Dynamics and subgroups of $\mathrm{SL}_N(\mathbb{Z})$

Master starter FMJH

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Abstract

The aim of this mini-course is to give an elementary proof of two consequential results about $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$. The first result concerns normal subgroups of $\mathrm{SL}_N(\mathbb{Z})$ and is due independently to Bass-Lazard-Serre and Mennicke. It states that the normal subgroups of $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$ are either $\{I_N\}$, $\{\pm I_n\}$ (provided N is even) or have finite index. The second result is a rigidity result that concerns stabilizers of measurable actions of $\mathrm{SL}_N(\mathbb{Z})$ that preserves an ergodic probability measure and is due to Nevo-Stuck-Zimmer. To prove these results, we follow the recent approach of *boomerang subgroups* due to Glasner-Lederle and provide the proof of Glasner-Lederle's theorem, which gives an explicit list of the boomerang subgroups of $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$.

Introduction

Throughout the mini-course, the letter G will always denote a countably infinite group.

Definition ([GL22]). — *Let G be a group. A subgroup $H \leq G$ is a* ***boomerang*** *if for all $g \in G$, there exists a sequence of integers $k_n \underset{n\to+\infty}{\longrightarrow} +\infty$ such that*

$$\liminf_{n\to+\infty} g^{k_n} H g^{-k_n} = \limsup_{n\to+\infty} g^{k_n} H g^{-k_n} = H,$$

Here we used the standard notations of liminf and limsup of sets, namely

$$\liminf_{n\to+\infty} A_n = \bigcup_{n\geq 1} \bigcap_{N\geq n} A_N, \quad \limsup_{n\to+\infty} = \bigcap_{n\geq 1} \bigcup_{N\geq n} A_N.$$

Recall that:

- $x \in \liminf_{n\to+\infty} A_n$ if and only if x belongs to A_n eventually.
- $x \in \limsup_{n\to+\infty} A_n$ if and only if x belongs to infinitely many A_n.

The definition of boomerang subgroup may seem to be pulled out of a hat, but it has (more natural) characterization in terms of topological recurrence that we will discuss later in the mini-course. The main theorem that we will prove in this mini-course concerns boomerang subgroups of $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$. Recall that $\mathrm{SL}_N(\mathbb{Z})$ denotes the group of $n \times n$ matrices with integer coefficients of determinant 1. This is indeed a group: if $A, B \in \mathrm{SL}_N(\mathbb{Z})$, then AB is a $n \times n$ matrix with integer coefficients and $\det(AB) = \det(A)\det(B) = 1$. Moreover, if $A \in \mathrm{SL}_N(\mathbb{Z})$, then A is invertible and $A^{-1} = \frac{1}{\det(A)} {}^t\mathrm{Com}(A)$, which is indeed a matrix with integer coefficients.

Given a subgroup H of a group G, a (left) coset of H in G is a set of the form $gH = \{gh \colon h \in H\}$ for some $g \in G$ and that H has ***finite index*** in G if the number of cosets of H in G is finite. We denote by $[G : H]$ the number of left cosets of H in G.

Main Theorem ([GL22]). — *Fix an integer $N \geq 3$. Then the boomerang subgroups of $\mathrm{SL}_N(\mathbb{Z})$ are $\{I_N\}$, $\{\pm I_N\}$ (provided that N is even) and the finite index subgroups.*

Outline of the mini-course

This mini-course consists in three parts.

- Part I: Proof of Glasner-Lederle's theorem

  After providing preliminaries on boomerang subgroups, we will dive into the proof of Glasner-Lederle's theorem. Two of the main tools that we will use along the proof are the Bruhat decomposition of $\mathrm{SL}_N(\mathbb{R})$ and a criterion on subgroups of $\mathrm{SL}_N(\mathbb{Z})$ that ensure finite index.

- Part II : Measurable actions for $\mathrm{SL}_N(\mathbb{Z})$ with $N \geq 3$.

  We will proof the celebrated Poincaré's recurrence theorem and use it to prove – as a consequence of Glasner-Lederle's theorem – a theorem due to Nevo-Stuck-Zimmer, which concerns stabilizers of measurable actions of $\mathrm{SL}_N(\mathbb{Z})$ that preserves an ergodic probability measure.

- Part III : The case of $\mathrm{SL}_2(\mathbb{Z})$.

  The aim of this part is to show that Glasner-Lederle's theorem fails for $\mathrm{SL}_2(\mathbb{Z})$. We will prove that the second derived subgroup $\mathrm{SL}_2(\mathbb{Z})''$ is indeed a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which is infinite and has finite index.

Part I : Proof of Glasner-Lederle's theorem

This part follows the lines of the paper [GL22].

1 Preliminaries of boomerang subgroups

The first examples of boomerang subgroups are given in the following lemma. Recall that a subgroup H of a group G is ***normal*** if for all $g \in G$, we have $gHg^{-1} = H$. We denote it by $H \trianglelefteq G$.

Lemma 1. — *Normal subgroups are boomerang subgroups.*

Proof. Let $H \leq G$ be a normal subgroup. Then for all $g \in G$, it suffices to set $k_n = n$ to get $\liminf\limits_{n\to+\infty} g^{k_n}Hg^{-k_n} = \liminf\limits_{n\to+\infty} H = H$ and $\limsup\limits_{n\to+\infty} g^{k_n}Hg^{-k_n} = \limsup\limits_{n\to+\infty} H = H$. ☐

As a corollary of Lemma 1 and the Main Theorem, we get the following description of normal subgroups of $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$.

Theorem 2 ([BLS64], [Men65]). — *Fix an integer $N \geq 3$. Then the normal subgroups of $\mathrm{SL}_N(\mathbb{Z})$ are $\{I_N\}$, $\{\pm I_N\}$ (provided that N is even) and the finite index normal subgroups.*

Remark. — In Part III, we will prove that $\mathrm{SL}_2(\mathbb{Z})$ admits normal subgroups that are infinite and of infite index.

Lemma 3. — *Finite index subgroups are boomerang.*

Proof. Let H be a finite index subgroup of G. We first show that H contains a non trivial power of any element of G. Fix $g \in G$ and let n be the number of cosets of H in G. Among $H, gH, \dots, g^nH$ two cosets are equal. The equality $g^iH = g^jH$ implies that $g^{i-j} \in H$. Therefore H contains a non trivial power of any element of G. Let us now show that H is a boomerang. Fix $g \in G$ and let $k \in \mathbb{N}$ such that $g^k \in H$. Then for any integer $n \geq 0$, we have $g^{nk} \in H$ and thus $g^{nk}Hg^{-nk}$. Thus $\liminf_{n\to+\infty} g^{nk}Hg^{-nk} = \limsup_{n\to+\infty} g^{nk}Hg^{-nk} = H$ which shows that H is a boomerang. ☐

With Lemmas 1 and 3, we have now checked that all the subgroups in the Main Theorem are indeed boomerang subgroups.

We now develop one result on boomerang subgroups that will be useful several times in the sequel. Given a group G and two group elements $g, h \in G$, the ***commutator*** of g and h is $[g, h] = ghg^{-1}h^{-1}$.

Lemma 4. — *Let H be a boomerang subgroup of G. Then for all $g \in G$ and $h \in H$, there exists infinitely many integers $k > 0$ such that $[h, g^k] \in H$.*

Proof. We apply the definition of boomerang with g^{-1}. There exists a sequence of integers $k_n \underset{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\liminf_{n\to+\infty} g^{-k_n} H g^{k_n} = \limsup_{n\to+\infty} g^{-k_n} H g^{k_n} = H.$$

Since $h^{-1} \in H$, then $h^{-1} \in g^{-k_n} H g^{k_n}$ eventually. In other words, we have $g^{k_n} h^{-1} g^{-k_n} \in H$ eventually. Multiplying by h, we obtain that $[h, g^{k_n}]$ belongs H eventually. $\square$

2 Bruhat decomposition

Let us fix some standard notation. Fix $i, j \in \{1, \dots, N\}$ and $\lambda \in \mathbb{R}$. We denote by $E_{i,j}$ the matrix $(\delta_{i,k}\delta_{k,l})_{k,l\in\{1,\dots,N\}}$ where δ is the Kronecker delta. Whenever $i \neq j$, we denote $T_{i,j}(\lambda)$ the ***transvection matrix*** $I_N + \lambda E_{i,j}$. We recall that:

- multiplying a matrix $M \in \mathrm{GL}_N(\mathbb{R})$ *on the left* by $T_{i,j}(\lambda)$ (that means looking at $T_{i,j}(\lambda)M$) is the same as making the elementary operation $L_i \leftarrow L_i + \lambda L_j$,
- multiplying a matrix $M \in \mathrm{GL}_N(\mathbb{R})$ *on the right* by $T_{i,j}(\lambda)$ (that means looking at $MT_{i,j}(\lambda)$) is the same as making the elementary operation $C_j \leftarrow C_j + \lambda C_i$.

The Bruhat decomposition is a decomposition of $\mathrm{SL}_N(K)$ that holds for any field K. Here, we will work with the field $K = \mathbb{Q}$. Let $\mathcal{U} \leq \mathrm{SL}_N(\mathbb{Q})$ be the group of upper triangular unipotent matrix (diagonal coefficients are equal to 1). For $\sigma \in \mathfrak{S}_N$ and $m_1, \dots, m_N \in \mathbb{Q}$ satisfying $m_1 \dots m_N = \mathrm{sgn}(\sigma)$, we denote by $P_\sigma(m_1, \dots, m_N)$ the matrix of $\mathrm{SL}_N(\mathbb{Q})$ defined by

$$P_\sigma(m_1, \dots, m_N) = (m_j \delta_{i,\sigma(j)})_{i,j\in\{1,\dots,N\}}.$$

Notice that $P_\sigma(m_1, \dots, m_N) \in \mathrm{SL}_N(\mathbb{Q})$ because of the assumption $m_1 \dots m_n = \mathrm{sgn}(\sigma)$.

Theorem 5 (Bruhat decomposition). — *For all matrix $M \in \mathrm{SL}_N(\mathbb{Q})$, there exists two matrices $U, V \in \mathcal{U}$, a permutation $\sigma \in \mathfrak{S}_N$ and $m_1, \dots, m_N \in \mathbb{Q}$ satisfying $m_1 \dots m_N =$ $\mathrm{sgn}(\sigma)$, such that such that $M = UP_\sigma(m_1, \dots, m_N)V$.*

Proof. Let $M \in \mathrm{SL}_N(\mathbb{Q})$ with $M = (m_{i,j})_{i,j\in\{1,\dots,N\}}$.
Step 1: the first column of M is non null. Let i_1 be the greatest index such that $m_{i_1,1} \neq 0$. We make the following elementary operations.

- We multiply M on the left by $T_{k,i_1}(-m_{k,1}/m_{i_1,1})$ for all $k < i_1$. This is equivalent to making the elementary operation $L_k \leftarrow L_k - (m_{k,1}/m_{i_1,1})L_{i_1}$ for all $k < i_1$. All the coefficients of the first column, expect the ${i_1}^{\text{th}}$ one, becomes zero.
- We multiply M on the right by $T_{1,k}(-m_{i_1,k}/m_{i_1,1})$. This is equivalent to making the elementary operation $C_k \leftarrow C_k - (m_{i_1,k}/m_{i_1,1})C_1$ for all $k > 1$. All the coefficients on the ${i_1}^{\text{th}}$ row becomes zero, except the first one.

We have multiplied on the left and on the right by elements in $\mathcal{U}$ and we now get a matrix of the form

$$\begin{bmatrix} 0 & & & \\ \vdots & & * & \\ 0 & & & \\ m_{i_1,1} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}.$$

Step 2: We make similar elementary operations on the second column, which is non null. By construction, we have $i_2 \neq i_1$.

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Step N: We have obtained two matrices $T, T' \in \mathcal{U}$ and a permutation $\sigma : k \mapsto i_k$ in $\mathfrak{S}_N$ such that $TMT' = P_\sigma(m_1, \dots, m_N)$. □

Exercise 6. — Prove that the permutation $\sigma \in \mathfrak{S}_N$ in a Bruhat decomposition of a matrix $M \in \mathrm{SL}_N(\mathbb{Q})$ is uniquely determined.

3 A criterion of finite index for boomerang subgroups

In order to prove Glasner-Lederle's theorem, we will use this criterion, which is a necessary condition for a subgroup of $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$ to have finite index. We won't give the proof of it but we refer to [Ben21, Prop. 3.18] for a modern proof.

Lemma 7 ([Tit76], [Vas73]). — *Fix $N \geq 3$. Let $H \leq \mathrm{SL}_N(\mathbb{Z})$ be a subgroup such that for all $i, j \in \{1, \dots, N\}$ distinct, there exists $k \in \mathbb{Z}^*$ with $T_{i,j}(k) \in H$. Then $[\mathrm{SL}_N(\mathbb{Z}) : H] < +\infty$*

Remark. — The converse of Lemma 7 holds. In fact, for $N \geq 3$, $\mathrm{SL}_N(\mathbb{Z})$ satisfies the congruence subgroup property: any finite index subgroup is contained in the kernel of the group homomorphism $\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}/k\mathbb{Z})$ induced by the reduction modulo some integer $k \geq 1$.

When the subgroup is a boomerang, the above criterion is easier to satisfy.

Lemma 8. — *Fix $N \geq 3$. Let $H \leq \mathrm{SL}_N(\mathbb{Z})$ be a boomerang subgroup. If there exists $i, j \in \{1, \dots, N\}$ distinct and $k \in \mathbb{Z}^*$ such that $T_{i,j}(k) \in H$, then H has finite index.*

Proof. Assume that $T_{i,j}(k)$ belongs to H. Let $i', j' \in \{1, \dots, N\}$ distinct, with $i \neq i'$ and $j \neq j'$. By Lemma 4, there exists (infinitely many and therefore) one integer $l \in \mathbb{Z}^*$ such that $[T_{i,j}(k), T_{j,j'}(l)] \in H$. But this commutator is equal to $T_{i,j'}(kl)$. We apply again Lemma 4 to get an integer $m \in \mathbb{Z}^*$ such that $[T_{i,j'}(kl), T_{i',i}(m)] \in H$. But this commutator is equal to $T_{i',j'}(klm)$. We conclude by using Lemma 7. □

4 The proof of Glasner-Lederle theorem

In this section, we denote by $(X_1, \dots, X_N)$ the canonical basis of $\mathbb{R}^N$.

LEMMA 9. — *Let $H \leq \mathrm{SL}_N(\mathbb{Z})$ be a boomerang subgroup. If $HX_1 \subseteq \mathrm{span}(X_1)$ then for all $i \in \{1, \dots, N\}$, we have $HX_i \subseteq \mathrm{span}(X_i)$.*

Proof. There is nothing to prove when $i = 1$. Let $i \in \{2, \dots, N\}$ and $M \in H$. We apply the definition of boomerang subgroup for the group element $T_{1,i}(1)$: there exists a sequence of integers $k_n \longrightarrow +\infty$ as $n \to +\infty$ such that the matrix $M_n = T_{1,i}(k_n)MT_{1,i}(-k_n)$ belongs to H for all $n \in \mathbb{N}$. By assumption on H, we have $M_nX_1 \subseteq \mathrm{span}(X_1)$. Since M is invertible, then so is M_n for all $n \in \mathbb{N}$ and thus X_1 is an eigenvector of M_n. Notice that $T_{1,i}(-k_n)X_1 = X_1 - k_nX_i$. We therefore get that $X_1 - k_nX_i$ is an eigenvector of M. Thus, there exists $\lambda_n \in \mathbb{R}^*$ such that

$$M(X_1/k_n - X_i) = \lambda_n(X_1/k_n - X_i). \tag{1}$$

Since the norm of $M(X_1/k_n - X_i)$ is bounded from above and that of $(X_1/k_n - X_i)$ is bounded from below by a strictly positive constant, we obtain that the sequence $(\lambda_n)_{n\geq 0}$ is bounded. Up to a subsequence, one can assume without loss of generality that it converges to $\lambda \in \mathbb{R}$. At the limit in Equation (1), we get $MX_i = \lambda X_i$, which finishes the proof. □

LEMMA 10. — *Let $H \leq \mathrm{SL}_N(\mathbb{Z})$ be a boomerang subgroup. Assume that for every $M \in H$, the permutation $\sigma \in \mathfrak{S}_N$ given in a Bruhat decomposition of M satisfies $\sigma(1) = 1$. Then H is a subgroup of $\{\pm I_N\}$.*

Proof. Notice that if for every $M \in H$, the permutation $\sigma \in \mathfrak{S}_N$ given in a Bruhat decomposition of M satisfies $\sigma(1) = 1$, then we have $HX_1 \subseteq \mathrm{span}(X_1)$. Thus, by Lemma 9, for all $i \in \{2, \dots, N\}$, we have $HX_i \subseteq \mathrm{span}(X_i)$. This implies that H is a subgroup of $\{\mathrm{diag}(d_1, \dots, d_N) \colon d_i \in \{\pm 1\}\}$. Assume that there exists $d_1, \dots, d_N \in \{\pm 1\}$ such that $M = \mathrm{diag}(d_1, \dots, d_N)$ and $d_i \neq d_j$ for some $i \neq j$. Then for all $k \in \mathbb{Z}^*$, the matrix $T_{i,j}(k)MT_{i,j}(-k)$ is not diagonal and therefore not in H. Thus, $H \leq \{\pm I_N\}$. □

Proof of the Main theorem. Let $H \leq \mathrm{SL}_N(\mathbb{Z})$ be a boomerang subgroup and assume that H is not a subgroup of $\{\pm I_N\}$. By Lemma 10, there exists $M \in H$ with a Bruhat decomposition $M = UP_\sigma(m_1, \dots, m_N)V$ satisfying $\sigma(1) \neq 1$. There are two cases.
Case 1: if $\sigma(N) \neq 1$. Fix $l \in \mathbb{Z}^*$ such that $UT_{1,\sigma(1)}(l)U^{-1} \in \mathrm{SL}_N(\mathbb{Z})$. By Lemma 4, there are infinitely many integers $k, l' \in \mathbb{Z}$ such that

$$[[M, T_{1,N}(k)], UT_{1,\sigma(1)}(ll')U^{-1}] \in H$$

By Exercise 13 and the fact that $[T_{\sigma(1),\sigma(N)}(km_1/m_N), T_{1,\sigma(1)}(ll')] = T_{1,\sigma(N)}(-kll'm_1/m_N)$, we therefore obtain that $UT_{1,\sigma(N)}(-kll'm_1/m_N)U^{-1} \in H$. Since $m_1/m_N \in \mathbb{Q}$, one can choose k and l large enough so that $U^{-1}HU$ contains a transvection $T_{i,j}(\lambda)$ with $i \neq j$ and $\lambda \in \mathbb{Z}^*$.

Case 2: if $\sigma(N) = 1$, choose any $j \neq 1, \sigma(1)$. Fix $l \in \mathbb{Z}^*$ such that $UT_{1,j}(l)U^{-1} \in \mathrm{SL}_N(\mathbb{Z})$. By Lemma 4, there are infinitely many integers $k, l' \in \mathbb{Z}$ such that

$$[[M, T_{1,N}(k)], UT_{1,j}(ll')U^{-1}] \in H$$

By Exercise 13 and the fact that $[T_{\sigma(1),\sigma(N)}(km_1/m_N), T_{1,j}(ll')] = UT_{\sigma(1),j}(kll'm_1/m_N)U^{-1}$, we therefore obtain that $UT_{\sigma(1),j}(kll'm_1/m_N)U^{-1} \in H$. Since $m_1/m_N \in \mathbb{Q}$, one can choose k and l large enough so that $U^{-1}HU$ contains a transvection $T_{i,j}(\lambda)$ with $i \neq j$ and $\lambda \in \mathbb{Z}^*$
Now by Exercise 14, the group $\mathrm{SL}_N(\mathbb{Z}) \cap U^{-1}\mathrm{SL}_N(\mathbb{Z})U$ has finite index in $\mathrm{SL}_N(\mathbb{Z})$. By Exercise 12, we deduce that $\mathrm{SL}_N(\mathbb{Z}) \cap U^{-1}HU$ is a boomerang subgroup of $\mathrm{SL}_N(\mathbb{Z}) \cap U^{-1}\mathrm{SL}_N(\mathbb{Z})U$ and therefore of $\mathrm{SL}_N(\mathbb{Z})$ by Exercise 11. Since it contains a transvection $T_{i,j}(\lambda)$ with $i \neq j$ and $\lambda \in \mathbb{Z}^*$, we deduce by Lemma 8 that $\mathrm{SL}_N(\mathbb{Z}) \cap U^{-1}HU$ has finite index in $\mathrm{SL}_N(\mathbb{Z})$. It is now an easy to show that it implies that H has finite index in $\mathrm{SL}_N(\mathbb{Z})$. □

We close this section with some exercises that are used in the proof of the Main theorem.

Exercise 11. — Let $H \leq G$ be a finite index subgroup of a group G. If $K \leq H$ is a boomerang subgroup of H, then K is a boomerang subgroup of G.

Hint. — *Use the fact (proved in the proof of Lemma 3) that H contains a nontrivial power of every element of G.*

Exercise 12. — Let $H \leq G$ be a subgroup of a group G. Let $g \in G$ such that $gHg^{-1} \cap H$ is a finite index subgroup of H. If K is a boomerang subgroup of H, then $gKg^{-1} \cap H$ is a boomerang subgroup of H.

Exercise 13. — Fix a matrix $M \in \mathrm{SL}_N(\mathbb{Q})$ and let $M = UP_\sigma(m_1, \ldots, m_N)V$ be a Bruhat decomposition. Let $k \in \mathbb{Z}$.

- Prove that for all $k \in \mathbb{Z}$, the matrix $T_{1,N}(k)$ commutes with every element of $\mathcal{U}$.
- Prove that $[U^{-1}MU, T_{1,N}(k)] = T_{\sigma(1),\sigma(N)}(k\frac{m_1}{m_N})T_{1,N}(-k)$.
- Deduce that for all $1 \leq i < j \leq N$ and $l \in \mathbb{Z}$, we have $[[M, T_{1,N}(k)], UT_{i,j}(l)U^{-1}] = U[T_{\sigma(1),\sigma(N)}(k\frac{m_1}{m_N}), T_{i,j}(l)]U^{-1}$.

Exercise 14 (Not that easy). — Let $M \in \mathrm{SL}_N(\mathbb{Q})$. Prove that there exists an integer $k \geq 1$ such that if Γ_k denotes the kernel of the group homomorphism $\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}/k\mathbb{Z})$ induced by the reduction modulo k, then $M\Gamma_k M^{-1} \subseteq \mathrm{SL}_N(\mathbb{Z})$. Deduce that the index of $\mathrm{SL}_N(\mathbb{Z}) \cap M\mathrm{SL}_N(\mathbb{Z})M^{-1}$ in $\mathrm{SL}_N(\mathbb{Z})$ is finite.

Part II : Measurable actions for $\mathrm{SL}_N(\mathbb{Z})$ with $N \geq 3$.

In this part, we prove a dynamical consequence of the Main theorem. It concerns measurable actions of $\mathrm{SL}_N(\mathbb{Z})$ for $N \geq 3$ which preserves an ergodic probability measure. Before stating it, we develop some elementary notions of ergodic theory.

1 Poincaré recurrence theorem

Let G be a group and X a set. An ***action*** of G on X is a group homomorphism $G \to \mathfrak{S}(X)$. Such an action will be denoted $G \curvearrowright X$. Given an action $G \curvearrowright X$, we denote by $x \mapsto g \cdot x$ the image of g in $\mathfrak{S}(X)$. The ***stabilizer*** of a point $x \in X$ is the subgroup $G_x = \{g \in G \colon g \cdot x = x\}$.

If $(X, \mathcal{B})$ is a measurable space, a ***measurable action*** of G on X is an action $G \curvearrowright X$ such that for all $g \in G$, the map $x \mapsto g \cdot x$ is a measurable bijection, whose inverse is also measurable (we call such a map a ***bi-measurable bijection***). When G is isomorphic to $\mathbb{Z}$, any measurable action of G corresponds to the action generated by the iteration of a bi-measurable bijection on X. A bi-measurable bijection $f : X \to X$ preserves a probability measure μ on $(X, \mathcal{B})$ if for all $A \in \mathcal{B}$, we have $\mu(f^{-1}(A)) = \mu(A)$. A measurable action $G \curvearrowright X$ preserves a probability measure μ on X if for all group element $g \in G$, the bi-measurable bijection $x \mapsto g \cdot x$ preserves μ. A measurable action $G \curvearrowright X$ that preserves a probability measure μ is ***ergodic*** if the sets $A \in \mathcal{B}$ that satisfy $g \cdot A = A$ for all $g \in G$ have measure 0 or 1.

Example 1. — Let $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Given a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$, we define a bijection of $\hat{\mathbb{R}}$ (still denoted by M) by

$$M : x \mapsto \begin{cases} a/c & \text{if } x = \infty, \\ \infty & \text{if } x = -d/c, \\ (ax+b)/(cx+d) & \text{else.} \end{cases}$$

One checks that it defines an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\hat{\mathbb{R}}$.

Exercise 2. — Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Let $\hat{\mathcal{B}} = \mathcal{B} \cup \{A \cup \{\infty\} \colon A \in \mathcal{B}\}$. Prove that $\hat{\mathcal{B}}$ is a σ-algebra on $\hat{\mathbb{R}}$ and that the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \hat{\mathbb{R}}$ from Example 1 is a measurable action.

We recall Fatou's lemma. Given a measurable space $(X, \mathcal{B})$ and a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable non-negative functions, we have

$$\int_X \liminf_{n \to +\infty} f_n(x) d\mu(x) \leq \liminf_{n \to +\infty} \int_X f_n(x) d\mu(x).$$

Exercise 3 (Reverse Fatou lemma). — Let $(X, \mathcal{B}, \mu)$ be a measure space, $(f_n)_{n \geq 0}$ a sequence of measurable real valued functions. If there exists a measurable real valued function g such that $f_n \leq g$ for all $n \geq 0$, then prove that

$$\limsup_{n\geq 0} \int_X f_n(x)d\mu(x) \leq \int_X \limsup_{n\geq 0} f_n(x)d\mu(x).$$

Theorem 4 (Poincaré recurrence theorem, measurable version). — *Let $(X, \mathcal{B})$ be a measurable space and let μ be a probability measure on it. Let $f : X \to X$ be a bi-measurable bijection that preserves μ. Then for all $A \in \mathcal{B}$ and for μ-a.e. $x \in A$, the set $\{n \in \mathbb{N} \colon f^n(x) \in A\}$ is infinite.*

Proof. For $x \in X$ and $n \in \mathbb{N}^*$, we define

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_A(f^k(x)).$$

Set $S(x) = \limsup_{n\geq 0} S_n(x)$. Then S is measurable as a limsup of measurable functions. Let us compute $S \circ f$.

$$\begin{aligned} S \circ f(x) &= \limsup_{n\to+\infty} S_n(f(x)) \\ &= \limsup_{n\to+\infty} \Big(\frac{n+1}{n} S_{n+1}(x) - \frac{1}{n} \mathbb{1}_A(x) \Big) \\ &= \limsup_{n\to+\infty} S_n(x). \end{aligned}$$

Therefore, we obtain that $S \circ f = S$. We now compute the following:

$$\begin{aligned} \mu(A \cap (\mathbb{1}_{S=0})) &= \int_X \mathbb{1}_A(x) \mathbb{1}_{S=0}(x) d\mu(x) \\ &= \int_X \mathbb{1}_A(f^k(x)) \mathbb{1}_{S=0}(x) d\mu(x) \quad \text{since } f \text{ preserves } \mu \text{ and } S \circ f = S \\ &= \int_X S_n(x) \mathbb{1}_{S=0} d\mu(x) \\ &\leq \int_X S(x) \mathbb{1}_{S=0}(x) d\mu(x) \quad \text{by the “reverse” Fatou lemma} \\ &= 0. \end{aligned}$$

This shows that for μ-a.e. point $x \in A$, the frequency $S(x)$ is strictly positive, which shows the result. □

Let (X, d) be a compact metric space and $f : X \to X$ an homeomorphism. A point $x \in X$ is ***recurrent*** for f if there exists a sequence of integers $k_n \to +\infty$ as $n \to +\infty$ such that $f^{k_n}(x) \underset{n\to+\infty}{\longrightarrow} x$.

Theorem 5 (Poincaré recurrence theorem, topological version). — *Let X be a compact metric space, μ a probability measure on X and $f : X \to X$ an homeomorphism* *that preserves μ. Then μ-a.e. point of X is recurrent for f.*

Proof. Since X is a compact metric space, it is separable: let $C \subseteq X$ be countable and dense subset. Let $\mathcal{C}$ be the (countable) set of balls centered at an element of C and with rational radius. For all $A \in \mathcal{C}$, we apply the measurable version of Poincaré recurrence theorem: the set N_A of points in A which come back only finitely many times in A is μ-negligible. Thus, the set $N = \bigcup_{A \in \mathcal{C}} N_A$ is also μ-negligible. By construction, every point in $X \setminus N$ is recurrent. $\square$

2 Boomerang subgroups and recurrence

In this section, we will apply the topological version of Poincaré recurrence theorem to give a characterization of the notion of boomerang subgroup in terms of recurrence. Let G be a (countably infinite, as usual) group. Let $X = \{0,1\}^G$. In the sequel, we will identify without mentioning it X with the set $\mathcal{P}(G)$ of subsets of G via the bijective map $E \mapsto \mathbb{1}_E$. We endow E with the product topology, which is the coarsest topology (that is the topology with the fewest open sets) for which for all $g \in G$, the projection onto the g^{th} coordinate $pb_g : X \to \{0,1\}$ is continuous. We denote by $\mathcal{B}$ the Borel σ-algebra on X. We will use in the sequel the following two basic facts.

- Given a σ-algebra $(X', \mathcal{B}')$, a map $f : X' \to X$ is measurable if and only if for all $g \in G$, the map $p_g \circ f : X' \to \{0,1\}$ is measurable.
- Given a topological space X', a map $f : X' \to X$ is continuous if and only if for al $g \in G$, the map $p_g \circ f : X' \to \{0,1\}$ is continuous.

The set X is compact and metrizable: one can for instance fix an enumeration $(g_n)_{n \geq 0}$ of all the elements of G and define the distance

$$d(x, x') = \sup_{n \in \mathbb{N}} \frac{\delta_{x_{g_n}, x'_{g_n}}}{2^n},$$

where δ is the Kronecker delta. Let $\mathrm{Sub}(G) \subseteq X$ denotes the set of subgroups of G (seen as subsets of G).

Lemma 6. — *The set* $\mathrm{Sub}(G)$ *is a closed, and therefore compact, subset of* $\{0,1\}^G$.

Proof. Fix $(H_n)_{n \geq 0} \in \mathrm{Sub}(G)^{\mathbb{N}}$ a sequence which converges in X. By definition of the product topology, all the projections $p_g : X \to \{0,1\}$ are continuous. Projecting onto the neutral element e_G, we obtain that $e_G \in H_n$ for all $n \geq 0$ and therefore $e_G \in H_n$ by continuity. Fix two elements $g, h \in H$. Then using the continuity of the projections p_g and p_h, we deduce that g and h belongs to H_n eventually. Since the H_n's are groups, we deduce that $gh^{-1} \in H_n$ eventually. Using the continuity of the projection $p_{gh^{-1}}$, we obtain that $gh^{-1} \in H$ and therefore H is a subgroup. $\square$

Lemma 7. — *For all $g \in G$, the conjugacy map $H \mapsto gHg^{-1}$ from* $\mathrm{Sub}(G)$ *to itself is continuous.*

Proof. Denote by $f : \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ the map $H \mapsto gHg^{-1}$. Then f is continuous if and only if for all $k \in G$, the map $p_k \circ f$ is continuous. But notice that for all $H \in \mathrm{Sub}(G)$, we have $p_k \circ f = p_{g^{-1}kg}$, which is continuous by definition of the product topology. □

Exercise 8. — Let $H \in \mathrm{Sub}(G)$ and $(H_n)_{n\geq 0} \in \mathrm{Sub}(G)^{\mathbb{N}}$. Prove that $H_n \underset{n\to+\infty}{\longrightarrow} H$ if and only if $\liminf_{n\to+\infty} H_n = \limsup_{n\to+\infty} H_n = H$.

We can now state a characterization of boomerang subgroups in terms of recurrent points.

Lemma 9. — *A subgroup H of G is a boomerang subgroup if and only if for all group element $g \in G$, the subgroup H is a recurrent point for the conjugacy map* $\mathrm{Sub}(G) \to \mathrm{Sub}(G)$ *given by $K \mapsto gKg^{-1}$.*

3 Measurable actions of $\mathrm{SL}_N(\mathbb{Z})$ preserving a probability measure

Lemma 10. — *Let G be a group, X a compact metric space and $\mathcal{B}$ its Borel σ-algebra. If $G \curvearrowright (X, \mathcal{B})$ is a measurable action, then the map* $\mathrm{Stab} : X \to \mathrm{Sub}(G)$ *defined by* $\mathrm{Stab}(x) = G_x$ *is measurable.*

Proof. By definition of the product σ-algebra, Stab is measurable if and only if for all $g \in G$, the map $p_g \circ \mathrm{Stab}$ is measurable. Fix $g \in G$ and notice that $p_g \circ \mathrm{Stab}^{-1}(\{1\}) = \{x \in X : g \cdot x = x\}$. So we need to show that the latter is measurable. Since the action is measurable, the map $f : X \to X \times X$ given by $f(x) = (x, g \cdot x)$ is measurable. Since X is a compact metric space, the set $\Delta = \{(x,x) : x \in X\}$ is measurable in $X \times X$ for the product σ-algebra (beware that this is not true in general, but this holds when X is a compact metric space endowed with its Borel σ-algebra). We finally have

$$\{x \in X : g \cdot x = x\} = f^{-1}(\Delta),$$

which is measurable. □

We are now ready to state and prove that measurable actions of $\mathrm{SL}_N(\mathbb{Z})$ that preserves some ergodic probability measure are very rigid in some sense. The result below was first proved by Stuck-Zimmer and is sometimes refered to as the Nevo-Stuck-Zimmer's theorem, because it uses in a crutial way a powerful result due to Nevo-Zimmer.

Theorem 11 ([SZ94]). — *Let $N \geq 3$. Let X be a compact metric and $\mathcal{B}$ its Borel σ-algebra. Let $\mathrm{SL}_N(\mathbb{Z}) \curvearrowright (X, \mathcal{B})$ be a measurable action which preserves a probability measure μ on X. If the action is ergodic, then*

– *either* $\mathrm{supp}(\mu)$ *is finite and the action is $G \curvearrowright \mathrm{supp}(\mu)$ is transitive,*

– *or there exists* $H \leq \{\pm I_N\}$ *such that the stabilizer* $G_x = \{g \in G \colon g \cdot x = x\}$ *of* μ*-a.e.* $x \in X$ *is equal to* H*.*

Proof. The map Stab is measurable, so one can consider the pushforward measure $\nu = \mathrm{Stab}_*\mu$ on $\mathrm{Sub}(G)$. Notice that the action by conjugation $\mathrm{SL}_N(\mathbb{Z}) \curvearrowright \mathrm{Sub}(\mathrm{SL}_N(\mathbb{Z}))$ is a measurable action by Lemma 7. Moreover, it preserves the probability measure ν: indeed, given A a measurable subset of $\mathrm{Sub}(G)$ and $g \in G$, we have

$$\begin{aligned}
\nu(g^{-1} \cdot A) &= \mu(\{x \in X \colon G_x \in g \cdot A)\\
&= \mu(\{x \in X \colon G_{g \cdot x} \in A\})\\
&= \mu(g^{-1} \cdot \{x \in X \colon G_x \in A\})\\
&= \nu(A)
\end{aligned}$$

One can now use the topological version of Poincaré recurrence theorem: for all $g \in G$, ν-a.e. point $H \in \mathrm{Sub}(\mathrm{SL}_N(\mathbb{Z}))$ is a recurrent point for the conjugacy map $K \mapsto gKg^{-1}$. Since G is countable, we deduce using Lemma 9 that ν-a.e. point $H \in \mathrm{Sub}(\mathrm{SL}_N(\mathbb{Z}))$ is a boomerang subgroup. By the Main theorem and using the definition of ν, we deduce that for μ-a.e. point $x \in X$, the stabilizer G_x is either of finite index, or a subgroup of $\{\pm I_N\}$. But the measurable set $A = \{x \in X \colon G_x \text{ has finite index}\}$ satisfies $g \cdot A = A$ for all $g \in \mathrm{SL}_N(\mathbb{Z})$. By ergodicity, $\mu(A) = 0$ or $\mu(A) = 1$. The first case exactly says that $\mathrm{supp}(\mu)$ is finite and that the action $G \curvearrowright \mathrm{supp}(\mu)$ is transitive, whereas the second case says that the set $\{x \in X \colon G_x \leq \{\pm I_N\}\}$ has μ-measure one. □

Part III : The case of $\mathrm{SL}_2(\mathbb{Z})$

Given a group G and a subset $E \subseteq G$, we denote by $\langle E \rangle$ the smallest subgroup of G that contains E. We denote it by $G = \langle E \rangle$. Notice that the set

$$\{g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n} \colon n \geq 0, g_i \in E, \varepsilon_i \in \{\pm 1\}\}$$

is indeed a subgroup of G, which is equal to $\langle E \rangle$. Therefore, any element $g \in \langle E \rangle$ can be written as a product $g = g_1 \ldots g_n$ with $g_1, \ldots, g_n \in E$. We say that G is generated by E if $G = \langle E \rangle$. We define two matrices of $\mathrm{SL}_2(\mathbb{Z})$

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ et } T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Notice that $S^2 = T^3 = -I_2$ (S for “second”, T for “third”).

Lemma 1. — *The group* $\mathrm{SL}_2(\mathbb{Z})$ *is generated by* S *and* T*.*

Proof. We know that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two transvections

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(L for "left" and R right "right"). In order to prove that $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T\rangle$, we need to write L and R as a product of powers of L and R. One can check that $L = (TS)^{-1}$ and $R = ST$. $\square$

If G is a group, we denote by G' the ***derived subgroup*** of G, which is the group generated by all the commutators $[g, h] = g^{-1}h^{-1}gh$ for $g, h \in G$. Notice that this is a normal subgroup: if $[g_1, h_1] \dots [g_n, h_n] \in G'$ and $g \in G$, then

$$g[g_1, h_1] \dots [g_n, h_n]g^{-1} = [gg_1g^{-1}, gh_1g^{-1}] \dots [gg_ng^{-1}, gh_ng^{-1}] \in G'.$$

Exercise 2. — Let G be a group and E a subset of G, such that $G = \langle E\rangle$. Let F be a subset of G and let $H = \langle E\rangle$. Prove that H is normal in G if and only if for all $g \in E$, $h \in F$, we have $ghg^{-1} \in H$ and $g^{-1}hg \in H$.

Lemma 3. — *The derived subgroup* $\mathrm{SL}_2(\mathbb{Z})'$ *is generated by the two matrices*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Proof. One can check that $[S, T] = LR = A$ and $[S^{-1}, T^{-1}] = RL = B$. Therefore, we have $\langle A, B\rangle \leq \mathrm{SL}_2(\mathbb{Z})'$. In order to prove the other inclusion, we first prove that $\langle A, B\rangle$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Since we know that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, it by Exercise 2 to check the following equalities:

$$\begin{aligned} SAS^{-1} &= A^{-1} & SBS^{-1} &= B^{-1} \\ S^{-1}AS &= A^{-1} & S^{-1}BS &= B^{-1} \\ TAT^{-1} &= B^{-1} & TBT^{-1} &= B^{-1}A \\ T^{-1}AT &= A^{-1}B & T^{-1}BT &= A^{-1} \end{aligned}$$

Therefore $\langle A, B\rangle$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. Moreover, we have the identity $SR^3 = A^{-1}B$, which implies that S can be written as a word in A, B and R. Since L can also be written as a word in A and R by the formula $LR = A$, we deduce that $\mathrm{SL}_2(\mathbb{Z}) = \langle A, B, R\rangle$. Thus, the quotient group $\mathrm{SL}_2(\mathbb{Z})/\langle A, B\rangle$ is a cyclic group, generated by the image of R. This is an abelian group, thus $\mathrm{SL}_2(\mathbb{Z})' \leq \langle A, B\rangle$. $\square$

Remark. — For $N \geq 3$, the group $\mathrm{SL}_N(\mathbb{Z})$ is perfect : it is equal to its derived subgroup.

Exercise 4. — Compute $[A, B^{-1}]$. Deduce that the index of $\mathrm{SL}_2(\mathbb{Z})'$ in $\mathrm{SL}_2(\mathbb{Z})$ is at least 12. One could prove that we indeed have $[\mathrm{SL}_2(\mathbb{Z}) : \mathrm{SL}_2(\mathbb{Z})'] = 12$.

Lemma 5. — *The group* $\mathrm{SL}_2(\mathbb{Z})'$ *is a* ***free group****, which is* ***freely generated*** *by* A *et* B. *We denote it by* $\mathrm{SL}_2(\mathbb{Z})' = \langle A\rangle * \langle B\rangle$. *This means that* $\mathrm{SL}_2(\mathbb{Z})'$ *is generated by* A *and* B *and that for all integers* $n \geq 0$ *and all finite sequence of integers* $a_0, a_1, \dots, a_n$ *and* $b_1, \dots, b_n$ *all non null, except possibly* a_0 *or* a_n, *the matrix* $A^{a_0}B^{b_1}A^{a_1} \dots B^{b_n}A^{a_n}$ *is not equal to* I_2.

Proof. We will use the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \hat{\mathbb{R}}$ from Example 1. We denote by $A : x \mapsto (2x+1)/(x+1)$ and $B : (x+1)/(x+2)$ the bijections of $\hat{\mathbb{R}}$ corresponding to the matrices A and B. One checks that:

- the fixed points of A are $(1-\sqrt{5})/2$ and $(1+\sqrt{5})/2$ and are respectively repulsive and attractive,
- the fixed points of B are $(-1-\sqrt{5})/2$ and $(-1+\sqrt{5})/2$ and are respectively repulsive and attractive.

We define the following two sets:

$$
\begin{aligned}
X &= (\infty, -1) \sqcup (0, 1), \\
Y &= (-1, 0) \sqcup (1, \infty).
\end{aligned}
$$

One checks that for all $n \in \mathbb{Z} \setminus \{0\}$,

$$
\begin{aligned}
A^n(X) &\subseteq Y \\
B^n(Y) &\subseteq X.
\end{aligned}
$$

We now apply a standard method to prove that $\mathrm{SL}_2(\mathbb{Z})'$ is a free group, freely generated

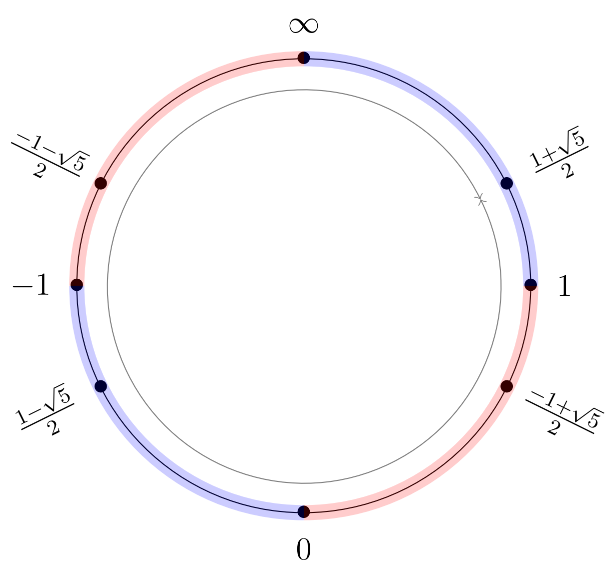

Figure 1: The sets X (red) and Y (blue).

by A and B: ping-pong lemma. Fix $a_0, a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ all non null, except possibly a_0 or a_n and let M be the matrix $A^{a_0}B^{b_1}A^{a_1}\ldots B^{b_n}A^{a_n}$. There are four cases.

1. If a_0 and a_n are non null, then for all $x \in X$, we have

$$\begin{aligned} A^{a_n}(x) &\in Y \\ B^{b_n}A^{a_n}(x) &\in X \\ &\dots \\ A^{a_0}B^{b_1}A^{a_1}\dots B^{b_n}A^{a_n}(x) &\in Y \end{aligned}$$

   Since $M(x) \in Y$ and $x \in X$, we deduce that $M(x) \neq x$. Thus $M \neq I_2$.

2. If a_0 is null and a_n is non null. We apply the first case to the matrix $A^{-a_n}MA^{a_n}$. It implies that $A^{-a_n}MA^{a_n} \neq I_2$, and thus $M \neq I_2$.

3. If a_0 is non null and a_n is null. We apply the first case to the matrix $A^{a_0}MA^{-a_0}$. It implies that $M \neq I_2$.

4. If a_0 and a_n are null, then we apply the first case to the matrix AMA^{-1}. Again, it implies that $M \neq I_2$. □

Lemma 6. — *Any matrix $M \in \mathrm{SL}_2(\mathbb{Z})'$ can be uniquely written as a product $M = A^{a_0}B^{b_1}A^{a_1}\dots B^{b_n}A^{a_n}$ where $a_0, a_1, \dots, a_n$ and $b_1, \dots, b_n$ are two finite sequences of integers, which are all non null except possibly a_0 and a_n.*

Proof. The existence of such a product decomposition is a consequence of Lemma 3 whereas the uniqueness is a consequence of Lemma 5 □

Given $M = A^{a_0}B^{b_1}A^{a_1}\dots B^{b_n}A^{a_n}$, we define the ***syllabic norm*** of M and denote it by $|M|_{syl}$ as follows:

$$|M|_{syl} = \begin{cases} 2n-1 & \text{if } a_0 \text{ and } a_n \text{ are null} \\ 2n+1 & \text{if } a_0 \text{ and } a_n \text{ are non-null} \\ 2n & \text{else.} \end{cases}$$

As a direct consequence of Lemma 6, the map $\varphi : \mathrm{SL}_2(\mathbb{Z})' \to \mathbb{Z}^2$ defined by

$$\varphi(A^{a_0}B^{b_1}A^{a_1}\dots B^{b_n}A^{a_n}) = (a_0 + \dots + a_n, b_1 + \dots + b_n).$$

is a well-defined group homomorphism.

Lemma 7. — *The kernel of the group homomorphism $\varphi : \mathrm{SL}_2(\mathbb{Z})' \to \mathbb{Z}^2$ is equal to $\mathrm{SL}_2(\mathbb{Z})''$, the second derived subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. First of all, for all $g, h \in \mathrm{SL}_2(\mathbb{Z})'$, we have $\varphi(g)\varphi(h) = \varphi(h)\varphi(g)$. In other words, $[g,h] \in \ker(\varphi)$ and thus $\mathrm{SL}_2(\mathbb{Z})'' \subseteq \ker(\varphi)$. Let us prove the reverse inclusion by induction on the syllabic norm. First, notice that any matrix $M \in \mathrm{SL}_2(\mathbb{Z})'$ whose syllabic norm is ≥ 3 is not in the kernel of φ. So for this initial step, take $M \in \ker(\varphi)$ whose syllabic norm is $|M|_{syl} = 4$. Then either $M = A^{a_0}B^{b_1}A^{a_1}B^{b_2}$ or $M = B^{b_1}A^{a_1}B^{b_2}A^{a_2}$. In the first case, we get that $a_0 + a_1 = b_1 + b_2 = 0$ and therefore $M =$

$[A^{a_0}, B^{b_1}] \in \mathrm{SL}_2(\mathbb{Z})''$. In the second case, we get that $a_1 + a_2 = b_1 + b_2 = 0$ and therefore $M = [A^{a_1}, B^{b_1}] \in \mathrm{SL}_2(\mathbb{Z})''$. For the inductive step, fix $m \geq 4$ and assume that any $M \in \ker(\varphi)$ with syllabic norm $|M|_{syl} \leq m$ belongs to $\mathrm{SL}_2(\mathbb{Z})''$. Fix $M \in \ker(\varphi)$ whose syllabic norm is $|M|_{syl} = m + 1$. Write $M = A^{a_0} B^{b_1} A^{a_1} \dots B^{b_n} A^{a_n}$. We distinguish two cases. If $a_0 \neq 0$, then the syllabic norm of $M' = B^{b_1} A^{a_0 + a_1} B^{b_2} A^{a_2} \dots B^{b_n} A^{a_n}$ is $\leq m$, and $M' \in \ker(\varphi)$ so $M' \in \mathrm{SL}_2(\mathbb{Z})''$ by the induction hypothesis. But $M = [A^{a_0}, B^{b_1}]M'$ and thus $M \in \mathrm{SL}_2(\mathbb{Z})''$. If $a_0 = 0$, then write $M = [B^{b_1}, A^{a_1}]M'$ with $|M|_{syl} \leq m$ and use the same argument. □

We can now provide normal subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which are neither finite, nor of finite index.

LEMMA 8. — *The second derived subgroup* $\mathrm{SL}_2(\mathbb{Z})''$ *is an infinite normal subgroup, of infinite index.*

Proof. Let us prove that $\mathrm{SL}_2(\mathbb{Z})''$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. Fix $g \in \mathrm{SL}_2(\mathbb{Z})$ and $g_1, \dots, g_n$, $h_1, \dots, h_n \in \mathrm{SL}_2(\mathbb{Z})'$. Then

$$g[g_1, h_1] \dots [g_n, h_n]g^{-1} = [gg_1g^{-1}, gh_1g^{-1}] \dots [gg_ng^{-1}, gh_ng^{-1}]$$

But $\mathrm{SL}_2(\mathbb{Z})'$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, so for all $1 \leq i \leq n$, the elements gg_ig^{-1} and gh_ig^{-1} belongs to $\mathrm{SL}_2(\mathbb{Z})'$. This shows that the second derived subgroup $\mathrm{SL}_2(\mathbb{Z})''$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. It is infinite as the elements $[A^k, B^l]$ for $k, l \in \mathbb{Z}$ form a family of pairwise disjoint (by Lemma 6) elements of $\mathrm{SL}_2(\mathbb{Z})''$. Finally, it has infinite index because Lemma 7 implies for instance that the cosets $A^k\mathrm{SL}_2(\mathbb{Z})''$ for $k \in \mathbb{Z}$ are pairwise disjoint. □

REMARK. — One can use a more general argument to prove that the second derived subgroup $\mathrm{SL}_2(\mathbb{Z})''$ is normal in $\mathrm{SL}_2(\mathbb{Z})$: the derived subgroup G' of a group G is always a characteristic subgroup (that is for any group automorphism $\psi : G \to G$, $\psi(G')$ is a subgroup of G') and use the fact that a normal subgroup $K \trianglelefteq H$ of a characteristic subgroup $H \leq G$ is normal in G.

References

[BLS64] H. Bass, M. Lazard, and Jean-Pierre Serre. "Subgroups of finite index in $\mathrm{SL}(n, \mathbb{Z})$." French. In: *Bull. Am. Math. Soc.* 70 (1964), pp. 385–392. ISSN: 0002-9904. DOI: `10.1090/S0002-9904-1964-11107-1` (cit. on p. 3).

[Ben21] Yves Benoist. "Arithmeticity of discrete subgroups". English. In: *Ergodic Theory Dyn. Syst.* 41.9 (2021), pp. 2561–2590. ISSN: 0143-3857 (cit. on p. 5).

[GL22] Yair Glasner and Waltraud Lederle. *Strong subgroup recurrence and the Nevo-Stuck-Zimmer theorem.* 2022. arXiv: `2208.14758 [math.GR]` (cit. on pp. 1–3).

[Men65] J. L. Mennicke. "Finite factor groups of the unimodular group". English. In: *Ann. Math. (2)* 81 (1965), pp. 31–37. ISSN: 0003-486X. DOI: 10.2307/1970380 (cit. on p. 3).

[SZ94] Garrett Stuck and Robert J. Zimmer. "Stabilizers for Ergodic Actions of Higher Rank Semisimple Groups". In: *Annals of Mathematics* 139.3 (1994), pp. 723–747. (Visited on 08/11/2023) (cit. on p. 11).

[Tit76] Jacques Tits. "Systemes générateurs de groupes de congruence". French. In: *C. R. Acad. Sci., Paris, Sér. A* 283 (1976), pp. 693–695. ISSN: 0366-6034 (cit. on p. 5).

[Vas73] L N Vaseršteĭn. "THE STRUCTURE OF CLASSICAL ARITHMETIC GROUPS OF RANK GREATER THAN ONE". In: *Mathematics of the USSR-Sbornik 1973-apr 30 vol. 20 iss. 3* 20 (3 Apr. 1973). DOI: 10.1070/sm1973v020n03abeh001885 (cit. on p. 5).