.10.

Lemma 4.6 (Bound Length). *Let n be the index of a return such that $\xi_n(\omega) \subset I_{r+1} \cup I_r \cup I_{r-1}$, and suppose that (BA) and (BE)(3) are satisfied up to time n. Then there exists a constant $\kappa_1 > 0$ such that the corresponding bound period satisfy*

$$\kappa_1^{-1} r \le p \le \kappa_1 r. \tag{8}$$

Proof. By the mean value theorem and Lemma 4.5 we have that

$$\begin{aligned}|\xi_j(a) - F^j(\eta;a)| &= |F^{j-1}(1;a) - F^{j-1}(1-a\eta^2;a)| \\ &= a\eta^2|\partial_x F^{j-1}(1-a\eta'^2;a)| \\ &\ge \frac{a\eta^2}{2}|\partial_x F^{j-1}(1;a)|, \end{aligned} \tag{9}$$

as long as $j \le p$. (Here, $0 < \eta' < \eta$.) Furthermore, as long as we also have $j \le (\log n)^2$, say, we can use the induction assumptions: using (BE)(3) we find that

$$\frac{1}{2}e^{-2(r+1)}Ce^{(\gamma/9)(j-1)} \le \frac{a\eta^2}{2}|\partial_x F^{j-1}(1;a)| \le \frac{|\xi_j(a)|}{10j^2} \le 1.$$

Taking the logarithm, using (BA), and making sure that m_0 is large enough, we therefore have

$$j \le 1 + \frac{9}{\gamma}\left(2r + 2 + \log 2 - \log C\right) \lesssim r \lesssim \log n \le (\log n)^2,$$

as long as $j \le p$ *and* $j \le (\log n)^2$. This tells us that $j \le p$ must break before $j \le (\log n)^2$; in particular there is a constant $\kappa_1 > 0$ such that $p \le \kappa_1 r$.

For the lower bound consider $j = p + 1$ and the equality (9). With $a \in \omega$ being the parameter for which the inequality in the binding condition is reversed, using Lemma 4.5 we find that

$$\frac{|\xi_{p+1}(a)|}{10(p+1)^2} \le |\xi_{p+1}(a) - F^{p+1}(\eta; a)| \le 4e^{-2r}|\partial_x F^p(1; a)| \le 4e^{-2r}4^p.$$

Using the upper bound for p we know that (BA) (or (PR)) is valid at time $p+1$, hence

$$\frac{|\xi_{p+1}(a)|}{10(p+1)^2} \ge \frac{1}{30(p+1)^{2+\hat{e}}},$$

where $\hat{e} = \max(\overline{e}, \sigma)$. Therefore

$$\frac{1}{30(p+1)^{2+\hat{e}}} \le 4e^{-2r}4^p,$$

and taking the logarithm proves the lower bound.

□

Remark 4.7. Notice that the lower bound is true without assuming the upper bound (which in our proof requires (BE)(3) at time n) as long as we assume (BA) to hold at time $p+1$.

The next result will concern the growth of $\xi_n(\omega)$ during the bound period.

Lemma 4.8 (Bound Growth). *Let n be the index of a return such that $\xi_n(\omega) \subset I$ with $I_{rl} \subset I \subset I_{r+1} \cup I_r \cup I_{r-1}$, and suppose that (BA) and (BE)(3) are satisfied up to time n. Then there exists a constant $\kappa_2 > 0$ such that*

$$|\xi_{n+p+1}(\omega)| \ge \frac{1}{r^{\kappa_2}} \frac{|\xi_n(\omega)|}{|I|}.$$

Proof. Denote $\Omega = \xi_{n+p+1}(\omega)$ and notice that for any two given parameters $a, b \in \omega$ we have

$$\begin{aligned}
|\Omega| &\geq |F^{n+p+1}(0;a) - F^{n+p+1}(0;b)| \\
&= |F^{p+1}(\xi_n(a);a) - F^{p+1}(\xi_n(b);b)| \\
&\geq |F^{p+1}(\xi_n(a);a) - F^{p+1}(\xi_n(b);a)| \\
&\quad - |F^{p+1}(\xi_n(b);a) - F^{p+1}(\xi_n(b);b)|.
\end{aligned} \tag{10}$$

Due to exponential increase of the phase derivative along the critical orbit, the dependence on parameter is inessential in the following sense:

$$|F^{p+1}(\xi_n(b);a) - F^{p+1}(\xi_n(b);b)| \leq e^{-(\gamma/18)n}|\xi_n(\omega)|. \tag{11}$$

To realise this, first notice that we have the following (somewhat crude) estimate for the parameter derivative:

$$|\partial_a F^j(x;a)| \leq 5^j \qquad (j = 1, 2, \dots).$$

Indeed, $|\partial_a F(x;a)| \leq 1 < 5$, and by induction

$$\begin{aligned}
|\partial_a F^{j+1}(x;a)| &= |\partial_a(1 - aF^j(x;a)^2)| \\
&= |-F^j(x;a)^2 - 2aF^j(x;a)\partial_a F^j(x;a)| \\
&\leq 1 + 4 \cdot 5^j \\
&\leq 5^{j+1}.
\end{aligned}$$

Using the mean value theorem twice, Lemma 3.2 and (BE)(3) we find that

$$|F^{p+1}(\xi_n(b);a) - F^{p+1}(\xi_n(b);b)| \leq [(1-A)T]^{-1} 5^{p+1} C^{-1} e^{-(\gamma/9)(n-1)} |\xi_n(\omega)|.$$

In view of (8) and (BA), making m_0 larger if needed, the inequality (11) can be achieved.

Assume now that at time $p+1$ (BC) is broken for parameter a, and let b be an endpoint of ω such that

$$|\xi_n(a) - \xi_n(b)| \geq \frac{|\xi_n(\omega)|}{2}.$$

Continuing the estimate of $|\Omega|$, using (11), we find that

$$\begin{aligned}
|\Omega| &\geq |F^p(1 - a\xi_n(a)^2;a) - F^p(1 - a\xi_n(b)^2;a)| \\
&\qquad - |F^{p+1}(\xi_n(b);a) - F^{p+1}(\xi_n(b);b)| \\
&\geq \left(a|\xi_n(a) + \xi_n(b)||\partial_x F^p(1 - a\xi_n(a')^2;a)| - 2e^{-(\gamma/18)n}\right) \frac{|\xi_n(\omega)|}{2} \\
&\geq \left(2ae^{-r}|\partial_x F^p(1 - a\xi_n(a')^2;a)| - 2e^{-(\gamma/18)n}\right) \frac{|\xi_n(\omega)|}{2}.
\end{aligned} \tag{12}$$

Using Lemma 4.5 twice and the equality in (9) (with $p+1$ instead of p) together with (BC) (now reversed inequality) we continue the estimate in (12) to find that

$$
\begin{aligned}
|\Omega| &\geq \left(2ae^{-r}\frac{1}{4a\eta^2}\frac{|\xi_{p+1}(a)|}{10(p+1)^2} - 2e^{-(\gamma/18)n}\right)\frac{|\xi_n(\omega)|}{2} \\
&\geq \left(e^r\frac{|\xi_{p+1}(a)|}{20(p+1)^2} - 2e^{-(\gamma/18)n}\right)\frac{|\xi_n(\omega)|}{2}. \qquad (13)
\end{aligned}
$$

In either case of $p \leq m_0$ or $p > m_0$ we have that (using (BA), (PR), and the assumption that our recurrence rate is admissible)

$$
\frac{|\xi_{p+1}(a)|}{(p+1)^2} \geq \frac{K}{3(p+1)^{2+\hat{e}}},
$$

where $\hat{e} = \max(\bar{e}, \sigma)$. We can make sure that the second term in the parenthesis in (13) is always less than a fraction, say $1/2$, of the first term and therefore, using (BC), (8), and that $e^r \geq 1/(2r^2|I|)$, we finish the estimate as follows

$$
\begin{aligned}
|\Omega| &\geq \frac{K}{240}\frac{1}{(p+1)^{2+\hat{e}}}|\xi_n(\omega)|e^r \\
&\geq \frac{K}{480}\frac{1}{r^2(p+1)^{2+\hat{e}}}\frac{|\xi_n(\omega)|}{|I|} \\
&\geq \frac{K}{480(2\kappa_1)^{2+\hat{e}}}\frac{1}{r^{4+\hat{e}}}\frac{|\xi_n(\omega)|}{|I|} \\
&\geq \frac{1}{r^{\kappa_2}}\frac{|\xi_n(\omega)|}{|I|}, \qquad (14)
\end{aligned}
$$

where we can choose $\kappa_2 = 5 + \hat{e}$ as long as δ is sufficiently small. □

Remark 4.9. Using the lower bound for p, the upper bound

$$
|\xi_{n+p+1}(\omega)| \leq \frac{1}{r}\frac{|\xi_n(\omega)|}{|I|}
$$

can be proved similarly.

This finishes the analysis of the bound period, and we continue with describing the *free period*. A free period will always follow a bound period, and during this period we will be iterating outside $(-\delta, \delta)$. We let L denote the length of this period, i.e. L is the smallest integer for which

$$
\xi_{n+p+L}(\omega) \cap (-\delta, \delta) \neq \emptyset.
$$

The following lemma gives an upper bound for the length of the free period, following the bound period of a complete return, or an essential return.

Lemma 4.10 (Free length). *Let $\xi_n(\omega) \subset I_{r+1} \cup I_r \cup I_{r-1}$ with n being the index of a complete return or an essential return, and suppose that (BA) and (BE)(3) are satisfied up to time n. Let p be the associated bound period, and let L be the free period. Then there exists a constant $\kappa_3 > 0$ such that*

$$L \leq \kappa_3 r.$$

Proof. Assuming $j \leq L$ and $j \leq (\log n)^2$, similar calculations as in the proof of Lemma 4.8 gives us parameter independence (see (11) and notice that from (7) we are allowed to apply Lemma 3.2); using Lemma 4.8 and Lemma 3.1 we find that

$$2 \geq |\xi_{n+p+j}(\omega)| \geq \frac{\delta C_M}{2} e^{\gamma_M (j-1)} \frac{1}{r^{\kappa_2}}.$$

Taking the logarithm, using (BA), and making sure that m_0 is large enough, we therefore have

$$j \leq 1 + \frac{1}{\gamma_M}(\kappa_2 \log r + \Delta + \log 4 - \log C_M) \lesssim r \lesssim \log n < \frac{1}{2}(\log n)^2,$$

as long as $j \leq L$ and $j \leq (\log n)^2$. This tells us that $j \leq L$ must break before $j \leq (\log n)^2$; in particular there is a constant $\kappa_3 > 0$ such that $L \leq \kappa_3 r$. □

Remark 4.11. If the return $\xi_{n+p+L}(\omega)$ is inessential or essential, then there is no dependence on δ in the growth factor; more generally, if the prerequisites of Lemma 4.8 are satisfied, then

$$|\xi_{n+p+L}(\omega)| \geq \frac{C_M}{2} e^{\gamma_M (L-1)} \frac{1}{r^{\kappa_2}} \frac{|\xi_n(\omega)|}{|I|}.$$

Before considering iterations of $\omega = \omega_k^{rl} \subset \Delta_k$ from $m = m_k^{rl}$ to m_{k+1}^{rl}, we make the following observation that as long as (BA) is assumed in a time window $[n, 2n]$, the derivative will not drop too much.

Lemma 4.12. *Suppose that a is a parameter such that*

$$|\partial_x F^j(1; a)| \geq C e^{\gamma' j} \qquad (j = 0, 1, \dots, n-1), \tag{15}$$

with $\gamma' \geq \gamma/3$. Then, if (BA) is satisfied up to time $2n$, we have

$$|\partial_x F^{n+j}(1; a)| \geq C e^{(\gamma'/3)(n+j)} \qquad (j = 0, 1, \dots, n-1).$$

In other words, if (BA) and (BE)(1) [(BE)(2)] are satisfied up to time n then (BE)(2) [(BE)(3)] is satisfied up to time $2n$, as long as (BA) is.

Proof. The proof is based on the fact that we trivially have no loss of derivative during the bound and free periods. Indeed suppose $\xi_{n'}(a) \sim e^{-r}$, with $n' \geq n$ and let p be the bound period (here we use pointwise binding, see Remark 4.4), and L the free period. Moreover we assume that $n' + p + L < 2n$; in particular this implies $p < n$ and we can use (15) during this period. Introducing $D_p = |\partial_x F^p(1;a)|$ and using similar calculations as in Lemma 4.8 (e.g. the equality in (9) and reversed inequality in (BC)) we find that

$$e^{-2r} D_p \gtrsim a\eta^2 |\partial_x F^p(1 - a\eta^2; a)| \geq \frac{|\xi_{p+1}(a)|}{10(p+1)^2} \gtrsim \frac{1}{(p+1)^{2+\hat{e}}},$$

where we used (BA) (or (PR)). Since $p < n$ we are free to use (15) and therefore the above inequalities yield

$$e^{-r} D_p \gtrsim D_p^{1/2} \frac{1}{\sqrt{(p+1)^{2+\hat{e}}}} \gtrsim \frac{e^{(\gamma'/2)p}}{\sqrt{(p+1)^{2+\hat{e}}}} \geq C_M^{-1},$$

provided δ is small enough. Here in the last inequality we used the lower bound in (8) (see Remark 4.7). Assuming $\xi_{n'+p+L}(a)$ is a return (and that $n' + p + L < 2n$), we therefore have

$$\begin{aligned} |\partial_x F^{p+L}(\xi_{n'}(a); a)| &\geq 2a|\xi_{n'}(a)||\partial_x F^p(1 - a\xi_{n'}(a)^2; a)||\partial_x F^{L-1}(\xi_{n'+p+1}(a); a)| \\ &\gtrsim e^{-r} D_p C_M e^{\gamma_M (L-1)} \\ &\geq 1. \end{aligned}$$

We conclude that the combination of a return, a bound period, and a free period does not decrease the derivative.

Let us now follow a parameter a satisfying (15) and (BA) up to time $2n$. If the iterates $\xi_{n+j}(a)$ are always outside $(-\delta, \delta)$ then

$$\begin{aligned} |\partial_x F^{n+j}(1;a)| &= |\partial_x F^{n-1}(1;a)||\partial_x F^{j+1}(\xi_n(a); a)| \\ &\geq C e^{\gamma'(n-1)} \delta C_M e^{\gamma_M (j+1)} \\ &\geq C e^{(\gamma'/3)(n+j)} \delta C_M e^{(2\gamma'/3)(n+j)} \\ &\geq C e^{(\gamma'/3)(n+j)} \qquad (j = 0, 1, \dots, n-1), \end{aligned}$$

provided m_0 is big enough.

Otherwise, the worst case is if we have a short free period followed by a return, a bound period, a free period, and so on, and which ends with a return together with a short bound period. In this case, using the above argument, the estimate is as follows:

$$\begin{aligned}
|\partial_x F^{n+j}(1;a)| &\geq |\partial_x F^{n-1}(1;a)| \cdot C_M \cdot 1 \cdot 1 \cdots 1 \cdot 2a|\xi_{n+j}(a)| \cdot C \\
&\geq Ce^{\gamma'(n-1)} C_M C 2a \frac{\delta_{n+j}}{3} \\
&\geq Ce^{(\gamma'/3)(n+j)} C_M C \frac{2}{3a} e^{(\gamma'/3)n - \bar{\epsilon}\log(2n)} \\
&\geq Ce^{(\gamma'/3)(n+j)} \qquad (j = 0, 1, \dots, n-1),
\end{aligned}$$

provided m_0 is big enough. This proves the lemma. □

4.6 From the k^{th} complete return to the first inessential return

If $\omega \subset T_k$ then we have already reached an escape situation and proceed accordingly as is described below in the section about escape. We therefore assume $\omega \subset N_k$ and $\xi_m(\omega) = I_{r_0 l} \subset (-4\delta, 4\delta)$.

If it happens that for some $j \leq p$

$$\xi_{m+j}(\omega) \cap (-\delta_{m+j}/3, \delta_{m+j}/3) \neq \emptyset,$$

then we stop and consider this return complete. If not, we notice that $\xi_{m+p}(\omega)$ can not be a return, unless it is escape or complete; indeed we would otherwise have $|\xi_{m+p+1}(\omega)| < |\xi_{m+p}(\omega)|$, due to the fact that we return close to the critical point, thus contradicting the definition of the bound period. We therefore assume that $\xi_{m+p}(\omega)$ does not intersect $(-\delta, \delta)$.

Up until the next return we will therefore experience an orbit outside of $(-\delta, \delta)$, i.e. we will be in a free period. After the free period, our return is either inessential, essential, escape, or complete. In the next section we consider the situation of an inessential return.

4.7 From the first inessential return to the first essential return

Let $i_1 = m + p_0 + L_0$ denote the index of the first inessential return to $(-\delta, \delta)$. We will keep iterating $\xi_{i_1}(\omega)$ until we once again return. If this next return is again inessential, we denote its index by $i_2 = i_1 + p_1 + L_1$, where p_1 and L_1 are the associated bound period and free period, respectively. Continuing like this, let i_j be the index of the j^{th} inessential return.

The following lemma gives an upper bound for the total time spent doing inessential returns (compare with Lemma 2.3 in [BC91]).

Lemma 4.13 (Inessential Length). *Let $\xi_n(\omega) \subset I_{r+1} \cup I_r \cup I_{r-1}$ with n being the index of a complete return or an essential return, and suppose that (BA) and (BE)(2) are satisfied up to time n. Then there exists a constant* $\kappa_4 > 0$ *such that the total time o spent doing inessential returns satisfy*

$$o \leq \kappa_4 r.$$

Proof. Let $i_1 = n + p + L$ be the index of the first inessential return, i.e. $\xi_{i_1}(\omega) \subset I_{r_1}$, with I_{r_1} being the host interval. From Lemma 4.6 and Lemma 4.10, together with (BA), we have that

$$i_1 = n + p + L \leq n + (\kappa_1 + \kappa_3)r \leq 2n,$$

provided m_0 is large enough. We can therefore apply Lemma 4.12 and conclude that (BE)(3) is satisfied at time i_1. To this first inessential return we associate a bound period of length p_1 (satisfying $p_1 \leq \kappa_1 r_1$ due to the fact that (BE)(3) is satisfied time i_1) and a free period of length L_1. We let $i_2 = i_1 + p_1 + L_1$ denote the index of the second inessential return. Continuing like this, we denote by $i_j = i_{j-1} + p_{j-1} + L_{j-1}$ the index of the j^{th} inessential return. With o_j denoting the total time spent doing inessential returns up to time i_j, we have that $o_j = i_j - i_1 = \sum_{k=1}^{j-1}(p_k + L_k)$. Suppose that the return with index i_s is the first that is not inessential. We estimate $o = o_s$ as follows. Suppose that o_j is as above and that $p_k \leq \kappa_1 r_k$ for $k = 1, 2, \ldots, j-1$. Using Remark 4.11 we find that

$$\frac{|\xi_{i_{k+1}}(\omega)|}{|\xi_{i_k}(\omega)|} \geq \frac{C_M e^{\gamma_M(L_k-1)}}{2r_k^{\kappa_2}|I_{r_k}|} \geq \frac{C_M}{2}\frac{e^{\gamma_M(L_k-1)+r_k}}{r_k^{\kappa_2}}, \tag{16}$$

and therefore

$$2 \geq |\xi_{i_j}(\omega)| = |\xi_{i_1}(\omega)| \prod_{k=1}^{j-1} \frac{|\xi_{i_{k+1}}(\omega)|}{|\xi_{i_k}(\omega)|} \geq \frac{\delta C_M e^{\gamma_M}}{2r^{\kappa_2}} \prod_{k=1}^{j-1} \frac{C_M}{2}\frac{e^{\gamma_M(L_k-1)+r_k}}{r_k^{\kappa_2}}. \tag{17}$$

Here the δ is added to make sure that the estimate also holds for the last free orbit, when the return can be escape or complete. This gives us a rather poor estimate, but since $p \lesssim r$ it is good enough.

Taking the logarithm of (17) we find that

$$\sum_{k=1}^{j-1} \left(\log C_M - \log 2 + \gamma_M(L_k - 1) + r_k - \kappa_2 \log r_k\right) \leq \kappa_2 \log r + \Delta + \text{const.}.$$

Provided δ is small enough we have $r_k \geq 4\kappa_2 \log r_k$ and $r_k \geq -\log\delta > -2(\log C_M + \gamma_M + \log 2)$. Therefore, using $p_k \leq \kappa_1 r_k$, we find that

$$o_j = i_j - i_1 = \sum_{k=1}^{j-1}(p_k + L_k) \leq \kappa_4 r,$$

with κ_4 being an absolute constant. In particular

$$i_j = i_1 + o_j \leq 2n,$$

and therefore (BE)(3) is still valid at time i_j. Consequently the associated bound period satisfies $p_j \leq \kappa_1 r_j$, and the above argument can therefore be repeated. With this we conclude that $o_s \leq \kappa_4 r$. □

We proceed in the next section with describing the situation if our return is assumed to be essential.

4.8 From the first essential return to the first escape return

With n_1 denoting the index of the first essential return, we are in the following situation

$$\begin{aligned} &\xi_{n_1}(\omega) \cap I_{rl} \neq \emptyset, \quad |\xi_{n_1}(\omega)| \geq |I_{rl}|, \\ &\text{and } \xi_{n_1}(\omega) \subset (-4\delta, 4\delta) \setminus (-\delta_{n_1}/3, \delta_{n_1}/3), \end{aligned}$$

for some r, l. At this point, in order not to lose too much distortion, we will make a partition of as much as possible, and keep iterating what is left. That is, we will consider iterations of larger partition elements $I_r = (e^{-r-1}, e^{-r}) \subset (-4\delta, 4\delta)$, and we establish an upper bound for the *number* of essential returns needed to reach an escape return or a complete return.

Let $\Omega_1 = \xi_{n_1}(\omega)$ and let $I_1 = I_{r_1} \subset \Omega_1$, for smallest such r_1. (In fact, we extend I_1 to the closest endpoint of Ω_1, and therefore have $I_1 \subset I_{r_1} \cup I_{r_1 - 1}$.) If there is no such r, we instead let $I_1 = \Omega_1$. Moreover, let ω^1 be the interval in parameter space for which $\xi_{n_1}(\omega^1) = I_1$. The interval I_1 is referred to as the essential interval, and this is the interval we will iterate. If $\hat{\omega} = \omega \setminus \omega^1$ is nonempty we make a partition

$$\hat{\omega} = \bigcup_{r,l} \omega^{rl} \subset \Delta_{k+1},$$

where each ω^{rl} is such that $I_{rl} \subset \xi_{n_1}(\omega^{rl}) = I_{rl} \cup I_{r'l'} \subset (-4\delta, 4\delta)$. (If there is not enough left for a partition, we extend I_1 further so that $I_1 \subset I_{r_1+1} \cup I_{r_1} \cup I_{r_1-1}$.) Notice that, since the intervals I_r are dyadic, the proportion of what remains after partitioning satisfies

$$\frac{|I_1|}{|\Omega_1|} \geq 1 - \frac{1}{e} \geq \frac{1}{2}. \tag{18}$$

We associate for each partitioned parameter interval ω^{rl} the complete return time n_1 (even though nothing is removed from these intervals). From the conclusions made in the previous sections we know that

$$n_1 = m + p_0 + L_0 + o_0 \leq m + (\kappa_1 + \kappa_3 + \kappa_4) r_0 \leq 2m,$$

provided m_0 is large enough. In particular Lemma 4.12 tells us that (BE)(2) is satisfied up to time n_1 for all $a \in \omega$. At this step, to make sure that (BE)(1) is satisfied for our partitioned parameter intervals $\omega^{rl} \subset \Delta_{k+1}$, we make the following rule (compare with the initial iterates at the beginning of the induction step). If there is no $a' \in \omega$ such that

$$|\partial_x F^{n_1-1}(1; a')| \geq C_B e^{\gamma_B (n_1 - 1)},$$

then we remove the entire interval. If there is such a parameter, on the other hand, using Lemma 5.1 we have that

$$\begin{aligned} |\partial_x F^{n_1-1}(1; a)| &\geq D_1^{-(\log^* m)^2} |\partial_x F^{n_1-1}(1; a')| \\ &\geq C_B \exp\left\{ \left(\gamma_B - \frac{(\log^* m)^2}{n_1 - 1} \log D_1 \right) (n_1 - 1) \right\} \\ &\geq C e^{\gamma (n_1 - 1)}, \end{aligned}$$

provided m_0 is large enough.

With the above rules applied at each essential return to come, we now describe the iterations. Since $\xi_m(\omega) = I_{r_0 l}$, using Lemma 4.8 we know that the length of Ω_1 satisfies

$$|\Omega_1| \geq \frac{C_M e^{\gamma_M}}{2} \frac{1}{r_0^{\kappa_2}} \geq \frac{1}{r_0^{\kappa_2+1}}.$$

Notice that since $e^{-r_1+1} \geq |\Omega_1|$ we have that $r_1 \leq 2\kappa_2 \log r_0$. Iterating I_1 with the same rules as before, we will eventually reach a second noninessential return, and if this return is essential we denote its index by n_2. This index constitutes the addition of a bound period, a free period, and an inessential period: $n_2 = n_1 + p_1 + L_1 + o_1$. Similarly as before, we let $\Omega_2 = \xi_{n_2}(\omega^1)$, and let $I_2 \subset \Omega_2$ denote the essential interval of Ω_2. Let $\omega^2 \subset \omega^1$ be such that $\xi_{n_2}(\omega^2) = I_2$, and make a complete partition of $\omega^1 \setminus \omega^2$. By applying Lemma 4.8 again, we find that

$$|\Omega_2| \geq \frac{1}{r_1^{\kappa_2+1}} \geq \frac{1}{(2\kappa_2 \log r_0)^{\kappa_2+1}}.$$

If we have yet to reach an escape return or a complete return, let n_j be the index of the j^{th} essential return, and realise that we are in the following situation

$$\xi_{n_j}(\omega^j) = I_j \subset \Omega_j = \xi_{n_j}(\omega^{j-1}) \quad \text{and} \quad |\Omega_j| \geq \frac{1}{r_{j-1}^{\kappa_2+1}}. \tag{19}$$

Introducing the function $r \mapsto 2\kappa_2 \log r = \varphi(r)$, we see from the above that $r_j \leq \varphi^j(r_0)$. The orbit $\varphi^j(r_0)$ will tend to the attracting fixed point $\hat{r} = -2\kappa_2 W(-1/(2\kappa_2))$, where W is the Lambert W function. The following simple lemma gives an upper bound for the number of essential returns needed to reach an escape return or a complete return.

Lemma 4.14. *Let $\varphi(r) = 2\kappa_2 \log r$, and let $s = s(r)$ be the integer defined by*

$$\log_s r \leq 2\kappa_2 \leq \log_{s-1} r.$$

Then

$$\varphi^s(r) \leq 12\kappa_2^2.$$

Proof. Using the fact that $3 \leq 2\kappa_2 \leq \log_j r$, for $j = 0, 1, \ldots, s-1$, it is straightforward to check that

$$\varphi^j(r) \leq 6\kappa_2 \log_j r. \tag{20}$$

Therefore

$$\varphi^s(r) \leq 2\kappa_2 \log\left(6\kappa_2 \log_{s-1} r\right) = 2\kappa_2 \left(\log 3 + \log 2\kappa_2 + \log_s r\right) \leq 12\kappa_2^2.$$

□

Given $s = s(r_0)$ as in the above lemma we have that $r_s \leq \varphi^s(r_0) \leq 12\kappa_2^2$. By making sure δ is small enough we therefore conclude that

$$|\Omega_{s+1}| \geq \frac{1}{(12\kappa_2^2)^{\kappa_2+1}} \geq 4\delta.$$

To express s in terms of r_0 we introduce the so-called iterated logarithm, which is defined recursively as

$$\log^* x = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 + \log^* \log x & \text{if } x > 1. \end{cases}$$

That is, $\log^* x$ is the number of times one has to apply to logarithm to x in order for the result to be less than or equal to one.

Since s satisfies $\log_s r_0 \leq 2\kappa_2 \leq \log_{s-1} r_0$ and since $2\kappa_2 > 1$, we have

$$s \leq \log^* r_0 \leq \log^* m. \tag{21}$$

We finish by giving an upper bound for the index of the first escape return (or $(k+1)^{\text{th}}$ complete return), i.e. we wish to estimate

$$n_{s+1} = m + \sum_{j=0}^{s} \left(p_j + L_j + o_j\right).$$

From Lemma 4.6, Lemma 4.10, and Lemma 4.13, we have that

$$p_j \le \kappa_1 r_j, \quad L_j \le \kappa_3 r_j, \quad \text{and} \quad o_j \le \kappa_4 r_j.$$

Together with the inequalities $r_j \le \varphi^j(r_0)$ and (20), we find that

$$\begin{aligned}\sum_{j=0}^{s} \left(p_j + L_j + o_j\right) &\lesssim r_0 + \sum_{j=1}^{s} \varphi^j(r_0) \\ &\lesssim r_0 + \sum_{j=1}^{s} \log_j r_0 \\ &\lesssim r_0.\end{aligned}$$

Using (BA) we conclude that $n_{s+1} - m \lesssim \log m$, provided m_0 is large enough.

4.9 From the first escape return to the $(k+1)^{\text{th}}$ complete return

Keeping the notation from the previous section, $\Omega_{s+1} = \xi_{n_{s+1}}(\omega^s)$ is the first escape return, satisfying

$$\begin{aligned}&\Omega_{s+1} \cap (-\delta, \delta) \ne \emptyset, \quad \Omega_{s+1} \cap (-\delta_{n_s}/3, \delta_{n_s}/3) = \emptyset, \\ &\text{and} \quad |\Omega_{s+1} \setminus (-\delta, \delta)| \ge 3\delta.\end{aligned}$$

We will keep iterating ω^s until we get a complete return, and we show below that this must happen within finite (uniform) time. In order to not run into problems with distortion we will, as in the case of essential returns, whenever possible make a partition of everything that is mapped inside of $(-\delta, \delta)$, and the corresponding parameter intervals will be a part of Δ_{k+1}; i.e. at time $n_{s+1+j} = n_{s+1} + j$ let $I_{s+1+j} = \Omega_{s+1+j} \setminus \left(\Omega_{s+1+j} \cap (-\delta, \delta)\right)$, let ω^{s+1+j} be such that $\xi_{n_{s+1+j}}(\omega^{s+1+j}) = I_{s+1+j}$, and make a partition of $\omega^{s+j} \setminus \omega^{s+1+j}$. As in the case of essential returns, we associate to each partitioned parameter interval the complete time n_{s+1+j}, and as before we make sure that at these times (BE)(1) is satisfied.

Let $\omega_e = \omega_L \cup \omega_M \cup \omega_R$ be the disjoint union of parameter intervals for which

$$\xi_{n_{s+1}}(\omega_L) = (\delta, 2\delta), \quad \xi_{n_{s+1}}(\omega_M) = (2\delta, 3\delta), \quad \text{and} \quad \xi_{n_{s+1}}(\omega_R) = (3\delta, 4\delta).$$

Clearly it is enough to show that ω_e reaches a complete return within finite time. Let t_* be the smallest integer for which

$$C_M e^{\gamma_M t_*} \geq 4.$$

If δ is small enough, and if $|\omega_0| = 2\epsilon$ is small enough, we can make sure that

$$\xi_{n_{s+1}+j}(\omega_e) \cap (-2\delta, 2\delta) = \emptyset \qquad (1 \leq j \leq t_*).$$

Suppose that, for some $j \geq t_*$, $\xi_{n_{s+1+j}}(\omega_e) \cap (-\delta, \delta) \neq \emptyset$, and that this return is not complete. Assuming that ω_L returns we can not have $\xi_{n_{s+1+j}}(\omega_L) \subset (-2\delta, 2\delta)$. Indeed, if this was the case, then (using Lemma 3.1 and parameter independence)

$$|\xi_{n_{s+1+j}}(\omega_L)| > 2|\xi_{n_{s+1}}(\omega_L)| > 2\delta,$$

contradicting the return not being complete. We conclude that after partitioning what is mapped inside of $(-\delta, \delta)$, what is left is of size at least δ, and we are back to the original setting. In particular, ω_M did not return to $(-\delta, \delta)$. Repeating this argument, ω_L and ω_R will return, but ω_M will stay outside of $(-\delta, \delta)$. (Here we abuse the notation: if ω_L returns we update it so that it maps onto $(\delta, 2\delta)$, and similarly if ω_R returns.) Due to Lemma 3.1 we therefore have

$$2 \geq |\xi_{n_{s+1+j}}(\omega_M)| \gtrsim |\xi_{n_{s+1}}(\omega_M)|\delta C_M e^{\gamma_M j} \geq \delta^2 C_M e^{\gamma_M j} \qquad (j \geq 0),$$

and clearly we must reach a complete return after $j = t$ iterations, with

$$t \lesssim \frac{2\Delta - \log C_M}{\gamma_M}.$$

With this we conclude that if m_0 is large enough then there exists a constant $\kappa > 0$ such that

$$m_{k+1}^{rl} \leq m_k^{rl} + \kappa \log m_k^{rl}. \tag{22}$$

We finish by estimating how much of Ω_{s+1+j} is being partitioned at each iteration. By definition of an escape return we have that $|\Omega_{s+1}| \geq 3\delta$, and since it takes a long time for ω_e to return, the following estimate is valid:

$$\frac{|I_{s+1+j}|}{|\Omega_{s+1+j}|} \geq \frac{|\Omega_{s+1+j}| - \delta}{|\Omega_{s+1+j}|} \geq 1 - \frac{1}{3} = \frac{2}{3}. \tag{23}$$

4.10 Parameter exclusion

We are finally in the position to estimate how much of ω is being removed at the next complete return. Up until the first free return, nothing is removed (unless we have a bound return, for which we either remove nothing, or remove enough to consider the return complete). Let E be what is removed in parameter space, and write $\omega = \omega^0$. Taking into account what we partition in between m_k and m_{k+1} we have that

$$\frac{|E|}{|\omega^0|} = \frac{|E|}{|\omega^{s+t}|} \prod_{\nu=0}^{t-1} \frac{|\omega^{s+1+\nu}|}{|\omega^{s+\nu}|} \prod_{\nu=0}^{s-1} \frac{|\omega^{1+\nu}|}{|\omega^{\nu}|}$$

Using the the mean value theorem we find that for each factor in the above expression

$$\begin{aligned}
\frac{|\omega^j|}{|\omega^{j-1}|} &= \frac{|a_j - b_j|}{|a_{j-1} - b_{j-1}|} \\
&= \frac{|a_j - b_j|}{|\xi_{n_j}(a_j) - \xi_{n_j}(b_j)|} \frac{|\xi_{n_j}(a_{j-1}) - \xi_{n_j}(b_{j-1})|}{|a_{j-1} - b_{j-1}|} \frac{|\xi_{n_j}(a_j) - \xi_{n_j}(b_j)|}{|\xi_{n_j}(a_{j-1}) - \xi_{n_j}(b_{j-1})|} \\
&= \frac{1}{|\partial_a \xi_{n_j}(c_j)|} |\partial_a \xi_{n_j}(c_{j-1})| \frac{|I_j|}{|\Omega_j|} \\
&= \frac{|\partial_x F^{n_j-1}(1; c_j)|}{|\partial_a \xi_{n_j}(c_j)|} \frac{|\partial_a \xi_{n_j}(c_{j-1})|}{|\partial_x F^{n_j-1}(1; c_{j-1})|} \frac{|\partial_x F^{n_j-1}(1; c_{j-1})|}{|\partial_x F^{n_j-1}(1; c_j)|} \frac{|I_j|}{|\Omega_j|}.
\end{aligned}$$

Making use of Lemma 3.2 and Lemma 5.1, we find that

$$\frac{|\omega^j|}{|\omega^{j-1}|} : \frac{|I_j|}{|\Omega_j|} \sim D_A D_1^{(\log^* m_k)^2},$$

and therefore, using (21), (18), and (23), there is, provided m_0 is large enough, an absolute constant $0 < \tau < 1$ such that

$$\frac{|E|}{|\omega^0|} \geq \frac{(\delta_{m_{k+1}}/3)}{1} \left(\frac{1}{3} D_A^{-1} D_1^{-(\log^* m_k)^2} \right)^{t + \log^* m_k} \geq \delta_{m_{k+1}} \tau^{(\log^* m_{k+1})^3}.$$

In particular, for the remaining interval $\hat{\omega} = \omega \setminus E$ we have that

$$|\hat{\omega}| \leq |\omega| (1 - \delta_{m_{k+1}} \tau^{(\log^* m_{k+1})^3}). \tag{24}$$

5 Main Distortion Lemma

Before giving a proof of Theorem B, we give a proof of the much important distortion lemma that, together with Lemma 3.2, allow us to restore derivative and to estimate what is removed in parameter space at the $(k+1)^{\text{th}}$ complete return. The proof is similar to that of Lemma 5 in [BC85], with the main difference being how we proceed at essential returns. As will be seen, our estimate is unbounded.

If not otherwise stated, the notation is consistent with that of the induction step. Recall that

$$\Delta_k = N_k \cup T_k,$$

with $\omega_k \subset N_k$ being mapped onto some $I_{rl} \subset (-4\delta, 4\delta)$, and $\omega_k \subset T_k$ being mapped onto an interval $\pm(\delta, x)$ with $|x - \delta| \geq 3\delta$. Moreover, we let $m_{k+1}(a, b)$ denote the largest time for which parameters $a, b \in \omega_k$ belong to the same parameter interval $\omega_k^j \subset \omega_k$, e.g. if $a, b \in \omega_k^j$ then $m_{k+1}(a, b) \geq n_{j+1}$.

Lemma 5.1 (Main Distortion Lemma). *Let $\omega_k \subset \Delta_k$, and let m_k be the index of the k^{th} complete return. There exists a constant $D_1 > 1$ such that, for $a, b \in \omega_k$ and $j < m_{k+1} = m_{k+1}(a, b)$,*

$$\frac{|\partial_x F^j(1; a)|}{|\partial_x F^j(1; b)|} \leq D_1^{(\log^* m_k)^2}.$$

Proof. Using the chain rule and the elementary inequality $x + 1 \leq e^x$ we have

$$\begin{aligned}
\frac{|\partial_x F^j(1; a)|}{|\partial_x F^j(1; b)|} &= \prod_{\nu=0}^{j-1} \frac{|\partial_x F(F^\nu(1; a); a)|}{|\partial_x F(F^\nu(1; b); b)|} \\
&= \left(\frac{a}{b}\right)^j \prod_{\nu=1}^{j} \frac{|\xi_\nu(a)|}{|\xi_\nu(b)|} \\
&\leq \left(\frac{a}{b}\right)^j \prod_{\nu=1}^{j} \left(\frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} + 1\right) \\
&\leq \left(\frac{a}{b}\right)^j \exp\left(\sum_{\nu=1}^{j} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|}\right).
\end{aligned}$$

We claim that the first factor in the above expression can be made arbitrarily close to 1. To see this, notice that

$$\left(\frac{a}{b}\right)^j \leq \left(1 + |\omega_k|\right)^j.$$

Using (BE)(1) and Lemma 3.2 we have that $|\omega_k| \lesssim e^{-\gamma m_k}$, and for m_0 large enough we have from (22) that $j < m_{k+1} \leq m_k + \kappa \log m_k \leq 2m_k$; therefore

$$\left(1 + |\omega_k|\right)^j \leq (1 + e^{-(\gamma/2)m_k})^{2m_k}.$$

Since

$$(1 + e^{-(\gamma/2)m_k})^{2m_k} \leq (1 + e^{-(\gamma/2)m_0})^{2m_0} \longrightarrow 1 \quad \text{as} \quad m_0 \longrightarrow \infty,$$

making m_0 larger if needed proves the claim. It is therefore enough to only consider the sum

$$\Sigma = \sum_{\nu=1}^{m_{k+1}-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|}.$$

With $m_k^* \leq m_{k+1}$ being the last index of a return, i.e. $\xi_{m_{k^*}}(\omega_k) \subset I_{r_k^*} \subset (-4\delta, 4\delta)$, we divide Σ as

$$\Sigma = \sum_{\nu=1}^{m_k^*-1} + \sum_{\nu=m_k^*}^{m_{k+1}-1} = \Sigma_1 + \Sigma_2,$$

and begin with estimating Σ_1.

The history of ω_k will be that of $\omega_0, \omega_1, \dots, \omega_{k-1}$. Let $\{t_j\}_{j=0}^N$ be all the inessential, essential, escape, and complete returns. We further divide Σ_1 as

$$\sum_{\nu=1}^{m_k^*-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} = \sum_{j=0}^{N-1} \sum_{\nu=t_j}^{t_{j+1}-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} = \sum_{j=0}^{N-1} S_j.$$

The contribution to S_j from the bound period is

$$\sum_{\nu=0}^{p_j} \frac{|\xi_{t_j+\nu}(a) - \xi_{t_j+\nu}(b)|}{|\xi_{t_j+\nu}(b)|} \lesssim \frac{|\xi_{t_j}(\omega)|}{|\xi_{t_j}(b)|} + \frac{|\xi_{t_j}(\omega)|}{|\xi_{t_j}(b)|} \sum_{\nu=1}^{p_j} \frac{e^{-2r_j}|\partial_x F^{\nu-1}(1;a)|}{|\xi_\nu(b)|}.$$

Let $\iota = (\kappa_1 \log 4)^{-1}$ and further divide the sum in the above right hand side as

$$\sum_{\nu=1}^{\iota p_j} + \sum_{\nu=\iota p_j+1}^{p_j}.$$

To estimate the first sum we use the inequalities $|\partial_x F^\nu| \leq 4^\nu$ and $|\xi_\nu(b)| \geq \delta_\nu/3 \gtrsim \nu^{-\bar{e}}$, and that $p_j \leq \kappa_1 r_j$, to find that

$$\begin{aligned}
\sum_{\nu=1}^{\iota p_j} \frac{e^{-2r_j}|\partial_x F^{\nu-1}(1;a)|}{|\xi_\nu(b)|} &\lesssim e^{-2r_j} \sum_{\nu=1}^{\iota p_j} 4^\nu \nu^{\bar{e}} \\
&\lesssim e^{-2r_j} 4^{\iota p_j} p_j^{\bar{e}} \\
&\lesssim \frac{r_j^{\bar{e}}}{e^{r_j}}.
\end{aligned}$$

To estimate the second sum we use (BC) and the equality (9), and find that

$$\sum_{\nu=\iota p_j+1}^{p_j} \frac{e^{-2r_j}|\partial_x F^{\nu-1}(1;a)|}{|\xi_\nu(b)|} \lesssim \frac{1}{r_j^2}.$$

Therefore the contribution from the bound period adds up to

$$\sum_{\nu=t_j}^{t_j+p_j} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} \lesssim \frac{|\xi_{t_j}(\omega)|}{|\xi_{t_j}(b)|} + \frac{|\xi_{t_j}(\omega)|}{|\xi_{t_j}(b)|}\left(\frac{1}{r_j^2} + \frac{r_j^{\bar{e}}}{e^{r_j}}\right)$$
$$\lesssim \frac{|\xi_{t_j}(\omega)|}{|\xi_{t_j}(b)|}.$$

After the bound period and up to time t_{j+1} we have a free period of length L_j during which we have exponential increase of derivative. We wish to estimate

$$\sum_{\nu=t_j+p_j+1}^{t_{j+1}-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} = \sum_{\nu=1}^{L_j-1} \frac{|\xi_{t_j+p_j+\nu}(a) - \xi_{t_j+p_j+\nu}(b)|}{|\xi_{t_j+p_j+\nu}(b)|}.$$

Using the mean value theorem, parameter independence and Lemma 3.1, we have that for $1 \le \nu \le L_j - 1$

$$\begin{aligned}
|\xi_{t_{j+1}}(a) - \xi_{t_{j+1}}(b)| &= |\xi_{t_j+p_j+L_j}(a) - \xi_{t_j+p_j+L_j}(b)| \\
&\simeq |F^{L_j-\nu}(\xi_{t_j+p_j+\nu}(a); a) - F^{L_j-\nu}(\xi_{t_j+p_j+\nu}(b); a)| \\
&= |\partial_x F^{L_j-\nu}(\xi_{t_j+p_j+\nu}(a'); a)||\xi_{t_j+p_j+\nu}(a) - \xi_{t_j+p_j+\nu}(b)| \\
&\gtrsim e^{\gamma_M(L_j-\nu)}|\xi_{t_j+p_j+\nu}(a) - \xi_{t_j+p_j+\nu}(b)|,
\end{aligned}$$

and therefore

$$|\xi_{t_j+p_j+\nu}(a) - \xi_{t_j+p_j+\nu}(b)| \lesssim \frac{|\xi_{t_{j+1}}(a) - \xi_{t_{j+1}}(b)|}{e^{\gamma_M(L_j-\nu)}}, \tag{25}$$

provided $\xi_{t_{j+1}}(\omega)$ does not belong to an escape interval. If $\xi_{t_{j+1}}(\omega)$ belongs to an escape interval, then we simply extend the above estimate to $t_{j+2}, t_{j+3}, \ldots$, until we end up inside some $I_{rl} \subset (-4\delta, 4\delta)$ (which will eventually happen, per definition of m_k^*). Hence we may disregard escape returns, and see them as an extended free period.

Since $|\xi_{t_{j+1}}(b)| \le |\xi_{t_j+p_j+\nu}(b)|$ for $1 \le \nu \le L_j - 1$, it follows from the above inequality that

$$\sum_{\nu=1}^{L_j-1} \frac{|\xi_{t_j+p_j+\nu}(a) - \xi_{t_j+p_j+\nu}(b)|}{|\xi_{t_j+p_j+\nu}(b)|} \le \frac{|\xi_{t_{j+1}}(a) - \xi_{t_{j+1}}(b)|}{|\xi_{t_{j+1}}(b)|} \sum_{\nu=1}^{L_j-1} e^{-\gamma_M(L_j-\nu)}$$
$$\lesssim \frac{|\xi_{t_{j+1}}(a) - \xi_{t_{j+1}}(b)|}{|\xi_{t_{j+1}}(b)|},$$

thus the contribution from the free period is absorbed in S_{j+1}.

What is left is to give an estimate of

$$\sum_{\nu=m_0}^{m_k^*-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} \lesssim \sum_{j=0}^{N} \frac{|\xi_{t_j}(\omega)|}{|\xi_{t_j}(b)|} \lesssim \sum_{j=0}^{N} \frac{|\xi_{t_j}(\omega)|}{|I_{r_j}|},$$

where, with the above argument, $\{t_j\}_{j=0}^{N}$ are now considered to be indices of inessential, essential, and complete returns only. Because of the rapid growth rate, we will see that among the returns to the same interval, only the last return will be significant. From Lemma 4.8 we have that $|\xi_{t_j+1}(\omega)| \gtrsim (e^{r_j}/r_j^{\kappa_2})|\xi_{t_j}(\omega)| \gg 2|\xi_{t_j}(\omega)|$, hence with $J(\nu)$ the last j for which $r_j = \nu$,

$$\sum_{j=0}^{N} \frac{|\xi_{t_j}(\omega)|}{|I_{r_j}|} = \sum_{\nu\in\{r_j\}} \frac{1}{|I_\nu|} \sum_{r_j=\nu} |\xi_{t_j}(\omega)| \lesssim \sum_{\nu\in\{r_j\}} \frac{|\xi_{t_{J(\nu)}}(\omega)|}{|I_\nu|}.$$

If $t_{J(\nu)}$ is the index of an inessential return, then $|\xi_{t_{J(\nu)}}(\omega_k)|/|I_\nu| \lesssim \nu^{-2}$, and therefore the contribution from the inessential returns to the above left most sum is bound by some small constant. It is therefore enough to only consider the contribution from essential returns and complete returns. To estimate this contribution we may assume that $m_k^* \geq m_k$, and that $\xi_{m_k}(\omega) = I_{r_k l}$. Moreover, we assume that $\xi_{m_j}(\omega) = I_{r_j l}$ for all j.

With $n_{j,0} = m_j$ being the index of the j^{th} complete return, and $n_{j,\nu} \in (m_j, m_{j+1})$ being the index of the ν^{th} essential return for which $\xi_{n_{j,\nu}}(\omega) \subset I_{r_{j,\nu}}$, we write

$$\sum_{\nu\in\{r_j\}} \frac{|\xi_{t_{J(\nu)}}(\omega)|}{|I_\nu|} \lesssim \sum_{j=0}^{k} \sum_{\nu=0}^{\nu_j} \frac{|\xi_{n_{j,\nu}}(\omega)|}{|I_{r_{j,\nu}}|} = \sum_{j=0}^{k} S_{m_j}.$$

For the last partial sum we use the trivial estimate $S_{m_k} \leq \log^* m_k$. To estimate S_{m_j}, for $j \neq k$, we realise that between any two free returns $n_{j,\nu}$ and $n_{j,\nu+1}$ the distortion is uniformly bound by some constant $C_1 > 1$. Therefore

$$\frac{|\xi_{n_{k-1,\nu_{k-1}-j}}(\omega)|}{|I_{r_{k-1,\nu_{k-1}-j}}|} \leq \frac{C_1^j}{r_k^2},$$

and consequently, since $\nu_j \leq \log^* r_j$ (see (21)),

$$S_{m_{k-1}} \leq \frac{C_2^{\log^* r_{k-1}}}{r_k^2},$$

for some uniform constant $C_2 > 1$. Continuing like this, we find that

$$\begin{aligned}
S_{m_{k-j}} &\le \frac{C_2^{\log^* r_{k-j}} C_2^{\log^* r_{k-j+1}} \cdots C_2^{\log^* r_{k-1}}}{r_{k-j+1}^2 r_{k-j+2}^2 \cdots r_k^2} \\
&\le \frac{C_2^{\log^* r_{k-j}}}{r_{k-j+1}^{3/2} r_{k-j+2}^{3/2} \cdots r_k^2},
\end{aligned}$$

where we in the last inequality used the (very crude) estimate

$$C_2^{\log^* x} \le \sqrt{x}.$$

Let us call the estimate of $S_{m_{k-j}}$ *good* if $C_2^{\log^* r_{k-j}} \le r_{k-j+1}$. For such $S_{m_{k-j}}$ we clearly have

$$S_{m_{k-j}} \le \frac{1}{\Delta^{j/2}}.$$

Let $j_1 \ge 1$ be the smallest integer for which $S_{m_{k-j_1}}$ is not good, i.e.

$$\log^* r_{k-j_1} \ge (\log C_2)^{-1} \log r_{k-j_1+1} \ge (\log C_2)^{-1} \log \Delta.$$

We call this the first *bad* estimate, and for the contribution from $S_{m_{k-j_1}}$ to the distortion we instead use the trivial estimate

$$S_{m_{k-j_1}} \le \log^* r_{k-j_1} \le \log^* m_k.$$

Suppose that $j_2 > j_1$ is the next integer for which

$$C_2^{\log^* r_{k-j_2}} \ge r_{k-j_2+1}.$$

If it turns out that

$$C_2^{\log^* r_{k-j_2}} \le r_{k-j_1},$$

then

$$S_{m_k-j_2} \le \frac{1}{\Delta^{j/2}},$$

and we still call this estimate good. If not, then

$$\log^* r_{k-j_2} \ge (\log C_2)^{-1} \log r_{k-j_1},$$

and j_2 is the index of the second bad estimate. Continuing like this, we get a number s of bad estimates and an associated sequence $R_i = \eta_{k-j_i}$ satisfying

$$\begin{aligned}
\log^* R_1 &\geq (\log C_2)^{-1} \log \Delta, \\
\log^* R_2 &\geq (\log C_2)^{-1} \log R_1, \\
&\vdots \\
\log^* R_s &\geq (\log C_2)^{-1} \log R_{s-1}.
\end{aligned}$$

This sequence grows incredibly fast, and its not difficult to convince oneself that

$$R_s \gg \underbrace{e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}}_{s \text{ copies of } e}.$$

In particular, since $R_s \leq m_k$ we find that

$$s \ll \log^* R_s \leq \log^* m_k.$$

We conclude that

$$\begin{aligned}
\left(\sum_{\text{good}} + \sum_{\text{bad}} \right) S_{m_j} &\leq \sum_{j=1}^{\infty} \frac{1}{\Delta^{j/2}} + s \log^* m_k \\
&\lesssim (\log^* m_k)^2,
\end{aligned}$$

hence

$$\sum_{\nu=1}^{m_k^*-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} \lesssim (\log^* m_k)^2.$$

From m_{k^*} to $m_{k+1} - 1$, the assumption is that we only experience an orbit outside $(-\delta, \delta)$. By a similar estimate as (25) we find that for $\nu \geq 1$

$$|\xi_{m_{k^*}+\nu}(a) - \xi_{m_{k^*}+\nu}(b)| \lesssim \frac{|\xi_{m_k-1}(a) - \xi_{m_k-1}(b)|}{\delta e^{\gamma_M (m_k - 1 - m_{k^*} - \nu)}},$$

and therefore

$$\sum_{\nu=m_k^*}^{m_{k+1}-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(b)|} \lesssim 1 + \frac{1}{\delta^2} \leq (\log^* m_k)^2,$$

provided m_0 is large enough. This proves the lemma. □

6 Proof of Theorem B

Returning to the more cumbersome notation used in the beginning of the induction step, let $\omega_k^{rl} \subset \Delta_k$. We claim that a similar inequality as (24) is still true if we replace $\hat{\omega}$ and ω with Δ_{k+1} and Δ_k, respectively. To realise this write Δ_{k+1} as the disjoint union

$$\Delta_{k+1} = \bigcup \omega_{k+1}^{rl} = \bigcup \hat{\omega}_k^{rl}.$$

With m_0 being the start time, consider the sequence of integers defined by the equality

$$m_{k+1} = \lceil m_k + \kappa \log m_k \rceil \qquad (k \geq 0),$$

where $\lceil x \rceil$ denotes the smallest integer satisfying $x \leq \lceil x \rceil$. By induction, using (22),

$$m_{k+1}^{rl} \leq m_k^{r'l'} + \kappa \log m_k^{r'l'} \leq m_k + \kappa \log m_k \leq m_{k+1}.$$

Hence the sequence (m_k) dominates every other sequence (m_k^{rl}), and therefore it follows from (24) that

$$\begin{aligned}
|\Delta_{k+1}| &= \sum |\hat{\omega}_k^{rl}| \\
&= \sum |\omega_k^{rl}| (1 - \delta_{m_{k+1}^{rl}} \tau^{(\log^* m_{k+1}^{rl})^3}) \\
&\leq \left(\sum |\omega_k^{rl}|\right) (1 - \delta_{m_{k+1}} \tau^{(\log^* m_{k+1})^3}) \\
&= |\Delta_k| (1 - \delta_{m_{k+1}} \tau^{(\log^* m_{k+1})^3}).
\end{aligned}$$

By construction

$$\Lambda_{m_0}(\delta_n) \cap \mathcal{CE}(\gamma_B, C_B) \cap \omega_0 \subset \Delta_\infty = \bigcap_{k=0}^{\infty} \Delta_k,$$

and therefore, to prove Theorem B, it is sufficient to show that

$$\prod_{k=0}^{\infty} (1 - \delta_{m_k} \tau^{(\log^* m_k)^3}) = 0.$$

By standard theory of infinite products, this is the case if and only if

$$\sum_{k=0}^{\infty} \delta_{m_k} \tau^{(\log^* m_k)^3} = \infty.$$

To evaluate the above sum we make use of the following classical result, due to Schlömilch (see [BK06], for instance).

Proposition 6.1 (Schlömilch Condensation Theorem). *Let $q_0 < q_1 < q_2 \cdots$ be a strictly increasing sequence of positive integers such that there exists a positive real number α such that*

$$\frac{q_{k+1} - q_k}{q_k - q_{k-1}} < \alpha \qquad (k \geq 0).$$

Then, for a nonincreasing sequence a_n of positive nonnegative real numbers,

$$\sum_{n=0}^{\infty} a_n = \infty \quad \textit{if and only if} \quad \sum_{k=0}^{\infty} (q_{k+1} - q_k) a_{q_k} = \infty.$$

Proof. We have

$$(q_{k+1} - q_k) a_{q_{k+1}} \leq \sum_{n=0}^{q_{k+1}-q_k-1} a_{q_k+n} \leq (q_{k+1} - q_k) a_{q_k},$$

and therefore

$$\alpha^{-1} \sum_{k=0}^{\infty} (q_{k+2} - q_{k+1}) a_{q_{k+1}} \leq \sum_{n=q_0}^{\infty} a_n \leq \sum_{k=0}^{\infty} (q_{k+1} - q_k) a_{q_k}.$$

□

Since $m_{k+1} - m_k \sim \log m_k$ is only dependent on m_k, we can easily apply the above result in a backwards manner. Indeed we have that

$$\begin{aligned} \frac{m_{k+1} - m_k}{m_k - m_{k-1}} &\leq \frac{\kappa \log m_k + 1}{\kappa \log m_{k-1}} \\ &\leq \frac{\kappa \log \left(m_{k-1} + \kappa \log m_{k-1} + 1\right) + 1}{\kappa \log m_{k-1}} \\ &\leq 1 + \frac{\text{const.}}{\log m_0} \qquad (k \geq 0), \end{aligned}$$

and therefore with $q_k = m_k$ and $a_n = \delta_n \tau^{(\log^* n)^3} / \log n$, the prerequisites of Schlömilch result are satisfied. We conclude that

$$\sum_{n=m_0}^{\infty} \frac{\delta_n}{\log n} \tau^{(\log^* n)^3} = \infty$$

if and only if

$$\sum_{k=0}^{\infty} (m_{k+1} - m_k) \frac{\delta_{m_k}}{\log m_k} \tau^{(\log^* m_k)^3} \sim \sum_{k=0}^{\infty} \delta_{m_k} \tau^{(\log^* m_k)^3} = \infty.$$

This proves Theorem B.

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Paper II

Slowly recurrent Collet–Eckmann maps with non-empty Fatou set

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Abstract

In this paper we study rational Collet–Eckmann maps for which the Julia set is not the whole sphere and for which the critical points are recurrent at a slow rate. In families where the orders of the critical points are fixed, we prove that such maps are Lebesgue density points of hyperbolic maps. In particular, if all critical points are simple, they are Lebesgue density points of hyperbolic maps in the full space of rational maps of any degree $d \geq 2$.

1 Introduction

Uniformly expanding maps have the property that nearby points on the Julia set repel each other at a uniform rate (with respect to some smooth metric). One of the central problems in complex dynamics is to prove that the set of these so called hyperbolic maps is open and dense in the parameter space of rational maps (or other complex analytic families of maps). This *conjecture by P. Fatou* in the 1920s has been proven in the real case [GŚ97, Lyu97, KSvS07], but is still open in the complex setting. In recent years, a great deal of attention has been focused on maps which are non-hyperbolic but satisfy a certain non-uniformly expanding condition, like the Misiurewicz condition (critically non-recurrent or even postcritically finite maps), the Collet–Eckmann condition or other more general summability conditions, semi-hyperbolicity etc. Conjecturally, almost every map is hyperbolic or satisfies such a non-uniformly expanding condition. This would also imply that the Fatou conjecture is true. In this paper, we focus on maps which satisfy the Collet–Eckmann condition. Our result demonstrates that any such map, for which the critical points are allowed to be recurrent at a slow rate (slowly recurrent maps), can be perturbed into hyperbolic maps in a strong sense; they are Lebesgue density points of hyperbolic maps. We discuss the (rather weak) condition on slow recurrence more below.

The Collet–Eckmann condition stems from the pioneering works by P. Collet and J.-P. Eckmann in the 1980s [CE80]. In the real setting, there are many works on the perturbation of such maps, see e.g. the seminal papers [BC85, BC91] by M. Benedicks and L. Carleson. M. Tsujii generalised these results for real maps in [Tsu93], see also the more recent work of B. Gao and W. Shen [GS14]. We are going to study perturbations of such maps in the complex setting. For the quadratic family and other unicritical families, J. Graczyk and G. Świątek recently made an extensive study of perturbations of typical Collet–Eckmann maps with respect to harmonic measure, in a series of papers [GS17, GS, GS00, GS15]. M. Benedicks and J. Graczyk also have an unpublished work on perturbations on such (quadratic or, more generally, unicritical) maps. The maps there and in the recent papers [GS17, GS] are also slowly recurrent, and hence the results in this paper is partially a generalisation of some of those results. We will not use harmonic measure, but develop the classical Benedicks–Carleson parameter exclusion techniques and combining it with strong results on transversality, by G. Levin [Lev14]. Technically, this paper is closely related to [Asp].

Let f be a rational map. As usual let $\mathcal{J}(f)$ and $\mathcal{F}(f)$ denote the Julia and Fatou set of f respectively. Let $\mathrm{Crit}(f)$ be the set of critical points of f, i.e. the set of points with vanishing spherical derivative. With $\mathrm{Jrit}(f)$ we mean the the set of critical points of f contained in the Julia set, i.e. $\mathrm{Jrit}(f) = \mathrm{Crit}(f) \cap \mathcal{J}(f)$. As is standard, we let f^n denote the n-th iterate of f.

In this paper we will consider perturbations of rational maps satisfying the following two properties. Recall that a rational map is called *hyperbolic* if it is expanding on the Julia set or, equivalently, if every critical point belongs to the Fatou set and is attracted to an attracting cycle. If a rational map is not hyperbolic, it is called *non-hyperbolic*. Derivatives are always with respect to the spherical metric on the Riemann sphere.

Definition 1.1 (Collet–Eckmann condition). A non-hyperbolic rational map f without parabolic periodic points satisfies the *Collet–Eckmann condition*, if there exist $C > 0$ and $\gamma > 0$ such that for each critical point c in the Julia set of f, one has

$$|Df^n(f(c))| \geq Ce^{\gamma n} \quad \text{for all } n \geq 0.$$

We will often refer to the constant γ appearing in the above definition as the *Lyapunov exponent* or simply the *exponent*. Notice that the Collet–Eckmann condition is equivalent to requiring the lower Lyapunov exponent at critical values (in the Julia set) to be strictly positive.

Definition 1.2 (Slow recurrence). A point z is said to be *slowly recurrent* if for any $\alpha > 0$, there exists $K > 0$ such that

$$\operatorname{dist}(f^n(z), \operatorname{Jrit}(f)) \geq K e^{-\alpha n} \quad \text{for all } n \geq 0. \tag{1}$$

Moreover, we say that f is slowly recurrent if every point in $\operatorname{Jrit}(f)$ is slowly recurrent.

This condition is conjecturally generic, as for example in the real quadratic family [AM05]. In fact, every Collet–Eckmann map has *some* $\alpha > 0$ for which (1) holds; see [DPU96, Lemma 2.2 or Lemma 2.3].

We denote by $\mathcal{R}_d$, the space of rational maps of degree d. In this paper we always assume that $d \geq 2$. If we write $f(z) = p(z)/q(z)$, where p and q are polynomials, and the maximal degree of p and q is d, there are two local charts on the coefficients; one for the case when $\deg(p) = d$ and another for $\deg(q) = d$. The Lebesgue measure on each of these charts are not equal but mutually absolutely continuous. So talking about sets of positive measure is independent of the chart used. We also mention that the Fubini-Study metric on $\mathcal{R}_d$ (which is a measure on the projective space $\mathbb{P}^{2d+1}(\mathbb{C})$) is mutually absolutely continuous with respect to the Lebesgue measure on each chart.

We will also consider a certain normalisation of the space $\mathcal{R}_d$ as follows, following G. Levin [Lev14, Lev11]. We say that two maps f and g are equivalent if they are conjugate by a Möbius transformation. Then we can consider the space $\Lambda_{d,\overline{p'}} \subset \mathcal{R}_d$, (see [Lev14]) up to equivalence, as the set of rational maps f of degree $d \geq 2$ with precisely p' critical points, i.e. $\operatorname{Crit} = \{c_1, \dots, c_{p'}\}$, with corresponding multiplicities $\overline{p'} = \{m_1, \dots, m_{p'}\}$ (in the same order). This means in particular that all critical points move analytically inside $\Lambda_{d,\overline{p'}}$.

We will prove the following result.

Theorem 1.3. *Any slowly recurrent rational Collet–Eckmann map $f \in \Lambda_{d,\overline{p'}}$ of degree $d \geq 2$, for which the Julia set is not the entire sphere, is a Lebesgue density point of hyperbolic maps in $\Lambda_{d,\overline{p'}}$.*

If all critical points are simple, then $\Lambda_{d,\overline{p'}}$ is locally equal to $\mathcal{R}_d$ (up to Möbius conjugacy), and we immediately get the following corollary.

Corollary 1.4. *Any slowly recurrent rational Collet–Eckmann map f of degree $d \geq 2$ with only simple critical points, and for which the Julia set is not the entire sphere, is a Lebesgue density point of hyperbolic maps in $\mathcal{R}_d$.*

Note that if f is Collet–Eckmann and $\mathcal{F}(f) \neq \emptyset$, then the Fatou set $\mathcal{F}(f)$ consists only of attracting cycles and the Julia set of f has Lebesgue measure zero (and actually the Hausdorff dimension is strictly less than 2) [GS98].

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2 Preliminaries

We will consider one-dimensional complex analytic families in $\Lambda_{d,\overline{p'}} \subset \mathcal{R}_d$ and prove the corresponding main result in almost all such families (where 'almost all' means almost all tangents in $\Lambda_{d,\overline{p'}}$ in the sense of Levin, see Section 3.1). The main result will then follow by Fubini's theorem. Throughout the paper $\mathcal{Q} = \mathcal{Q}(\varepsilon)$ will denote a fixed one-dimensional parameter square with sidelength ε, centred around a slowly recurrent Collet–Eckmann map f_0. So f_a, $a \in \mathcal{Q}$ is a one-dimensional analytic family of rational maps in $\Lambda_{d,\overline{p'}}$. Let C_0 and γ_0 be the associated constant and exponent of f_0 appearing in Definition 1.1 for the *starting map* f_0. We denote by $c_l(0)$ and $v_l(0)$ (or simply c_l and v_l if it is clear from the context) the critical points and critical values of f_0 in $\mathcal{J}(f_0)$. In other words, $c_l \in \mathrm{Jrit}(f_0)$. The corresponding critical points and critical values for f_a with $a \in \mathcal{Q}$ are denoted by $c_l(a)$ and $v_l(a)$, respectively, and we see that $c_l(a)$ (and consequently $v_l(a)$) are analytic in $\mathcal{Q}$ (since we are considering $\Lambda_{d,\overline{p'}}$). For simplicity, we write Jrit_0 instead of $\mathrm{Jrit}(f_0)$, while Jrit_a denotes the set of $c_l(a)$, for $a \in \mathcal{Q}$. (Note that we are not claiming that $c_l(a)$ lies in the Julia set $\mathcal{J}(f_a)$.) For a connected set $A \subset \mathcal{Q}$, we let Jrit_A denote the union of Jrit_a over $a \in A$.

For $a \in \mathcal{Q}$, we are going to study the evolution of the critical points $c_l(a)$, and for this we introduce the functions

$$\xi_{n,l}(a) = f_a^n(c_l(a)) \quad \text{for all } n \geq 0.$$

With $x \lesssim y$ (or $x \gtrsim y$) we will mean that there exists a constant $C > 0$ (not dependent on the dynamics) such that $x \leq Cy$ (or $x \geq Cy$). If both $x \lesssim y$ and $x \gtrsim y$ then we will write $x \sim y$.

Since f_0 satisfies the Collet–Eckmann condition, nearby parameters inherit expansion for some time, and therefore the image of the parameter square $\mathcal{Q}$ will expand under $\xi_{n,l}$. Once the image of $\mathcal{Q}$ gets close to $\mathrm{Jrit}_{\mathcal{Q}}$, the derivative will decrease (depending on the distance to $\mathrm{Jrit}_{\mathcal{Q}}$). To ensure that we still have good expansion after getting close to $\mathrm{Jrit}_{\mathcal{Q}}$, a local analysis is needed. Let $0 < \Delta' < \Delta$ be two large numbers, and let

$$\delta = e^{-\Delta}, \quad \delta' = e^{-\Delta'},$$
$$U_l = D(c_l, \delta), \quad U_l' = D(c_l, \delta'),$$

and define

$$U = \bigcup_l U_l \quad \text{and} \quad U' = \bigcup_l U_l' \tag{2}$$

to be neighbourhoods of the critical points of f_0 belonging to $\mathcal{J}_0$. By continuity, one can choose ε sufficiently small such that U is also a neighbourhood of $c_l(a)$ for all parameters $a \in \mathcal{Q}$. In fact, we want to have $\operatorname{diam}(U_l) \gg \operatorname{diam}(c_l(\mathcal{Q}))$. We will frequently use local Taylor expansion

$$f_a(z) = f_a(c_l) + B(z - c_l)^{d_l} + \mathcal{O}\left((z - c_l)^{d_l+1}\right),$$

and U' is chosen to be some fixed neighbourhood where first order Taylor expansions are sufficiently good around any c_l. Considering the multiplicity at critical points, we let $\hat{d} = \max_l d_l$. (Note that we assume that the critical points do not split under perturbation, i.e. $d_l = d_l(a)$ is constant for $a \in \mathcal{Q}$.) The smaller neighbourhood U should be thought of as a neighbourhood that could be as small as one likes to fit into the construction. Furthermore, in the section on large deviations, we will also make use of a smaller neighbourhood $U^2 = \cup_l U_l^2$, where $U_l^2 = D(c_l, \delta^2) \subset U_l$. By choosing ε small enough we make sure that U_l^2 is still a neighbourhood of $\operatorname{Jrit}_{\mathcal{Q}}$.

As time evolves, we will discard parameters that come too close to Jrit_a. For this reason we define the *basic approach rate assumption* (or simply the *basic assumption*) as follows.

Definition 2.1. Let $\alpha > 0$ and $K > 0$ be the constants from the slow recurrence condition (Definition 1.2). We say that $c_l(a)$ satisfies the basic assumption up to time n with exponent α, if

$$\operatorname{dist}(\xi_{k,l}(a), \operatorname{Jrit}_a) \geq Ke^{-2\alpha k} \quad \text{for all } k \leq n. \tag{3}$$

For our starting map f_0 which is assumed to be slowly recurrent, the basic assumption is, per definition, always satisfied for all n and for all l with exponent α. By making the perturbations sufficiently small, i.e. choosing ε small enough, each parameter $a \in \mathcal{Q}$ will also satisfy the basic assumption up to some time. However, as the number n of iterates grows, $\xi_{n,l}(\mathcal{Q})$ becomes a comparatively large set, so that we shall need to partition our parameter square $\mathcal{Q}$ in the following way. Let U be as defined in (2).

Definition 2.2 (Partition element)**.** Let $S > 0$ be given. A connected set $A \subset \mathcal{Q}$ is called a partition element at time n if the following holds for all $k \leq n$:

$$\operatorname{diam} \xi_{k,l}(A) \leq \begin{cases} \dfrac{\operatorname{dist}(\xi_{k,l}(A), \operatorname{Jrit}_{\mathcal{Q}})}{(\log \operatorname{dist}(\xi_{k,l}(A), \operatorname{Jrit}_{\mathcal{Q}}))^2} & \text{if } \xi_{k,l}(A) \cap U \neq \emptyset, \\ S & \text{if } \xi_{k,l}(A) \cap U = \emptyset. \end{cases}$$

For convenience, the partition elements are going to be squares in our situation, since we start with a square $\mathcal{Q}$, but in principle this is not needed. The reason to make partitions according to the above rule is that we have distortion control of $\xi_{n,l}(a)$ for $a \in A$. So as time evolves, the partition gets finer. The constant S appearing in the above definition is usually referred to as the *large scale*, and we say that a partition element has *escaped* when it reaches size S under the action of the function $\xi_{n,l}$.

Our main task in the paper is to show that almost all partition elements will reach the large scale within a bounded (but not necessarily uniform) amount of time. This is relatively easy if $\xi_{n,l}(A)$ never comes close to critical points (which is true if our starting map is, say, of Misiurewicz type). In our case, however, it can happen that $\xi_{n,l}(A)$ approaches critical points, since we are starting with a (slowly) recurrent map. Although the approach rate is controlled by the basic assumption, we may still lose derivative. To restore this loss, we shall use the ideas from [BC85, BC91] (see also [Asp] which is similar to the setting here).

The fundamental concepts for dealing with the above situations are the so-called *bound periods* and *free periods*. To define them, we first introduce the notion of *returns* which can be defined for single parameters and also for partition elements.

Recall that U and U' are defined in (2). For a partition element A we say that $\xi_{n,l}(A)$ is a *return* if $\xi_{n,l}(A) \cap U' \neq \emptyset$ or $\xi_{n,l}(A) \cap U \neq \emptyset$. We speak of a *pseudo-returns* if $\xi_{n,l}(A)$ is a return into U' but not U. For a parameter $a \in \mathcal{Q}$, we say that $\xi_{n,l}(a)$ is a return if $\xi_{n,l}(a) \in U'$ or $\xi_{n,l}(a) \in U$.

Definition 2.3 (Bound period for parameters)**.** Let α be as in the basic assumption (3). Let $\xi_{n,l}(a) \in U'_k$ be a return, where U'_k is the component of U' containing $c_k(0)$. The bound period for this return is defined as the indices $j > 0$ such that the following holds:

$$\left|\xi_{n+j,l}(a) - \xi_{j,k}(a)\right| \leq e^{-\alpha j}\operatorname{dist}(\xi_{j,k}(a), \mathrm{Jrit}_a).$$

The largest number $p = p(a) > 0$ for which the above inequality holds is called the length of the bound period.

During the bound period, the growth of derivative is inherited from its early orbit, regardless of whether or not there are more returns during this period. Such returns are called bound returns. Because of the binding condition in the above definition, these returns are harmless. As soon as the bound period ends, we enter into a *free period*, which means that this piece of orbit stays away from critical points. During the free period, derivative growth is guaranteed by the classical result of Mañé (see the next section for a more precise statement). If p is the length of the bound period, when $\xi_{n+p+L,l}(a) \in U'$ for the least possible $L > 0$, we speak of a *free return*. The number L is the length of the free period. Since bound returns are harmless we only speak of returns, and thereby mean free returns.

We will also need a corresponding notion of bound period for partition elements. To define this, let $A \subset Q$ be a partition element at time n. We say that a return $\xi_{\nu,l}(A)$, $\nu > n$, into U is *essential* if

$$\operatorname{diam} \xi_{\nu,l}(A) \geq \frac{1}{3} \frac{\operatorname{dist}(\xi_{\nu,l}(A), \mathrm{Jrit}_Q)}{(\log \operatorname{dist}(\xi_{\nu,l}(A), \mathrm{Jrit}_Q))^2}.$$

Otherwise, it is called an *inessential return*. When an essential return occurs we will have to make partitions according to Definition 2.2. Because of strong bounds on distortion, we will see that $\xi_{n,l}$ is almost affine on each partition element A and if A is a perfect square then $\xi_{n,l}(A)$ is also almost a perfect square. If A has side length d, simply partition A into four subsquares of equal length. If all these four subsquares are partition elements according to Definition 2.2, we are done; the new partition is thereby defined. If a subsquare is not a partition element, continue partitioning it into four new subsquares of equal length, and continue like this until all the new subsquares are partition elements. We get a collection of squares of sidelength of the form $2^{-k}d$, for some $k \geq 0$ (note that we can have different values of k). No partition is made at an inessential return. We can now define the bound period for partition elements.

Definition 2.4 (Bound period for partition elements)**.** Let A be a partition element at time n and $\xi_{n,l}(A)$ an essential, inessential or pseudo- return to U'_k, the component of U'. The bound period for this return is defined as indices $j > 0$ such that the following holds for all $a, b \in A$ and for all $z \in \xi_{n,l}(A)$:

$$\left| f_a^j(z) - \xi_{j,k}(b) \right| \leq e^{-\alpha j} \operatorname{dist}(\xi_{j,k}(b), \mathrm{Jrit}_b).$$

The largest number $p = p(A) > 0$ for which the above inequality holds is called the length of the bound period.

With the above notions, we will follow the parameter exclusion technique originated by M. Benedicks and L. Carleson [BC85, BC91]. However, we have to deal with the situation caused by the presence of several critical points (again following an idea due to M. Benedicks). What can happen is that a critical orbit might get close to a critical point other than itself. In this case, to use induction we need to use the binding information of this latter critical point. To handle this we make the following definition. Let $\gamma_I > 0$ be a constant to be defined later.

Definition 2.5. Given $\gamma > 0$, we say that a parameter a belongs to $\mathcal{E}_{n,l}(\gamma)$ if

$$|Df_a^k(v_l(a))| \geq C_0 e^{\gamma k} \quad \text{for all } k \leq n-1,$$

and

$$|Df_a^k(v_j(a))| \geq C_0 e^{\gamma k} \quad \text{for all } k \leq 2\hat{d}\alpha n/\gamma_I, \text{ and all } j \neq l. \tag{4}$$

We say that a parameter a belongs to $\mathcal{B}_{n,l}$ if

$$\operatorname{dist}(\xi_{k,l}(a), \operatorname{Jrit}_a) \geq Ke^{-2\alpha k} \quad \text{for all} \ \ k \leq n-1,$$

and

$$\operatorname{dist}(\xi_{k,j}(a), \operatorname{Jrit}_a) \geq Ke^{-2\alpha k} \quad \text{for all} \ \ k \leq 2\hat{d}\alpha n/\gamma_l, \ \text{and all} \ \ j \neq l.$$

3 Lemmas

In this section we present several lemmas on distortion and transversality. The transversality property says that phase and parameter derivatives can be compared if the phase derivative grows at a certain rate. In our new situation with recurrent critical points, this property is inherited from quite recent powerful results by G. Levin [Lev21, Lev14]. Together with a strong distortion lemma in the phase space (the main distortion lemma), we get strong control of the geometry of $\xi_{n,l}(A)$ on partition elements.

3.1 Phase-parameter relations

3.1.1 Transversality

Using a result by G. Levin we state a transversality result for Collet–Eckmann parameters, relating phase and parameter derivatives. In the following there is a notion of degenerate families of rational maps, following [Lev14, Lev21]. We consider one-dimensional complex families of rational maps in $\Lambda_{d,\overline{p}'}$ through the starting map f_0 such that this family has a non-zero tangent at f_0, i.e. such that $f_a(z) = f_0(z) + au(z) + \mathcal{O}(a^2)$, for some non-zero $u(z)$. For almost all directions of this tangent in the parameter space, it is shown that we have a certain transversality property (see [Lev21], Corollary 2.1, part (8)), namely that the limit

$$\lim_{n\to\infty} \frac{\xi'_{n,l}(0)}{(f_0^{n-1})'(f_0(c_l))} = L_l,$$

exists and is different from 0 and ∞. (With $\xi'_{n,l}(a)$ we mean the parameter derivative of f_a^n evaluated at $c_l(a)$, i.e. $\xi'_{n,l}(a) = \partial_a f_a^n(c_l(a))$.) Families satisfying this condition are called *non-degenerate in the sense of Levin*. Based on this we get the following, see Proposition 4.1 in [Asp].

Lemma 3.1. *Let $f = f_0$ be a slowly recurrent Collet–Eckmann map with exponent γ_0 and f_a, $a \in \mathcal{Q}$, an analytic non-degenerate family in the sense of Levin. Then for any $q \in (0,1)$ and* *any $\gamma \in (0, \gamma_0)$ there exists $N > 0$ and $\varepsilon > 0$ such that*

$$\left| \frac{\xi'_{n,l}(a)}{(f_a^{n-1})'(v_l(a))} - L_l \right| \leq q|L_l|$$

provided that f_a satisfies the Collet–Eckmann condition up to time $n \geq N$ with exponent γ for all $a \in \mathcal{Q}$.

Recall that our starting map f_0 satisfies the Collet–Eckmann condition with exponent γ_0. With γ_H being the exponent from Lemma 3.10 below (see Remark 3.11), we shall apply the above lemma for

$$\gamma_L := \frac{1}{6} \min\{\gamma_0, \gamma_H\}(1 - \tau),$$

where $\tau \in (0, 1)$ is some constant to be determined later. This choice of γ_L also dictates the choices of the corresponding N and ε in Lemma 3.1, which we denote by N_L and ε_L correspondingly. We choose the size ε of our domain of perturbation (i.e., the parameter square $\mathcal{Q}$) to comply with Lemma 3.1, e.g. $\varepsilon < \varepsilon_L$. For later convenience, we also define

$$\gamma_I := 2\gamma_L, \quad \text{and } \gamma_B := \frac{9}{2}\gamma_L.$$

We thus have that $\gamma_B > \gamma_I > \gamma_L$.

3.1.2 Weak parameter dependence

The following lemma tells us that the dependence on parameter is weak as long as we have exponential growth of the derivative. As a matter of fact, the dependence is even weaker, as will be seen after the proof of the main distortion lemma.

Lemma 3.2. *Let N_L and γ_L be as in Lemma 3.1, and let $\gamma_1 > (3/2)\gamma_L$. Suppose that $a, b \in \mathcal{Q}$. If ε and δ are small enough, and if there is an integer $k_1 \geq 0$ such that*

i) $|Df_a^n(v_l(a))| \geq C_1 e^{\gamma_1 n}$ *for all $n \leq N_L + k_1$;*

ii) for all $n \leq N_L + k_1$, if $\xi_{n,l}(a), \xi_{n,l}(b) \notin U$ then $|\xi_{n,l}(a) - \xi_{n,l}(b)| \leq S$, and otherwise if $\xi_{n,l}(a) \in U$ or $\xi_{n,l}(b) \in U$ then

$$|\xi_{n,l}(a) - \xi_{n,l}(b)| \leq \frac{\operatorname{dist}(\xi_{n,l}(a'), \operatorname{Jrit}_{a'})}{\left(\log\left(\operatorname{dist}(\xi_{n,l}(a'), \operatorname{Jrit}_{a'})\right)\right)^2},$$

with $a' \in \{a, b\}$ minimising $\operatorname{dist}(\xi_{n,l}(a'), \operatorname{Jrit}_{a'})$*;*

then there exists $Q > 1$ *(arbitrarily close to* 1 *if* N_L *is large enough) such that for all* $N_L \le n \le N_L + k_1$

$$|\xi_n(a) - \xi_n(b)| \ge Q^{-(n-1)} |Df_a^{n-1}(v_l(a))||a-b|.$$

Moreover, for all $0 \le j \le n - N_L$

$$|\xi_{n,l}(a) - \xi_{n,l}(b)| \sim_{Q^j} |Df_a^j(\xi_{n-j,l}(a))||\xi_{n-j,l}(a) - \xi_{n-j,l}(b)|.$$

Proof. We fix l and write $\xi_n = \xi_{n,l}$, $N = N_L$. We begin with proving that there is a $Q > 1$ close to 1 such that

$$|\xi_n(a) - \xi_n(b)| \ge Q^{-(n-1)} |Df_a^{n-1}(v_l(a))||a-b| \tag{5}$$

is true for all $N \le n \le N + k_1$.

By making ε small enough we can make sure that

$$|\xi_N(a) - \xi_N(b)| \ge \frac{1}{2}|\xi_N'(a)||a-b|.$$

Using Lemma 3.1 we find that

$$\begin{aligned}|\xi_N(a) - \xi_N(b)| &\ge \frac{1}{2}|L_l|(1-q)|Df_a^{N-1}(v_l(a))||a-b| \\ &\ge Q^{-(N-1)}|Df_a^{N-1}(v_l(a))||a-b|,\end{aligned}$$

where $Q > 1$ can be made arbitrarily close to 1 by increasing N. Assume that the above inequality holds for some $N \le n \le N + k_1 - 1$. We have that

$$\begin{aligned}|\xi_{n+1}(a) - \xi_{n+1}(b)| &\ge |f_a(\xi_n(a)) - f_a(\xi_n(b))| - |f_a(\xi_n(b)) - f_b(\xi_n(b))| \\ &\ge Q_0^{-1}|Df_a(\xi_n(a))||\xi_n(a) - \xi_n(b)| - 2|\partial_a f_a(\xi_n(a))||a-b| \\ &\ge Q_0^{-1}Q^{-(n-1)}\left(|Df_a^n(v_l(a))| - 2BQ_0Q^{n-1}\right)|a-b|,\end{aligned}$$

where $B = \sup|\partial_a f_a|$ and $Q_0 > 1$ can be made arbitrarily close to 1 by making ϵ_1 small enough in $S = \partial\epsilon_1$, and N large enough.

From assumption i) since if Q is such that $\log Q < \gamma_1/2$, say, then

$$2BQ_0Q^{n-1} \le \frac{Q_0 - 1}{Q_0}C_1 e^{\gamma_1 n} \le \frac{Q_0 - 1}{Q_0}|Df_a^n(v_l(a))|, \tag{6}$$

for N large enough. Combining this with with the above, taking $Q_0 = \sqrt{Q}$, we find that

$$|\xi_{n+1}(a) - \xi_{n+1}(b)| \ge Q^{-n}|Df_a^n(v_l(a))||a-b|,$$

proving the first conclusion of the lemma.

The proof of the second claim of the lemma is very similar to the proof of the first claim above. We use an inductive argument as follows. For $n = N$ (and thus $j = 0$) the result is trivial. Suppose therefore that for some $N \le n \le N + k_1 - 1$ the conclusion in the statement of the lemma is true, and consider the case $n + 1$. Pick some $0 \le j \le n - N$. Using (5) we find that

$$\begin{aligned}
&|\xi_{n+1}(a) - \xi_{n+1}(b)| \\
&\geq |f_a(\xi_n(a)) - f_a(\xi_n(b))| - |f_a(\xi_n(b)) - f_b(\xi_n(b))| \\
&\geq Q_0^{-1}|Df_a(\xi_n(a))||\xi_n(a) - \xi_n(b)| - 2|\partial_a f_a(\xi_n(a))||a - b| \\
&\geq Q_0^{-1}\left(|Df_a(\xi_n(a))| - \frac{2BQ_0Q^{n-1}}{|Df_a^{n-1}(v_l(a))|}\right)|\xi_n(a) - \xi_n(b)|.
\end{aligned}$$

It follows from inequality (6) that

$$\frac{2BQ_0Q^{n-1}}{|Df_a^{n-1}(v_l(a))|} \leq \frac{Q_0 - 1}{Q_0}|Df_a(\xi_n(a))|.$$

We continue now, using the induction assumption that the lemma is true for n, to conclude that, for $0 \le j \le n - N$,

$$\begin{aligned}
|\xi_{n+1}(a) - \xi_{n+1}(b)| &\geq Q_0^{-2}|Df_a(\xi_n(a))||\xi_n(a) - \xi_n(b)| \\
&\geq Q_0^{-2}|Df_a(\xi_n(a))|Q^{-j}|Df_a^j(\xi_{n-j}(a))||\xi_{n-j}(a) - \xi_{n-j}(b)| \\
&\geq Q_0^{-2}Q^{-j}|Df_a^{j+1}(\xi_{n-j}(a))||\xi_{n-j}(a) - \xi_{n-j}(b)|.
\end{aligned}$$

Choosing $Q_0 = \sqrt{Q}$ close enough to 1, we get

$$|\xi_{n+1}(a) - \xi_{n+1}(b)| \geq Q^{-(j+1)}|Df_a^{j+1}(\xi_{n-j}(a))||\xi_{n-j}(a) - \xi_{n-j}(b)|.$$

The case $j = 0$ in the second claim of the lemma is trivial. Hence, this proves one of the inequalities of the second claim. We can achieve the other inequality in a completely analogous way. □

With the above lemma we immediately get the following result, telling us that the analytic dependence of critical points on the parameters are negligible.

Lemma 3.3. *Under the assumptions of Lemma 3.2 there exists a constant* $C > 0$ *such that*

$$|\xi_{n,l}(a) - \xi_{n,l}(b)| \geq Ce^{\gamma_2 n}|c_l(a) - c_l(b)|,$$

for any $N_L \le n \le N_L + k_1$.

Proof. Since we assume that the critical points $c_l(a)$ move analytically in a we have

$$c_l(a) = c_l(b) + K_l(a-b)^{k_l} + \mathcal{O}\left((a-b)^{k_l+1}\right).$$

From (5) and the the conclusion of Lemma 3.2 we find that

$$\begin{aligned}|c_l(a) - c_l(b)| &\leq 2|K_l||k_l||a-b|^{k_l-1}|a-b| \\ &\lesssim \frac{Q^n}{|Df_a^n(v_l(a))|}|\xi_n(a) - \xi_n(b)| \\ &\lesssim e^{-\gamma_2 n}|\xi_n(a) - \xi_n(b)|,\end{aligned}$$

where γ_2 is slightly smaller than γ_1. □

With these lemmas we can neglect the parameter dependence on each partition element A. In particular, for returns into U', $\text{dist}(\xi_{n,l}(A), c_l(A))$ is very close to $|\xi_{n,l}(a) - c_l(a)|$ for all $a \in A$, so we can almost view $c_l(A)$ as one single critical point.

3.1.3 Distortion estimate

In the sequel we will frequently use the following distortion estimate, which we for convenience formulate as a lemma.

Lemma 3.4. *If z and w stay sufficiently close to each other under iteration by f_a up to time n, then*

$$\left|\frac{Df_a^n(z)}{Df_a^n(w)} - 1\right| \leq \exp\left(C\sum_{j=0}^{n-1}\frac{|f_a^j(z) - f_a^j(w)|}{\text{dist}(f_a^j(w), \text{Jrit}_a)}\right) - 1, \tag{7}$$

for some constant $C > 0$ dependent on f_0 and ε. Moreover, if also $z = \xi_\nu(a)$ and $w = \xi_\nu(b)$, with $a, b \in \mathcal{Q}$, and if the assumptions of Lemma 3.2 are satisfied, then

$$\left|\frac{Df_a^n(\xi_\nu(a))}{Df_b^n(\xi_\nu(b))} - 1\right| \leq \exp\left(C\sum_{j=0}^{n-1}\frac{|\xi_{\nu+j}(a) - \xi_{\nu+j}(b)|}{\text{dist}(\xi_{\nu+j}(b), \text{Jrit}_b)}\right) - 1. \tag{8}$$

Proof. Given any complex numbers $z_1, \dots, z_n$, the following inequality is standard

$$\left|\prod_{j=1}^{n} z_j - 1\right| \leq \exp\left(\sum_{j=1}^{n}|z_j - 1|\right) - 1,$$

and therefore, using the chain rule, we conclude that

$$\left|\frac{Df_a^n(z)}{Df_b^n(w)} - 1\right| \leq \exp\left(\sum_{j=0}^{n-1}\frac{|Df_a(f_a^j(z)) - Df_b(f_b^j(w))|}{|Df_b(f_b^j(w))|}\right) - 1.$$

We begin to prove (7); (8) will then follow from Lemma 3.2.

Let us write $z_j = f_a^j(z)$ and $w_j = f_a^j(w)$. If one (or both) of z_j and w_j does not belong to U', we are away from critical points and the estimate follows easily. Indeed we have that

$$|Df_a(z_j) - Df_a(w_j)| \le 2 \sup_{z\in\widehat{\mathbb{C}}} |D^2 f_a(z)||z_j - w_j|,$$

and also

$$|Df_a(w_j)| \ge \frac{1}{2} \frac{\inf_{z\notin U'} |Df_a(z)|}{\sup_{z\notin U'} \operatorname{dist}(z, \operatorname{Jrit}_a)} \operatorname{dist}(w_j, \operatorname{Jrit}_a).$$

If both z_j and w_j belong to $U_l' \subset U'$, using local Taylor expansion

$$f_a(x) = f(c_l) + B(x - c_l)^{d_l} + \mathcal{O}\left((x - c_l)^{d_l+1}\right),$$

and the fact that $|Df(z_j) - Df(w_j)| \le |D^2 f(x)||z_j - w_j|$ for some $x \in [z_j, w_j]$ we find that

$$\begin{aligned} |Df_a(z_j) - Df_a(w_j)| &\le d_l^2 |B||x - c|^{d_l-2} |1 + \mathcal{O}\left((x - c)\right)||z_j - w_j| \\ &\le 2d_l^2 |B||w - c|^{d_l-2} |z_j - w_j|, \end{aligned}$$

where we used that $|z_j - w_j|$ and δ' is very small.

For the derivative we have the estimate

$$|Df_a(w_j)| = |Bd_l(w_j - c_l)^{d_l-1} + \mathcal{O}\left((w_j - c_l)^{d_l}\right)| \ge \frac{1}{2} d_l |B||w_j - c|^{d_l-1}.$$

We conclude that

$$\sum_{j=0}^{n-1} \frac{|Df_a(f_a^j(z)) - Df_a(f_a^j(w))|}{|Df_a(w)|} \lesssim \sum_{j=0}^{n-1} \frac{|f_a^j(z) - f_a^j(w)|}{\operatorname{dist}(w, \operatorname{Jrit}_a)}.$$

To prove (8) we use the previous discussion together with Lemma 3.2 to conclude that

$$\begin{aligned} &|Df_a(\xi_{\nu+j}(a)) - Df_b(\xi_{\nu+j}(b))| \\ &\le |Df_a(\xi_{\nu+j}(a)) - Df_a(\xi_{\nu+j}(b))| + |Df_a(\xi_{\nu+j}(b)) - Df_b(\xi_{\nu+j}(b))| \\ &\lesssim |\xi_{\nu+j}(a) - \xi_{\nu+j}(b)| + 2|\partial_a Df_a(\xi_{\nu+j}(a))||a - b| \\ &\lesssim \left(1 + \frac{2|\partial_a Df_a(\xi_{\nu+j}(a))|Q^{\nu+j-1}}{|Df_a^{\nu+j-1}(v(a))|}\right) |\xi_{\nu+j}(a) - \xi_{\nu+j}(b)| \\ &\lesssim |\xi_{\nu+j}(a) - \xi_{\nu+j}(b)|. \end{aligned}$$

□

3.2 Expansion during bound periods

We study in this section how an orbit preserves certain expansion during bound periods (although some loss is unavoidable at returns). To achieve this, we first need some distortion control during these periods. Recall that U' is defined in (2).

Lemma 3.5. *Let $\varepsilon' > 0$. Let $\delta' > 0$ be sufficiently small and N sufficiently large. Let also $\gamma \geq \gamma_I$. Suppose that $\xi_{\nu,l}(a)$ is a free return to U_i' with $a \in \mathcal{E}_{\nu,l}(\gamma) \cap \mathcal{B}_{\nu,l}$. Then we have*

$$\left| \frac{Df_a^j(\xi_{\nu+1,l}(a))}{Df_a^j(\xi_{1,i}(a))} - 1 \right| \leq \varepsilon'$$

for all $j \leq p$, where p is the length of the bound period.

Proof. Since $\xi_{\nu,l}(a)$ is a free return to U_i', we can thus assume that for some $r > 0$

$$|\xi_{\nu,l}(a) - c_i(a)| \sim e^{-r}.$$

By Lemma 3.4, it suffices to prove that the following sum can be made sufficiently small

$$\sum_{j=1}^{p} \frac{|Df_a(\xi_{\nu+j,l}(a)) - Df_a(\xi_{j,i}(a))|}{|Df_a(\xi_{j,i}(a))|} \leq C_1 \sum_{j=1}^{p} \frac{|\xi_{\nu+j,l}(a) - \xi_{j,i}(a)|}{\operatorname{dist}(\xi_{j,i}(a), \operatorname{Jrit}_a)},$$

where $C_1 > 0$ is some constant. Assume that d_i is the local degree of f_0 at $c_i(0)$. So we have, for some constant $C_2 > 0$,

$$|\xi_{\nu+1,l}(a) - \xi_{1,i}(a)| \leq C_2 e^{-d_i r}. \tag{9}$$

Put $J = d_i r/10(\Gamma + 2\alpha)$, where $\Gamma = \sup_{z \in \widehat{\mathbb{C}},\, a \in Q} \log |f_a(z)|$. We can divide the above sum into two parts $[1, J]$ and $[J+1, p]$, and estimate them separately.

For the first sum, we have

$$|\xi_{\nu+j,l}(a) - \xi_{j,i}(a)| \leq e^{\Gamma(j-1)} |\xi_{\nu+1,l}(a) - \xi_{1,i}(a)|.$$

Therefore, combining with the basic assumption (3) and (9) we have

$$\begin{aligned}
\sum_{j=1}^{J} \frac{|\xi_{\nu+j,l}(a) - \xi_{j,i}(a)|}{\operatorname{dist}(\xi_{j,i}(a), \operatorname{Jrit}_a)} &\leq \sum_{j=1}^{J} \frac{e^{\Gamma(j-1)} |\xi_{\nu+1,l}(a) - \xi_{1,i}(a)|}{K e^{-2\alpha j}} \\
&\leq C_3 e^{-9 d_i r/10} \leq C_3 e^{-9 d_i \Delta'/10}.
\end{aligned}$$

Here C_3 depends only on C_2 and K.

For the second sum we can use Definition 2.4 directly to see that

$$\sum_{j=J+1}^{p} \frac{|\xi_{\nu+j,l}(a) - \xi_{j,i}(a)|}{\operatorname{dist}(\xi_{j,i}(a), \mathrm{Jrit}_a)} \leq \sum_{j=J+1}^{p} e^{-\alpha j} \leq C_4 e^{-\alpha d_i r/(\Gamma+2\alpha)} \leq C_4 e^{-\alpha d_i \Delta'/(\Gamma+2\alpha)}$$

for some constant $C_4 > 0$. As both of the above sums can be made sufficiently small by choosing Δ' large enough (i.e., δ' sufficiently small), we reach the conclusion. □

Lemma 3.6 (Expansion and lengths for bound periods). *Let $\gamma \geq \gamma_I$ and $a \in \mathcal{E}_{\nu,l}(\gamma) \cap \mathcal{B}_{\nu,l}$, where $\nu \geq N$ (N as in Lemma 3.1). Assume that $\xi_{\nu,l}(a)$ is a return to U_i' whose length of bound period is p. Then if N is sufficiently large we have*

$$\left|Df_a^{p+1}(\xi_{\nu,l}(a))\right| \geq e^{\gamma p/(2d_i)},$$

where d_i is the degree of f at c_i. Moreover, if $\operatorname{dist}(\xi_{\nu,l}(a), \mathrm{Jrit}_a) \sim_{\sqrt{\varepsilon}} e^{-r}$, then

$$\frac{d_i r}{2\Gamma} \leq p \leq \frac{2d_i r}{\gamma}.$$

In particular, $p \leq 2\alpha d_i \nu/\gamma$.

Proof. Recall that $\hat{d}$ is the maximal multiplicity of critical points of f_0. First we show that $p \leq 2\hat{d}\alpha\nu/\gamma_I$ so that we can use the expansion along the orbit of $v_i(a)$ up to time p. It follows from Lemma 3.5 that, for $j \leq p+1$

$$\begin{aligned} \left|\xi_{\nu+j,l}(a) - \xi_{j,i}(a)\right| &\sim |Df_a^{j-1}(\xi_{\nu+1,l}(a))||\xi_{\nu+1,l}(a) - v_i(a)| \\ &\sim |Df_a^{j-1}(v_i(a))||\xi_{\nu+1,l}(a) - v_i(a)|. \end{aligned} \tag{10}$$

The above relation (10), combined with the definition of bound period (i.e., Definition 2.3), gives us for some constant $C > 0$

$$|Df_a^{j-1}(v_i(a))||\xi_{\nu+1,l}(a) - v_i(a)| \leq Ce^{-\alpha j} \operatorname{dist}(\xi_{j,i}(a), \mathrm{Jrit}_a) \leq C' e^{-\alpha j}. \tag{11}$$

Now we see that $p \leq 2\hat{d}\alpha\nu/\gamma_I$. Otherwise, we put $j = 2\hat{d}\alpha\nu/\gamma_I$ in (11) and use the fact that $a \in \mathcal{E}_{\nu,l}(\gamma)$ (cf. (4)) to obtain that

$$C_0 e^{\gamma(2\hat{d}\alpha\nu/\gamma_I - 1)} e^{-d_i r} \leq C' e^{-\alpha 2\hat{d}\alpha\nu/\gamma_I}.$$

This means that

$$\frac{2\hat{d}\alpha\nu}{\gamma_I} \leq \frac{2d_i r}{\gamma},$$

which is impossible, since $a \in \mathcal{B}_{\nu,l}$, $d_i \leq \hat{d}$ and $\gamma > \gamma_I$. This also proves that

$$p \leq \frac{2d_i r}{\gamma}.$$

Now it follows from Lemma 3.5 and $a \in \mathcal{E}_{\nu,l}(\gamma)$ that

$$E_j := |Df_a^j(\xi_{\nu+1,l}(a))| \sim |Df_a^j(v_i(a))| \geq C_0 e^{\gamma j} \tag{12}$$

for $j \leq p$.

From (10) we also have that, for $j \leq p+1$,

$$|\xi_{\nu+j,l}(a) - \xi_{j,i}(a)| \sim |Df_a^j(\xi_{\nu,l}(a))||\xi_{\nu,l}(a) - c_i(a)|. \tag{13}$$

With $D_j := |Df_a^j(\xi_{\nu,l}(a))|$ and (13) we see that

$$D_{p+1}e^{-r} \geq e^{-\alpha(p+1)} \operatorname{dist}(\xi_{p+1,i}(a), \mathrm{Jrit}_a) \geq Ke^{-3\alpha(p+1)}, \tag{14}$$

where the first inequality follows from the definition of bound periods and the second one holds since $a \in \mathcal{B}_{\nu,l}$. By the definition of Γ we see from (14) that

$$e^{\Gamma p} e^{-d_i r} \geq D_{p+1}e^{-r} \geq Ke^{-3\alpha(p+1)}.$$

So we get that

$$p \geq \frac{d_i r}{2\Gamma}.$$

It remains to estimate D_p. By (14),

$$e^{-r} \geq KD_{p+1}^{-1}e^{-3\alpha(p+1)}$$

and thus

$$e^{-r(d_i-1)} \geq K^{d_i-1}D_{p+1}^{-(d_i-1)}e^{-3\alpha(p+1)(d_i-1)}.$$

As $D_{p+1} \sim e^{-r(d_i-1)}E_p$, we have, by (12),

$$D_{p+1} \geq K^{d_i-1}D_{p+1}^{-(d_i-1)}e^{-3\alpha(p+1)(d_i-1)}e^{\gamma p},$$

which means that

$$D_{p+1}^{d_i} \geq K^{d_i-1}e^{-3\alpha(p+1)(d_i-1)}e^{\gamma p}.$$

So we obtain

$$D_{p+1} \geq K^{(d_i-1)/d_i}e^{-3\alpha(p+1)(d_i-1)/d_i}e^{\gamma p/d_i} \geq e^{\gamma p/(2d_i)}.$$

□

3.3 Expansion during free periods

Roughly speaking, outside expansion means that the derivative of f^n grows exponentially if the orbit stays away from a neighbourhood of the critical points. For slowly recurrent Collet–Eckmann maps, this was proved in [Asp] which will also be crucial in our case. We provide the same statement here (with some modifications).

We begin with stating the following classical lemma by Mañé [Mañ93].

Lemma 3.7. *Let f be a rational map. Provided δ is small enough, there exist a constant $C_M > 0$, dependent on δ, and an exponent $\lambda_M > 1$ such that if $z \in \mathcal{J}(f)$ and $f^k(z) \notin U$ for $k = 1, 2, \dots n-1$ then*

$$|Df^n(z)| \geq C_M \lambda_M^n.$$

If $f^n(z)$ is a return to U, one can say something stronger. To state our next lemma we recall the definition of the second Collet–Eckmann condition.

Definition 3.8 (Second Collet–Eckmann condition)**.** A non-hyperbolic rational map f without parabolic periodic points is said to satisfy the *second Collet–Eckmann condition*, if there exist $C > 0$ and $\gamma > 0$ such that for every $n \geq 1$ and $w \in f^{-n}(c)$, for $c \in \mathrm{Jrit}(f)$ not in the forward orbit of other critical points,

$$|Df^n(w)| \geq Ce^{\gamma n}.$$

In general, the Collet–Eckmann condition does not imply the second Collet–Eckmann condition, and vice versa. However, within the family of slowly recurrent rational maps, these two conditions are equivalent [Byl]. By the assumptions imposed on our starting map f_0, we therefore have that it satisfies the second Collet–Eckmann condition. The following lemma ensures strong expansion for orbits outside of U. We give an outline of the proof which technically is similar to the proof of Lemma 2.3 in [PRLS03]. For a detailed proof we refer to Lemma 3.1 in [Asp].

Lemma 3.9. *Let f be a rational map satisfying the second Collet–Eckmann map with exponent $\gamma > 1$, and let $U = \bigcup_l U_l$ be such that $U_l = D(c_l, \delta)$ is a neighbourhood of $c_l \in \mathrm{Jrit}(f)$. If δ is small enough there exists a constant $C > 0$, not dependent on δ, such that if*

$$z, f(z), \dots, f^{n-1}(z) \notin U, \ \ f^n(z) \in U_k,$$

with c_k not in the forward orbit of other critical points, then

$$|Df^n(z)| \geq Ce^{\gamma n}.$$

Proof. Let W_j denote the connected component of $f^{-j}(U_k)$ containing $z_j = f^{n-j}(z)$, and let (with some abuse of notation) c_j denote the j-th preimage of the critical point $c = c_k$ contained in W_j, i.e. $f^j(c_j) = c$ and $c_j \in W_j$. Following the proof of Lemma 2.3 in [PRLS03], once a small $\delta_0 > 0$ is fixed there is $\ell \geq 1$ (dependent on δ_0) such that, if $\delta < \delta_0$ is small enough,

$$\operatorname{diam}(W_j) \leq e^{-\gamma' j} \operatorname{diam}(U_k) = e^{-\gamma' j}\delta, \tag{15}$$

for all $j > \ell$. Here $\gamma' > 0$ is slightly smaller than the exponent from the second Collet–Eckmann condition. We now consider the quotient

$$\left|\frac{Df^n(c_n)}{Df^n(z)}\right| = \left|\frac{Df^{n-\ell}(c_n)}{Df^{n-\ell}(z)}\right| \left|\frac{Df^{\ell}(c_\ell)}{Df^{\ell}(z_\ell)}\right|.$$

By making δ small, the second factor in the above is bounded by some small constant $C' \geq 1$. For the first factor, using (7), (15), and the assumption that $\operatorname{dist}(z_j, \operatorname{Jrit}(f)) \geq \delta$ for $j \geq 1$, we find that

$$\begin{aligned}
\left|\frac{Df^{n-l}(c_n)}{Df^{n-l}(z)}\right| &\leq \exp\left(C'' \sum_{j=0}^{n-l-1} \frac{|f^j(c_n) - f^j(z)|}{\operatorname{dist}(f^j(z), \operatorname{Jrit}(f))}\right) \\
&\leq \exp\left(C'' \sum_{j=l+1}^{n} \frac{\operatorname{diam}(W_j)}{\delta}\right) \\
&\leq \exp\left(\frac{C''}{e^{\gamma'} - 1}\right).
\end{aligned}$$

This proves the result, since

$$|Df^n(z)| \geq \frac{1}{C'} \exp\left(-\frac{C''}{e^{\gamma'} - 1}\right) |Df^n(c_n)| \gtrsim e^{\gamma n}.$$

□

We are now in position to prove our desired outside expansion lemma, satisfied for any small perturbation f_a of f_0, and also valid in an ε_0-neighbourhood $\mathcal{N}_{\varepsilon_0}$ of $\mathcal{J}(f_0)$. We make U so small that $U \subset \mathcal{N}_{\varepsilon_0}$.

Lemma 3.10. *Suppose $f = f_0$ is a rational second Collet–Eckmann with non-empty Fatou set. If ε_0, δ and ε are small enough, there exist constants $C_\delta > 0$ (dependent on δ) and $\gamma > 0$ such that, for all $a \in \mathcal{Q}$, if*

$$z, f_a(z), \ldots f_a^{n-1}(z) \in \mathcal{N}_{\varepsilon_0} \setminus U$$

then

$$|Df_a^n(z)| \geq C_\delta e^{\gamma n} \tag{16}$$

Moreover, there exists a constant $C > 0$ (not dependent on δ) such that if we also have $f_a^n(z) \in U$ then

$$|Df_a^n(z)| \geq Ce^{\gamma n}. \tag{17}$$

Remark 3.11. For later convenience, we put $\gamma_H = \gamma$, where γ is as in (17).

Proof. Fix $\gamma > 0$ such that $3\gamma \in (0, \min\{\gamma_M, \gamma_H\})$, where γ_M and γ_H comes from Lemma 3.7 and Lemma 3.9, respectively. We will establish the result with this γ.

From Lemma 3.7, provided δ is small enough, we can find $\hat{n}$ large enough such that if $f^k(z) \in \mathcal{J}(f) \setminus U$ for $k = 0, 1, \ldots, \hat{n} - 1$ then

$$|Df^{\hat{n}}(z)| \geq C_M e^{\gamma_M \hat{n}} \geq e^{3\gamma\hat{n}}.$$

If $\varepsilon_0 > 0$ is small enough, we therefore conclude by continuity that if $f^k(z) \in \mathcal{N}_{\varepsilon_0} \setminus U$ for $k = 0, 1, \ldots, \hat{n} - 1$ then

$$|Df^{\hat{n}}(z)| \geq e^{2\gamma\hat{n}}.$$

Using continuity again, now in the parameter variable, we conclude that if ε is small enough, if for $a \in \mathcal{Q}$ we have that $f_a^k(z) \in \mathcal{N}_{\varepsilon_0} \setminus U$ for $k = 0, 1, \ldots, \hat{n} - 1$ then

$$|Df_a^{\hat{n}}(z)| \geq e^{\gamma\hat{n}}.$$

Suppose now that $f_a^k(z) \in \mathcal{N}_{\varepsilon_0} \setminus U$ for $k = 0, 1, \ldots n - 1$ and write $n = q\hat{n} + r$, with q and r positive integers and $0 \leq r \leq \hat{n} - 1$. We find that

$$\begin{aligned} |Df_a^n(z)| &= |Df_a^r(f_a^{q\hat{n}}(z))||Df_a^{\hat{n}}(f_a^{(q-1)\hat{n}}(z))| \cdots |Df_a^{\hat{n}}(z)| \\ &\geq |Df_a^r(f^{q\hat{n}}(z))|e^{\gamma q\hat{n}}, \end{aligned}$$

and (16) now follows with constant $C_\delta = \inf_{a\in\mathcal{Q}} \inf_{z\in\mathcal{N}_{\varepsilon_0}\setminus U} |Df_a^{\hat{n}}(z)|$.

If we have a return to U then we are in the situation $f_a^k(f_a^{q\hat{n}}) \notin U$ for $k = 0, \ldots, r - 1$ and $f_a^r(f_a^{q\hat{n}}(z)) \in U$. In the case of the unperturbed map we get from Lemma 3.9 that

$$|Df^r(f^{q\hat{n}}(z))| \geq Ce^{\gamma_H r},$$

with C not depending on δ. Once again, due to continuity in a, a similar estimate holds for the perturbed map f_a if ε is small enough (recall that $r \leq \hat{n} - 1$). This proves (17).

□

3.4 At the next free return

Combining the results above we draw the following conclusions at the next free return.

Using Lemma 3.6 and Lemma 3.10, we can obtain longer time of exponential growth for derivatives. More precisely, we have the following.

Lemma 3.12. *Let N be sufficiently large. Let $\nu \geq N$ be a return such that $\xi_{\nu,l}(a) \in U'$ for $a \in \mathcal{E}_{\nu,l}(\gamma) \cap \mathcal{B}_{\nu,l}$, where $\gamma \geq \gamma_I$. Let also ν' be the next free return time. Then we have*

$$|Df_a^n(v_l(a))| \geq e^{\gamma_1 n},$$

for all $0 \leq n \leq \nu' - 1$, with $\gamma_1 \geq (9/10)\min\{\gamma, \gamma_H\}$.

Proof. Recall that $\hat{d}$ is the maximal multiplicity of critical points of f_0. Let p be the length of the bound period for the return $\xi_{\nu,l}(a)$, and suppose $n = \nu + j$ with $1 \leq j \leq p$. By the chain rule, the fact that $a \in \mathcal{E}_{\nu,l}(\gamma)$ and Lemma 3.6, using the notation from the proof of Lemma 3.6,

$$\begin{aligned}
|Df_a^{\nu+j}(v_l(a))| &= |Df_a^{\nu-1}(v_l(a))|D_{j+1} \\
&\gtrsim e^{\gamma(\nu-1)}|Df_a(\xi_\nu(a))|E_j \\
&\gtrsim e^{\gamma(\nu-1)}e^{-2\alpha\hat{d}\nu}e^{\gamma j} \\
&\geq e^{\gamma_1(\nu+j)},
\end{aligned}$$

by the choice of α and provided N is large enough. If $n = \nu + p + j$, with $1 \leq j \leq L-1$, it also follows from Lemma 3.10 (here with respect to U') and the above that

$$\begin{aligned}
|Df_a^{\nu+p+j}(v_l(a))| &= |Df_a^{\nu-1}(v_l(a))|D_{p+1}|Df_a^j(\xi_{\nu+p+1}(a))| \\
&\gtrsim e^{\gamma(\nu-1)}e^{-2\alpha\hat{d}\nu}e^{\gamma p}C_{\delta'}e^{\gamma_H j} \\
&\geq e^{\gamma_1(\nu+p+j)},
\end{aligned}$$

again provided that N is large enough. □

By the weak parameter dependence (Lemma 3.2) and using similar methods as above we can see that parameters belonging to the same partition element repel each other in the following sense.

Lemma 3.13. *Let $a, b \in \mathcal{E}_{\nu,l}(\gamma) \cap \mathcal{B}_{\nu,l}$ be in the same partition element for $\gamma \geq \gamma_I$. Let also ν be a return time for this partition element. If ν' is the next free return, then*

$$|\xi_{\nu',l}(a) - \xi_{\nu',l}(b)| \geq 2|\xi_{\nu,l}(a) - \xi_{\nu,l}(b)|.$$

Proof. By Lemma 3.12 above we see that for $a \in \mathcal{E}_{\nu,l}(\gamma) \cap \mathcal{B}_{\nu,l}$ we have exponential growth of phase derivative up to the next free return ν', i.e.

$$|Df_a^n(v_l(a))| \geq C_1 e^{\gamma_1 n},$$

for $0 \leq n \leq \nu' - 1$. Since $\gamma_1 \geq (9/10)\min\{\gamma, \gamma_H\}$, one can use the weak parameter dependence property to get

$$|\xi_{\nu',l}(a) - \xi_{\nu',l}(b)| \geq Q^{-(\nu'-\nu)} |Df_a^{\nu'-\nu}(\xi_{\nu,l}(a))| |\xi_{\nu,l}(a) - \xi_{\nu,l}(b)|.$$

Since $\nu' - \nu = p + L$, with p and L being the associated bound period and free period, respectively, we get from Lemma 3.6 and Lemma 3.10 that

$$\begin{aligned} |Df_a^{\nu'-\nu}(\xi_{\nu,l}(a))| &= |Df_a^{p+1}(\xi_{\nu,l}(a))| |Df_a^{L-1}(\xi_{\nu+p+1}(a))| \\ &\geq C e^{\gamma p/(2\hat{d}) + \gamma_H(L-1)} \\ &\geq e^{\gamma_2(\nu'-\nu)}, \end{aligned}$$

for some $\gamma_2 > 0$, provided δ' is small enough (hence p is large). Notice that the constant C coming from the outside expansion lemma is not dependent on δ' since we have an actual return. As Q is chosen very small we have $\log Q < \alpha \ll \gamma_2$, and we get the conclusion. □

3.5 Main distortion lemma (MDL)

The following lemma is the main result of this section, and it tells us that we have strong distortion estimates for parameters belonging to the same partition element. An essential ingredient in the proof is the weak parameter dependence proved earlier (Lemma 3.2).

Lemma 3.14 (Main distortion lemma). *Let $\varepsilon' > 0$. Then there exists N large enough such that the following holds: If $A \subset \mathcal{E}_{\nu,l}(\gamma) \cap \mathcal{B}_{\nu,l}$ is a partition element for $\gamma \geq \gamma_I$ and $\nu \geq N$ is a return time or does not belong to the bound period, and ν' is the next free return, then we have*

$$\left| \frac{Df_a^n(v_l(a))}{Df_b^n(v_l(b))} - 1 \right| \leq \varepsilon'$$

for $a, b \in A$ and for $\nu \leq n \leq \nu'$, provided A is still a partition element at the time n.

Proof. By Lemma 3.2 and Lemma 3.4, it reduces to check whether the following sum can be made arbitrarily small:

$$\Upsilon := \sum_{j=1}^{n-1} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)}.$$

Let (ν_k) be the free returns before time n, where $k \leq s$. In other words, $\nu = \nu_s$ and $\nu' = \nu_{s+1}$. Let also p_k be the length of the associated bound period of the return ν_k. The estimate of Υ is divided into several parts:

$$\begin{aligned}\Upsilon = \sum_{k=1}^{s} \sum_{j=\nu_{k-1}}^{\nu_{k-1}+p_{k-1}} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} &+ \sum_{k=1}^{s} \sum_{j=\nu_{k-1}+p_{k-1}+1}^{\nu_k - 1} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} \\ &+ \sum_{j=\nu_s}^{n-1} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} \\ &=: \Upsilon_B + \Upsilon_F + \Upsilon_T.\end{aligned}$$

Here Υ_B denotes the contribution from bound periods, while Υ_F the contribution from free periods, and Υ_T the contribution from the last return ν_s up until time n.

Contribution from bound periods: the estimate of Υ_B. Let ν_k be one of the free returns, with $k \leq s-1$. We would like to estimate the following

$$\Upsilon_B^k := \sum_{j=\nu_k}^{\nu_k+p_k} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} = \sum_{j=0}^{p_k} \frac{|\xi_{\nu_k+j,l}(a) - \xi_{\nu_k+j,l}(b)|}{\operatorname{dist}(\xi_{\nu_k+j,l}(b), \mathrm{Jrit}_b)}.$$

Assume also that $\xi_{\nu_k,l}(a) \in U_i'$ is a return and $\operatorname{dist}(\xi_{\nu_k,l}(b), \mathrm{Jrit}_b) \sim e^{-r}$. It then follows from the distortion in Lemma 3.5 and the definition of bound periods that

$$\begin{aligned}\Upsilon_B^k &\leq \frac{|\xi_{\nu_k,l}(a) - \xi_{\nu_k,l}(b)|}{e^{-r}} \left(1 + \sum_{j=1}^{p_k} \frac{|Df_a^j(\xi_{\nu_k,l}(a))| e^{-r}}{\operatorname{dist}(\xi_{\nu_k+j,l}(b), \mathrm{Jrit}_b)}\right) \\ &\lesssim \frac{|\xi_{\nu_k,l}(a) - \xi_{\nu_k,l}(b)|}{e^{-r}} \left(1 + \sum_{j=1}^{p_k} \frac{|Df_a^j(\xi_{\nu_k,l}(a))||\xi_{\nu_k,l}(a) - c_i(a)|}{\operatorname{dist}(\xi_{\nu_k+j,l}(b), \mathrm{Jrit}_b)}\right) \\ &\lesssim \frac{|\xi_{\nu_k,l}(a) - \xi_{\nu_k,l}(b)|}{e^{-r}} \left(1 + \sum_{j=1}^{p_k} \frac{|\xi_{\nu_k+j,l}(a) - \xi_{j,i}(a)|}{\operatorname{dist}(\xi_{\nu_k+j,l}(b), \mathrm{Jrit}_b)}\right) \\ &\lesssim \frac{|\xi_{\nu_k,l}(a) - \xi_{\nu_k,l}(b)|}{e^{-r}} \left(1 + \sum_{j=1}^{p_k} e^{-\alpha j}\right) \\ &\lesssim \frac{|\xi_{\nu_k,l}(a) - \xi_{\nu_k,l}(b)|}{e^{-r}}.\end{aligned}$$

Given $r \geq \Delta$, let $K(r)$ be the set of indices k such that $\operatorname{dist}(\xi_{\nu_k,l}(A), \mathrm{Jrit}_A) \sim e^{-r}$, and let $\hat{k}(r)$ be the largest index contained in $K(r)$. Then it follows from Lemma 3.13 that

$$\Upsilon_B = \sum_{k=1}^{s-1} \Upsilon_B^k \leq \sum_{r \geq \Delta} \sum_{k \in K(r)} \Upsilon_B^k \lesssim \sum_{r \geq \Delta} \Upsilon_B^{\hat{k}(r)} \lesssim \sum_{r \geq \Delta} \frac{1}{r^2} \lesssim \frac{1}{\Delta},$$

where we used that for the last return associated with this r

$$|\xi_{\nu_{\hat{k}(r)},l}(a) - \xi_{\nu_{\hat{k}(r)},l}(b)| \lesssim \frac{e^{-r}}{r^2}.$$

Contribution from free periods: the estimate of Υ_F. Similar as above, we define for $k \leq s-1$,

$$\Upsilon_F^k := \sum_{j=\nu_k+p_k+1}^{\nu_{k+1}-1} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\mathrm{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)}.$$

By the weak parameter dependence and Lemma 3.10 we see that

$$\begin{aligned}
|\xi_{\nu_{k+1},l}(a) - \xi_{\nu_{k+1},l}(b)| &\geq \frac{1}{Q^{\nu_{k+1}-j}} |Df_a^{\nu_{k+1}-j}(\xi_{j,l}(a))| |\xi_{j,l}(a) - \xi_{j,l}(b)| \\
&\gtrsim \left(\frac{e^{\gamma_H}}{Q}\right)^{\nu_{k+1}-j} |\xi_{j,l}(a) - \xi_{j,l}(b)|
\end{aligned}$$

for $\nu_k + p_k + 1 \leq j \leq \nu_{k+1} - 1$. Since ν_{k+1} is the index of a return, we assume that $\mathrm{dist}(\xi_{\nu_{k+1},l}(b), \mathrm{Jrit}_b) \sim e^{-r}$. So we see that, for $\nu_k + p_k + 1 \leq j \leq \nu_{k+1} - 1$,

$$|\xi_{j,l}(a) - \xi_{j,l}(b)| \lesssim \left(\frac{Q}{e^{\gamma_H}}\right)^{\nu_{k+1}-j} |\xi_{\nu_{k+1},l}(a) - \xi_{\nu_{k+1},l}(b)|.$$

Since $\mathrm{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b) \geq \mathrm{dist}(\xi_{\nu_{k+1},l}(b), \mathrm{Jrit}_b) \sim e^{-r}$ this gives

$$\Upsilon_F^k \lesssim \frac{|\xi_{\nu_{k+1},l}(a) - \xi_{\nu_{k+1},l}(b)|}{e^{-r}} \sum_{j=\nu_k+p_k+1}^{\nu_{k+1}-1} \left(\frac{Q}{e^{\gamma_H}}\right)^{\nu_{k+1}-j} \lesssim \frac{|\xi_{\nu_{k+1},l}(a) - \xi_{\nu_{k+1},l}(b)|}{e^{-r}},$$

where we have used the fact that $\log Q$ is much smaller than γ_H. Using the same argument as in the estimate of the contribution from the bound periods, we find that

$$\Upsilon_F = \sum_{k=1}^{s-1} \Upsilon_F^k \leq \sum_{r\geq\Delta} \sum_{k\in K(r)} \Upsilon_F^k \lesssim \sum_{r\geq\Delta} \Upsilon_F^{\hat{k}(r)} \lesssim \sum_{r\geq\Delta} \frac{1}{r^2} \lesssim \frac{1}{\Delta}.$$

Estimate of tail Υ_T. It remains to estimate the sum between the last free return ν_s up to time n. As $\nu_s \leq n \leq \nu_{s+1}$, we need to consider the different situations that can occur in this time interval. If $n \leq \nu_s + p_s$ (i.e. n belongs to the bound period immediately after ν_s), then the tail Υ_T can be estimated in the same as Υ_B by reducing to the return at time ν_s. If $n = \nu_{s+1}$, then the tail consists of a bound period following ν_s and a free period before the return $n = \nu_{s+1}$ happens. In this case, the estimate Υ_T can be estimated again as above.

The remaining case is when $\nu_s + p_s + 1 \le n < \nu_{s+1}$. For this purpose, we consider pseudo-returns, and we let $\nu_s + p_s + 1 \le q_1 \le \cdots \le q_t \le n$ be the indices of these returns. By definition, $\xi_{q_k,l}(a) \cap U' \neq \emptyset$ and $\xi_{q_k,l}(a) \cap U = \emptyset$. For pseudo-returns, bound periods and free periods are defined in a similar way. As in the previous estimates, the contribution to the distortion between any two pseudo-returns of index q_k and q_{k+1} is a constant times

$$\frac{|\xi_{q_k,l}(a) - \xi_{q_k,l}(b)|}{e^{-r_k}} + \frac{|\xi_{q_{k+1},l}(a) - \xi_{q_{k+1},l}(b)|}{e^{-r_{k+1}}},$$

where r_k and r_{k+1} such that $\operatorname{dist}(\xi_{q_k,l}(b), \mathrm{Jrit}_a) \sim e^{-r_k}$ and $\operatorname{dist}(\xi_{q_{k+1},l}(b), \mathrm{Jrit}_a) \sim e^{-r_{k+1}}$. The difference here is that, at a pseudo-return, the only thing we know about the length of our interval is that $|\xi_{q_k,l}(a) - \xi_{q_k,l}(b)| \le S$, where $S = \varepsilon_1 \delta$ is the large scale. With similar methods and notation used for estimating the bound and free contributions, we have

$$\begin{aligned}
\Upsilon_T &= \left(\sum_{j=\nu_s}^{q_1} + \sum_{k=1}^{t-1} \sum_{j=q_k}^{q_{k+1}-1} + \sum_{j=q_t}^{n-1} \right) \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} \\
&\lesssim \frac{1}{r_s^2} + \sum_{k=1}^{t} \frac{|\xi_{q_k,l}(a) - \xi_{q_k,l}(b)|}{\operatorname{dist}(\xi_{q_k,l}(b), \mathrm{Jrit}_b)} + \sum_{j=q_t}^{n-1} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} \\
&\lesssim \frac{1}{\Delta^2} + \sum_{r=\Delta'}^{\Delta} \frac{|\xi_{q_{\hat{k}(r)},l}(a) - \xi_{q_{\hat{k}(r)},l}(b)|}{\operatorname{dist}(\xi_{q_{\hat{k}(r)},l}(b), \mathrm{Jrit}_b)} + \frac{S}{\delta'} \\
&\lesssim \frac{1}{\Delta^2} + \varepsilon_1 \sum_{r=\Delta'}^{\Delta} e^{r-\Delta} + \varepsilon_1 \\
&\lesssim \frac{1}{\Delta^2} + \varepsilon_1,
\end{aligned}$$

where we in the sum from q_t to $n-1$ used Lemma 3.10 (inequality (16), now with respect to U') and that $\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b) > \delta' > \delta$ during this time.

Combining all these estimate above we arrive at

$$\Upsilon = \sum_{j=1}^{n-1} \frac{|\xi_{j,l}(a) - \xi_{j,l}(b)|}{\operatorname{dist}(\xi_{j,l}(b), \mathrm{Jrit}_b)} \lesssim \frac{1}{\Delta} + \varepsilon_1,$$

and if δ and ε_1 are small enough, we reach the desired conclusion of strong distortion. □

3.6 Consequences of MDL

With Lemma 3.14 in hand, we can conclude that the previously obtained weak parameter dependence of Lemma 3.2 can be promoted to a stronger form:

$$|\xi_{n+j,l}(a) - \xi_{n+j,l}(b)| \sim |Df_a^n(\xi_{j,l}(a))| |\xi_{j,l}(a) - \xi_{j,l}(b)|,$$

provided that a and b belong to the same partition element.

Another direct consequence of Lemma 3.14 is that for a sufficiently small parameter square $\mathcal{Q}$, we have the following dichotomy for each critical point c_l: there exists N_l such that either $\xi_{N_l,l}(\mathcal{Q})$ grows to some definite size or $\xi_{N_l,l}(\mathcal{Q})$ is the first essential return.

Lemma 3.15. *Let f_0 be a slowly recurrent Collet–Eckmann rational map. Let N_L be as in Lemma 3.1, and let $\varepsilon' > 0$ be sufficiently small. Then there is a neighbourhood U of Jrit_0 and $S > 0$ (depending on U) such that, for each sufficiently small $\varepsilon > 0$ and for each critical point $c_l(0) \in \mathrm{Jrit}_0$, there is $N_l \geq N_L$ such that for all $a \in \mathcal{Q}$ we have the following:*

(i) For some $\gamma_l \geq \gamma_0(1-\varepsilon')$, one has

$$\left|Df_a^k(v_l(a))\right| \geq Ce^{\gamma_l k} \quad for \quad k \leq N_l - 1;$$

(ii) for $k \leq N_l - 1$, one has

$$\operatorname{diam} \xi_{k,l}(\mathcal{Q}) \leq \begin{cases} \dfrac{\operatorname{dist}(\xi_{k,l}(\mathcal{Q}), \mathrm{Jrit}_{\mathcal{Q}})}{(\log \operatorname{dist}(\xi_{k,l}(\mathcal{Q}), \mathrm{Jrit}_{\mathcal{Q}}))^2}, & if\ \xi_{k,l}(\mathcal{Q}) \cap U \neq \emptyset, \\ S, & if\ \xi_{k,l}(\mathcal{Q}) \cap U = \emptyset; \end{cases}$$

(iii) for $k = N_l$, one has

$$\operatorname{diam} \xi_{k,l}(\mathcal{Q}) \geq \begin{cases} \dfrac{\operatorname{dist}(\xi_{k,l}(\mathcal{Q}), \mathrm{Jrit}_{\mathcal{Q}})}{(\log \operatorname{dist}(\xi_{k,l}(\mathcal{Q}), \mathrm{Jrit}_{\mathcal{Q}}))^2}, & if\ \xi_{k,l}(\mathcal{Q}) \cap U \neq \emptyset, \\ S, & if\ \xi_{k,l}(\mathcal{Q}) \cap U = \emptyset; \end{cases}$$

(iv) for all $a, b \in \mathcal{Q}$ one has

$$\left|\frac{Df_a^{n-N}(\xi_{N,l}(a))}{Df_a^{n-N}(\xi_{N,l}(b))} - 1\right| \leq \varepsilon' \quad for \quad n \leq N_l.$$

Proof. By the choice of N_L, we can choose $\varepsilon > 0$ sufficiently small such $\mathcal{Q} \subset \mathcal{E}_{N_L,l}(\gamma) \cap \mathcal{B}_{N_L,l}$ for all l and for some γ arbitrarily close to γ_0.

Now we fix any $c_l(0) \in \mathrm{Jrit}_0$ and assume that (ii) is always satisfied up to some time, denoted by $N_l - 1$. Then this implies that all other parameters in $\mathcal{Q}$ will inherit expansion from our starting map:

$$\left|Df_a^k(v_l(a))\right| \geq C^{-k} e^{\gamma_0 k} \geq C_0 e^{\gamma_1 k}.$$

Since we assumed that (ii) is satisfied, we see that all parameters will be slowly recurrent up to time $N_l - 1$. Then by the definition of partition element we can use Lemma 3.14 repeatedly starting from the time N_L up to $N_l - 1$ to get the distortion claimed in (iv). $\square$

To prove our main result, we would like to see if a small parameter square will grow to the large scale S under the action of $\xi_{n,l}$. For this purpose, let N_l be as in the above lemma and suppose without loss of generality that $N_1 \leq N_2 \leq \cdots$. Then by Lemma 3.15, we have the situation that $\xi_{N_1,1}(\mathcal{Q})$ either reaches the large scale S or is the first essential return. If the first case happens, we stop and consider the next critical point. If the second case occurs, we partition the parameter square $\mathcal{Q}$ into small dyadic squares inductively as follows. Since the partition rule should be valid for all returns, let us consider a given partition element $A \subset \mathcal{Q}$ (instead of $\mathcal{Q}$), which is assumed to be a perfect square. So suppose that $\xi_{n,l}(A)$ is an essential return, and A is not a partition element according to Definition 2.2, then partition A into four perfect squares $A_{j_1} \subset A$, $j_1 = 1, 2, 3, 4$ of equal size. If each of these subsquares satisfies Definition 2.2, then stop. If not, for each subsquare $A_{j_1} \subset A$ that is not a partition element, partition A_{j_1} into four new subsquares $A_{j_1 j_2}$, $j_2 = 1, 2, 3, 4$ of equal size. If they are partition elements, then stop. Otherwise go on until all subsquares are partition elements. In this way we obtain a partition of A into subsquares of possibly different sizes of the form 2^{-k} times the side length of A. We get a collection of dyadic subsquares $A_n^i \subset A$ such that $A = \cup_i A_n^i$ and

$$\frac{1}{3}\frac{\operatorname{dist}(\xi_{n,l}(A_n^i), \mathrm{Jrit}_{A_n^i})}{(\log \operatorname{dist}(\xi_{n,l}(A_n^i), \mathrm{Jrit}_{A_n^i}))^2} \leq \operatorname{diam} \xi_{n,l}(A_n^i) \leq \frac{\operatorname{dist}(\xi_{n,l}(A_n^i), \mathrm{Jrit}_{A_n^i})}{(\log \operatorname{dist}(\xi_{n,l}(A_n^i), \mathrm{Jrit}_{A_n^i}))^2}.$$

By construction, each A_n^i is a partition element, as defined in 2.2 (the constant $1/3$ is chosen because of small distortion; in an completely affine situation, $1/2$ would suffice). At this point we will need to delete parameters which violate the basic assumption. But it turns out that these deleted parameters constitute only a small portion. After (possibly) deleting parameters not satisfying the basic assumption, we continue to iterate each partition element individually.

4 Large deviations and escape of partition elements

We now consider a partition element in $\mathcal{Q}$ and follow it in a time window of the type $[m, (1+\iota)m]$, for some (small) $\iota > 0$. This section is very similar to [BC91] where the original ideas were developed. See also [Asp13, Asp]. Let $A_n(a) \subset \mathcal{Q}$ be a partition element at time n, containing the parameter a. Since the proofs are very similar to these earlier papers, we are not going through all the proofs here again but instead give references.

Definition 4.1. We say that $\xi_{n,l}(A_n(a))$ has escaped or is in escape position, if n does not belong to a bound period and $\operatorname{diam}(\xi_{n,l}(A_n(a))) \geq S$ before partitioning. We also speak of escape situation for $A_n(a)$ and say that $A_n(a)$ has escaped if $\xi_{n,l}(A_n(a))$ has escaped.

The first observation is that the measure of parameters deleted between two consecutive essential returns is exponentially small in terms of the return time of the former return. See Lemma 8.1 in [Asp].

Lemma 4.2. *Let $\xi_{\nu,l}(A)$ be an essential return, $A \subset \mathcal{E}_{\nu,l}(\gamma_I) \cap \mathcal{B}_{\nu,l}$ and let $\xi_{\nu'}(A)$ be the next essential return. Then if $\hat{A}$ is the set of parameters in A that satisfy the basic assumption at time ν', we have*

$$m(\hat{A}) \geq (1 - e^{-\alpha\nu})m(A).$$

Proof. We first show that $\xi_{\nu,l}(A)$ grows rapidly during the bound period p. By Lemmas 3.5 and 3.14 and the definition of the bound period, we get, for any $a \in A$,

$$\begin{aligned}
\operatorname{diam}(\xi_{\nu+p+1,l}(A)) &\sim \frac{e^{-rd_i}}{r^2}|Df_a^p(\xi_{\nu+1,l}(a))| \\
&\sim \frac{|\xi_{\nu+p+1,l}(a) - \xi_{p+1,i}(a)|}{r^2} \\
&\geq Ce^{-\alpha(p+1)-2\log r}\operatorname{dist}(\xi_{p+1,i}(a), \mathrm{Jrit}_a) \\
&\geq CKe^{-2\alpha(p+1)-\alpha(p+1)-2\log r} \geq e^{-(7/2)\alpha p - 2\log r},
\end{aligned}$$

if p is large. So, by Lemma 3.10 and Lemma 3.6,

$$\begin{aligned}
\operatorname{diam}(\xi_{\nu',l}(A)) &\geq \operatorname{diam}(\xi_{\nu+p+1,l}(A))C'e^{\gamma_H(\nu'(\nu+p+1))} \\
&\geq C'e^{-(7/2)\alpha p - 2\log r} \\
&\geq e^{-7\alpha dr/\gamma - 2\log r} \\
&\geq e^{-\frac{8\alpha d}{\gamma}r}.
\end{aligned} \tag{18}$$

So by the main distortion Lemma 3.14 together with Lemma 3.1, we see that the measure of parameters deleted at time ν' is

$$\frac{m(A) - m(\hat{A})}{m(A)} \leq 2\frac{(e^{-2\alpha\nu'})^2}{\operatorname{diam}(\xi_{\nu'}(A))^2} \leq 2e^{-\alpha(4-\frac{16\alpha d}{\gamma})\nu} \leq e^{-\alpha\nu},$$

since $\alpha\hat{d}/\gamma \leq 1/100$ ($\hat{d}$ is the maximal multiplicity of the critical points). □

We next state the following lemma, which is a correspondence to Lemma 8.3 in [Asp]

Lemma 4.3. *Let $\xi_{\nu,l}(A)$ be an essential return, $A \subset \mathcal{E}_{\nu,l}(\gamma_I) \cap \mathcal{B}_{\nu,l}$, with* $\operatorname{dist}(\xi_{\nu,l}(A), \operatorname{Jrit}_A) \sim e^{-r}$. *If q is the time after this return spent on inessential returns up until $\xi_{n,l}(A_n(a))$ either escapes, or makes an essential return, $n > \nu$, whichever comes first, then*

$$q \le \frac{1}{2} hr,$$

where $h = 8\hat{d}^2/r$.

We now assume that we have a partition element $A \subset \mathcal{Q}$ at time m. We follow a parameter $a \in A$ in the time window $[m, (1+\iota)m]$, for some (small) $\iota > 0$. Suppose that $\nu_0, \nu_1, \nu_2, \dots, \nu_s$ are the essential returns in this time window for a. In addition we assume that $a \in \mathcal{E}_{k,l}(\gamma_I)$ for $k \le (1+\iota)m$ (this will be satisfied a posteriori). At each return ν_j the basic approach rate assumption may force us to delete a fraction of parameters.

Let now $A_j = A_{\nu_j}(a)$ and suppose that $\operatorname{dist}(\xi_{\nu_j,l}(A_j), \operatorname{Jrit}_{A_j}) \sim e^{-r_j}$. Then by (18), we have,

$$\frac{m(A_{j+1})}{m(A_j)} \le C \frac{(e^{-r_{j+1}})^2}{(e^{-8\alpha\hat{d}r_j/\gamma_I})^2} = C \frac{e^{-2r_{j+1}}}{e^{-16\alpha\hat{d}r_j/\gamma_I}}.$$

So if we look at the sequence of parameter squares, the measure of A_s compared to A_1 is

$$\frac{m(A_s)}{m(A_0)} = \prod_{j=0}^{s-1} \frac{m(A_{j+1})}{m(A_j)} \le C^s \prod_{j=1}^{s-1} \frac{e^{-2r_{j+1}}}{e^{-16\alpha\hat{d}r_j/\gamma_I}}.$$

Now write $R = r_1 + \dots + r_s$. Then, putting $r_0 = r$, we have

$$\frac{m(A_s)}{m(A_0)} \le C^s e^{r_0 16\alpha\hat{d}/\gamma_I - \sum_{j=1}^{s-1} r_j(2-16\alpha\hat{d}/\gamma_I) - r_s} = C^s e^{r_0 16\alpha\hat{d}/\gamma_I - (3/2)R}.$$

So we suppose that $\xi_{\nu_0,l}(A_0)$ is an essential return and will estimate the measure of parameters that do not escape after a long time. If the parameter $a \in A = A_0$ has s essential returns before it has escaped then, with $\nu_0 = \nu$,

$$E_l(a,\nu) \le p + 1 + \sum_{j=0}^{s} (h/2) r_j \le hr + hR,$$

where we have included the first bound period p, which is bounded by $2\alpha\hat{d}/\gamma_I < \hat{d}^2/\gamma_I$ in hr.

The number of combinations of r_j's such that $R = r_1 + \dots + r_s$ is at most

$$\binom{R+s-1}{s-1}.$$

Since $\xi_{n,l}(A)$ is almost a small perfect square by the strong distortion lemma, there are maximum about $2\pi e^{-r}/r^2$ number of such disjoint squares at distance e^{-r} from the critical points, if $\mathrm{diam}(\xi_{n,l}(A)) \sim e^{-r}/r^2$. Note also that $s\Delta \leq R$. Let $s\Delta = qR$, for some $0 < q \leq 1$. Taking this into account we get by Stirling's formula that the number of combinations is

$$\begin{aligned}\binom{R+s-1}{s-1} &\leq C\frac{(R+s-1)^{R+s-1}e^{-R-s+1}}{R^R(s-1)^{s-1}e^{-R}e^{-s+1}}\sqrt{\frac{R(s-1)}{R+s-1}} \\ &\leq C\left(\frac{(1+(q/\Delta))^{1+q/\Delta}}{(q/\Delta)^{q/\Delta}}\right)^R\sqrt{R} \\ &\leq e^{R/32}(1+\eta(\Delta))^R,\end{aligned}$$

where $\eta(\Delta) \to 0$ as $\Delta \to \infty$, for R large enough (i.e., Δ large enough). Continuing following the the earlier papers, we let $A_{s,R} \subset A$ be the set of all parameters which have exactly s essential return in the time window $[m,(1+\iota)m]$, for some $\iota > 0$ and fixed R. If we let R and s vary, we get a partition of A into countably many subsquares. For fixed s and R, let $\hat{A}_s$ be the largest of all such subsquares. Then we get,

$$m(A_{s,R}) \leq m(\hat{A}_s)e^{R/32}(1+\eta(\Delta))^R.$$

Now, we go through the same type of calculations as in [Asp] et al.

$$\begin{aligned}&m(\{a \in A : E_l(a,\nu) = t\}) \\ &\qquad\leq \sum_{R\geq t/h-r_0, s\leq R/\Delta} m(A_{s,R}) \\ &\qquad\leq \sum_{R\geq t/h-r_0, s\leq R/\Delta} m(\hat{A}_s)e^{R/32}(1+\eta(\Delta))^R \\ &\qquad\leq m(A)\sum_{R=t/h-r_0}^{\infty}\sum_{s=1}^{R/\Delta} e^{R/32}(1+\eta(\Delta))^R C^s e^{r_0(16\alpha\hat{d}/\gamma)-(3/2)R} \\ &\qquad\leq C'm(A)\sum_{R=t/h-r_0}^{\infty} C^{R/\Delta}e^{R/32+R\log(1+\eta(\Delta))-(3/2)R+(16\hat{d}\alpha/\gamma)r_0} \\ &\qquad\leq C'm(A)e^{-(\frac{t}{h}-r_0)\frac{46}{32}+(16\hat{d}\alpha/\gamma)r_0} \\ &\qquad\leq C'm(A)e^{-\frac{t}{h}\frac{46}{32}+(\frac{46}{32}+\frac{16\hat{d}\alpha}{\gamma})r_0}.\end{aligned}$$

for some constant $C' > 0$.

By the condition on α, if $\gamma \geq \gamma_I$, we get an estimate of the measure of parameters for large escape times. Let us suppose that $t > 2hr_0$. Then

$$m(\{a \in A : E_l(a,\nu) = t\}) \leq Ce^{-\frac{t}{3h}}m(A).$$

Of course we may put

$$m(\{a \in A : E_l(a, \nu) \geq t\}) \leq Ce^{-\frac{t}{3h}} m(A). \tag{19}$$

for possibly another constant $C > 0$.

5 Conclusion and proof

Choose $\varepsilon_0 > 0$ and consider a ε_0-neighbourhood $\mathcal{N}_{\varepsilon_0}$ of the Julia set $\mathcal{J}(f_0)$. Then $\widehat{\mathbb{C}} \setminus \mathcal{N}_{\varepsilon_0}$ is a compact subset of the Fatou set. Hence there is $\varepsilon > 0$ such that $\mathcal{J}(f_a) \in \mathcal{N}_{\varepsilon_0}$ holds for all $a \in \mathcal{Q}$. Consequently, $\mathcal{F}(f_a) \supset \widehat{\mathbb{C}} \setminus \mathcal{N}_{\varepsilon_0}$, for $a \in \mathcal{Q}$.

Now suppose that $\xi_{n,l}(A)$ is in escape position, i.e. has diameter comparable to S. If we choose $\varepsilon_0 \ll S$, then by the strong distortion control:

$$m(\{a \in A : \xi_{n,l}(a) \in \mathcal{F}(f_a)\}) \geq m(A)(1 - \varepsilon_0'), \tag{20}$$

where $\varepsilon_0' \to 0$ as $\varepsilon_0 \to 0$.

For the $\varepsilon > 0$ chosen from the beginning, let α be such that $32\hat{d}^2\alpha/\gamma_I \leq \iota/2$. Then, given a first essential return $\xi_{\nu_0,l}(A)$ with $\mathrm{dist}(\xi_{\nu_0,l}(A), \mathrm{Jrit}_A) \sim e^{-r}$, we have that $2hr \leq 4h\alpha n = 32\hat{d}^2\alpha/\gamma_I \leq \iota n$. According to (19), parameters in A that have escape time longer than ιn are very few in measure, i.e. less than $Ce^{-2hr/3h}m(A) \leq e^{-r/2}m(A) < \varepsilon' m(A)$, for some $\varepsilon' > 0$, for $r \geq \Delta$ large enough. Let us disregard from them. The rest of the parameters makes escape before $n + \iota n$ and we can use the estimate (20), given that the Lyapunov exponent does not drop below γ_I. But since $\xi_{\nu_0,l}(A)$ is a first essential return, we have that all $a \in A$ have $a \in \mathcal{E}_{\nu_0,l}(\gamma_B)$, so that, at time $(1 + \iota)\nu$ we have, given that $a \in \mathcal{B}_{(1+\iota)\nu,l}$, that indeed $a \in \mathcal{E}_{(1+\iota)\nu,l}(\gamma_I)$ (see the definitions of γ_B and γ_I).

Now we know from Lemma 3.15 that, for each c_l there is an $N_l > 0$ such that $\xi_{N_l,l}(\mathcal{Q})$ satisfies the statements in Lemma 3.15, i.e. bounded distortion of $\xi_{N_l,l}(a)$ on $\mathcal{Q}$ and that $\xi_{N_l,l}(\mathcal{Q})$ is an essential return or escape situation. If it is an escape situation we are done, and can use (20). Suppose that $N_1 = \min\{N_l\}$ and $N_2 = \max\{N_l\}$. To be able to use the binding information for all critical points $c_l(a)$ for $a \in \mathcal{Q}$ we need to make sure that the bound periods for returns in a time window of the type $(N_l, (1 + \iota)N_l)$, where $\iota > 0$ is given above, are all smaller than N_1. Actually, we make α so small that all such bound periods satisfy

$$p \leq \frac{4\hat{d}\alpha}{\gamma_I}(1 + \iota)N_2 \leq N_1.$$

By doing this we can use the binding information of all critical points as said. We also delete parameters not satisfying the basic assumption. Using Lemma 4.2, from this we conclude that, for every $\varepsilon' > 0$ (depending on ι and α), we get sets $\Omega_l \subset \mathcal{Q}$ of measure

$$m(\Omega_l) \geq (1-\varepsilon')(1-e^{-\alpha N_l})m(\mathcal{Q})$$

such that every partition element in $\mathcal{A}$ in Ω_l has escaped and

$$\Omega_l \subset \mathcal{E}_{(1+\iota)N_2,l}(\gamma_l) \cap \mathcal{B}_{(1+\iota)N_2,l}\,.$$

Moreover, from (20) we get that

$$\begin{aligned} m(\{a \in \mathcal{Q} : \xi_{n,l}(a) \in \mathcal{F}(f_a)\}) &\geq m(\Omega_l)(1-\varepsilon_0') \\ &\geq (1-\varepsilon_0')(1-\varepsilon')(1-e^{-\alpha N_l}) \\ &\geq m(\mathcal{Q})(1-\varepsilon''), \end{aligned}$$

for some $\varepsilon'' > 0$ arbitrarily small. Taking the intersection of all critical points and noting that f_a is hyperbolic if all critical points belong to the Fatou set, we get, where d' are the number of critical points,

$$m(\{a \in \mathcal{Q} : \ f_a \text{ is hyperbolic }\}) \geq m(\mathcal{Q})(1-d'\varepsilon'').$$

Since the set of degenerate one-dimensional families in the parameter space $\Lambda_{d,\overline{p'}}$ of rational maps around f_0 has measure zero, we get by Fubini's theorem that f_0 is a Lebesgue density point of hyperbolic maps in $\Lambda_{d,\overline{p'}}$.

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Paper III

Equivalence of Collet–Eckmann conditions for slowly recurrent rational maps

Mats Bylund

Abstract

In this short note we observe that within the family of slowly recurrent rational maps on the Riemann sphere, the Collet–Eckmann, second Collet–Eckmann, and topological Collet–Eckmann conditions are equivalent and also invariant under topological conjugacy.

1 Introduction

The Collet–Eckmann condition first appeared in the seminal papers by P. Collet and J.-P. Eckmann [CE80, CE83] where they studied the chaotic behaviour of certain non-uniformly expanding maps on the interval. This condition, which requires exponential growth of the derivative along the critical orbit(s), was later introduced in [Prz98] to the study of holomorphic (rational) maps on the Riemann sphere. The Collet–Eckmann condition, which often implies the existence of absolutely continuous invariant measures with strong ergodic properties, is known to be abundant in both the real [BC85, BC91, AM05] and complex [Ree86, Asp04] settings. A related and purely topological condition was introduced in [PR98], where it was proved to be implied by the Collet–Eckmann condition. Much work has been made to identify the relationships between the *Collet–Eckmann condition* (abbr. CE), the *second Collet–Eckmann condition* (abbr. CE2), and the *topological Collet–Eckmann condition* (abbr. TCE) (see below for definitions). Notably, these conditions are known to be equivalent within the family of unicritical maps (see [PRLS03] and references therein). In [PRLS03] examples are given of maps which satisfy TCE but not CE and/or CE2, and maps which satisfy CE but not CE2, and vice versa. The main problem that arises is when critical points come close to other critical points of high multiplicity. By assuming a recurrence condition of the critical orbits, known as the *slow recurrence condition* (abbr. SR), we observe in this note that these conditions become equivalent; in a sense slow recurrence takes the rôle of unicritical. The slow recurrence condition is defined as follows.

Definition 1.1. A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 is said to satisfy the *slow recurrence condition* if for each $\alpha > 0$ there exists $C > 0$ such that, for every critical point $c \in \mathrm{Crit}(f) \cap J(f)$,

$$\mathrm{dist}\left(f^n(c), \mathrm{Crit}(f) \cap J(f)\right) \geq Ce^{-\alpha n} \quad (n \geq 1).$$

Remark 1.2. Note that if f satisfies SR then no critical point is mapped onto another critical point.

The SR condition is generally believed to be a typical property among rational Collet–Eckmann maps, and it should be noted that within the real quadratic family this is known to be true due to a result by A. Avila and C. G. Moreira [AM05]. In fact they proved that for a typical non-hyperbolic (the critical point does not tend to an attractive cycle) real quadratic map F one has

$$\mathrm{dist}(F^n(c), c) \geq \frac{C}{n^{1+\epsilon}} \quad (n \geq 1),$$

for any $\epsilon > 0$ and $C = C(\epsilon) > 0$ a constant. Moreover, in the multimodal setting, B. Gao and W. Shen [GS14] proved that for one-parameter families the slow recurrence condition is satisfied on a set of positive Lebesgue measure. We also mention that for complex unicritical polynomials $z \mapsto z^d + c$, it follows from a result by J. Graczyk and G. Świątek [GS15] that the slow recurrence condition is satisfied for a typical parameter c with respect to harmonic measure on the boundary of the connectedness locus.

The SR condition is also natural in the sense that CE+SR is invariant under topological conjugacy, as was observed by H. Li (Theorem A.1 in [Li17], see also [LW06]). The short proof of the invariance is given at the end of this note.

The following is our main observation.

Proposition 1.3. *Within the family of slowly recurrent rational maps of degree ≥ 2 on the Riemann sphere, CE, CE2, and TCE are equivalent. Moreover, these conditions are invariant under topological conjugacy.*

In [PRLS03] examples of real polynomials of degree 5 that satisfy CE but not CE2 (and vice versa) are given, and also examples of real polynomials of degree 3 that satisfy TCE but neither CE nor CE2. We therefore conclude that none of these examples satisfy SR.

We make a final remark that it would be interesting to investigate the set of rational maps satisfying TCE+SR. Indeed if almost every topological Collet–Eckmann map is slowly recurrent then TCE and CE are equivalent up to a set of measure zero.

Below we indicate the changes in three already existing lemmas in order to reach the above stated result of Proposition 1.3. For completeness we provide the minimal of definitions and proofs, but refer to the relevant articles for greater detail. Throughout this note the standing assumption is that f is a slowly recurrent rational map on the Riemann sphere $\widehat{\mathbb{C}}$ of degree ≥ 2, and with f^n we mean f composed with itself n times. We let $B(z,r) = \{w : \operatorname{dist}(z,w) < r\} \subset \widehat{\mathbb{C}}$ denote the disk of radius $r > 0$ centred at z, and we let $\operatorname{Crit}'(f) = \operatorname{Crit}(f) \cap J(f)$ with $\operatorname{Crit}(f)$ the set of critical points of f, and $J(f)$ the Julia set of f. Distances, diameters, and derivatives are taken with respect to the spherical metric on $\widehat{\mathbb{C}}$.

2 Equivalence of CE+SR and CE2+SR

The Collet–Eckmann condition and second Collet–Eckmann condition are defined as follows.

Definition 2.1. A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 without parabolic periodic points is said to satisfy the *Collet–Eckmann condition* (CE) if there exist constants $\lambda_1 > 1$ and $C_1 > 0$ such that, for each critical point $c \in \operatorname{Crit}'(f)$,

$$|(f^n)'(f(c))| \geq C_1\lambda_1^n \quad (n \geq 0).$$

Definition 2.2. A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 is said to satisfy the *second Collet–Eckmann condition* (CE2) if there exist constants $\lambda_2 > 1$ and $C_2 > 0$ such that, for every $n \geq 1$ and every $w \in f^{-n}(c)$ for $c \in \operatorname{Crit}'(f)$ not in the forward orbit of other critical points,

$$|(f^n)'(w)| \geq C_2\lambda_2^n.$$

In [GS98] it was proved that these two conditions are equivalent for critical points of maximal (dynamical) multiplicity. This was achieved through the so-called *(reversed) telescope construction*. At the heart of these techniques lies the *shrinking neighbourhoods* (first introduced in [Prz98]) which are defined as follows. Fix a decreasing sequence of positive real numbers (δ_n) satisfying $\prod_n(1-\delta_n) > 1/2$. Let $B_r = B(z,r)$, and consider a sequence $(f^{-n}(z))$ of consecutive preimages of z. With $\Delta_n = \prod_{k<n}(1-\delta_k)$, the n^{th} shrinking neighbourhoods of z are now defined as

$$U_n = \operatorname{Comp}_{f^{-n}(z)} f^{-n} B_{r\Delta_n} \quad \text{and} \quad U_n' = \operatorname{Comp}_{f^{-n}(z)} f^{-n} B_{r\Delta_{n+1}}.$$

Here, Comp_w denotes the connected component containing w. With the right *scale* around each critical point, using these shrinking neighbourhoods, one gets distortion and expansion estimates. The scale is defined by the choice of two positive numbers $R' \ll R \ll 1$, and the correct choice of these two numbers is crucial for the local analysis. We refer to [GS98] for details; for our purposes it is enough to keep in mind that these two numbers are fixed throughout the analysis.

2.1 CE+SR $\Longrightarrow$ CE2+SR

Let $(c_{-k})_{k=1}^n$ be a sequence of consecutive preimages of $c_0 = c \in \text{Crit}'(f)$ of length $n \geq 1$, i.e. $f(c_{-k}) = c_{-k+1}$ and $f^k(c_{-k}) = c$. In [GS98], the authors inductively define an increasing sequence of numbers $0 = n_0 < n_1 < \cdots < n_m = n$, and each (backward) orbit of length $n_{k+1} - n_k$ is classified as either a *type* 1, *type* 2, or *type* 3 orbit. For orbits of type ... 2, ... 3, or 1 ... 13 (one reads from the right), one has exponential growth of the derivative (a 12 block is not allowed by construction). The only problem thus arise when a given backward orbit begins with a block of 1's which is not preceded by a 3. For clarity we give the definition of a type 1 orbit.

Definition 2.3. A sequence $z_0 = z, z_{-1} \in f^{-1}(z), \ldots, z_{-n} \in f^{-n}(z)$ of consecutive preimages of z is of the first type with respect to the critical points c' and c'' if

1) Shrinking neighbourhoods U_k for $B(z, r)$, $1 \leq k \leq n$, avoid critical points for some $r < 2R'$,
2) The critical point $c'' \in \partial U_n$,
3) The critical value of c' is close to z with $f(c') \in B(f(z), R)$.

The situation of having a block of 1's not preceded by a 3 can only happen in the beginning, and given such a situation the authors prove that there is a constant $\lambda > 1$ such that

$$|(f^n)'(c_{-n})|^{\mu_{\max}} \geq \text{const}\, \lambda^n r_1^{\mu_{\max} - \mu(c)}, \tag{1}$$

where $\mu(c)$ is the multiplicity of c, $\mu_{\max} = \max_{c \in \text{Crit}'(f)} \mu(c)$, and $r_1 < 2R'$ is the radius of a disk centred at c. Here r_1 can not be chosen freely in order for the inductive definition of the n_k's to work, thus for large n (1) might not yield expansion. The authors assume $\mu(c) = \mu_{\max}$ and in doing so prove that CE implies CE2 for critical points of maximal multiplicity (Proposition 1 in [GS98]). If one assumes SR this problem is easily seen to vanish since the slow recurrence condition dictates how small r_1 can be.

Lemma 2.4. *If a slowly recurrent rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 satisfies CE then it satisfies CE2.*

Proof. Suppose the situation is as described above, and let n_1 be the length of the first type 1 orbit. Per definition of a type 1 orbit there exists a critical point $c'' \in \partial U_{n_1}$ which is mapped into $B(c, r_1)$. From SR we get that

$$r_1 \geq \operatorname{dist}(f^{n_1}(c''), c) \geq Ce^{-\alpha n_1}.$$

Since $n_1 \leq n$, inserting the above in (1) we find that

$$|(f^n)'(c_{-n})|^{\mu_{\max}} \geq \text{const}\, \lambda^n \, (Ce^{-\alpha n})^{\mu_{\max}-\mu(c)} \geq C_2 \lambda_2^n,$$

where we can make λ_2 arbitrarily close to λ by decreasing α (and thus also decreasing C_2). □

2.2 CE2+SR $\Longrightarrow$ CE+SR

Pick $c \in \operatorname{Crit}'(f)$, fix n, and consider a sequence of images

$$z_0 = f^n(f(c)), z_{-1} = f^{n-1}(f(c)), \dots, z_{-(n+1)} = c.$$

Similarly as in the previous case, the authors inductively define an increasing subsequence $n_0 < n_1 < \cdots < n_m = n$. Here n_0 is the smallest positive integer such that $z_{-(n_0+1)}$ is in the R-neighbourhood of some critical point. Due to the *exponential shrinking of components* (see below for a definition), which is implied by CE2 (see [PRLS03]), one can prove that during this last orbit of length n_0 one has expansion. (In [GS98] *R-expansion* was taken as an assumption.) The conditions imposed on n_j, $j \neq 0$, are as follows:

I) The sequence $z_{-n_{j-1}}, \dots, z_{-n_j}$ is of the *first reversed type*,

II) Some critical point $c^{(j)} \in B(z_{-(n_j+1)}, R)$.

The definition of a *first reversed type* orbit is as follows.

Definition 2.5. A sequence $z_0 = z, z_{-1} \in f^{-1}(z), \dots, z_{-n} \in f^{-n}(z)$ of consecutive preimages of z is of the reversed first type with respect to two critical points c' and c'' if

1) Shrinking neighbourhoods U_k for $B(z_{-1}, r)$, $1 \leq k \leq n-1$, avoid critical points,

2) $\operatorname{dist}(z_{-1}, c') = r/2 < R$,

3) $c'' \in U_n$.

The authors continue and prove (Proposition 5 in [GS98]) that there is a constant $\lambda > 1$ such that

$$|(f^n)'(f(c))| \geq \text{const}\,\lambda^n \left(\operatorname{diam}(U_m)\right)^{\mu_{\max}-\mu(c)}. \tag{2}$$

Here $\operatorname{diam}(U_m)$ is the diameter of a shrinking neighbourhood around c. As in the previous case, this factor might interfere with expansion for large n unless c is assumed to be a critical point of maximal multiplicity $\mu(c) = \mu_{\max}$. Again, assuming SR, we get a lower bound for the diameter.

Lemma 2.6. *If a slowly recurrent rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 satisfies CE2 then it satisfies CE.*

Proof. It is given that $f^m(c) \in B(c', R)$ for a critical point c', and U_m is the shrinking neighbourhood of $B(f^m(c), r)$ of radius $r = 2\operatorname{dist}(f^m(c), c')$. By definition of a reversed type 1 orbit $f^{-(m-1)}\colon B(f^m(c), r/2) \to f(U_m)$ is univalent, and with an application of Koebe's $\frac{1}{4}$-lemma we find that

$$\operatorname{diam}(U_m) \geq \operatorname{diam}(f(U_m)) \geq \frac{1}{C} r |(f^{m-1})'(f(c))|^{-1}.$$

(Here $C > 1$ is a constant depending on the scale R we are working with, and it shows up since we are adapting Koebe's $\frac{1}{4}$-lemma to the spherical metric.) The first inequality follows since $c \in U_m$ and thus the image of U_m under f is contracted. Since $r/2 = \operatorname{dist}(f^m(c), c')$, invoking SR and that $m \leq n$, we find by inserting the above in (2) that

$$|(f^n)'(f(c))| \geq \text{const}\left[\frac{1}{C}|(f^{m-1})'(f(c))|^{-1}\right]^{\mu_{\max}-\mu(c)} \lambda^n \left(Ce^{-\alpha n}\right)^{\mu_{\max}-\mu(c)}.$$

We observe that $|(f^{m-1})'(f(c))| \leq K$ with $K = K(R)$ an absolute constant depending on the choice of R. Indeed, for each critical point c under consideration, and for a fixed R, there exists a unique smallest integer $m = m(c, R)$ for which $f^m(c) \in B(c'', R)$, for some critical point c''. We simply let

$$K = \max_{c \in \operatorname{Crit}'(f)} |(f^{m(c,R)-1})'(f(c))|.$$

Thus we get that

$$|(f^n)'(f(c))| \geq C_1 \lambda_1^n,$$

where we can make λ_1 arbitrarily close to λ by decreasing α (and thus also decreasing C_1). □

3 Equivalence of CE+SR and TCE+SR

The *topological Collet–Eckmann condition* for rational maps on the Riemann sphere was first introduced in [PR98] and is defined as follows. Recall that for a connected set Ω, $\mathrm{Comp}_w\, g^{-1}(\Omega)$ denotes the connected component of $g^{-1}(\Omega)$ containing w.

Definition 3.1. A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 is said to satisfy the *topological Collet–Eckmann condition* (TCE) if there exist $M \geq 0$, $P \geq 0$ and $r > 0$ such that for every $z \in J(f)$ there exists a strictly increasing sequence of positive integers n_j, for $j = 1, 2, \ldots$, such that $n_j \leq P \cdot j$, and for each j

$$\#\left\{i : 0 \leq k < n_j, \mathrm{Comp}_{f^k(z)} f^{-(n_j-k)} \left(B(f^{n_j}(z), r)\right) \cap \mathrm{Crit} \neq \emptyset\right\} \leq M.$$

Since TCE is formulated purely in topological terms it is a topological invariant. One of the useful properties of this condition is its many equivalent formulations (see [PRLS03] and also [PRL07, RL10]). Here we make use of the following equivalent condition.

Definition 3.2. A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 is said to satisfy *exponential shrinking of components* (ExpShrink) if there exists $\lambda_{\mathrm{Exp}} > 1$ and $r_{\mathrm{Exp}} > 0$ such that for every $x \in J(f)$, every $n > 0$, and every connected component W of $f^{-n}(B(x, r_{\mathrm{Exp}}))$

$$\mathrm{diam}(W) \leq (\lambda_{\mathrm{Exp}}^{-1})^n.$$

It was first proved in [PR98] that CE implies TCE, and in [PR99] it was proved that under the assumption that for every $c \in \mathrm{Crit}'(f)$ whose forward trajectory does not meet any other critical point

$$\mathrm{cl} \bigcup_{n>0} f^n(c) \cap \left(\mathrm{Crit}(f) \setminus \{c\}\right) = \emptyset, \tag{3}$$

TCE implies CE. This latter result clearly implies that CE+(3) is a topological invariant; in particular CE is a topological invariant in the case of unicritical maps. Another proof of this result was obtained in [Prz00]. We will effectively replace condition (3) with SR, thus proving that TCE+SR implies CE+SR. This constitutes an obvious modification in the proof of Lemma 4.5 in [Prz00]; for completeness we give a sketch of this proof. (See also Proposition 3.4 in [Li17].)

Lemma 3.3. *If a slowly recurrent rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree ≥ 2 satisfies ExpShrink then it satisfies CE.*

Proof. Let α be the exponent in SR and let $n_0 = n_0(\alpha)$ be large enough such that for every $n \geq n_0$

$$\operatorname{dist}(f^j(f(c)), \operatorname{Crit}(f)) > e^{-2\alpha n} \quad (j = 0, 1, \dots, n-1).$$

This condition is assumed in Lemma 4.5 [Prz00], and the proof now continues as follows. Fix $\epsilon > 0$ arbitrary and let

$$s = \left[\frac{-\log \epsilon}{\log \lambda_{\text{Exp}}} + \frac{2\alpha n}{\log \lambda_{\text{Exp}}} \right] + 1,$$

where $[x]$ denotes the integral part of x. By ExpShrink we have that for all $0 \leq j \leq n$

$$\begin{aligned} \operatorname{diam}\left(\operatorname{Comp}_{f^{n-j}(f(c))} f^{-s-j}\left(B(f^{n+s}(f(c)), r_{\text{Exp}}) \right) \right) &\leq (\lambda_{\text{Exp}}^{-1})^{s+j} \\ &\leq (\lambda_{\text{Exp}}^{-1})^{s} \\ &\leq \epsilon e^{-2\alpha n}. \end{aligned}$$

Let $B = B(f^n(f(c)), r_{\text{Exp}} e^{-3\alpha M n})$, where

$$M = \left[\frac{\log \sup_{\widehat{\mathbb{C}}} |f'|}{\log \lambda_{\text{Exp}}} \right] + 1.$$

Then for n large enough we get that

$$B \subset \operatorname{Comp}_{f^n(f(c))} f^{-s}\left(B(f^{n+s}(f(c)), r_{\text{Exp}}) \right).$$

Let $W_n = \operatorname{Comp}_{f(c)} f^{-n}(B)$. Then there exists $w \in W_n$ such that

$$|(f^n)'(w)| \geq \frac{\operatorname{diam} B}{\operatorname{diam} W_n} \geq (2 r_{\text{Exp}} e^{-3\alpha M n}) \lambda_{\text{Exp}}^n.$$

If ϵ is sufficiently small we have distortion and can switch from w to $f(c)$, hence

$$|(f^n)'(f(c))| \geq \lambda_1^n,$$

where we can make λ_1 arbitrarily close to λ_{Exp} by decreasing α and ϵ (and thus increasing n_0). $\square$

4 Topological invariance

We finish by giving the short proof of the topological invariance, as outlined in Theorem A.1 [Li17].

Lemma 4.1. *Let f and g be topologically conjugated rational maps on the Riemann sphere of degree ≥ 2. If f satisfies TCE+SR then so does g.*

Proof. Since f is TCE+SR it is CE+SR and therefore, by Theorem A in [PR99], the conjugacy is quasi-conformal and therefore bi-Hölder. Let h denote this conjugacy, and let $A > 0$ and $B > 0$ be the associated constant and exponent from the Hölder condition, respectively. Let c_1' and c_2' be distinct critical points of g; then $c_1 = h^{-1}(c_1')$ and $c_2 = h^{-1}(c_2')$ are distinct critical points of f. Since h preserves TCE, g is at least TCE. The fact that g is also SR follows from

$$\begin{aligned} A \operatorname{dist}\left(g^n(c_1'), c_2'\right)^B &\geq \operatorname{dist}\left(h^{-1}(g^n(c_1')), h^{-1}(c_2')\right) \\ &= \operatorname{dist}\left(f^n(c_1), c_2\right) \\ &\geq Ce^{-\alpha n}. \end{aligned}$$

□

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