Numéro National de Thèse : 2022LYSEN021

THÈSE DE DOCTORAT DE L'UNIVERSITÉ DE LYON

opérée par

l'École Normale Supérieure de Lyon

École Doctorale n°512
École Doctorale en Informatique et Mathématiques de Lyon

Discipline : Mathématiques

Soutenue publiquement le 16/06/2022, par :

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Dynamique topologique et mesurée : allostérie, équivalence orbitale quantitative

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"Tout ça n'aurait jamais pu fleurir si cela n'avait pas été planté sur un terreau merveilleusement fertile."

Clément Viktorovitch

Remerciements

Je voudrais exprimer ma sincère gratitude à Damien Gaboriau, mon directeur de thèse. Merci de m'avoir partagé tant de belles mathématiques, d'avoir orienté mes recherches de manière décisive durant ma thèse, tout en me laissant une liberté considérable. Je te remercie d'avoir été à l'initiative d'une multitude de rencontres mathématiques qui m'ont permis de me sentir pleinement intégré dans notre communauté mathématique.

I am thankful to the referees Yair Glasner and Roman Sauer, who accepted to review this thesis. Your valuable feedback helped me to improve this thesis. Je remercie aussi Nathalie Aubrun, María Isabel Cortez, Tatiana Nagnibeda, Mikael de la Salle et Romain Tessera d'avoir accepté de faire partie du jury de cette thèse.

J'ai la chance de travailler avec des personnes particulièrement stimulantes. François et Alessandro, c'est un réel plaisir de pouvoir travailler avec ses "grands frères". Romain, je te remercie pour ton mini-cours au CIRM à l'automne 2019 : il m'a ouvert les yeux sur la beauté de la théorie mesurée *quantitative* des groupes. Andy, Colin et Gianluca, je garde un souvenir très plaisant de nos discussions mathématiques en ligne au gré des confinements successifs. Colin, les recherches que nous menons ensemble sont très motivantes ! Amandine, j'espère que nous parviendrons à avancer sur cette question de couplage optimal.

L'UMPA m'a fourni un cadre de travail idéal ces dernières années. C'est grâce à l'ensemble de ses membres que l'atmosphère y est aussi chaleureuse. Je tiens à remercier particulièrement les non-permanents du laboratoire avec qui j'ai partagé beaucoup de moments conviviaux. Le groupe de travail « Actions ! » est un formidable rendez-vous pour les amoureux des groupes et des actions, merci à Damien ainsi qu'à ses membres de le faire vivre. During my thesis, I spent a few months in Indiana University Bloomington. I am thankful to Russell Lyons for his warm hospitality, as well as to Homin Lee for our conversations of great interest.

Quentin, Christopher, Thomas, Colin et Marie, Camille et Lucille, Thomas et Cloé, merci les amis, pour les randonnées, les jeux de sociétés, les discussions, les Champclauses, et bien d'autres choses encore.

Merci à ma famille, qui a su me donner le goût du savoir et des sciences. L'épigraphe de cette thèse vous est dédié.

Axelle, merci pour tout le bonheur que tu m'apportes. Quant à Sirius, bienvenue à toi ✳

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Introductions

Introduction en français

La théorie des systèmes dynamiques étudie le comportement d'un espace sur lequel un groupe de transformations agit. Cette thèse se concentre sur deux branches importantes de cette théorie : la dynamique topologique et la théorie ergodique des actions de groupes. La première consiste en l'étude d'actions par homéomorphismes sur des espaces compacts. La seconde consiste en l'étude des actions de groupes sur des espaces mesurés qui admettent des mesures invariantes.

Ce manuscript est découpé en trois parties indépendantes. Dans cette introduction, qui se veut illustrative, nous expliquons les principales notions étudiées au cours de cette thèse et détaillons les principaux résultats obtenus.

Plan de la thèse :

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Partie I : Allostérie

Minimalité et ergodicité

Cette première partie mêle dynamique topologique et théorie ergodique des actions de groupes. On s'intéresse à des actions de groupes dénombrables sur des espaces compacts, pour lesquelles il existe une mesure de probabilité invariante. Plus précisément, une action ***minimale ergodique*** d'un groupe dénombrable Γ est une action $\Gamma \curvearrowright (C, \mu)$ où

- $\Gamma \curvearrowright C$ est une action par homéomorphismes sur un espace compact C, qui est minimale (toutes les orbites sont denses).

– μ est une mesure de probabilité borélienne Γ-invariante sur C, qui est ergodique (les ensembles mesurables Γ-invariants sont de mesure 0 ou 1).

Les actions minimales ergodiques apparaissent naturellement dans différents contextes. Par exemple, soit Γ un sous-groupe dense d'un groupe compact G et soit μ la mesure de probabilité de Haar sur G. Alors l'action par multiplication à gauche $\Gamma \curvearrowright (G, \mu)$ est une action minimale ergodique. Concrètement, on peut penser à une rotation irrationnelle $x \mapsto x + \theta \pmod 1$ qui fournit une action minimale ergodique de $\mathbb{Z}$ sur le tore $\mathbb{T}$ muni de la mesure de Lebesgue. Les actions profinies fournissent elles aussi des actions minimales ergodiques. Soit Γ un groupe dénombrable et soit $\Gamma \geq \Gamma_1 \geq \dots \geq \Gamma_n \geq \dots$ une suite décroissante de sous-groupes d'indice fini. Le groupe Γ agit sur chacun des ensembles finis Γ/Γ_n et donc sur la limite profinie $\varprojlim \Gamma/\Gamma_n$ des quotients Γ/Γ_n. Cette action est une action minimale lorsque l'on munit la limite profinie de la topologie profinie. Par ailleurs, il existe une unique mesure de probabilité μ sur la limite profinie qui soit invariante par cette action. Alors l'action $\Gamma \curvearrowright (\varprojlim \Gamma/\Gamma_n, \mu)$ est une action minimale ergodique qui est appelée ***action profinie***. Les groupes qui admettent des actions profinies *fidèles* sont les groupes ***résiduellement finis***, c'est-à-dire les groupes Γ qui possèdent une suite décroissante $\Gamma \geq \Gamma_1 \geq \dots \geq \Gamma_n \geq \dots$ de sous-groupes d'indice fini telle que

$$\bigcap_{n \geq 1} \Gamma_n = \{1_\Gamma\}.$$

En fait, tout groupe dénombrable admet des actions minimales ergodiques qui sont fidèles. Plus précisément, on a le résultat suivant.

THÉORÈME ([Ele21]). — *Tout groupe dénombrable Γ admet une action minimale ergodique qui est libre.*

Ici, une action est *libre* si tout point a un stabilisateur trivial. Dans la suite, ce sont plutôt les actions non-libres qui vont nous intéresser. Un des intérêts de telles actions réside dans le fait qu'elles fournissent des familles intéressantes de sous-groupes qui sont donnés par les stabilisateurs des points. De façon équivalente, lorsque le groupe est de type fini, cela fournit des familles de graphes de Schreier du groupe, qui sont des objets d'études fondamentaux en théorie spectrale des graphes. On pourra consulter l'article de survol [Gri11] qui illustre l'utilisation de telles familles de graphes de Schreier.

Notions de liberté

Plusieurs notions de liberté *générique* existent. Une action minimale ergodique $\Gamma \curvearrowright (C, \mu)$ est

– ***topologiquement libre*** si l'ensemble des points de C dont le stabilisateur est trivial est G_δ dense.

– ***essentiellement libre*** si l'ensemble des points de C dont le stabilisateur est trivial est de mesure pleine.

Ces deux propriétés sont des propriétés *génériques*, la première du point de vue topologique, la seconde dans un sens mesuré. En effet, un point d'un ensemble G_δ dense est un point générique au sens topologique et un point d'un ensemble de mesure pleine est un point générique du point de vue de la mesure. Ces deux notions sont reliées dans le lemme qui suit. Énonçons tout d'abord une observation utile.

FAIT. — *Soit $\Gamma \curvearrowright C$ une action par homéomorphismes sur un compact. Alors l'ensemble des $x \in C$ dont le stabilisateur est trivial est un ensemble G_δ.*

Preuve. Soit $x \in C$. Alors $\mathrm{Stab}(x) = \{1\}$ si et seulement si pour tout $\gamma \in \Gamma \setminus \{1\}$, on a $\gamma x \neq x$, ce qui termine la preuve. □

Ainsi, pour qu'une action minimale soit topologiquement libre, il suffit qu'il existe un point dont le stabilisateur est trivial.

LEMME. — *Une action minimale ergodique essentiellement libre est topologiquement libre.*

Preuve. Soit $\Gamma \curvearrowright (C, \mu)$ une action minimale ergodique essentiellement libre. On sait déjà que l'ensemble des points dont le stabilisateur est trivial est un ensemble G_δ. Puisque c'est aussi un ensemble de mesure pleine, il est en particulier non vide. Donc il existe $x \in C$ tel que $\mathrm{Stab}(x)$ est trivial, ce qui conclut la preuve par minimalité de l'action. □

La réciproque est fausse en général, ce qui nous amène à introduire la notion d'allostérie.

Allostérie

DÉFINITION. — Soit Γ un groupe dénombrable. Une action minimale ergodique de Γ est ***allostérique***[1] si elle est topologiquement libre mais pas essentiellement libre. Un groupe Γ est ***allostérique*** s'il admet une action minimale ergodique qui est allostérique.

Exemples de groupes non-allostériques

La notion d'allostérie est intimement liée à la dynamique sur l'espace des sous-groupes. Si Γ est un groupe dénombrable, on note $\mathrm{Sub}(\Gamma)$ l'espace des sous-groupes de Γ, sur lequel Γ agit par conjugaison. Les résultats de ce paragraphe illustrent la maxime suivante : si l'action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ n'est pas suffisamment riche, alors Γ n'est pas allostérique.

PROPOSITION. — *Le groupe $\mathbb{Z}$ n'est pas allostérique.*

[1] ἄλλος : autre, στερεός : exprime une idée de fixité.

Preuve. Soit $\mathbb{Z} \curvearrowright (C, \mu)$ une action minimale ergodique qui n'est pas essentiellement libre. Par définition

$$\mu(\{x \in C \colon \mathrm{Stab}(x) \neq \{0\}\}) > 0.$$

En particulier, on peut trouver un point $x \in C$ dont le stabilisateur n'est pas trivial. Puisque ce dernier est un sous-groupe non-trivial de $\mathbb{Z}$, il s'agit d'un $n\mathbb{Z}$ pour un certain entier $n \geq 1$ et donc le cardinal de l'orbite d'un tel point x est fini. Par minimalité de l'action, on en déduit que $C = \mathrm{Orb}(x)$ est fini et que le stabilisateur d'aucun point de C n'est trivial. L'action $\mathbb{Z} \curvearrowright (C, \mu)$ n'est donc pas topologiquement libre. □

Proposition. — *Si* $\mathrm{Sub}(\Gamma)$ *est dénombrable, alors* Γ *n'est pas allostérique.*

Preuve. Soit Γ un tel groupe. Soit $\Gamma \curvearrowright (C, \mu)$ une action minimale ergodique qui n'est pas essentiellement libre. Alors $\mu(\{x \in C \colon \mathrm{Stab}(x) \neq \{1_\Gamma\}\}) > 0$. Pour tout sous-groupe $\Lambda \leq \Gamma$, notons

$$P_\Lambda \coloneqq \{x \in C \colon \mathrm{Stab}(x) = \Lambda\}.$$

La famille des P_Λ pour Λ sous-groupe de Γ forme une partition de C. Par hypothèse sur Γ, cette partition n'admet qu'un nombre dénombrable de pièces. Il existe alors un sous-groupe $\Lambda \leq \Gamma$ tel que $\mu(P_\Lambda) > 0$. Par ailleurs, pour tout $\gamma \in \Gamma$, on a $\gamma P_\Lambda = P_{\gamma\Lambda\gamma^{-1}}$, ce qui implique que $\mu(P_\Lambda) = \mu(P_{\gamma\Lambda\gamma^{-1}})$. Donc Λ ne possède qu'un nombre fini de conjugués.

Soit $x \in C$. Par minimalité de l'action et puisque Λ n'admet qu'un nombre fini de conjugués, il existe $\gamma \in \Gamma$ et une suite $(x_n)_{n\geq 0} \in C^{\mathbb{N}}$ tels que $x_n \to x$ et $\mathrm{Stab}(x_n) = \gamma\Lambda\gamma^{-1}$ pour tout $n \geq 0$. On en déduit que $\gamma\Lambda\gamma^{-1} \leq \mathrm{Stab}(x)$. Ainsi, le stabilisateur d'aucun point de C n'est trivial. Donc l'action n'est pas topologiquement libre, ce qui termine la preuve. □

On trouvera dans la Figure 1.2 des exemples explicites de groupes dont l'ensemble des sous-groupes est dénombrable. On peut démontrer un résultat similaire pour les groupes qui ne possèdent pas suffisamment de sous-groupes aléatoires invariants. Soit Γ un groupe dénombrable et soit $\mathrm{Sub}(\Gamma)$ l'espace des sous-groupes de Γ. C'est un espace compact lorsqu'on l'identifie à un sous-ensemble de $\{0,1\}^\Gamma$, sur lequel Γ agit par conjugaison. Un ***sous-groupe aléatoire invariant (SAI)*** de Γ est une mesure de probabilité Γ-invariante sur $\mathrm{Sub}(\Gamma)$. On peut adapter la preuve ci-dessus pour démontrer qu'un groupe dont les SAI ergodiques sont tous atomiques n'est pas allostérique. On trouvera des exemples concrets de tels groupes dans la Figure 1.2.

Sous-groupes aléatoires invariants / Sous-groupes uniformésments récurrents

Dans ce paragraphe, nous expliquons plus en détail les liens entre l'allostérie et la dynamique sur l'espace des sous-groupes.

Soit $\Gamma \curvearrowright (C, \mu)$ une action minimale ergodique. L'application mesurable

$$\mathrm{Stab} : x \mapsto \{\gamma \in \Gamma \colon \gamma \cdot x = x\}$$

associe à tout point de C un sous-groupe de Γ. Par ailleurs Stab est équivariante puisque pour tout $x \in C$ et $\gamma \in \Gamma$, on a

$$\mathrm{Stab}(\gamma \cdot x) = \gamma \mathrm{Stab}(x) \gamma^{-1}.$$

Ainsi, la mesure image de μ par Stab fournit une mesure de probabilité ergodique Γ-invariante sur $\mathrm{Sub}(\Gamma)$, c'est-à-dire un SAI ergodique. Par définition, l'action $\Gamma \curvearrowright (C, \mu)$ est essentiellement libre si et seulement si ce SAI ergodique est égal à la mesure $\delta_{\{1_\Gamma\}}$.

Les sous-groupes aléatoires invariants admettent un pendant topologique. Un ***sous-groupe uniformément récurrent (SUR)*** est un sous-ensemble fermé, minimal et Γ-invariant de $\mathrm{Sub}(\Gamma)$. À toute action minimale ergodique $\Gamma \curvearrowright (C, \mu)$, on peut associer un SUR, défini comme l'unique SUR inclus dans la clôture de l'ensemble $\{\mathrm{Stab}(x) \colon x \in C\}$, voir [GW15]. On peut montrer que ce SUR est inclus dans le support du SAI associé à l'action [Jos21, Lem. 2.2]. Par ailleurs, ce SUR est égal à $\{\{1_\Gamma\}\}$ si et seulement si l'action est topologiquement libre [LBMB18, Prop. 2.7]. On obtient ainsi la caractérisation suivante de l'allostérie.

Lemme. — *Une action minimale ergodique $\Gamma \curvearrowright (C, \mu)$ est allostérique si et seulement si le groupe trivial $\{1_\Gamma\}$ est strictement contenu dans le support du SAI associé à cette action.*

Exemples de groupes allostériques

L'existence de groupes allostériques a été posée par Grigorchuk, Nekrashevich et Suschanskii. Plus précisément, ils posent la question suivante.

Question ([GNS00, Prob. 7.3.3]). — Existe-t il un groupe dénombrable Γ qui admet des actions profinies allostériques ?

Les premiers exemples de groupes allostériques ont été découverts par Bergeron et Gaboriau. Ils répondent de plus à la question de [GNS00].

Théorème ([BG04]). — *Soient Γ et Λ deux groupes non-triviaux qui sont résiduellement finis. Alors le produit libre $\Gamma * \Lambda$ est allostérique (sauf si Γ et Λ sont isomorphes au groupe cyclique C_2). Plus précisément, le groupe $\Gamma * \Lambda$ admet des actions profinies qui sont allostériques.*

Une preuve indépendante de ce résultat pour $\Gamma = \Lambda = \mathbb{Z}$ est donnée par Abért et Elek dans [AE12]. Dans le chapitre 1, nous nous intéressons à une autre classe de groupes : les groupes fondamentaux $\pi_1(\Sigma_g)$ de surfaces fermées, connexes et orientables Σ_g. Ce sont des groupes à un relateur, dont une présentation est donnée par

$$\pi_1(\Sigma_g) := \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \dots [x_g, y_g] = 1 \rangle,$$

où $[x,y] = xyx^{-1}y^{-1}$ désigne le commutateur de x et y, et $g \geq 1$ est un entier. Le résultat principal du chapitre 1 est le théorème suivant.

THÉORÈME. — *Soit Σ_g une surface fermée, connexe et orientable de genre $g \geq 2$. Alors $\pi_1(\Sigma_g)$est allostérique. Plus précisément, il admet des actions profinies qui sont allostériques.*

Une construction d'actions allostériques pour $\mathbf{F}_2$

Pour conclure cette introduction à la partie I de cette thèse, nous proposons une construction explicite d'actions allostériques pour le groupe libre $\mathbf{F}_2$ de rang deux. Pour ce faire, on construit d'abord des actions de $\mathbf{F}_2$ sur des ensembles finis avec certaines propriétés.

Pour tout entier $n \geq 0$ et $p \geq 2$ premier, soit

$$C_{p,n} := \bigcup_{l=0}^{2n} [\![0, p-1]\!] \times [\![0, p-2]\!]^l.$$

Un point x de $C_{p,n}$ s'écrit donc sous la forme $(x^0, \ldots, x^l)$ avec $x^0 \in [\![0, p-1]\!]$ et $x^i \in [\![0, p-2]\!]$ pour tout $1 \leq i \leq l$. L'entier l s'appelle la ***profondeur*** de x. On définit une permutation A sur l'ensemble $C_{p,n}$ de la façon suivante. Soit $x \in C_{p,n}$ de profondeur l.

– Si l est pair, alors $A(x) := \begin{cases} (x^0, \ldots, x^l, 0) & \text{si } l < 2n, \\ x & \text{si } l = 2n. \end{cases}$

– Si l est impair, alors $A(x) := \begin{cases} (x^0, \ldots, x^l + 1) & \text{si } x^l < p-2, \\ (x^0, \ldots, x^{l-1}) & \text{sinon.} \end{cases}$

Puis, on définit une autre permutation B sur l'ensemble $C_{p,n}$ de la façon suivante. Soit $x \in C_{p,n}$ de profondeur l.

– Si l est impair, alors $B(x) := (x_0, \ldots, x_l, 0)$.

– Si $l = 0$, alors $B(x) := x^0 + 1 \pmod p$. Si l est pair > 0, alors

$$B(x) := \begin{cases} (x^0, \ldots, x^l + 1) & \text{si } x^l < p-2, \\ (x^0, \ldots, x^{l-1}) & \text{sinon.} \end{cases}$$

Notons a et b les deux générateurs de $\mathbf{F}_2$. Les permutations A et B définissent une action du groupe libre $\mathbf{F}_2$ sur $C_{p,n}$ en posant $a \cdot x := A(x)$ et $b \cdot x := B(x)$ pour tout $x \in C_{p,n}$. Cette action est illustrée dans la FIGURE 1.1.

Appelons ***bord*** de $C_{p,n}$ l'ensemble $\partial C_{p,n}$ des points de profondeur $2n$. Ce sont exactement les points fixes de a. Par ailleurs, un calcul élémentaire donne

$$\frac{|\partial C_{p,n}|}{|C_{p,n}|} \underset{n \to +\infty}{\longrightarrow} \frac{p-2}{p-1}.$$

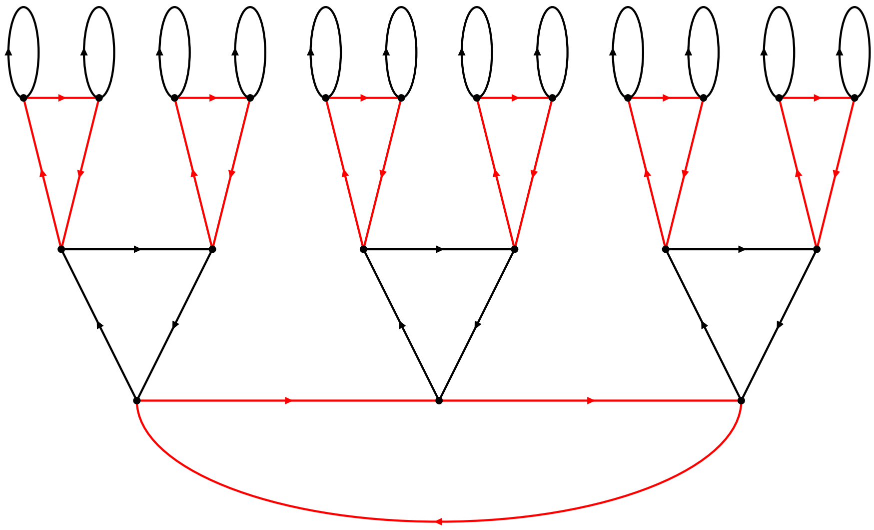

FIGURE 1.1. — L'ensemble $C_{3,1}$ et les permutations A (en noir) et B (en rouge).

Puisque $(p-2)/(p-1) \to 1$ lorsque $p \to +\infty$, on peut fixer une suite de nombres premiers deux à deux distincts $(p_k)_{k\geq 0}$ et une suite de nombres entiers $(n_k)_{k\geq 0}$ qui tend vers $+\infty$, telles que

$$\prod_{k=0}^{+\infty} \frac{|\partial C_{p_k,n_k}|}{|C_{p_k,n_k}|} > 0.$$

Muni de la topologie produit, l'espace $C := \prod_{k\geq 0} C_{p_k,n_k}$ est compact. On le munit de la mesure de probabilité μ, produit des mesures de probabilité uniformes sur chaque C_{p_k,n_k}.

PROPOSITION. — *L'action diagonale de* $\mathbf{F}_2$ *sur* C *est allostérique.*

Esquisse de preuve. Tout d'abord $\mathbf{F}_2 \curvearrowright (C,\mu)$ est une action minimale ergodique. En effet, une application du lemme des restes chinois permet de montrer que pour tout $K \geq 0$, l'action diagonale $\mathbf{F}_2 \curvearrowright \prod_{k\leq K} C_{p_k}$ est transitive. Ainsi, les cylindres de C de la forme

$$O_{(y_1,\dots,y_K)} := \{(x_k)_{k\geq 0} \in C \colon x_0 = y_0, \dots, x_K = y_K\}$$

sont tous de la même mesure, et intersectent toute orbite de l'action $\mathbf{F}_2 \curvearrowright C$. Cela implique que

– la mesure μ est l'unique mesure $\mathbf{F}_2$-invariante sur C, qui est donc ergodique,

– l'action $\mathbf{F}_2 \curvearrowright C$ est minimale.

Démontrons que $\mathbf{F}_2 \curvearrowright (C,\mu)$ est topologiquement libre, mais pas essentiellement libre. Pour tout $k \in \mathbb{N}$, fixons x_k un élément de longueur 0 dans C_{p_k,n_k} et soit $x := (x_k)_{k\geq 0} \in C$. Par construction, on peut remarquer que $\mathrm{Stab}(x) = \{1_{\mathbf{F}_2}\}$. Puisque $\mathbf{F}_2 \curvearrowright C$ est minimale, on en déduit que l'ensemble des points de C avec stabilisateur trivial est dense dans X. Puisque c'est un ensemble G_δ, on en déduit que $\mathbf{F}_2 \curvearrowright C$ est topologiquement libre. Par ailleurs, par définition de la mesure μ, on a

$$\mu(\{x \in C \colon a \in \mathrm{Stab}(x)\}) = \prod_{k=0}^{+\infty} \frac{|\partial C_{p_k,n_k}|}{|C_{p_k,n_k}|} > 0.$$

Donc $\mathbf{F}_2 \curvearrowright (C, \mu)$ n'est pas essentiellement libre, ce qui implique que c'est une action allostérique. □

Quelques groupes allostériques/non-allostériques

Groupe	Allostérie	Raison ou Référence
$\Gamma * \Lambda$ ($\neq C_2 * C_2$) avec $\Gamma, \Lambda \neq \{1\}$ et résiduellement fini	✓	[BG04]
$\pi_1(\Sigma_g)$, $g \geq 2$	✓	[Jos21]
$\mathbb{Z} \wr \mathbb{Z}$	✓	Travail en cours
Γ nilpotent de type fini	✗	$\lvert\mathrm{Sub}(\Gamma)\rvert = \aleph_0$
$\mathrm{BS}(1, n)$	✗	$\lvert\mathrm{Sub}(\mathrm{BS}(1, n))\rvert = \aleph_0$
$\mathrm{FSym}(\mathbb{N})$, groupe des permutations à support fini de $\mathbb{N}$	✗	[Ver12]
Réseaux de $\mathrm{SL}_n(\mathbb{R})$ $n \geq 3$	✗	Les SAI ergodiques sont atomiques [SZ94]
Réseaux de $\mathrm{PSL}_n(\mathbb{Q}_p)$ $n \geq 2$, p premier	✗	Les SAI ergodiques sont atomiques [PT16]

FIGURE 1.2. — Une liste de quelques groupes allostériques et non-allostériques. Cette liste n'est pas exhaustive. En particulier, les deux dernières lignes restent valides en plus grande généralité.

Partie II : Équivalence orbitale quantitative des actions de $\mathbb{Z}$

Équivalence orbitale

Le sujet principal de cette seconde partie est celui de l'équivalence orbitale des actions de groupes, avec un accent sur les actions de $\mathbb{Z}$. Soit Γ un groupe dénombrable. Une action p.m.p. $\Gamma \curvearrowright (X,\mu)$ sur un espace de probabilité (X,μ) est une action par bijections bimesurables sur X qui préservent la mesure μ. L'action de $\gamma \in \Gamma$ sur $x \in X$ est alors notée $\gamma \cdot x$. Lorsqu'il est nécessaire de donner un nom à l'action, on écrira $\Gamma \curvearrowright^{\alpha} (X,\mu)$ et l'action de $\gamma \in \Gamma$ sur $x \in X$ est notée $\alpha(\gamma)x$.

Soient Γ et Λ deux groupes dénombrables. Deux actions p.m.p. $\Gamma \curvearrowright (X,\mu)$ et $\Lambda \curvearrowright (Y,\nu)$ sont ***orbitalement équivalentes*** s'il existe une ***équivalence orbitale***, c'est-à-dire une bijection bimesurable $\Phi : X \to Y$ telle que $\Phi_*\mu = \nu$ et pour μ-presque tout $x \in X$,

$$\Phi(\Gamma \cdot x) = \Lambda \cdot \Phi(x).$$

Exemple d'une équivalence orbitale

Soit $X := \{0,1\}^{\mathbb{N}}$ et $\mu := (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes \mathbb{N}}$. ***L'odomètre dyadique*** est la transformation $T : X \to X$ donnée par

$$T(1,\dots,1,0,x_{k+1},x_{k+2},\dots) = (0,\dots,0,1,x_{k+1},x_{k+2},\dots)$$

et $T(1,1,\dots) = (0,0,\dots)$. Cette transformation est une bijection bimesurable qui préserve la mesure μ. Ainsi, les itérés de T fournissent une action p.m.p. $\mathbb{Z} \curvearrowright (X,\mu)$. On définit deux transformations $T_{pair} : X \to X$ et $T_{impair} : X \to X$, appelées respectivement ***odomètre dyadique pair*** et ***odomètre dyadique impair***, de la façon suivante

$$T_{pair}(x_0,x_1,\dots) = (y_0,y_1,\dots),$$
$$T_{impair}(x_0,x_1,\dots) = (z_0,z_1,\dots),$$

où pour tout $n \geq 0, x_{2n+1} = y_{2n+1}, x_{2n} = z_{2n}$ et

$$(y_0,y_2,\dots) = T(x_0,x_2,\dots),$$
$$(z_1,z_3,\dots) = T(x_1,x_3,\dots).$$

Les transformations T_{pair} et T_{impair} préservent la mesure μ et commutent. Elles définissent donc une action p.m.p. $\mathbb{Z}^2 \curvearrowright (X,\mu)$.

Lemme. — *Ces deux actions $\mathbb{Z} \curvearrowright (X,\mu)$ et $\mathbb{Z}^2 \curvearrowright (X,\mu)$ sont orbitalement équivalentes*

Esquisse de preuve. Soit X' l'ensemble des suites $(x_n)_{n\geq 0} \in X$ telles que au moins l'une des sous-suites $(x_{2n})_{n\geq 0}$ ou $(x_{2n+1})_{n\geq 0}$ soit constante à partir d'un certain rang. Alors $\mu(X \setminus X') = 1$ et pour tout $x \in X \setminus X'$, les deux assertions suivantes sont vérifiées :

– pour tout $n \in \mathbb{Z}$, il existe $(u, v) \in \mathbb{Z}^2$ tel que $T^n(x) = (T_{pair})^u (T_{impair})^v(x)$.

– pour tout $(u, v) \in \mathbb{Z}^2$, il existe $n \in \mathbb{Z}$ tel que $(T_{pair})^u (T_{impair})^v(x) = T^n(x)$.

Cela signifie que id_X est une équivalence orbitale entre ces deux actions. □

Si à la place de la partition $\{pair\} \sqcup \{impair\}$, on partitionne $\mathbb{N}$ en fonction du reste modulo $n, m \geq 1$, on obtient de manière similaire une équivalence orbitale entre deux actions ergodiques $\mathbb{Z}^n \curvearrowright (X, \mu)$ et $\mathbb{Z}^m \curvearrowright (X, \mu)$.

Équivalence orbitale et invariants

Dans l'exemple précédent, on a démontré que pour tout entier $n, m \geq 1$, les groupes $\mathbb{Z}^n$ et $\mathbb{Z}^m$ admettent des actions p.m.p. ergodiques qui sont orbitalement équivalentes. En réalité, un résultat bien plus fort est vrai.

Théorème ([OW80]). — *Les actions ergodiques de groupes moyennables infinis sont toutes orbitalement équivalentes.*

Ainsi, l'équivalence orbitale ne préserve aucun des invariants géométriques que l'on peut associer aux groupes moyennables. Par exemple, la dimension d du groupe $\mathbb{Z}^d$ n'est pas préservée. Parmi les invariants géométriques, on peut citer la fonction de croissance d'un groupe, la fonction de Følner, le cône asymptotique, etc. De même, aucun invariant ergodique des actions p.m.p. n'est en général préservé par équivalence orbitale. Parmi les invariants ergodiques, on peut citer le spectre de la représentation de Koopman, l'entropie de Kolmogorov-Sinai, etc. Ce manque d'invariant d'équivalence orbitale est l'une des motivations principales à l'introduction de raffinements quantitatifs à l'équivalence orbitale.

Équivalence orbitale quantitative

Soit Γ un groupe de type fini et soit $|\cdot|_\Gamma$ la longueur des mots associée à un système fini de générateurs $S_\Gamma \subseteq \Gamma$. Étant donnée une action p.m.p. $\Gamma \curvearrowright (X, \mu)$, on peut construire une structure métrique sur (X, μ), notée d_Γ, en déclarant que la d_Γ-distance entre deux points $x, y \in X$ dans la même Γ-orbite est égale à

$$d_\Gamma(x, y) := \inf\{|\gamma|_\Gamma \colon \gamma \cdot x = y\}.$$

Cela fournit une L'un des objectifs principaux de l'équivalence orbitale quantitative est de comprendre comment une équivalence orbitale entre deux actions p.m.p. peut *distordre* les structures métriques associées. Plus précisément, soit Γ et Λ deux groupes de type fini et soit $|\cdot|_\Gamma$ et $|\cdot|_\Lambda$ les longueurs des mots associées à un système fini de générateurs de Γ et Λ respectivement.

DÉFINITION. — Soit $p \in]0, +\infty]$. Une équivalence orbitale $\Phi : (X, \mu) \to (Y, \nu)$ entre deux actions p.m.p. $\Gamma \curvearrowright (X, \mu)$ et $\Lambda \curvearrowright (Y, \nu)$ est une ***équivalence orbitale*** $\mathbf{L}^p$ si pour tout $\gamma \in \Gamma$ et $\lambda \in \Lambda$, les applications

$$x \mapsto d_\Lambda(\Phi(x), \Phi(\gamma \cdot x)) \quad \text{et} \quad y \mapsto d_\Gamma(\Phi^{-1}(y), \Phi^{-1}(\lambda \cdot y))$$

appartiennent respectivement à $\mathrm{L}^p(X, \mu)$ et $\mathrm{L}^p(Y, \nu)$.

Par inégalité triangulaire, il suffit de vérifier ces conditions pour γ et λ appartenant respectivement à un système fini de générateurs de Γ et de Λ. Remarquons de plus que si l'on change les systèmes finis de générateurs, les nouvelles structures métriques obtenues restent bi-lipschitziennes aux anciennes structures métriques, ce qui ne change par le caractère L^p d'une équivalence orbitale. Ainsi, la notion d'équivalence orbitale L^p est indépendante du choix des systèmes finis de générateurs.

On dit que deux actions p.m.p. $\Gamma \curvearrowright (X, \mu)$ et $\Lambda \curvearrowright (Y, \nu)$ sont $\mathbf{L}^p$ ***orbitalement équivalentes*** ($p \in]0, +\infty]$) s'il existe une équivalence orbitale L^p entre elles. Finalement, on dit que deux actions p.m.p. sont $\mathbf{L}^{<p}$ ***orbitalement équivalentes*** s'il existe une équivalence orbitale entre elles qui est une équivalence orbitale L^q pour tout $q < p$.

Exemple d'une équivalence orbitale L^p

Soit $X := \{0, 1\}^{\mathbb{N}}$ et $\mu := (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes \mathbb{N}}$. Soit $T : X \to X$ l'odomètre dyadique et T_{pair}, T_{impair} les odomètres dyadiques pair et impair. Comme expliqué précédemment, les actions $\mathbb{Z} \curvearrowright (X, \mu)$ et $\mathbb{Z}^2 \curvearrowright (X, \mu)$ induites par T et le couple (T_{pair}, T_{impair}) sont orbitalement équivalentes, $\mathrm{id}_X : X \to X$ étant une équivalence orbitale. Nous allons calculer explicitement le degré d'intégrabilité de cette équivalence orbitale.

Dans les calculs qui suivent, on munit $\mathbb{Z}$ du système fini de générateurs $\{\pm 1\}$ et $\mathbb{Z}^2$ du système fini de générateurs $\{(\pm 1, 0), (0, \pm 1)\}$. On note $d_{\mathbb{Z}}$ et $d_{\mathbb{Z}^2}$ les structures métriques obtenues sur (X, μ).

Propriétés d'intégrabilité pour $x \mapsto d_{\mathbb{Z}^2}(x, T(x))$. Pour tout $n \geq 0$, soit $A_n \subseteq X$ l'ensemble des $x \in X$ tels que $x_0 = \cdots = x_{n-1} = 1$ et $x_n = 0$. Les ensembles $(A_n)_{n \geq 0}$ forment une partition de X sur laquelle on peut calculer explicitement la valeur de $d_{\mathbb{Z}^2}(x, T(x))$.

- Si $x \in A_{2n}$, alors $T(x) = T_{pair}(T_{impair})^{1-2^n}(x)$.
- Si $x \in A_{2n+1}$, alors $T(x) = (T_{pair})^{2^{n+1}-1} T_{impair}(x)$.

Puisque l'action $\mathbb{Z}^2 \curvearrowright (X, \mu)$ est libre, on en déduit que

$$d_{\mathbb{Z}^2}(x, T(x)) = \begin{cases} |1 - 2^n| + 1 & \text{si } x \in A_{2n}, \\ |2^{n+1} - 1| + 1 & \text{si } x \in A_{2n+1}. \end{cases}$$

Enfin, la mesure de A_n vaut $1/2^{n+1}$, donc l'intégrale

$$\int_X d_{\mathbb{Z}^2}(x, T(x))^p d\mu$$

a la même nature que $\sum_{n\geq 0} 2^{(p-2)n}$, qui converge si et seulement si $p < 2$.

Propriétés d'intégrabilité pour $x \mapsto d_{\mathbb{Z}}(x, T_{pair}(x))$. Pour tout $n \geq 0$, soit $B_n \subseteq X$ l'ensemble des $x \in X$ tels que $x_0 = x_2 = \cdots = x_{2=-2} = 1$ et $x_{2n} = 0$. Si $x \in B_n$, alors

$$T_{pair}(x) = T^{2^{2n}-(2^{2n-2}+2^{2n-4}+\cdots+1)}(x) = T^{(2^{2n+1}+1)/3}(x).$$

Puisque l'action $\mathbb{Z} \curvearrowright (X, \mu)$ est libre, on en déduit que

$$d_{\mathbb{Z}}(x, T_{pair}(x)) = \frac{2^{2n+1}+1}{3} \quad \text{si } x \in B_n.$$

Enfin, la mesure de B_n vaut $1/2^{n+1}$, donc l'intégrale

$$\int_X d_{\mathbb{Z}}(x, T_{pair}(x))^p d\mu$$

a la même nature que $\sum_{n\geq 0} 2^{(2p-1)n}$, qui converge si et seulement si $p < 1/2$. Un calcul similaire donne la même conclusion pour l'intégrale

$$\int_X d_{\mathbb{Z}}(x, T_{impair}(x))^p d\mu.$$

Comme il suffit de s'assurer de l'intégrabilité sur les générateurs des groupes $\mathbb{Z}$ et $\mathbb{Z}^2$, on en déduit que les actions $\mathbb{Z} \curvearrowright (X, \mu)$ et $\mathbb{Z}^2 \curvearrowright (X, \mu)$ sont $\mathrm{L}^{<1/2}$ orbitalement équivalentes.

Puisque ces actions sont libres et orbitalement équivalentes, on peut associer à presque tout point $x \in X$ une application $f_x : \mathbb{Z} \to \mathbb{Z}^2$ où $f_x(n)$ est l'unique élément $(u, v) \in \mathbb{Z}^2$ tel que $T(x) = (T_{pair})^u (T_{impair})^v(x)$. Lorsque l'on regarde précisément le graphe de f_x pour un $x \in X$ générique, on reconnaît la ***courbe de Lebesgue***, voir FIGURE 1.3. Cette condition "$p < 1/2$" n'est pas sans rappeler le fait qu'une courbe continue $[0, 1] \to [0, 1]^2$ qui remplit le carré ne peut pas être p-Hölder pour $p > 1/2$.

En partitionnant $\mathbb{N}$ en fonction du reste modulo $n, m \geq 2$, on obtient d'une façon similaire deux actions p.m.p. $\mathbb{Z}^n \curvearrowright (X, \mu)$ et $\mathbb{Z}^m \curvearrowright (X, \mu)$ qui sont $\mathrm{L}^{<p}$ avec $p = \min(n/m, m/n)$. On peut retrouver ces exemples (sous une forme moins explicite) dans l'article [DKLMT20, Thm. 6.9]. Nous expliquons ci-dessous en quoi ces conditions sur p sont optimales.

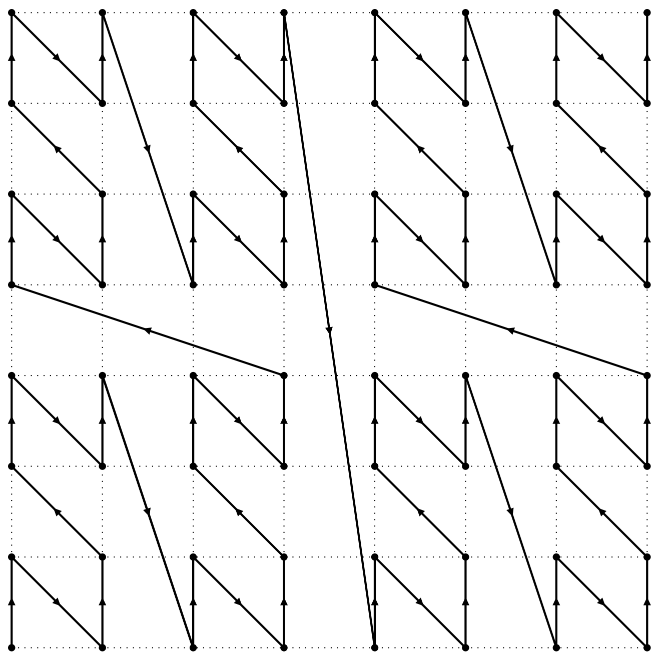

FIGURE 1.3. — Le graphe de $f_x : \mathbb{Z} \to \mathbb{Z}^2$ avec $x = 000000{*}{*}\ldots$

Le retour des invariants

L'équivalence orbitale quantitative permet de retrouver plusieurs invariants qui ne sont pas capturés par l'équivalence orbitale. Par exemple, Bowen démontre que deux groupes de type fini, qui admettent des actions p.m.p. essentiellement libres qui sont L^1 orbitalement équivalentes, possèdent la même fonction de croissance [Aus16b, Appen. B]. Plusieurs autres invariants sont préservés lorsque des actions essentiellement libres sont L^1 orbitalement équivalentes, comme par exemple le cône asymptotique pour les groupes à croissance polynomiale [Aus16b], ou encore l'entropie de Kolmogorov-Sinai pour les actions p.m.p. de groupes moyennables [Aus16].

La condition "$p < 1/2$" obtenue dans l'équivalence orbitale du paragraphe précédent s'explique grâce au théorème suivant.

THÉORÈME. — *Soient $n, n \geq 1$ deux entiers. Soient $\mathbb{Z}^n \curvearrowright (X,\mu)$ et $\mathbb{Z}^m \curvearrowright (Y,\nu)$ deux actions p.m.p. essentiellement libres qui sont L^p orbitalement équivalentes. Alors $p \leq \min(n/m, m/n)$.*

On pourra se rapporter à [DKLMT20, Thm. 3.2] pour un énoncé plus général. On ne sait pas s'il existe des actions p.m.p. essentiellement libres $\mathbb{Z}^n \curvearrowright (X,\mu)$ et $\mathbb{Z}^m \curvearrowright (Y,\nu)$ qui sont L^p orbitalement équivalentes avec $p = \min(n/m, m/n)$.

Le théorème de Belinskaya est optimal

Dans le cas d'actions de $\mathbb{Z}$, l'équivalence orbitale L^1 est extrêmement rigide. Deux actions p.m.p. $\mathbb{Z} \curvearrowright^{\alpha} (X,\mu)$ et $\mathbb{Z} \curvearrowright^{\beta} (Y,\nu)$ sont ***flip-conjuguées*** s'il existe une bijection bimesurable $\Phi : X \to Y$ telle que $\Phi_*\mu = \nu$ et

– soit $\Phi \circ \alpha(n) = \beta(n) \circ \Phi$ pour tout $n \in \mathbb{Z}$,

– soit $\Phi \circ \alpha(n) = \beta(-n) \circ \Phi$ pour tout $n \in \mathbb{Z}$.

Théorème ([Bel68]). — *Deux actions p.m.p. ergodiques $\mathbb{Z} \curvearrowright (X, \mu)$ et $\mathbb{Z} \curvearrowright (Y, \nu)$ sont* L^1 *orbitalement équivalentes si et seulement si elles sont flip-conjuguées.*

Parmi les différents résultats du chapitre 2, nous démontrons que le théorème de Belinskaya est optimal au sens suivant.

Théorème. — *Il existe des actions p.m.p. ergodiques de $\mathbb{Z}$ qui sont* $\mathrm{L}^{<1}$ *orbitalement équivalentes mais pas flip-conjuguées.*

Ce résultat est le fruit d'un travail en commun avec Carderi, Le Maître et Tessera [CJLMT22]. Le chapitre 3 est une annexe du chapitre 2 dans lequel nous traduisons un résultat probabiliste dû à Liggett [Lig02] dans le langage introduit dans le chapitre 2.

Involution non croisante

Si $k, l \in \mathbb{Z}$, on note $[\![k, l]\!]$ l'intervalle de $\mathbb{Z}$ dont les extrémités sont k et l. Une involution $P : \mathbb{Z} \to \mathbb{Z}$ est ***non-croisante*** si pour tout $x, y \in \mathbb{Z}$, soit les intervalles $[\![x, P(x)]\!]$ et $[\![y, P(y)]\!]$ sont disjoints, soit l'un contient l'autre. Cette définition a une interprétation géométrique. Pour tout $x \in \mathbb{Z}$ qui n'est pas un point fixe de P, on trace le demi-cercle dans le demi-plan supérieur, qui est perpendiculaire à l'axe horizontal et dont les extrémités sont x et $P(x)$. Alors P est non-croisante si et seulement si aucun de ces demi-cercles ne se coupe.

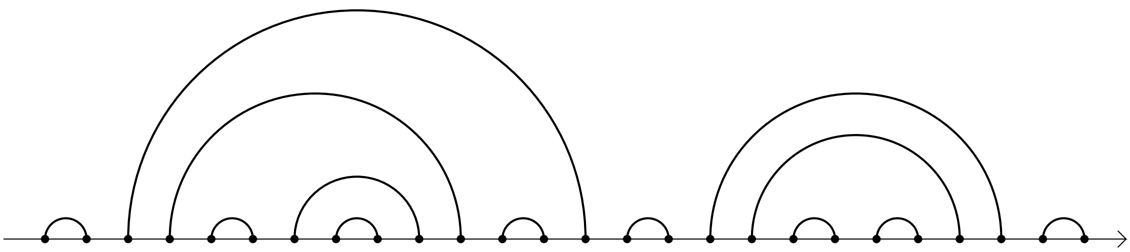

Figure 1.4. — Une portion d'une involution non-croisante.

Soit $P : \mathbb{Z} \to \mathbb{Z}$ une involution non-croisante *sans point fixe*. Alors l'application $Q : x \mapsto P(x-1) + 1$ est aussi une involution non-croisante sans point fixe. On peut à nouveau représenter géométriquement cette application, en traçant les demi-cercles dans le plan inférieur cette fois-ci. Ces demi-cercles sont obtenus en partant des demi-cercles qui représentent P, puis en effectuant une translation $x \mapsto x + 1$ puis une symétrie par rapport à l'axe horizontal. Dans le chapitre 4, nous démontrons que la réunion des demi-cercles associés à P et Q forme une courbe connexe, comme illustré dans la Figure 1.5.

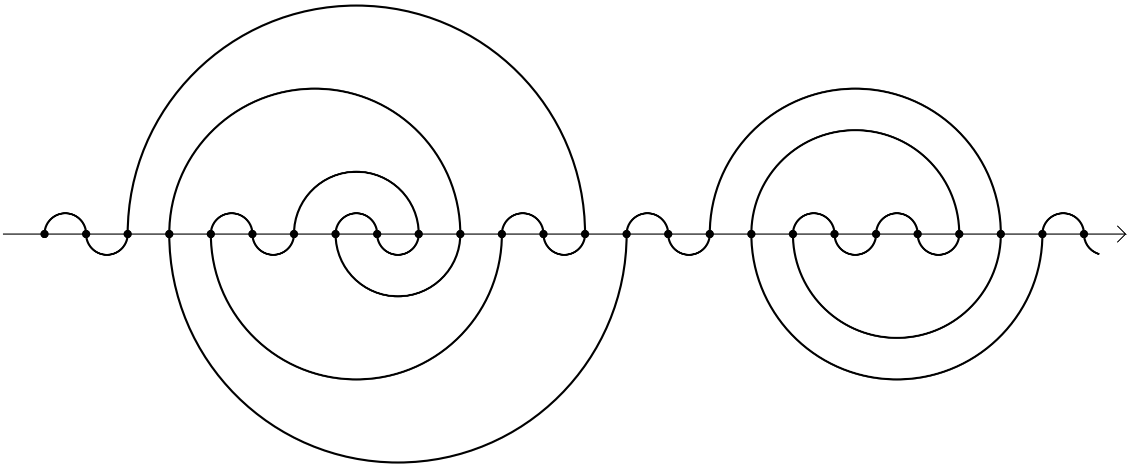

FIGURE 1.5. — Les demi-cercles associés à P et Q.

Équivalence orbitale quantitative entre $\mathbb{Z}$ et D_∞

Nous utilisons dans le chapitre 4 ce lemme combinatoire pour comparer les actions de $\mathbb{Z}$ et du groupe diédral infini D_∞ du point de vue de l'équivalence orbitale quantitative. Le groupe diédral infini est le groupe dont une présentation est donnée par

$$D_\infty := \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

Nous démontrons le résultat suivant.

THÉORÈME. — *Toute action p.m.p. essentiellement libre* $\mathbb{Z} \curvearrowright (X, \mu)$ *est* $\mathrm{L}^{<1}$ *orbitalement équivalente à une action p.m.p. du groupe diédral infini* D_∞.

À contrario, une action p.m.p. essentiellement libre $\mathbb{Z} \curvearrowright (X, \mu)$ *est* L^1 *orbitalement équivalente à une action de* D_∞ *si et seulement si le sous-groupe* $2\mathbb{Z}$ *n'agit pas ergodiquement sur* (X, μ).

Partie III : Équivalence orbitale quantitative et graphages

Graphage

L'équivalence orbitale quantitative étudie les distortions que peuvent subir les structures métriques données par les actions p.m.p. de groupes de type fini. Ici, nous introduisons une notion d'équivalence orbitale quantitative qui impose que les structures métriques ne subissent aucune distorsion, mais soient au contraire *isométriques*.

Soit (X, μ) un espace de probabilité. Un ***graphage*** sur (X, μ) est un graphe dont l'ensemble des sommets est X et l'ensemble des arêtes est un sous-ensemble symétrique mesurable de $X \times X$ qui vérifie la condition suivante : pour tous ensembles mesurables $A, B \subseteq X$,

$$\int_B \deg_A(x) d\mu = \int_A \deg_B(x) d\mu, \qquad (*)$$

où $\deg_E(x)$ désigne l'ensemble des voisins de x qui sont dans E. Les graphages sont fondamentaux en théorie des graphes limites, car ce sont les objets limites de graphes de valence uniformément bornée [Lov12, Chap. 18]. Les graphages sont aussi des objets fondamentaux dans la théorie des actions p.m.p. Ce sont par exemple les outils principaux de la théorie du coût, étudiée en détail par Gaboriau [Gab00].

Soit Γ un groupe de type fini et S_Γ un ***système fini de générateurs***, c'est-à-dire un sous-ensemble fini $S_\Gamma \subseteq \Gamma$, qui est symétrique ($S_\Gamma = S_\Gamma^{-1}$), qui ne contient pas l'élément neutre e_Γ et qui engendre le groupe ($\langle S_\Gamma \rangle = \Gamma$). À une action p.m.p. $\Gamma \curvearrowright (X, \mu)$, on peut associer un graphage dont l'ensemble des sommets est X et l'ensemble des arêtes est

$$\{(x, y) \in X \times X \colon \exists s \in S, s \cdot x = y\}.$$

La condition $(*)$ est automatiquement satisfaite puisque le groupe Γ agit sur (X, μ) en préservant la mesure. Lorsque l'action est essentiellement libre, ce graphage retient la géométrie du groupe, puisque pour μ-presque tout $x \in X$, la composante connexe du graphage qui contient x est isomorphe au graphe de Cayley du groupe. Ici, le ***graphe de Cayley***, noté (Γ, S_Γ), d'un groupe Γ muni d'un système fini de générateurs S_Γ est le graphe dont l'ensemble des sommets est Γ et l'ensemble des arêtes est

$$\{(\gamma, \delta) \in \Gamma \times \Gamma \colon \exists s \in S_\Gamma, \gamma s = \delta\}.$$

Équivalence orbitale isométrique

Dans le chapitre 5, on étudie une notion forte d'équivalence orbitale quantitative que l'on appelle équivalence orbitale isométrique.

Définition. — Soient Γ et Λ deux groupes de type fini. Fixons S_Γ et S_Λ des systèmes finis de générateurs pour Γ et Λ respectivement. Deux actions p.m.p. $\Gamma \curvearrowright^{\alpha} (X,\mu)$ et $\Lambda \curvearrowright^{\beta} (Y,\nu)$ sont ***isométriquement orbitalement équivalentes*** s'il existe une bijection bimesurable $\Phi : X \to Y$ telle que $\Phi_*\mu = \nu$ et pour μ-presque tout $x \in X$, l'application Φ induit une isométrie entre la composante connexe de x dans le graphage associé à α et la composante connexe de $\Phi(x)$ dans le graphage associé à β.

Contrairement aux notions d'équivalence orbitale L^p, la notion d'équivalence orbitale isométrique dépend du choix du système fini de générateurs de façon primordiale, puisque le graphage associé à une action en dépend.

Rigidité et flexibilité

La notion d'équivalence orbitale isométrique renforce celle d'équivalence orbitale L^∞. En effet, une équivalence orbitale L^∞ induit presque partout une bijection uniformément bi-lipschitzienne entre les composantes connexes des graphages associés aux actions, alors qu'une équivalence orbitale isométrique induit presque partout une isométrie.

Néanmoins, les différences entre ces notions sont notoires. Contrairement au théorème de Belinskaya, qui implique que l'équivalence orbitale L^1 pour les actions ergodiques de $\mathbb{Z}$ est peu ou prou triviale, l'équivalence orbitale L^1, et même L^∞, s'avère être riche pour les actions de $\mathbb{Z}^d$ lorsque $d \geq 2$. Par exemple, Fieldsteel et Friedman démontrent que pour tout $d \geq 2$ et toute action ergodique $\mathbb{Z}^d \curvearrowright^{\alpha} (X,\mu)$, il existe une action mélangeante $\mathbb{Z}^d \curvearrowright^{\beta} (Y,\nu)$ telle que α et β sont L^∞ orbitalement équivalentes. Nous démontrons qu'un tel résultat est faux en équivalence orbitale isométrique.

Théorème. — *Soit $d \geq 2$ et soit S un système de générateurs fini pour $\mathbb{Z}^d$. Soit $\mathbb{Z}^d \curvearrowright^{\alpha} (X,\mu)$ une action p.m.p. mélangeante. Toute action $\mathbb{Z}^d \curvearrowright (Y,\nu)$ qui est isométriquement orbitalement équivalente à α lui est conjuguée.*

En fait, nous démontrons un résultat de rigidité du même type pour les groupes Γ munis d'un système fini de générateurs S_Γ pour lesquels le groupe d'automorphismes du graphe de Cayley (Γ, S_Γ) est dénombrable. Un théorème de Trofimov permet de démontrer que le groupe d'automorphismes du graphe de Cayley de $\mathbb{Z}^d$, muni de n'importe quel système fini de générateurs S, est en effet dénombrable, voir [MS98, Thm. 4.3].

L'archétype d'un groupe qui admet un graphe de Cayley dont le groupe d'automorphismes est non-dénombrable est le groupe libre $\mathbf{F}_d$ à $d \geq 2$ générateurs. Nous démontrons que le phénomène de rigidité obtenu pour $\mathbb{Z}^d$ ne s'applique pas pour $\mathbf{F}_d$.

Théorème. — *Soit $\mathbf{F}_d$ le groupe libre à $d \geq 2$ générateurs $x_1, \ldots, x_d$ et soit S le système fini de générateurs $S := \{x_1^{\pm 1}, \ldots, x_d^{\pm 1}\}$. Alors il existe des actions p.m.p. isométriquement orbitalement équivalentes $\mathbf{F}_d \curvearrowright^{\alpha} (X,\mu)$ et $\mathbf{F}_d \curvearrowright^{\beta} (Y,\nu)$ telles que α est mélangeante mais β ne l'est pas.*

Références

[AE12] Miklós Abért and Gábor Elek. Hyperfinite actions on countable sets and probability measure spaces. In *Dynamical systems and group actions. Dedicated to Anatoli Stepin on the occasion of his 70th birthday*, pages 1–16. Providence, RI : American Mathematical Society (AMS), 2012.

[Aus16a] Tim Austin. Behaviour of Entropy Under Bounded and Integrable Orbit Equivalence. *Geom. Funct. Anal.*, 26(6) :1483–1525, 2016.

[Aus16b] Tim Austin. Integrable measure equivalence for groups of polynomial growth. *Groups Geom. Dyn.*, 10(1) :117–154, 2016.

[Bel68] Raisa M. Belinskaya. Partitions of Lebesgue space in trajectories defined by ergodic automorphisms. *Funkts. Anal. Prilozh.*, 2(3) :190–199, 1968.

[BG04] Nicolas Bergeron and Damien Gaboriau. Asymptotique des nombres de Betti, invariants ℓ^2 et laminations. *Comment. Math. Helv.*, 79(2) :362–395, 2004.

[CJLMT22] Alessandro Carderi, Matthieu Joseph, François Le Maître, and Romain Tessera. Belinskaya's theorem is optimal, 2022.

[DKLMT20] Thiebout Delabie, Juhani Koivisto, François Le Maître, and Romain Tessera. Quantitative measure equivalence. *arXiv :2002.00719*, 2020.

[Ele21] Gábor Elek. Free minimal actions of countable groups with invariant probability measures. *Ergodic Theory Dyn. Syst.*, 41(5) :1369–1389, 2021.

[Gab00] Damien Gaboriau. Coût des relations d'équivalence et des groupes. *Invent. Math.*, 139(1) :41–98, 2000.

[GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitaly I. Suschanskii. Automata, dynamical systems, and groups. In *Dynamical systems, automata, and infinite groups. Transl. from the Russian*, pages 128–203. Moscow : MAIK Nauka/Interperiodica Publishing, 2000.

[Gri11] Rostislav I. Grigorchuk. Some topics in the dynamics of group actions on rooted trees. *Proc. Steklov Inst. Math.*, 273 :64–175, 2011.

[GW15] Eli Glasner and Benjamin Weiss. Uniformly recurrent subgroups. In *Recent trends in ergodic theory and dynamical systems. International conference in honor of S. G. Dani's 65th birthday, Vadodara, India, December 26–29, 2012. Proceedings*, pages 63–75. Providence, RI : American Mathematical Society (AMS), 2015.

[Jos21] Matthieu Joseph. Continuum of allosteric actions for non-amenable surface groups, 2021.

[LBMB18] Adrien Le Boudec and Nicolás Matte Bon. Subgroup dynamics and C^*-simplicity of groups of homeomorphisms. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3) :557–602, 2018.

[Lig02] Thomas M. Liggett. Tagged particle distributions or how to choose a head at random. In *In and out of equilibrium. Probability with a physics flavor. Papers from the 4th Brazilian school of probability, Mambucaba, Brazil, August 14–19, 2000*, pages 133–162. 2002.

[Lov12] László Lovász. *Large networks and graph limits*, volume 60. Providence, RI : American Mathematical Society (AMS), 2012.

[MS98] Rögnvaldur G. Möller and Norbert Seifter. Digraphical regular representations of infinite finitely generated groups. *Eur. J. Comb.*, 19(5) :597–602, 1998.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I : The Rohlin lemma. *Bull. Am. Math. Soc., New Ser.*, 2 :161–164, 1980.

[PT16] Jesse Peterson and Andreas Thom. Character rigidity for special linear groups. *J. Reine Angew. Math.*, 716 :207–228, 2016.

[SZ94] Garrett J. Stuck and Robert J. Zimmer. Stabilizers for ergodic actions of higher rank semisimple groups. *Ann. Math. (2)*, 139(3) :723–747, 1994.

[Ver12] Anatoly M. Vershik. Totally nonfree actions and the infinite symmetric group. *Mosc. Math. J.*, 12(1) :193–212, 2012.

Introduction in English

Dynamical systems theory is the study of the behavior of a space on which a group of transformations acts. This PhD thesis focuses on two main branches of this theory: topological dynamics and ergodic theory of group actions. The former consists in the study of group actions by homeomorphisms on compact spaces, whereas the latter consists in the study of group actions on measured spaces which admit invariant measures.

This manuscript is divided into three independent parts. In this introduction, which is meant to be illustrative, we explain the main notions studied during this thesis as well as the main results that we obtained.

Outline of the thesis:

Part I: Allostery

Chapter 1: *Continuum of allosteric actions for non-amenable surface groups*
prepublished article arXiv:2110.01068

Part II: Quantitative orbit equivalence for $\mathbb{Z}$-actions

Chapter 2: *Belinskaya's theorem is optimal*
prepublished article with A. Carderi, F. Le Maître and R. Tessera, arXiv:2201.06662

Chapter 3: *Cycles in φ-integrable full groups*

Chapter 4: *Quantitative orbit equivalence between $\mathbb{Z}$ and D_∞*

Part III: Quantitative orbit equivalence and graphings

Chapter 5: *Isometric orbit equivalence for probability-measure preserving actions*, prepublished article arXiv:2203.14598

Table of contents of the introduction in English

Part I : Allostery

Minimality and ergodicity

This first part combines topological dynamics and ergodic theory of group actions. The main object of study is actions of countable groups on compact spaces, for which there exists an invariant probability measure. More precisely, a ***minimal ergodic*** action of a countable group Γ is an action $\Gamma \curvearrowright (C, \mu)$ such that

– $\Gamma \curvearrowright C$ is an action by homeomorphisms on a compact Hausdorff space C which is minimal, that is, every orbit is dense.

– μ is a Borel probability measure on C which is Γ-invariant and ergodic, that is, any measurable Γ-invariant has measure 0 or 1.

Minimal ergodic actions appear naturally in different contexts. For instance, let Γ be a countable dense subgroup of a compact group G and let μ be the Haar probability measure on G. Then the right multiplication action $\Gamma \curvearrowright (G, \mu)$ is a minimal ergodic action. Concretely, one can think of an irrational rotation $x \mapsto x + \theta \pmod 1$, which provides a minimal ergodic action of $\mathbb{Z}$ on the torus $\mathbb{T}$ endowed with the Lebesgue measure. Profinite actions form another class of minimal ergodic actions. Let Γ be a countable group and let $\Gamma > \Gamma_1 > \cdots > \Gamma_n > \ldots$ be a decreasing sequence of finite index subgroups. The group Γ acts on each finite space Γ/Γ_n and thus on the profinite limit $\varprojlim \Gamma/\Gamma_n$. This is a minimal action when the profinite limit is endowed with the profinite topology. Moreover, there is a unique Γ-invariant probability measure μ on the profinite limit. Then the action $\Gamma \curvearrowright (\varprojlim \Gamma/\Gamma_n, \mu)$ is a minimal ergodic action called a ***profinite action***. Countable groups which admit profinite *faithful* actions are ***residually finite***, that is, they admit a decreasing sequence $\Gamma_1 \geq \cdots \geq \Gamma_n \geq \ldots$ of finite index subgroups such that

$$\bigcap_{n\geq 1} \Gamma_n = \{1_\Gamma\}.$$

Actually, any countable group admits minimal ergodic actions that are faithful. More precisely, we have the following result.

Theorem ([Ele21]). — *Any countable group* Γ *admits a minimal ergodic action which is free.*

Here *free* means that the stabilizer of every point is trivial. In the sequel, we will mostly be interested in non-free actions. One of the interests of such actions lies in the fact that they give families of subgroups, given by the stabilizer of points. Equivalently, when the group is finitely generated, a non-free action provides a family of Schreier graphs of the groups. These are fundamental objects in spectral graph theory. We refer to the survey [Gri11] for the use of such families of Schreier graphs.

Notions of freeness

Several notions of *generic* freeness exist. A minimal ergodic action $\Gamma \curvearrowright (C, \mu)$ is

– ***topologically free*** if there is a G_δ dense set of points having trivial stabilizer.

– ***essentially free*** if there is a set of full measure of points having trivial stabilizer.

These two properties are *generic* properties of the action, the first one in the topological sense, the second one in the measure sense. Indeed, a point in a G_δ dense set is generic from the topological point of view and a point in a full measure set is generic from the measure point of view. We will relate these two notions in the following lemma. Let us first make a useful observation.

FACT. — *Let $\Gamma \curvearrowright C$ be an action by homeomorphisms on a compact Hausdorff space. Then the set of $x \in C$ whose stabilizer is trivial is a G_δ set.*

Proof. Let $x \in C$. Then $\mathrm{Stab}(x) = \{1\}$ if and only if for all $\gamma \in \Gamma \setminus \{1\}$, we have $\gamma x \neq x$, which proves the fact. □

Therefore, it is enough to have just one point with a trivial stabilizer for a minimal action to be topologically free.

LEMMA. — *A minimal ergodic action which is essentially free is topologically free.*

Proof. Let $\Gamma \curvearrowright (C, \mu)$ be a minimal ergodic action which is essentially free. We already know that the set of points with trivial stabilizer is G_δ. Since this is also a set of full measure, it is in particular nonempty. Thus, there exists $x \in C$ such that $\mathrm{Stab}(x)$ is trivial, which finishes the proof by minimality of the action. □

The converse is false in general, which brings us to introduce the notion of allostery.

Allostery

DEFINITION. — Let Γ be a countable group. A minimal ergodic Γ-action is ***allosteric*** [1] if it is topologically free but not essentially free. A group Γ is ***allosteric*** if it admits a minimal ergodic action which is allosteric.

Examples of non-allosteric groups

The notion of allostery is closely related to the dynamic on the space of subgroups. If Γ is a countable group, we denote by $\mathrm{Sub}(\Gamma)$ the space of subgroups of Γ, on which Γ acts by conjugation. The results in this paragraph illustrate the following maxim: if the action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ is not rich enough, then Γ is not allosteric.

PROPOSITION. — *The group $\mathbb{Z}$ is not allosteric.*

Proof. Let $\mathbb{Z} \curvearrowright (C, \mu)$ be a minimal ergodic action, which is not essentially free. By definition

$$\mu(\{x \in C \colon \mathrm{Stab}(x) \neq \{0\}\}) > 0.$$

In particular, there is a point $x \in C$ whose stabilizer is not the trivial group. Since it is a nontrivial subgroup of $\mathbb{Z}$, it is isomorphic to $n\mathbb{Z}$ for some $n \geq 1$ and thus the cardinal of the orbit of such a point x is finite. By minimality, we deduce that $C = \mathrm{Orb}(x)$ is finite and that the stabilizer of every point is nontrivial. Thus, the action $\mathbb{Z} \curvearrowright (C, \mu)$ is not topologically free and this concludes the proof. □

[1] ἄλλος: other, στερεός: a notion of fixity.

Proposition. — *If* Sub(Γ) *is countable, then* Γ *is not allosteric.*

Proof. Let Γ be a group with only countably many subgroups. Let $\Gamma \curvearrowright (C,\mu)$ be a minimal ergodic action which is not essentially free. Then we have $\mu(\{x \in C\colon \mathrm{Stab}(x) \neq \{1_\Gamma\}\}) > 0$. For any subgroup $\Lambda \leq \Gamma$, let

$$P_\Lambda := \{x \in C\colon \mathrm{Stab}(x) = \Lambda\}.$$

The family $(P_\Lambda)_{\Lambda \leq \Gamma}$ forms a partition of C. By assumption on Γ, this partition has only countably many pieces. Thus, there is a subgroup $\Lambda \leq \Gamma$ such that $\mu(P_\Lambda) > 0$. Moreover, for all $\gamma \in \Gamma$, we have $\gamma P_\Lambda = P_{\gamma\Lambda\gamma^{-1}}$. This implies that $\mu(P_\Lambda) = \mu(P_{\gamma\Lambda\gamma^{-1}})$. We deduce that Λ has only finitely many conjugates.

Let $x \in C$. By minimality and since Λ has only finitely many conjugates, there is $\gamma \in \Gamma$ and a sequence $(x_n)_{n\geq 0} \in C^{\mathbb{N}}$ such that $x_n \to x$ and $\mathrm{Stab}(x_n) = \gamma\Lambda\gamma^{-1}$ for all $n \geq 0$. We deduce that $\gamma\Lambda\gamma^{-1} \leq \mathrm{Stab}(x)$. Thus, the stabilizer of every point is nontrivial. Therefore, the action is not topologically free, which concludes the proof. □

Explicit examples of groups which admit only countably many subgroups can be found in Figure 2.2. One can show a similar result for countable groups which have few invariant random subgroups. Let Γ be a countable group and let Sub(Γ) be the space of subgroups of Γ. This is a compact subspace of $\{0,1\}^\Gamma$, on which Γ acts by conjugation. An ***invariant random subgroup (IRS)*** of Γ is a Γ-invariant probability measure on Sub(Γ). The above proof can be adapted to prove that any group whose ergodic IRS are all atomic is not allosteric. Concrete examples of such groups can be found in Figure 2.2.

Invariant random subgroups / Uniformly recurrent subgroups

In this paragraph, we explain in greater detail the connections between allostery and the dynamic on the space of subgroups.

Let $\Gamma \curvearrowright (C,\mu)$ be a minimal ergodic action. The map

$$\mathrm{Stab} : x \mapsto \{\gamma \in \Gamma\colon \gamma \cdot x = x\}$$

is a measurable map $C \to \mathrm{Sub}(\Gamma)$. Moreover, Stab is equivariant, because for all $x \in C$ and $\gamma \in \Gamma$,

$$\mathrm{Stab}(\gamma \cdot x) = \gamma \mathrm{Stab}(x)\gamma^{-1}.$$

Thus, the pushforward of μ by Stab yields a Γ-invariant ergodic probability measure on Sub(Γ), that is, an ergodic IRS. By definition, the action is essentially free if and only if this IRS is equal to the measure $\delta_{\{1_\Gamma\}}$.

Invariant random subgroups have a topological counterpart. A ***uniformly recurrent subgroup (URS)*** is a closed, minimal, Γ-invariant subset of Sub(Γ). Let $\Gamma \curvearrowright (C,\mu)$ be a minimal ergodic action. The URS associated with this action is the unique URS contained in the closure of $\{\mathrm{Stab}(x)\colon x \in C\}$, see [GW15]. One can show that this URS is contained in the support of the IRS associated with the action [Jos21, Lem. 2.2]. Moreover, this URS is equal to $\{\{1_\Gamma\}\}$ if and only if the action is topologically free [LBMB18, Prop. 2.7]. We thus get the following characterization of allostery.

Lemma. — *A minimal ergodic action $\Gamma \curvearrowright (C, \mu)$ is allosteric if and only if the trivial group $\{1_\Gamma\}$ is strictly contained in the support of the IRS associated with this action.*

Examples of allosteric groups

The existence of allosteric groups was raised by Grigorchuk, Nekrashevich and Suschanskii. More precisely, they asked the following question.

Question ([GNS00, Prob. 7.3.3]). — Does there exist countable groups which admit profinite allosteric actions?

The first examples of allosteric groups were found by Bergeron and Gaboriau. They moreover answer the question of [GNS00].

Theorem ([BG04]). — *Let Γ and Λ be two nontrivial, finitely generated groups. Then the free product $\Gamma * \Lambda$ is allosteric (unless Γ and Λ are isomorphic to the cyclic group C_2). More precisely, the group $\Gamma * \Lambda$ admits profinite allosteric actions.*

An independent proof of this result when $\Gamma = \Lambda = \mathbb{Z}$ was given by Abért and Elek in [AE12]. In Chapter 1, we are interested in another class of groups: fundamental groups $\pi_1(\Sigma_g)$ of closed, connected and orientable surfaces Σ_g. These are one relator groups, given by a group presentation of the form

$$\pi_1(\Sigma_g) := \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \dots [x_g, y_g] = 1 \rangle,$$

where $[x, y] = xyx^{-1}y^{-1}$ is the commutator of x and y and $g \geq 1$ is an integer. The main result of Chapter 1 is the following.

Theorem. — *Let Σ_g be a closed, connected and orientable surface of genus $g \geq 2$. Then $\pi_1(\Sigma_g)$ is allosteric. More precisely, it admits profinite allosteric actions.*

A construction of allosteric actions for $\mathbf{F}_2$

We conclude this introduction to Part I by an explicit construction of allosteric actions for the free group $\mathbf{F}_2$ of rank two. For this, we first construct actions of $\mathbf{F}_2$ on finite sets with specific properties.

For all integer $n \geq 0$ and prime $p \geq 2$, let

$$C_{p,n} := \bigcup_{l=0}^{2n} [\![0, p-1]\!] \times [\![0, p-2]\!]^l.$$

A point $x \in C_{p,n}$ can be written as $x = (x^0, \dots, x^l)$ with $x^0 \in [\![0, p-1]\!]$ and $x^i \in [\![0, p-2]\!]$ for all $1 \leq i \leq l$. The number l is called the ***depth*** of x. We define a permutation A on the set $C_{p,n}$. Let $x \in C_{p,n}$ be of depth l.

– If l is even, then $A(x) := \begin{cases} (x^0,\dots,x^l,0) & \text{if } l < 2n, \\ x & \text{if } l = 2n. \end{cases}$

– If l is odd, then $A(x) := \begin{cases} (x^0,\dots,x^l+1) & \text{if } x^l < p-2, \\ (x^0,\dots,x^{l-1}) & \text{otherwise.} \end{cases}$

We define another permutation B on the set $C_{p,n}$. Let $x \in C_{p,n}$ be of depth l.

– If l is odd, then $B(x) := (x^0,\dots,x^l,0)$.

– If $l=0$, then $B(x) := x^0+1 \pmod p$. If l is even and >0, then

$$B(x) := \begin{cases} (x^0,\dots,x^l+1) & \text{if } x^l < p-2, \\ (x^0,\dots,x^{l-1}) & \text{otherwise.} \end{cases}$$

Let a and b be the two generators of $\mathbf{F}_2$. The permutations A and B induce an action of $\mathbf{F}_2$ on $C_{p,n}$ by letting $a \cdot x := A(x)$ and $b \cdot x := B(x)$ for all $x \in C_{p,n}$. This action is illustrated in Figure 2.1.

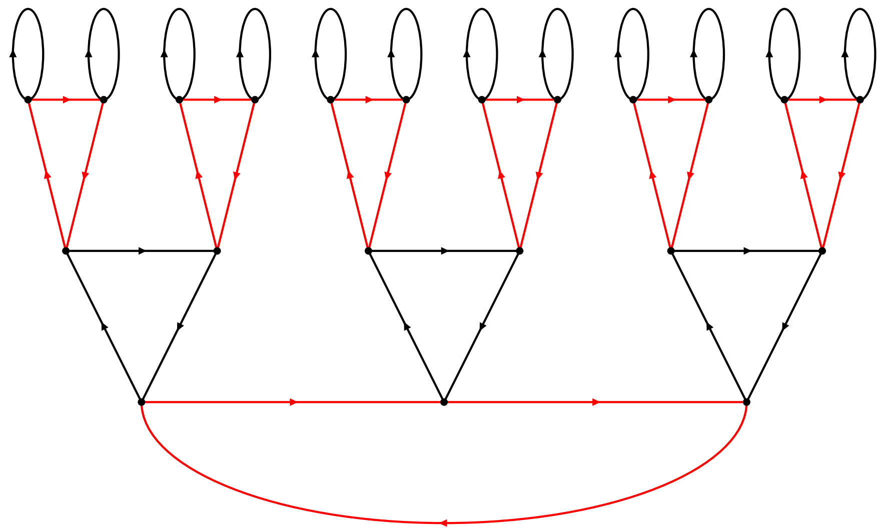

Figure 2.1. — The set $C_{3,1}$ and the permutations A in black, B in red.

We call ***boundary*** of $C_{p,n}$ the set $\partial C_{p,n}$ consisting of all points of depth $2n$. These are exactly the set of points that are fixed by a. Moreover, an elementary computation yields

$$\frac{|\partial C_{p,n}|}{|C_{p,n}|} \underset{n\to+\infty}{\longrightarrow} \frac{p-2}{p-1}.$$

Since $(p-2)/(p-1) \to 1$ as $p \to +\infty$, we can fix a sequence of pairwise distinct prime numbers $(p_k)_{k\geq 0}$ and a sequence of integers $(n_k)_{k\geq 0}$ which tends to $+\infty$, such that

$$\prod_{k=0}^{+\infty} \frac{|\partial C_{p_k,n_k}|}{|C_{p_k,n_k}|} > 0.$$

Endowed with the product topology, the space $C := \prod_{k\geq 0} C_{p_k,n_k}$ is compact. We let μ be the product of the uniform probability measure on each C_{p_k,n_k}.

Proposition 2.0.1. — *The diagonal action of* $\mathbf{F}_2$ *on* C *is allosteric.*

Sketch of the proof. First of all, $\mathbf{F}_2 \curvearrowright (C, \mu)$ is a minimal ergodic action. Indeed, an application of the Chinese remainder theorem allows us to show that for all $K \geq 0$, the diagonal action $\mathbf{F}_2 \curvearrowright \prod_{k \leq K} C_{p_k}$ is transitive. Thus, the cylinders of C of the form

$$U_{(y_1,\dots,y_K)} := \{(x_k)_{k\geq 0} \in C \colon x_0 = y_0, \dots, x_K = y_K\}$$

all have the same measure and intersect every orbit of the action $\mathbf{F}_2 \curvearrowright C$. This implies that

- the measure μ is the unique $\mathbf{F}_2$-invariant measure on C, which is thus ergodic,
- the action $\mathbf{F}_2 \curvearrowright C$ is minimal.

Let us prove that $\mathbf{F}_2 \curvearrowright (C, \mu)$ is topologically free, but not essentially free. For all $k \in \mathbb{N}$, fix $x_k \in C_{p_k, n_k}$ an element of depth 0 and let $x := (x_k)_{k \geq 0} \in C$. By construction, we observe that $\mathrm{Stab}(x) = \{1_{\mathbf{F}_2}\}$. Since $\mathbf{F}_2 \curvearrowright C$ is minimal, we deduce that the set of points with trivial stabilizer is dense in C. Since it is also a G_δ set, we deduce that $\mathbf{F}_2 \curvearrowright C$ is topologically free. Moreover, by definition of the measure μ, we have

$$\mu(\{x \in C \colon a \in \mathrm{Stab}(x)\}) = \prod_{k=0}^{+\infty} \frac{|\partial C_{p_k, n_k}|}{|X_{p_k, n_k}|} > 0.$$

Therefore $\mathbf{F}_2 \curvearrowright C$ is not essentially free, which implies that it is an allosteric action. $\square$

Some allosteric/non-allosteric groups

Group	Allostery	Reason or Reference
$\Gamma * \Lambda$ ($\neq C_2 * C_2$) with $\Gamma, \Lambda \neq \{1\}$ and residually finite	✓	[BG04]
$\pi_1(\Sigma_g)$, $g \geq 2$	✓	[Jos21]
$\mathbb{Z} \wr \mathbb{Z}$	✓	Work in progress
Γ nilpotent finitely generated	✗	$\lvert \mathrm{Sub}(\Gamma) \rvert = \aleph_0$
$\mathrm{BS}(1, n)$	✗	$\lvert \mathrm{Sub}(\mathrm{BS}(1, n)) \rvert = \aleph_0$
$\mathrm{FSym}(\mathbb{N})$, group of finitely supp. perm. on $\mathbb{N}$	✗	[Ver12]
Lattices in $\mathrm{SL}_n(\mathbb{R})$ $n \geq 3$	✗	Ergodic IRS are atomic [SZ94]
Lattices in $\mathrm{PSL}_n(\mathbb{Q}_p)$ $n \geq 2$, p prime	✗	Ergodic IRS are atomic [PT16]

FIGURE 2.2. — A list of some allosteric and non-allosteric groups. This list is not exhaustive. In particular, the last two lines are true in greater generality

Part II: Quantitative orbit equivalence for $\mathbb{Z}$-actions

Orbit equivalence

The main topic of this second part is orbit equivalence for actions of groups, especially for actions of $\mathbb{Z}$. Let Γ be a countable group. A p.m.p. action $\Gamma \curvearrowright (X,\mu)$ on a probability space (X,μ) is an action by bimeasurable bijections of X which preserve the measure μ. The action of $\gamma \in \Gamma$ on $x \in X$ is denoted by $\gamma \cdot x$. When it is necessary to give a name to the action, we will write $\Gamma \curvearrowright^{\alpha} (X,\mu)$ and the action of $\gamma \in \Gamma$ on $x \in X$ is denoted by $\alpha(\gamma) \cdot x$.

Let Γ and Λ be two countable groups. Two p.m.p. actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are ***orbit equivalent*** if there exists an ***orbit equivalence***, that is a bimeasurable bijection $\Phi : X \to Y$ such that $\Phi_*\mu = \nu$ and for μ-almost every $x \in X$,

$$\Phi(\Gamma \cdot x) = \Lambda \cdot \Phi(x).$$

Example of an orbit equivalence

Let $X := \{0,1\}^{\mathbb{N}}$ and $\mu := (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes \mathbb{N}}$. The ***dyadic odometer*** is the transformation $T : X \to X$ given by

$$T(1,\dots,1,0,x_{k+1},x_{k+2},\dots) = (0,\dots,0,1,x_{k+1},x_{k+2},\dots)$$

and $T(1,1,\dots) = (0,0,\dots)$. This is a bimeasurable bijection, which preserves the measure μ. This yields a p.m.p. action $\mathbb{Z} \curvearrowright (X,\mu)$. Let us define two transformations $T_{even} : X \to X$ and $T_{odd} : X \to X$, called the ***even dyadic odometer*** and the ***odd dyadic odometer***. They are defined by

$$T_{even}(x_0,x_1,\dots) = (y_0,y_1,\dots),$$
$$T_{odd}(x_0,x_1,\dots) = (z_0,z_1,\dots),$$

where for all $n \geq 0, x_{2n+1} = y_{2n+1}, x_{2n} = z_{2n}$ and

$$(y_0,y_2,\dots) = T(x_0,x_2,\dots),$$
$$(z_1,z_3,\dots) = T(x_1,x_3,\dots).$$

The transformations T_{even} and T_{odd} preserve the measure μ and commute. Thus they give rise to a p.m.p. action $\mathbb{Z}^2 \curvearrowright (X,\mu)$.

Lemma. — *These two actions $\mathbb{Z} \curvearrowright (X,\mu)$ and $\mathbb{Z}^2 \curvearrowright (X,\mu)$ are orbit equivalent.*

Sketch of the proof. Let X' be the set of all $(x_n)_{n\geq 0} \in X$ such that at least one of the subsequence $(x_{2n})_{n\geq 0}$ is eventually constant. Then $\mu(X \setminus X') = 1$ and for all $x \in X \setminus X'$, the following properties are true:

- for all $n \in \mathbb{Z}$, there exists $(u,v) \in \mathbb{Z}^2$ such that $T^n(x) = (T_{even})^u(T_{odd})^v(x)$.

– for all $(u,v) \in \mathbb{Z}^2$, there exists $n \in \mathbb{Z}$ such that $(T_{even})^u(T_{odd})^v(x) = T^n(x)$.

This means that id_X is an orbit equivalence between these two actions. □

If instead of the partition $\{odd\} \sqcup \{even\}$, we look at the partition of $\mathbb{N}$ given by the remainder modulo $n, m \geq 1$, then one obtains similarly two orbit equivalent ergodic actions $\mathbb{Z}^n \curvearrowright (X,\mu)$ and $\mathbb{Z}^m \curvearrowright (X,\mu)$.

Orbit equivalence and invariants

In the preceding example, we proved that for all $n, m \geq 1$, the groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ admit p.m.p. ergodic actions that are orbit equivalent. Much more is true.

Theorem ([OW80]). — *Any two p.m.p. ergodic actions of any two amenable infinite groups are orbit equivalent.*

Thus, orbit equivalence does not preserve any of the geometric invariants that one can associate to amenable groups. For instance, the dimension d of the group $\mathbb{Z}^d$ is not preserved. Among the geometric invariants, one can cite the growth rate of a group, the Følner function, the asymptotic cone, etc. Similarly, in general, no ergodic invariant of p.m.p. action is preserved under orbit equivalence. Among ergodic invariants, one can cite the spectrum of the Koopman representation, Kolmogorov-Sinai entropy, etc. This lack of orbit equivalence invariants is one of the main motivations for the introduction of quantitive refinement of orbit equivalence.

Quantitative orbit equivalence

Let Γ be a finitely generated group and let $|\cdot|_\Gamma$ be the word length associated with some finite generating system $S_\Gamma \subseteq \Gamma$. Given any p.m.p. action $\Gamma \curvearrowright (X,\mu)$, one can construct a metric structure on (X,μ), denoted by d_Γ, by letting the d_Γ-distance between two points $x, y \in X$ in the same Γ-orbit be equal to

$$d_\Gamma(x,y) := \inf\{|\gamma|_\Gamma \colon \gamma \cdot x = y\}.$$

One of the main goals of quantitative orbit equivalence is to understand how an orbit equivalence between two p.m.p. actions can distort the associated metric structures. More precisely, let Γ and Λ be two finitely generated groups and let $|\cdot|_\Gamma$ and $|\cdot|_\Lambda$ be the word length associated with finite generating systems for Γ and Λ respectively.

Definition. — Let $p \in]0,+\infty]$. An orbit equivalence $\Phi : (X,\mu) \to (Y,\nu)$ between two p.m.p. actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ is an $\mathbf{L}^p$ ***orbit equivalence*** if for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$, the maps

$$x \mapsto d_\Lambda(\Phi(x), \Phi(\gamma \cdot x)) \quad \text{and} \quad y \mapsto d_\Gamma(\Phi^{-1}(y), \Phi^{-1}(\lambda \cdot y))$$

belong to $\mathrm{L}^p(X,\mu)$ and $\mathrm{L}^p(Y,\nu)$ respectively.

By the triangle inequality, it is enough to check these conditions for γ and λ belonging to any finite generating system of Γ and Λ respectively. Let us mention that changing the finite generating systems leads to bilipschitz metric structures, which does not impact the integrability conditions. Thus, the notion of L^p orbit equivalence is independent of the choice of finite generating systems.

We say that two p.m.p. actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are **$\mathbf{L}^p$ *orbit equivalent*** ($p \in]0,+\infty]$) if there exists an L^p orbit equivalence between them. Finally, we say that two p.m.p. actions are **$\mathbf{L}^{<p}$ *orbit equivalent*** if there exists an orbit equivalence between them which is an L^q orbit equivalence for all $q < p$.

Example of an L^p orbit equivalence

Let $X := \{0,1\}^{\mathbb{N}}$ and $\mu := (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes\mathbb{N}}$. Let $T : X \to X$ be the dyadic odometer and T_{even}, T_{odd} be the even and odd dyadic odometers. The actions $\mathbb{Z} \curvearrowright (X,\mu)$ and $\mathbb{Z}^2 \curvearrowright (X,\mu)$ induced by T and the couple (T_{even}, T_{odd}) are orbit equivalent, the identity map $\mathrm{id}_X : X \to X$ being an orbit equivalence. We will compute explicitly the integrability degree of this orbit equivalence.

In the computations below, we endow $\mathbb{Z}$ with the finite generating system $\{\pm 1\}$ and $\mathbb{Z}^2$ with the finite generating system $\{(\pm 1, 0), (0, \pm 1)\}$. We denote by $d_{\mathbb{Z}}$ and $d_{\mathbb{Z}^2}$ the corresponding metric structures on (X,μ).

Integrability properties for $\boldsymbol{x \mapsto d_{\mathbb{Z}^2}(x, T(x))}$. For all $n \geq 0$, let $A_n \subseteq X$ be the set of $x \in X$ such that $x_0 = \cdots = x_{n-1} = 1$ and $x_n = 0$. The sets $(A_n)_{n \geq 0}$ form a partition of X on which one can compute explicitly the value of $d_{\mathbb{Z}^2}(x, T(x))$.

- If $x \in A_{2n}$, then $T(x) = T_{even}(T_{odd})^{1-2^n}(x)$.
- If $x \in A_{2n+1}$, then $T(x) = (T_{even})^{2^{n+1}-1} T_{odd}(x)$.

Since the action $\mathbb{Z}^2 \curvearrowright (X,\mu)$ is free, we then deduce that

$$d_{\mathbb{Z}^2}(x, T(x)) = \begin{cases} |1 - 2^n| + 1 & \text{if } x \in A_{2n}, \\ |2^{n+1} - 1| + 1 & \text{if } x \in A_{2n+1}. \end{cases}$$

Finally, the measure of A_n is equal to $1/2^{n+1}$, thus the behavior of the integral

$$\int_X d_{\mathbb{Z}^2}(x, T(x))^p d\mu$$

comes down to the behavior of the series $\sum_{n \geq 0} 2^{(p-2)n}$, which converges if and only if $p < 2$.

Integrability properties for $\boldsymbol{x \mapsto d_{\mathbb{Z}}(x, T_{even}(x))}$. For all $n \geq 0$, let $B_n \subseteq X$ be the set of $x \in X$ such that $x_0 = x_2 = \cdots = x_{2=-2} = 1$ and $x_{2n} = 0$. If $x \in B_n$, then

$$T_{even}(x) = T^{2^{2n} - (2^{2n-2} + 2^{2n-4} + \cdots + 1)}(x) = T^{(2^{2n+1}+1)/3}(x).$$

Since the action $\mathbb{Z} \curvearrowright (X, \mu)$ is free, we deduce that

$$d_{\mathbb{Z}}(x, T_{even}(x)) = \frac{2^{2n+1}+1}{3} \quad \text{if } x \in B_n.$$

Finally, the measure of B_n is equal to $1/2^{n+1}$, thus the behavior of the integral

$$\int_X d_{\mathbb{Z}}(x, T_{even}(x))^p d\mu$$

comes down to the behavior of the series $\sum_{n\geq 0} 2^{(2p-1)n}$, which is convergent if and only if $0 < p < 1/2$. A similar computation yields a similar conclusion for the integral

$$\int_X d_{\mathbb{Z}}(x, T_{odd}(x))^p d\mu.$$

Since it is enough to check the integrability conditions on the generators of the groups $\mathbb{Z}$ and $\mathbb{Z}^2$, we deduce that the p.m.p. actions $\mathbb{Z} \curvearrowright (X, \mu)$ and $\mathbb{Z}^2 \curvearrowright (X, \mu)$ are $\mathrm{L}^{<1/2}$ orbit equivalent.

Since these actions are free and orbit equivalent, to almost every point $x \in X$ one can associate a map $f_x : \mathbb{Z} \to \mathbb{Z}^2$, where $f_x(n)$ is the unique element $(u, v) \in \mathbb{Z}^2$ such that $T(x) = (T_{even})^u (T_{odd})^v (x)$. If one looks carefully at the graph of the map f_x for a generic $x \in X$, one recognizes the ***Lebesgue curve***, see FIGURE 2.3. This condition "$p < 1/2$" is not unconnected with the fact that a continuous curve $[0, 1] \to [0, 1]^2$ filling the square cannot be p-Hölder with $p > 1/2$.

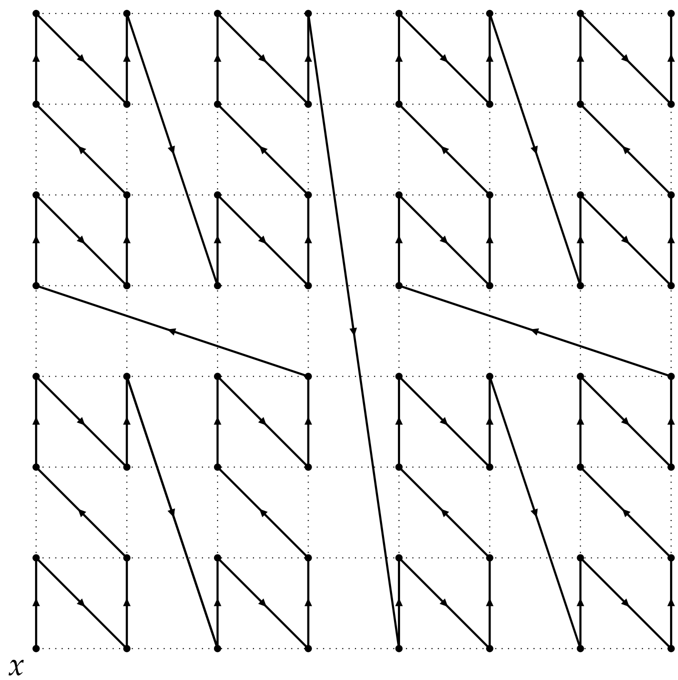

FIGURE 2.3. — The graph of $\sigma(-, x) : \mathbb{Z} \to \mathbb{Z}^2$ with $x = 000000{*}{*}\ldots$

By looking at the partitions of $\mathbb{N}$ given by the remainder modulo n and

$m \geq 2$, we get similarly two p.m.p. actions $\mathbb{Z}^n \curvearrowright (X, \mu)$ and $\mathbb{Z}^m \curvearrowright (X, \mu)$ which are $\mathrm{L}^{<p}$ orbit equivalent for $p = \min(n/m, m/n)$. One can find these examples (less explicitely) in the article [DKLMT20, Thm. 6.9]. We explain below why this condition on p is optimal.

Return of invariants

Quantitative orbit equivalence is one way to recover several invariants that were not captured by orbit equivalence. For instance, Bowen proved that any two finitely generated groups, which admit p.m.p. essentially free actions that are L^1 orbit equivalent, have the same growth function [Aus16b, Appen. B]. Several other invariants are preserved by L^1 orbit equivalence, such as asymptotic cones for groups of polynomial growth [Aus16b], or else Kolmogorov-Sinai entropy for p.m.p. actions of amenable groups [Aus16].

The condition "$p < 1/2$" obtained in the orbit equivalence of last paragraph is explained by the following theorem.

THEOREM. — *Let $n, n \geq 1$ be two integers. Let $\mathbb{Z}^n \curvearrowright (X, \mu)$ and $\mathbb{Z}^m \curvearrowright (Y, \nu)$ be two p.m.p. essentially free actions that are L^p orbit equivalent. Then $p \leq \min(n/m, m/n)$.*

We refer to [DKLMT20, Thm. 3.2] for a more general statement. It is unknown whether two p.m.p. essentially free actions $\mathbb{Z}^n \curvearrowright (X, \mu)$ and $\mathbb{Z}^m \curvearrowright (Y, \nu)$ can be L^p orbit equivalent with $p = \min(n/m, m/n)$.

Belinskaya's theorem is optimal

L^1 orbit equivalence is extremely rigid when it comes to $\mathbb{Z}$-actions. Two p.m.p. actions $\mathbb{Z} \curvearrowright^{\alpha} (X, \mu)$ and $\mathbb{Z} \curvearrowright^{\beta} (Y, \nu)$ are ***flip-conjugate*** if there exists a bimeasurable bijection $\Phi : X \to Y$ such that $\Phi_* \mu = \nu$ and

- either $\Phi \circ \alpha(n) = \beta(n) \circ \Phi$ for all $n \in \mathbb{Z}$,
- or $\Phi \circ \alpha(n) = \beta(-n) \circ \Phi$ for all $n \in \mathbb{Z}$.

THEOREM ([Bel68]). — *Two p.m.p. ergodic actions $\mathbb{Z} \curvearrowright (X, \mu)$ and $\mathbb{Z} \curvearrowright (Y, \nu)$ are L^1 orbit equivalent if and only if they are flip-conjugate.*

Among the results contained in Chapter 2, we prove that Belinskaya's theorem is optimal in the following sense.

THEOREM. — *There exists p.m.p. ergodic actions of $\mathbb{Z}$ which are $\mathrm{L}^{<1}$ orbit equivalent but not flip-conjugate.*

This result is the fruit of a joint work with Carderi, Le Maître and Tessera [CJLMT22]. Chapter 3 is an annex of Chapter 2 in which we translate a probabilistic result due to Liggett [Lig02] in the language introduced in Chapter 2.

Non-crossing involution

For $k, l \in \mathbb{Z}$, let $[\![k, l]\!]$ be the interval of $\mathbb{Z}$ whose extremities are k and l. An involution $P : \mathbb{Z} \to \mathbb{Z}$ is ***non-crossing*** if for all $x, y \in \mathbb{Z}$, the intervals $[\![x, P(x)]\!]$ and $[\![y, P(y)]\!]$ are either disjoint, or one contains the other. This definition has a geometric interpretation. For all $x \in \mathbb{Z}$ such that $P(x) \neq x$, we draw the circular arc in the upper half-plane whose extremities are x and $P(x)$, which is perpendicular to the horizontal axis. Then the involution P is non-crossing if and only if none of these circular arcs intersect.

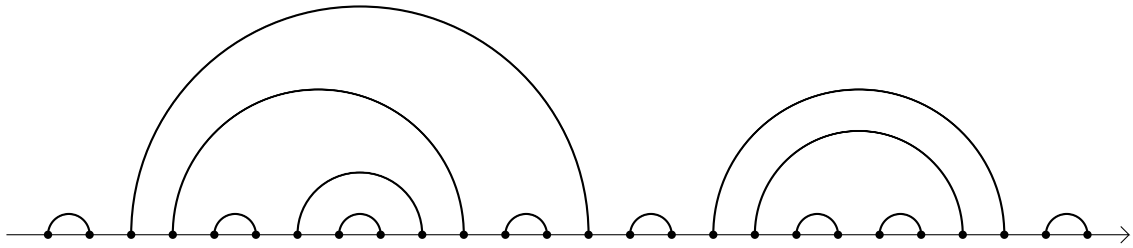

FIGURE 2.4. — A portion of a non-crossing involution.

Let $P : \mathbb{Z} \to \mathbb{Z}$ be a non-crossing involution which admits no fixed point. Then the map $Q : x \mapsto P(x-1) + 1$ is also a non-crossing involution without fixed point. We can again draw geometrically this involution with circular arcs, this time in the lower half plane. These circular arcs on the lower half plane are obtained by taking a reflexion across the horizontal line of the arcs on the upper half-plane, followed by a translation, see FIGURE 2.5.

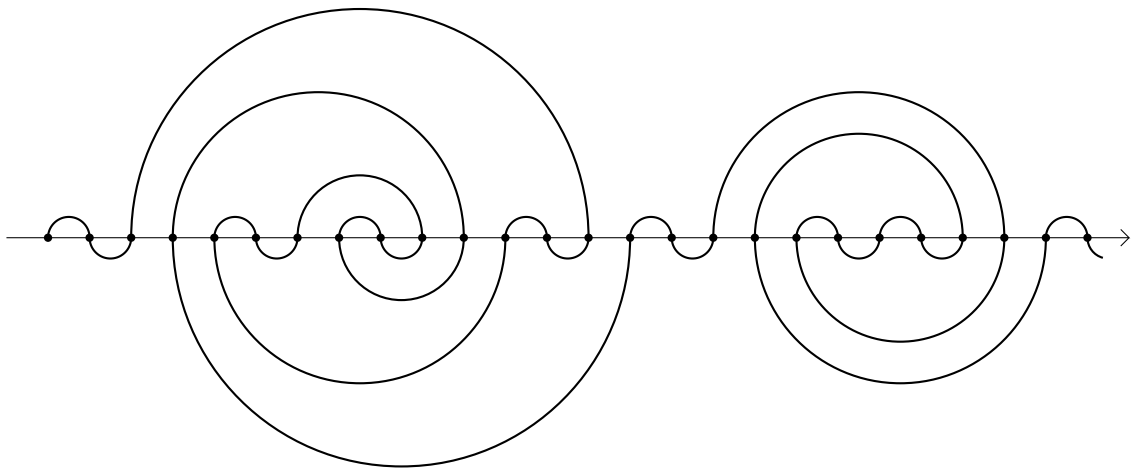

FIGURE 2.5. — The circular arcs associated with P and Q

Quantitative orbit equivalence between $\mathbb{Z}$ and D_∞

We use in Chapter 4 this combinatorial lemma to compare p.m.p. actions of $\mathbb{Z}$ and of the infinite dihedral group D_∞ up to quantitative orbit equivalence. The infinite dihedral group is the group which admits the following presentation:

$$D_\infty := \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

We prove the following result.

Theorem. — *Any p.m.p. essentially free action $\mathbb{Z} \curvearrowright (X,\mu)$ is $\mathrm{L}^{<1}$ orbit equivalent to some p.m.p. action of the infinite dihedral group D_∞.*

By contrast, a p.m.p. essentially free action $\mathbb{Z} \curvearrowright (X,\mu)$ is L^1 orbit equivalent to some p.m.p. action of the infinite dihedral group D_∞ if and only if the subgroup $2\mathbb{Z}$ does not act ergodically on (X,μ).

Part III: Quantitative orbit equivalence and graphings

Graphing

Quantitative orbit equivalence studies the distortions that can appear between metric structures associated with p.m.p. actions of finitely generated groups. Here, we introduce a notion of quantitative orbit equivalence which imposes isometries between metric structures.

Let (X,μ) be a probability space. A ***graphing*** on (X,μ) is a graph whose vertex set is X and whose edge set is a measurable symmetric subset of $X \times X$, which satisfies the following condition: for all measurable subsets $A, B \subseteq X$,

$$\int_B \deg_A(x) d\mu = \int_A \deg_B(x) d\mu, \qquad (*)$$

where $\deg_E(x)$ is the set of neighbors of x which belong to E. Graphings are fundamental in graph limit theory as they are the limit objects for sequences of bounded degree graphs, see [Lov12, Part 4] for an introduction to this theory. They are also one of the main objects in the cost theory of p.m.p. actions, which was extensively studied by Gaboriau [Gab00].

Let Γ be a finitely generated group and S_Γ be a ***finite generating system***, that is a finite subset $S_\Gamma \subseteq \Gamma$, which is symmetric ($S_\Gamma = S_\Gamma^{-1}$), which does not contain the identity element $e_\Gamma \in \Gamma$ and which generated the group: $\langle S_\Gamma \rangle = \Gamma$. To any p.m.p. action $\Gamma \curvearrowright (X,\mu)$, one can associate a graphing, whose vertex set is X and whose edge set is

$$\{(x,y) \in X \times X \colon \exists s \in S, \gamma x = y\}.$$

Condition $(*)$ is satisfied because the group Γ acts in a p.m.p. manner on (X,μ). When the action is essentially free, this graphing retains the geometry of the group, since the connected component of μ-almost every $x \in X$ is isomorphic to the Cayley graph of the group. Here, the ***Cayley graph*** (Γ, S_Γ) of a group Γ with a finite generating system S_Γ is the graph whose vertex set is Γ and whose edge set is

$$\{(\gamma,\delta) \in \Gamma \times \Gamma \colon \exists s \in S_\Gamma, \gamma s = \delta\}.$$

Isometric orbit equivalence

In Chapter 5, we study a strong notion of quantitative orbit equivalence that we call isometric orbit equivalence.

Definition. — Let Γ and Λ be two finitely generated groups. Fix S_Γ and S_Λ finite generating systems Γ and Λ respectively. Two p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X,\mu)$ and $\Lambda \curvearrowright^{\beta} (Y,\nu)$ are ***isometric orbit equivalent*** if there exists a bimeasurable bijection $\Phi : X \to Y$ such that $\Phi_*\mu = \nu$ and μ-almost every $x \in X$, the map Φ induces an isometry between the connected component of x in the graphing associated with α and the connected component of $\Phi(x)$ associated with β.

Contrary to L^p orbit equivalence, the notion of isometric orbit equivalence depends heavily on the finite generating systems, because the graphing associated with a p.m.p. action depends on them.

Rigidity and flexibility

Isometric orbit equivalence is a strengthening of L^∞ orbit equivalence. Indeed, an L^∞ orbit equivalence induces almost everywhere a biLipschitz bijection between connected components of the graphings associated with the actions, whereas an isometric orbit equivalence induces almost everywhere an isometry.

Nevertheless, the differences between these notions are considerable. Recall that Belinskaya's theorem implies that L^1 orbit equivalence is more or less trivial for ergodic actions of $\mathbb{Z}$. On the contrary, the theory of L^1 orbit equivalence (and even L^∞ orbit equivalence) for p.m.p. actions of $\mathbb{Z}^d$, $d \geq 2$, is much richer. For instance, Fieldsteel and Friedman proved that given any ergodic action $\mathbb{Z}^2 \curvearrowright^{\alpha} (X,\mu)$, there exists a mixing action $\mathbb{Z}^2 \curvearrowright^{\beta} (Y,\nu)$ such that α and β are L^∞ orbite equivalent.

We show that a similar result in the context of isometric orbit equivalence is false.

Theorem. — *Let $d \geq 2$ and let S be a finite generating system for $\mathbb{Z}^d$. Let $\mathbb{Z}^d \curvearrowright^{\alpha} (X,\mu)$ be a mixing action. Then any p.m.p. action $\mathbb{Z}^d \curvearrowright (Y,\nu)$ which is isometric orbit equivalent to α is conjugate to α.*

Actually, we prove a similar rigidity result for any group Γ equipped with a finite generating system S_Γ such that the automorphism group of the Cayley graph (Γ, S_Γ) is countable. A theorem due to Trofimov implies that this is indeed the case for any finite generating system S of $\mathbb{Z}^d$, see [MS98, Thm. 4.3].

The archetype of a group which admits a Cayley graph whose automorphism group is uncountable is the free group $\mathbf{F}_d$ on $d \geq 2$ generators. We show that the rigidity phenomenon obtained for $\mathbb{Z}^d$ cannot be true for $\mathbf{F}_d$.

Theorem. — *Let $\mathbf{F}_d$ be the free group of $d \geq 2$ generators $x_1,\dots,x_d$ and let S be the finite generating system $S := \{x_1^{\pm 1},\dots,x_d^{\pm 1}\}$. Then there exists ergodic actions $\mathbf{F}_d \curvearrowright^{\alpha} (X,\mu)$ and $\mathbf{F}_d \curvearrowright^{\beta} (Y,\nu)$ that are isometric orbit equivalent, such that α is mixing but β is not.*

References

[AE12] Miklós Abért and Gábor Elek. Hyperfinite actions on countable sets and probability measure spaces. In *Dynamical systems and group actions. Dedicated to Anatoli Stepin on the occasion of his 70th birthday*, pages 1–16. Providence, RI: American Mathematical Society (AMS), 2012.

[Aus16a] Tim Austin. Behaviour of Entropy Under Bounded and Integrable Orbit Equivalence. *Geom. Funct. Anal.*, 26(6):1483–1525, 2016.

[Aus16b] Tim Austin. Integrable measure equivalence for groups of polynomial growth. *Groups Geom. Dyn.*, 10(1):117–154, 2016.

[Bel68] Raisa M. Belinskaya. Partitions of Lebesgue space in trajectories defined by ergodic automorphisms. *Funkts. Anal. Prilozh.*, 2(3):190–199, 1968.

[BG04] Nicolas Bergeron and Damien Gaboriau. Asymptotique des nombres de Betti, invariants ℓ^2 et laminations. *Comment. Math. Helv.*, 79(2):362–395, 2004.

[CJLMT22] Alessandro Carderi, Matthieu Joseph, François Le Maître, and Romain Tessera. Belinskaya's theorem is optimal, 2022.

[DKLMT20] Thiebout Delabie, Juhani Koivisto, François Le Maître, and Romain Tessera. Quantitative measure equivalence. *arXiv:2002.00719*, 2020.

[Ele21] Gábor Elek. Free minimal actions of countable groups with invariant probability measures. *Ergodic Theory Dyn. Syst.*, 41(5):1369–1389, 2021.

[GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitaly I. Suschanskii. Automata, dynamical systems, and groups. In *Dynamical systems, automata, and infinite groups. Transl. from the Russian*, pages 128–203. Moscow: MAIK Nauka/Interperiodica Publishing, 2000.

[Gri11] Rostislav I. Grigorchuk. Some topics in the dynamics of group actions on rooted trees. *Proc. Steklov Inst. Math.*, 273:64–175, 2011.

[GW15] Eli Glasner and Benjamin Weiss. Uniformly recurrent subgroups. In *Recent trends in ergodic theory and dynamical systems. International conference in honor of S. G. Dani's 65th birthday, Vadodara, India, December 26–29, 2012. Proceedings*, pages 63–75. Providence, RI: American Mathematical Society (AMS), 2015.

[Jos21] Matthieu Joseph. Continuum of allosteric actions for non-amenable surface groups, 2021.

[Lig02] Thomas M. Liggett. Tagged particle distributions or how to choose a head at random. In *In and out of equilibrium. Probability with a physics flavor. Papers from the 4th Brazilian school of probability, Mambucaba, Brazil, August 14–19, 2000*, pages 133–162. 2002.

[MS98] Rögnvaldur G. Möller and Norbert Seifter. Digraphical regular representations of infinite finitely generated groups. *Eur. J. Comb.*, 19(5):597–602, 1998.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I: The Rohlin lemma. *Bull. Am. Math. Soc., New Ser.*, 2:161–164, 1980.

[PT16] Jesse Peterson and Andreas Thom. Character rigidity for special linear groups. *J. Reine Angew. Math.*, 716:207–228, 2016.

[SZ94] Garrett J. Stuck and Robert J. Zimmer. Stabilizers for ergodic actions of higher rank semisimple groups. *Ann. Math. (2)*, 139(3):723–747, 1994.

[Ver12] Anatoly M. Vershik. Totally nonfree actions and the infinite symmetric group. *Mosc. Math. J.*, 12(1):193–212, 2012.

Part I

Allostery

Chapter 1

Continuum of allosteric actions for non-amenable surface groups

The content of this chapter is the same as that of the article [Jos21].

Table of contents of Chapter 1

1.1 Introduction

Let Γ be a countable discrete group. Let α be a minimal action of Γ on a compact Hausdorff space C. The action α is ***topologically free*** if for every non-trivial element $\gamma \in \Gamma$, the set $\{x \in C \mid \alpha(\gamma)x = x\}$ has empty interior. This notion of freeness can be characterized by the triviality of the URS associated with the action α as follows. Let $\mathrm{Sub}(\Gamma)$ be the space of subgroups of Γ, and let $\mathrm{Stab}_\alpha : C \to \mathrm{Sub}(\Gamma)$ be the Borel map defined by

$$\mathrm{Stab}_\alpha(x) := \{\gamma \in \Gamma \mid \alpha(\gamma)x = x\}.$$

Here $\mathrm{Sub}(\Gamma)$ is equipped with the topology of pointwise convergence which turns it into a compact totally disconnected topological space on which Γ acts continuously by conjugation. Glasner and Weiss proved in [GW15] that there exists a unique closed, Γ-invariant, minimal subset in the closure of $\{\mathrm{Stab}_\alpha(x) \mid x \in C\}$, called the ***stabilizer Uniformly Recurrent Subgroup***, stabilizer URS for short, associated with the minimal action α, that we denote by URS(α). The stabilizer URS is trivial if it is equal to $\{\{1_\Gamma\}\}$. One of the feature of the stabilizer URS associated with a minimal action α is that its triviality is equivalent to the topological freeness of α, see LEMMA 1.2.1.

Let (X, μ) be a standard probability measure space, and let β be a probability measure preserving (hereafter p.m.p.) action of a countable group Γ on (X, μ). The action β is ***essentially free*** if for every non-trivial $\gamma \in \Gamma$, the set $\{x \in X \mid \alpha(\gamma)x = x\}$ is μ-negligible. The measurable counterpart of the stabilizer URS is the ***stabilizer Invariant Random Subgroup***, stabilizer IRS for short, associated with β. It is defined as the Γ-invariant Borel probability measure $(\mathrm{Stab}_\beta)_*\mu$ on Sub(Γ), and is denoted by IRS(β). A stabilizer IRS is the prototype of an ***IRS***, which is a Borel probability measure on Sub(Γ) that is invariant under the conjugation action of Γ. The ***trivial IRS*** is the Dirac measure at the trivial subgroup. Observe that IRS(β) is trivial if and only if β is essentially free. Abért, Glasner and Virág proved that every IRS is in fact a stabilizer IRS for some p.m.p. action, see [AGV14].

An ***ergodic minimal action*** $\Gamma \curvearrowright (C, \mu)$ is a minimal action of Γ on a compact Hausdorff space C together with a Γ-invariant ergodic Borel probability measure μ. Thus an ergodic minimal action has both a stabilizer URS and a stabilizer IRS. It is a classical result that the essential freeness of an ergodic minimal action implies its topological freeness, see LEMMA 1.2.2. In other words, if the stabilizer IRS of an ergodic minimal action is trivial, then its stabilizer URS is trivial. The present article provides new counterexamples in the study of the converse.

DEFINITION 1.1.1. — An ergodic minimal action is ***allosteric***[1] if it is topologically free but not essentially free. A group is allosteric if it admits an allosteric action.

MAIN QUESTION. — What is the class of allosteric groups?

First, let us discuss examples of groups that don't belong to this class. It is the case for groups whose ergodic IRS's are all atomic, i.e., equal to the uniform measure on the set of conjugates of a finite index subgroup. Indeed, we prove in PROPOSITION 1.2.3 that the IRS of an ergodic minimal action which is topologically free is either trivial, or has no atoms. Thus, if Sub(Γ) is countable, then Γ is not allosteric, see COROLLARY 1.2.4. Examples of groups with only countably many subgroups are: finitely generated nilpotent groups, more generally polycyclic groups, extensions of Noetherian groups by groups with only countably many subgroups (e.g. solvable Baumslag-Solitar groups BS$(1, n)$), see [BLT19], or Tarski monsters.

There are also groups whose ergodic IRS's are all atomic for other reasons. For instance, this is the case for lattices in simple higher rank Lie groups [SZ94], commutator subgroups of either a Higman-Thompson group or the full group of an irreducible shift of finite type [DM14], lattices in projective special linear

[1] ἄλλος: other, στερεός: fix, firm, solid, rigid

group $\mathrm{PSL}_n(k)$ over an infinite countable field k [PT16]. See also [Cre17], [CP] or [Bek20] for other examples of groups with few ergodic IRS's. Thus, none of these groups are allosteric, because of their lack of IRS's.

More surprisingly, there exists non-allosteric groups with plenty of ergodic IRS's, such as countable abelian groups which admit uncountably many subgroups. Indeed, if Γ is such a group, then any Borel probability measure on $\mathrm{Sub}(\Gamma)$ is an IRS, but Γ is not allosteric since any minimal Γ-action which is topologically free is actually essentially free for any invariant measure, see REMARK 1.4.4. Another example is given by the group $\mathrm{FSym}(\mathbb{N})$ of finitely supported permutations on $\mathbb{N}$, as well as its alternating subgroup $\mathrm{Alt}(\mathbb{N})$. They both admit a lot of ergodic IRS's, see [Ver12] and [TTD18]. However, an argument similar to that of Lemma 10.4 in [TTD18] implies that neither $\mathrm{FSym}(\mathbb{N})$ nor $\mathrm{Alt}(\mathbb{N})$ is allosteric.

Let us now discuss examples of allosteric groups. Bergeron and Gaboriau proved in [BG04] that if Γ is non-amenable and isomorphic to a free product of two non-trivial residually finite groups, then Γ is allosteric. We refer to REMARK 1.2.12 for a more precise statement of their results. In [AE07], Abért and Elek independently proved that finitely generated non-abelian free groups are allosteric, and in [AE12], they proved that the free product of four copies of $\mathbb{Z}/2\mathbb{Z}$ admits an allosteric action whose orbit equivalence relation is measure hyperfinite. In all [BG04], [AE07] and [AE12], the allosteric actions obtained are in fact profinite, see Section 1.2.2 for a definition. These were the first known examples answering a question of Grigorchuk, Nekrashevich and Sushchanskii in [GNS00, Problem 7.3.3] about the existence of profinite allosteric actions.

The main result of this article is to prove that non-amenable ***surface groups***, that is fundamental groups of closed surfaces other than the sphere, the torus, the projective plane or the Klein bottle, are allosteric. More precisely, we prove the following result.

THEOREM 1.1.2. — *Any non-amenable surface group admits a continuum of profinite allosteric actions that are pairwise topologically and measurably non-isomorphic.*

Moreover, we prove that the IRS's given by the non-isomorphic allosteric actions that we construct are pairwise distinct. We refer to THEOREM 1.4.1 and THEOREM 1.4.2 for a precise statement of our results. Let us mention that surface groups are known to have a large "zoo" of IRS's. For instance, Bowen, Grigorchuk and Kravchenko proved in [BGK17] that any non elementary Gromov hyperbolic group admits a continuum of IRS's which are weakly mixing when considered as dynamical systems on $\mathrm{Sub}(\Gamma)$. In an upcoming work (personal communication), Carderi, Le Maître and Gaboriau prove that non-amenable surface groups admit a continuum of IRS's whose support coincides with the ***perfect kernel*** of Γ, i.e., the largest closed subset without isolated points in $\mathrm{Sub}(\Gamma)$. However, our IRS's are drastically different from the latter ones: we show that they are not weakly mixing, and that their support is strictly smaller than the perfect kernel, see REMARK 1.4.4 and REMARK 1.4.5.

We develop in Section 1.2 the preliminary results needed about profinite actions and allosteric actions. In particular, we prove that allostery is invariant under commensurability. In order to build ergodic profinite allosteric actions of non-amenable surface groups, we rely on a residual property of non-amenable surface groups in order to prove in Section 1.3 that they admit special kinds of finite index subgroups. The proof of THEOREM 1.1.2 is completed in Section 1.4.

Acknowledgments. I would like to thank A. Le Boudec for various conversations related to this article, as well as T. Nagnibeda for letting me know that Vershik's work implies that the group of finitely supported permutations of the integers is not allosteric. I wish to thank particularly D. Gaboriau for his constant encouragement and support, as well as for his numerous remarks on this article.

1.2 Preliminaries

1.2.1 Topological dynamic and URS/IRS

Let C be a compact Hausdorff space, and let α be an action by homeomorphisms of a countable discrete group Γ on C. The action α is ***minimal*** if the orbit of every $x \in C$ is dense. Recall that α is topologically free if for every non-trivial element $\gamma \in \Gamma$, the closed set

$$\mathrm{Fix}_\alpha(\gamma) := \{x \in C \mid \alpha(\gamma)x = x\}$$

has empty interior. Since C is a Baire space, this is equivalent to saying that the set $\{x \in C \mid \mathrm{Stab}_\alpha(x) \neq \{1_\Gamma\}\}$ is meager, i.e., a countable union of nowhere dense sets.

The set $\mathrm{Sub}(\Gamma)$ of subgroups of Γ naturally identifies with a subset of $\{0,1\}^\Gamma$. It is closed for the product topology. Thus the induced topology on $\mathrm{Sub}(\Gamma)$ turns it into a compact totally disconnected space, on which Γ acts continuously by conjugation. A ***URS*** of Γ is a closed minimal Γ-invariant subset of $\mathrm{Sub}(\Gamma)$. The ***trivial URS*** is the URS that only contains the trivial subgroup. Recall that the stabilizer URS of a minimal action α of Γ on C is the unique closed, Γ-invariant minimal subset in the closure of $\{\mathrm{Stab}_\alpha(x) \mid x \in C\}$. If $C_0 \subset C$ denotes the locus of continuity of $\mathrm{Stab}_\alpha : C \to \mathrm{Sub}(\Gamma)$, then one can prove that $\mathrm{URS}(\alpha)$ is equal to the closure of the set $\{\mathrm{Stab}_\alpha(x) \mid x \in C_0\}$, see [GW15].

A proof of the following classical result can be found in [LBMB18, Prop. 2.7].

LEMMA 1.2.1. — *Let α be a minimal Γ-action on a compact Hausdorff space C. Then α is topologically free if and only if its stabilizer URS is trivial, if and only if there exists $x \in C$ such that $\mathrm{Stab}_\alpha(x)$ is trivial.*

The following lemma clarifies the relation between the stabilizer URS and the stabilizer IRS. Recall that the support of a Borel probability measure is the intersection of all closed subsets of full measure.

Lemma 1.2.2. — *Let α be a minimal Γ-action on a compact Hausdorff space C and μ be a Γ-invariant Borel probability measure on C. Then* $\mathrm{URS}(\alpha)$ *is contained in the support of* $\mathrm{IRS}(\alpha)$. *In particular, if* $\mathrm{IRS}(\alpha)$ *is trivial, then* $\mathrm{URS}(\alpha)$ *is trivial.*

Proof. Let F be a closed subset of $\mathrm{Sub}(\Gamma)$ such that $\mu(\mathrm{Stab}_\alpha^{-1}(F)) = 1$. By minimality of α, every non-empty open subset U of C satisfies $\mu(U) > 0$. Thus, $\mathrm{Stab}_\alpha^{-1}(F)$ is dense in C. Let $x \in C$ be a continuity point of Stab_α. Let $(x_n)_{n\geq 0}$ be a sequence of elements in $\mathrm{Stab}_\alpha^{-1}(F)$ that converges to x. Then $\mathrm{Stab}_\alpha(x) \in F$, and we thus obtain that $\mathrm{URS}(\alpha) \subset F$. By definition of the support of $\mathrm{IRS}(\alpha)$, this implies that $\mathrm{URS}(\alpha) \subset \mathrm{supp}(\mathrm{IRS}(\alpha))$. □

The following proposition gives a partial converse to Lemma 1.2.2.

Proposition 1.2.3. — *Let α be a minimal Γ-action on a compact Hausdorff space C, and μ be a Γ-invariant Borel probability measure on C. If* $\mathrm{URS}(\alpha)$ *is trivial, then* $\mathrm{IRS}(\alpha)$ *is either trivial or atomless.*

Proof. Assume that $\mathrm{IRS}(\alpha)$ has a non-trivial atom $\{\Lambda\}$. By invariance, the atoms $\{\gamma\Lambda\gamma^{-1}\}$ have equal measure for all $\gamma \in \Gamma$. Thus, Λ has only finitely many conjugates. Thus, the closure in $\mathrm{Sub}(\Gamma)$ of the set $\{\mathrm{Stab}_\alpha(x) \mid x \in C\}$ contains the finite set $\{\gamma\Lambda\gamma^{-1} \mid \gamma \in \Gamma\}$, which is closed, Γ-invariant and minimal. Thus, $\mathrm{URS}(\alpha)$ is non-trivial. □

This last result implies that the converse of Lemma 1.2.2 is actually true for groups admitting only countably many subgroups.

Corollary 1.2.4. — *Let α be a minimal Γ-action on a compact Hausdorff space and μ a Γ-invariant Borel probability measure on C. If* $\mathrm{Sub}(\Gamma)$ *is countable, then* $\mathrm{IRS}(\alpha)$ *is trivial iff* $\mathrm{URS}(\alpha)$ *is trivial.*

Thus, groups Γ such that $\mathrm{Sub}(\Gamma)$ is countable are not allosteric.

1.2.2 Profinite actions and their URS/IRS

Let Γ be a countable group. For every $n \geq 0$, let α_n be a Γ-action on a finite set X_n, and assume that for every $n \geq 0$, α_n is a quotient of α_{n+1}, i.e., there exists a Γ-equivariant onto map $q_n : X_{n+1} \twoheadrightarrow X_n$. The inverse limit of the finite spaces X_n is the space

$$\varprojlim X_n := \left\{ (x_n)_{n\geq 0} \in \prod_{n\geq 0} X_n \mid \forall n \geq 0, q_n(x_{n+1}) = x_n \right\}.$$

This space is closed, thus compact, and totally disconnected in the product topology. Let α be the Γ-action by homeomorphisms on $\varprojlim X_n$ defined by

$$\alpha(\gamma)(x_n)_{n\geq 0} := (\alpha_n(\gamma)x_n)_{n\geq 0}.$$

If each X_n is endowed with a Γ-invariant probability measure μ_n, we let μ be the unique Borel probability measure on $\varprojlim X_n$ that projects onto μ_k via the canonical projection $\pi_k : \varprojlim X_n \to X_k$, for every $k \geq 0$. The Γ-action α preserves μ, and is called the ***inverse limit*** of the p.m.p. Γ-actions α_n. A p.m.p. action of Γ is ***profinite*** if it is measurably isomorphic to an inverse limit of p.m.p. Γ-actions on finite sets. A proof of the following lemma can be found in [Gri11, Prop. 4.1].

LEMMA 1.2.5. — *The following are equivalent:*

(i) For every $n \geq 0, \alpha_n$ is transitive, and μ_n is the uniform measure on X_n.

(ii) The action α is minimal.

(iii) The action α is μ-ergodic.

(iv) The action α is uniquely ergodic, i.e., μ is the unique Γ-invariant Borel probability measure on $\varprojlim X_n$.

With the above notations, the following lemma is useful to compute the measure of a closed subset in an inverse limit (here, no group action is involved).

LEMMA 1.2.6. — *Let A be a closed subset of $\varprojlim X_n$. Then $A = \bigcap_{n\geq 0} \pi_n^{-1}(\pi_n(A))$. Thus*

$$\mu(A) = \lim_{n\to+\infty} \mu_n(\pi_n(A)).$$

Proof. First, A is contained in $\cap_{n\geq 0}\pi_n^{-1}(\pi_n(A))$ since it is contained in each $\pi_n^{-1}(\pi_n(A))$. Conversely, let x be in $\cap_{n\geq 0}\pi_n^{-1}(\pi_n(A))$. For every $n \geq 0$, there exists $y_n \in A$ such that $\pi_n(x) = \pi_n(y_n)$. By compactness of A, let $y \in A$ be a limit of some subsequence of $(y_n)_{n\geq 0}$. By definition of the product topology, for every $n \geq 0, \pi_n(x) = \pi_n(y)$, thus $x = y$ and x belongs to A. □

Let $(\Gamma_n)_{n\geq 0}$ be a ***chain*** in Γ, that is an infinite decreasing sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ of finite index subgroups. If $X_n = \Gamma/\Gamma_n$ and μ_n is the uniform probability measure on X_n, then we get a profinite action that is ergodic by LEMMA 1.2.5. Conversely, any ergodic (equivalently minimal) profinite Γ-action $\Gamma \curvearrowright \varprojlim X_n$ is measurably isomorphic to a profinite action of the form $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$ for some chain $(\Gamma_n)_{n\geq 0}$, by fixing a point $x \in \varprojlim X_n$, and letting Γ_n be the stabilizer of $\pi_n(x) \in X_n$.

LEMMA 1.2.7. — *Let $(\Gamma_n)_{n\geq 0}$ be a chain in Γ, and let α be the corresponding ergodic profinite Γ-action. Then* $\mathrm{URS}(\alpha)$ *is trivial if and only if there exists a sequence $(\gamma_n\Gamma_n)_{n\geq 0} \in \varprojlim \Gamma/\Gamma_n$ such that*

$$\bigcap_{n\geq 0} \gamma_n \Gamma_n \gamma_n^{-1} = \{1_\Gamma\}.$$

Proof. For all $x \in \varprojlim \Gamma/\Gamma_n$, if $x = (\gamma_n\Gamma_n)_{n\geq 0}$, then

$$\mathrm{Stab}_\alpha(x) = \bigcap_{n\geq 0} \gamma_n \Gamma_n \gamma_n^{-1}.$$

Thus, the result is a direct consequence of Lemma 1.2.1. □

Proposition 1.2.8. — *Let $(\Gamma_n)_{n\geq 0}$ be a chain in Γ, and let α be the corresponding ergodic profinite Γ-action. If* $\mathrm{URS}(\alpha)$ *is trivial, then either* $\mathrm{IRS}(\alpha)$ *is trivial, or there exists a finite index subgroup $\Lambda \leq \Gamma$ such that the p.m.p. Λ-action by conjugation on* $(\mathrm{Sub}(\Gamma), \mathrm{IRS}(\alpha))$ *is not ergodic.*

Proof. Assume that the p.m.p. Γ-action by conjugation on $(\mathrm{Sub}(\Gamma), \mathrm{IRS}(\alpha))$ remains ergodic under any finite index subgroup of Γ. Since $\mathrm{URS}(\alpha)$ is trivial, there exists by Lemma 1.2.7 a sequence $(\gamma_n)_{n\geq 0}$ of elements in Γ such that

$$\bigcap_{n\geq 0} \gamma_n \Gamma_n \gamma_n^{-1} = \{1_\Gamma\}.$$

For every $k \geq 0$, if $\pi_k : \varprojlim \Gamma/\Gamma_n \to \Gamma/\Gamma_k$ denotes the projection onto the k^{th} coordinate, then the set

$$\{\mathrm{Stab}_\alpha(x) \mid x \in \varprojlim \Gamma/\Gamma_n, \pi_k(x) = \gamma_k\Gamma_k\} \subset \mathrm{Sub}(\Gamma)$$

has positive measure for $\mathrm{IRS}(\alpha)$, is contained in $\mathrm{Sub}(\gamma_k\Gamma_k\gamma_k^{-1})$ and is invariant under the finite index subgroup $\mathrm{Stab}_{\alpha_k}(\gamma_k\Gamma_k) = \gamma_k\Gamma_k\gamma_k^{-1}$. By ergodicity, it is a full measure set. Thus, for a.e. $x \in \varprojlim \Gamma/\Gamma_n$, $\mathrm{Stab}_\alpha(x)$ is a subgroup of $\gamma_k\Gamma_k\gamma_k^{-1}$. Since this is true for every $k \geq 0$, we conclude that $\mathrm{IRS}(\alpha)$ is trivial. □

1.2.3 Allostery and commensurability

Two groups Γ_1 and Γ_2 are ***commensurable*** if there exists finite index subgroups $\Lambda_1 \leq \Gamma_1$ and $\Lambda_2 \leq \Gamma_2$ such that Λ_1 is isomorphic to Λ_2. In this section, we prove the following result.

Theorem 1.2.9. — *Allostery is invariant under commensurability.*

We prove Theorem 1.2.9 in two steps, by showing that allostery is inherited by finite index overgroups in Proposition 1.2.10 and by finite index subgroups in Proposition 1.2.11. Let Γ be a countable group and $\Lambda \leq \Gamma$ a finite index subgroup. Let $\alpha : \Lambda \curvearrowright (C, \mu)$ be an action by homeomorphisms on a compact Hausdorff space C with a Λ-invariant Borel probability measure μ on C. The group Γ acts on $X \times \Gamma$ trivially on the first factor and by left multiplication on the second factor. This action projects onto a Γ-action by homeomorphisms on the quotient of $X \times \Gamma$ by the Λ-action $\lambda \cdot (x, \gamma) = (\alpha(\lambda)x, \gamma\lambda)$, and the product of μ with the counting measure projects onto a Γ-invariant Borel probability measure. This action is the Γ-action ***induced*** by α.

Proposition 1.2.10. — *Let Γ be a countable group and $\Lambda \leq \Gamma$ a finite index subgroup. Then the Γ-action induced by any allosteric Λ-action is allosteric.*

Proof. Let $\alpha : \Lambda \curvearrowright (C, \mu)$ be an allosteric action. It is an exercise to prove that the Γ-action β induced by Λ is ergodic and minimal. Moreover, $\mathrm{IRS}(\beta)$ is non-trivial since the restriction of β to Λ is not essentially free. Finally, URS(α) is trivial, thus there exists by LEMMA 1.2.1 a point $x \in C$ such that $\mathrm{Stab}_\alpha(x) = \{1_\Lambda\}$. Let y be the projection of $(x, 1)$ onto the quotient $(C \times \Gamma)/\Lambda$, then $\mathrm{Stab}_\beta(y) = \{1_\Gamma\}$. Since β is minimal, this implies by LEMMA 1.2.1 that URS(β) is trivial. Thus β is allosteric. □

PROPOSITION 1.2.11. — *Any finite index subgroup of an allosteric group is allosteric.*

Proof. Let $\Lambda \leq \Gamma$ be a finite index subgroup. We recall the following two facts. If $\Gamma \curvearrowright (X, \mu)$ is an ergodic action, then any Λ-invariant measurable set $A \subset X$ of positive measure satisfies $\mu(A) \geq 1/[\Gamma : \Lambda]$. Moreover, for any Λ-invariant measurable set $B \subset X$ of positive measure, there exists a Λ-invariant measurable set $A \subset B$ of positive measure on which Λ acts ergodically.

Let Γ be an allosteric group, and let $\Lambda \leq \Gamma$ be a finite index subgroup. Let N be the normal core of Λ (the intersection of the conjugates of Λ). It is a finite index normal subgroup of Γ which is contained in Λ. We will prove that N is allosteric. PROPOSITION 1.2.10 will then imply that Λ is allosteric. We let $d = [\Gamma : N]$ and we fix $\gamma_1, \ldots, \gamma_d \in \Gamma$ a coset representative system for N in Γ. Let $\Gamma \curvearrowright^\alpha (C, \mu)$ be an allosteric action. For all $x \in C$, we define $\mathcal{O}_N(x) = \overline{\{\alpha(\gamma)x \mid \gamma \in N\}}$. This is a closed, N-invariant subset of C. By minimality of α, for all $x \in C$,

$$X = \bigcup_{i=1}^{d} \mathcal{O}_N(\alpha(\gamma_i)x).$$

Moreover, since N is normal in Γ, for all $x \in C$ and $\gamma \in \Gamma$, we have $\mathcal{O}_N(\alpha(\gamma)x) = \alpha(\gamma)\mathcal{O}_N(x)$. This implies that $\mu(\mathcal{O}_N(\alpha(\gamma)x)) = \mu(\mathcal{O}_N(x))$ and that $\mu(\mathcal{O}_N(x)) > 0$. Let y be a point in some closed, N-invariant and N-minimal set. Then $N \curvearrowright \mathcal{O}_N(y)$ is minimal. Let $A \subset \mathcal{O}_N(y)$ be a N-invariant measurable set of positive measure on which N acts ergodically. Let μ_A be the Borel probability measure on A induced by μ. Then $N \curvearrowright (\mathcal{O}_N(y), \mu_A)$ is an ergodic minimal action, which is still topologically free. Let us prove that it is not essentially free. Since α is allosteric, IRS(α) is atomless, see PROPOSITION 1.2.3. Thus, for μ-a.e. $x \in C$, $\mathrm{Stab}_\alpha(x)$ is infinite. Since N has finite index in Γ, this implies that for μ-a.e. $x \in C$, $\mathrm{Stab}_\alpha(x) \cap N$ is infinite. Thus $N \curvearrowright (\mathcal{O}_N(y), \mu_A)$ is not essentially free, and thus is allosteric. □

REMARK 1.2.12. — It is proved in [BG04, Théorème 4.1] that if Γ is isomorphic to a free product of two *infinite* residually finite groups, then Γ admits a continuum of profinite allosteric actions. Let Γ' be a non-amenable group which is isomorphic to a free product of two non-trivial residually finite groups. Then Kurosh's theorem [Ser77, Section 5.5] implies that Γ' admits a finite index subgroup Γ isomorphic to a free product of finitely many (and at least two) residually finite infinite groups. PROPOSITION 1.2.10 then implies that Γ' is allosteric.

1.3 Finite index subgroups of surface groups

1.3.1 Residual properties of surface groups

A surface group is the fundamental group of a closed connected surface. If the surface is orientable, then its fundamental group is called an ***orientable surface group***, and a presentation is given by

$$\langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \ldots [x_g, y_g] = 1 \rangle,$$

for some $g \geq 1$ called the genus of the surface (if $g = 0$, then the surface is a sphere, and its fundamental group is trivial). If the surface is non-orientable, we call its fundamental group a ***non-orientable surface group***. It has a presentation given by

$$\langle x_1, \ldots, x_g \mid x_1^2 \ldots x_g^2 = 1 \rangle,$$

for some $g \geq 1$ called the genus of the surface. A surface group is non-amenable if and only if it is the fundamental group of a surface other that the sphere, the torus (orientable surfaces of genus 0 and 1), the projective plane or the Klein bottle (non-orientable surfaces of genus 1 and 2).

Definition 1.3.1. — Let p be a prime number. A group Γ is a ***residually finite p-group*** if for every non-trivial element $\gamma \in \Gamma$, there exists a normal subgroup $N \trianglelefteq \Gamma$ such that Γ/N is a finite p-group and $\gamma \notin N$. Equivalently, Γ is a residually finite p-group if and only if there exists a chain $(\Gamma_n)_{n\geq 0}$ in Γ consisting of normal subgroups such that for every $n \geq 0$, the quotient Γ/Γ_n is a finite p-group, and

$$\bigcap_{n\geq 0} \Gamma_n = \{1_\Gamma\}.$$

Baumslag proved in [Bau62] that orientable surface groups are residually free, i.e., for every non-trivial element γ, there exists a normal subgroup $N \trianglelefteq \Gamma$ such that Γ/N is a free group and $\gamma \notin N$. Moreover, free groups are residually finite p-groups for every prime p, a result independently proved by Takahasi [Tak51] and by Gruenberg in [Gru57] (using a result of Magnus [Mag35]). This implies the following well-known result.

Theorem 1.3.2. — *Orientable surface groups are residually finite p-groups for every prime p.*

Remark 1.3.3. — By a result of Baumslag [Bau67], non-amenable non-orientable surface groups are also residually p-finite groups for every prime p. However, we leave as an exercise to the interested reader the fact that the fundamental group of a Klein bottle is not residually p for some prime p. We will not require these results.

1.3.2 Special kind of finite index subgroups in surface groups

Let A, B be two non-empty totally ordered finite sets. In what follows, when writing $\prod_{i\in A}$ or $\prod_{j\in B}$ we mean that the product is computed with respect to the increasing order of A or B respectively. We let $\Gamma_{A,B}$ be the group defined by the generators $(a_i, \alpha_i)_{i\in A}$ and $(b_j, \beta_j)_{j\in B}$, and the relation

$$\prod_{i\in A}[a_i, \alpha_i] = \prod_{j\in B}[b_j, \beta_j].$$

Then $\Gamma_{A,B}$ is isomorphic to a non-amenable orientable surface group, and every non-amenable orientable surface group is isomorphism to $\Gamma_{A,B}$ for some non-empty totally ordered finite sets A and B. The group $\Gamma_{A,B}$ naturally splits as an amalgamated product

$$\Gamma_{A,B} = \Gamma_A *_{\mathbb{Z}} \Gamma_B$$

where Γ_A and Γ_B are the free groups of rank $2|A|$ and $2|B|$ respectively, freely generated by $(a_i, \alpha_i)_{i\in A}$ and $(b_j, \beta_j)_{j\in B}$ respectively. If $A' \subset A$ and $B' \subset B$, there is a natural onto group homomorphism $\Gamma_{A,B} \twoheadrightarrow \Gamma_{A',B'}$ defined on the generators by

$$\begin{array}{llll} a_i \mapsto a'_i & \text{for every } i \in A', & b_j \mapsto b'_j & \text{for every } j \in B', \\ \alpha_i \mapsto \alpha'_i & \text{for every } i \in A', & \beta_j \mapsto \beta'_j & \text{for every } j \in B', \\ a_i, \alpha_i \mapsto 1 & \text{for every } i \in A \setminus A', & b_j, \beta_j \mapsto 1 & \text{for every } j \in B \setminus B'. \end{array}$$

We say that this morphism *erases* the generators $a_i, \alpha_i, b_j, \beta_j$ for $i \in A \setminus A'$ and $j \in B \setminus B'$, see FIGURE 1.1. Algebraically, $\Gamma_{A',B'}$ is isomorphic to the quotient of $\Gamma_{A,B}$ by the normal closure of the set $\{(\alpha_i, \beta_i) \mid i \in A \setminus A'\} \cup \{(b_j, \beta_j) \mid j \in B \setminus B'\}$ in $\Gamma_{A,B}$, and the homomorphism $\Gamma_{A,B} \twoheadrightarrow \Gamma_{A',B'}$ corresponds to the quotient group homomorphism.

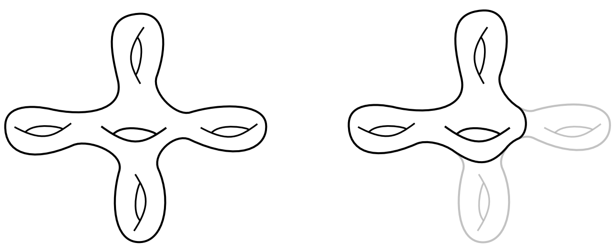

FIGURE 1.1. — An illustration of the morphism that erases generators.

Here is the main theorem of this section. In what follows, $\mathbb{Z}[1/p]$ denotes the set of rational numbers of the form k/p^n for $k, n \in \mathbb{Z}$.

THEOREM 1.3.4. — *Let Γ be a non-amenable orientable surface group, and fix a decomposition $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$ as above. Let p be a prime number, and $r \in]0,1[\cap \mathbb{Z}[1/p]$. Let $\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ be the normal closure of the amalgamated subgroup $\mathbb{Z}$ in Γ_B. For every non-trivial $\gamma \in \Gamma$ and for every element $\delta \in \Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$, there exists a finite index subgroup $\Lambda \leq \Gamma$ such that* *(i)* $\gamma \notin \Lambda$.

(ii) The index $[\Gamma : \Lambda]$ *is a power of* p.

(iii) The number of left cosets $x \in \Gamma/\Lambda$ *that are fixed by every element in* Γ_A *is equal to* $r[\Gamma : \Lambda]$.

(iv) None of the left coset $x \in \Gamma/\Lambda$ *is fixed by* δ.

Proof. Fix A, B two non-empty totally ordered finite sets, such that Γ is isomorphic to $\Gamma_{A,B}$. Let S be the set of generators $(a_i, \alpha_i)_{i\in A}$ and $(b_j, \beta_j)_{j\in B}$. Let j_0 be the smallest element in B. Let $\gamma \in \Gamma \setminus \{1_\Gamma\}$ and $\delta \in \Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. Let p be a prime number, and $r \in]0, 1[\cap \mathbb{Z}[1/p]$.

Step 1: Cyclic covering. Let $\varphi : \Gamma_{A,B} \twoheadrightarrow \mathbb{Z}$ be the onto homomorphism defined on the generators of $\Gamma_{A,B}$ by

$$\varphi(b_{j_0}) = 1,\ \varphi(\beta_{j_0}) = 0,$$
$$\varphi(a_i) = \varphi(\alpha_i) = \varphi(b_j) = \varphi(\beta_j) = 0 \text{ for every } i \in A, j \in B \setminus \{j_0\}.$$

For every $d \geq 1$, we let Λ_d be the kernel of the homomorphism $\Gamma \to \mathbb{Z}/d\mathbb{Z}$ obtained by composing φ with the homomorphism of reduction modulo d. Then Λ_d is a surface group. Let us describe a generating set for Λ_d. For every $0 \leq k \leq d-1$ and $i \in A$, let $a_{i,k}$ and $\alpha_{i,k}$ be the conjugates of a_i and α_i respectively, by $b_{j_0}^k$. Similarly, let $b_{j,k}$, and $\beta_{j,k}$ be the conjugate of b_j and β_j respectively, by $b_{j_0}^k$. Then Λ_d is generated by the set

$$\bigcup_{k=0}^{d-1} \{a_{i,k}, \alpha_{i,k} \mid i \in A\} \cup \bigcup_{k=0}^{d-1} \{b_{j,k}, \beta_{j,k} \mid j \in B \setminus \{j_0\}\} \cup \{b_{j_0}^d, \beta_{j_0}\}.$$

So far, every left coset $x \in \Gamma/\Lambda_d$ is fixed by every element of Γ_A, and either every or none of the left coset $x \in \Gamma/\Lambda_d$ is fixed by δ, depending on whether $\delta \in \Lambda_d$ or not.

Step 2: Erasing the right amount of generators. Let n be the length of $\gamma \in \Gamma \setminus \{1_\Gamma\}$ in the generating set S. In the sequel we let d be a (large enough) power of the prime p such that rd is an integer, and $rd + n \leq d$. Let $E \subset \{n+1, \ldots, d-1-n\}$ be a subset of cardinality rd, so that γ doesn't belong to the normal closure N of the set $\cup_{k\in E} b_{j_0}^k \Gamma_A b_{j_0}^{-k}$ in Λ_d. Let us prove that none of the conjugate of δ by a power of b_{j_0} belongs to N. Assume this is not the case, then this would imply that δ belongs to the normal closure of $\cup_{k=0}^{d-1} b_{j_0}^k \Gamma_A b_{j_0}^{-k}$ in Λ_d, which is easily seen to be equal to the normal closure $\langle\langle \Gamma_A \rangle\rangle^\Gamma$ of Γ_A in Γ. But the group $\Gamma/\langle\langle \Gamma_A \rangle\rangle^\Gamma$ is naturally isomorphic to $\Gamma_B/\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$, in such a way that the following diagram commutes

$$\begin{array}{ccc} \Gamma_B & & \\ \downarrow & \searrow & \\ \Gamma/\langle\langle\Gamma_A\rangle\rangle^\Gamma & \longrightarrow & \Gamma_B/\langle\langle\mathbb{Z}\rangle\rangle^{\Gamma_B}, \end{array}$$

which implies that $\Gamma_B \cap \langle\langle\Gamma_A\rangle\rangle^\Gamma$ is equal to $\langle\langle\mathbb{Z}\rangle\rangle^{\Gamma_B}$. This would thus imply that $\delta \in \langle\langle\mathbb{Z}\rangle\rangle^{\Gamma_B}$, a contradiction.

Step 3: The group Λ_d/N is a residually finite p-group. We let $\pi : \Lambda_d \twoheadrightarrow \Lambda_d/N$ be the quotient group homomorphism. Since Λ_d/N is an orientable surface group, it is a residually finite p-group by Theorem 1.3.2. Thus, there exists a normal subgroup $N' \trianglelefteq \Lambda_d/N$ whose index is a power of p, such that for every $k \in \{0,\dots,d-1\} \setminus E$, for every $i \in A$, $\pi(a_{i,k}) \notin N'$ and $\pi(\alpha_{i,k}) \notin N'$. If $\gamma \in \Lambda_d$, we also assume that $\pi(\gamma) \notin N$, and if $\delta \in \Lambda_d$, we also assume that for all $k \in \{0,\dots,d-1\}$, $\pi(b_{j_0}^k \delta b_{j_0}^{-k}) \notin N'$. Let us prove that the subgroup $\Lambda := \pi^{-1}(N')$ of Γ satisfies the four conclusions of the theorem.

Proof of (i). Either $\gamma \notin \Lambda_d$ and thus $\gamma \notin \Lambda$, or $\gamma \in \Lambda_d$ and $\pi(\gamma) \notin N$.

Proof of (ii). Since the index of N' in Λ_d/N is a power of p, $[\Lambda_d : \Lambda]$ is also a power of p. Thus $[\Gamma : \Lambda] = [\Gamma : \Lambda_d][\Lambda_d : \Lambda]$ is a power of p.

Proof of (iii). By construction, $x \in \Gamma/\Lambda$ is fixed by every element in Γ_A if and only its image under the canonical $[\Lambda_d : \Lambda]$-to-one map $\Gamma/\Lambda \mapsto \Gamma/\Lambda_d$ is equal to $b_{j_0}^k \Lambda_d$ for some $k \in E$. Since $|E| = rd$, there are exactly $rd[\Lambda_d : \Lambda] = r[\Gamma : \Lambda]$ such $x \in \Gamma/\Lambda$.

Proof of (iv). If $\delta \notin \Lambda_d$, then none of the coset $x \in \Gamma/\Lambda$ is fixed by δ. If $\delta \in \Lambda_d$, then for all $k \in \{0,\dots,d-1\}$, we have $\pi(b_{j_0}^k \delta b_{j_0}^{-k}) \notin N'$, and thus $\delta b_{j_0}^{-k}\Lambda \neq b_{j_0}^{-k}\Lambda$. By normality of Λ in Λ_d, we deduce that none of the coset $x \in \Gamma/\Lambda$ is fixed by δ. □

1.4 Proof of the main theorem

In this section, we give the proof of Theorem 1.1.2. More precisely, we prove the following results.

Theorem 1.4.1 (Orientable case). — *Let Γ be a non-amenable orientable surface group, and fix a decomposition $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$ as above. Let $\langle\langle\mathbb{Z}\rangle\rangle^{\Gamma_B}$ be the normal closure of the amalgamated subgroup $\mathbb{Z}$ in Γ_B. Then there exists a continuum $(\alpha^t)_{0<t<1}$ of ergodic profinite allosteric actions of Γ such that for all $0 < t < 1$,*

1. *The set of points whose stabilizer for α^t contains Γ_A has measure t.*

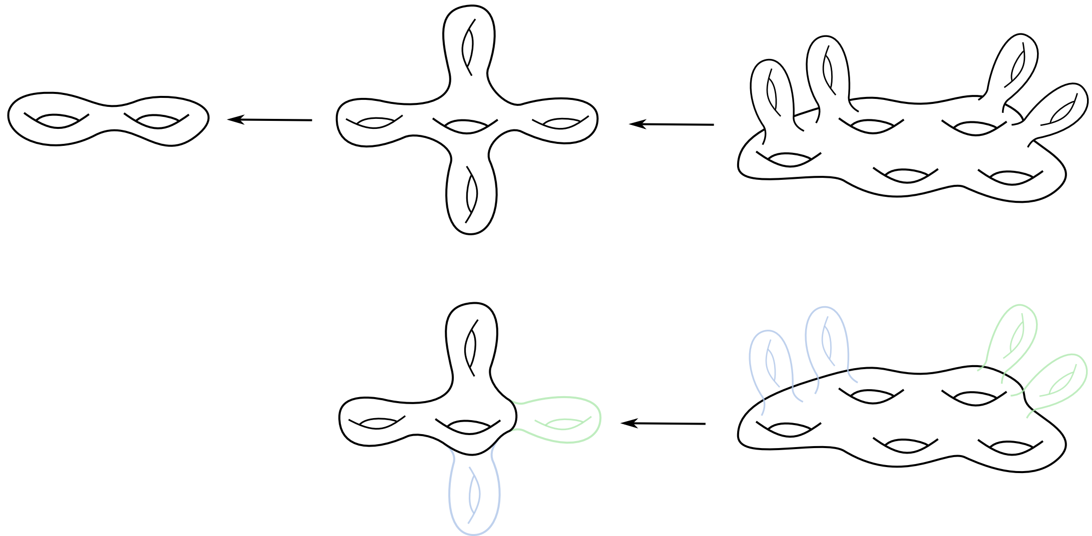

FIGURE 1.2. — Illustrations of the proof of THEOREM 1.3.4. The above line illustrates the coverings corresponding to the inclusions $\Lambda \leq \Lambda_d \leq \Gamma$. The bottom line illustrates the covering corresponding to the inclusion $N' \leq \Lambda_d/N$.

2. *Each element of* $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ *acts essentially freely for* α^t.

In particular, for all $0 < s < t < 1$, *the actions* α^s *and* α^t *are neither topologically nor measurably isomorphic, and the probability measures* $\mathrm{IRS}(\alpha^s)$ *and* $\mathrm{IRS}(\alpha^t)$ *are distinct.*

THEOREM 1.4.2 (Non-orientable case). — *Let* Γ' *be a non-amenable non-orientable surface group. Then there exists an index two subgroup* $\Gamma \leq \Gamma'$ *which is isomorphic to an orientable surface group, and which decomposes as* $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$, *and a continuum* $(\beta^t)_{0<t<1}$ *of ergodic profinite allosteric actions of* Γ' *such that for all* $0 < t < 1$, *the set of points whose stabilizer for* β^t *contains* Γ_A *has measure* $t/2$. *In particular, for all* $0 < s < t < 1$, *the actions* β^s *and* β^t *are neither topologically nor measurably isomorphic, and the probability measures* $\mathrm{IRS}(\beta^s)$ *and* $\mathrm{IRS}(\beta^t)$ *are distinct.*

During the proof of these theorems, we will need the following lemma.

LEMMA 1.4.3. — *Let* Γ *be a group, and* $\Lambda_1, \ldots, \Lambda_n$ *be finite index subgroups of* Γ. *If the indices* $[\Gamma : \Lambda_i]$, $i \in \{1, \ldots, n\}$, *are pairwise coprime integers, then the left coset action* $\Gamma \curvearrowright \Gamma/(\Lambda_1 \cap \cdots \cap \Lambda_n)$ *is isomorphic to the diagonal action* $\Gamma \curvearrowright \Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$ *of the left coset actions* $\Gamma \curvearrowright \Gamma/\Lambda_i$.

Proof. The kernel of the group homomorphism $\Gamma \to \Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$ defined by $\gamma \mapsto (\gamma\Lambda_1, \ldots, \gamma\Lambda_n)$ is equal to $\Lambda_1 \cap \cdots \cap \Lambda_n$. Thus $\Gamma/(\Lambda_1 \cap \cdots \cap \Lambda_n)$ is isomorphic to a subgroup of $\Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$. Moreover, for every $1 \leq i \leq n$,

$$[\Gamma : \Lambda_1 \cap \cdots \cap \Lambda_n] = [\Gamma : \Lambda_i][\Lambda_i : \Lambda_1 \cap \cdots \cap \Lambda_n],$$

and since the indices $[\Gamma : \Lambda_i]$ are pairwise coprime, this implies that $[\Gamma : \Lambda_1 \cap \cdots \cap \Lambda_n]$ is divisible by $[\Gamma : \Lambda_1] \ldots [\Gamma : \Lambda_n]$. Thus, the group homomorphism $\Gamma/\Lambda_1 \cap \cdots \cap \Lambda_n \to \Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$ is an isomorphism, and it is Γ-equivariant. □

We are now ready to prove THEOREM 1.4.1 and THEOREM 1.4.2.

Proof of THEOREM *1.4.1.* Let Γ be a non-amenable orientable surface group, and we fix a decomposition $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$. Let $0 < t < 1$ be a real number. Let $(p_n)_{n\geq 1}$ be a sequence of pairwise distinct prime numbers. We fix a sequence $(r_n)_{n\geq 1}$ such that each r_n belongs to $]0,1[\cap\mathbb{Z}[1/p_n]$ and $\prod_{n\geq 1} r_n = t$. Such a sequence exists because each $\mathbb{Z}[1/p_n]$ is dense in $\mathbb{R}$. Finally, let $(\gamma_n)_{n\geq 0}$ be an enumeration of the elements in Γ with $\gamma_0 = 1$, and $(\delta_n)_{n\geq 1}$ be an enumeration of the elements in $\Gamma_B \setminus \langle\langle Z\rangle\rangle^{\Gamma_B}$. For every $n \geq 1$, there exists by THEOREM 1.3.4 a finite index subgroup $\Lambda_n^t \leq \Gamma$ which doesn't contain γ_n, whose index $[\Gamma : \Lambda_n^t]$ is a power of p_n, such that the number of left cosets $x \in \Gamma/\Lambda_n^t$ that are fixed by any element of Γ_A is equal to $r_n[\Gamma : \Lambda_n^t]$, and such that none of the left coset $x \in \Gamma/\Lambda_n^t$ is fixed by δ_n. For every $n \geq 1$, let $\Gamma_n^t := \Lambda_1^t \cap \cdots \cap \Lambda_n^t$. The sequence $(\Gamma_n^t)_{n\geq 1}$ forms a chain in Γ and we denote by α^t the corresponding ergodic profinite action, and by μ_t the profinite Γ-invariant probability measure on $\varprojlim \Gamma/\Gamma_n^t$. This is a p.m.p. ergodic minimal action and we will prove that it is allosteric. By construction of Λ_n^t, we have that

$$\bigcap_{n\geq 1} \Gamma_n^t = \{1_\Gamma\}.$$

This implies by LEMMA 1.2.7 that $\mathrm{URS}(\alpha^t)$ is trivial. Let us prove that each element of $\Gamma_B \setminus \langle\langle \mathbb{Z}\rangle\rangle^{\Gamma_B}$ acts essentially freely for α^t. Let $\delta \in \Gamma_B \setminus \langle\langle \mathbb{Z}\rangle\rangle^{\Gamma_B}$. By LEMMA 1.4.3, the number of $x \in \Gamma/\Gamma_n^t$ such that $\delta x = x$ is equal to the number of $(x_1,\ldots,x_n) \in \Gamma/\Lambda_1^t \times \cdots \times \Gamma/\Lambda_n^t$ such that $(\delta x_1,\ldots,\delta x_n) = (x_1,\ldots,x_n)$. If n is large enough, then this last number is zero by construction of Λ_n^t. Thus, LEMMA 1.2.6 implies that $\mathrm{Fix}_{\alpha^t}(\delta)$ is μ_t-negligible.

Finally, let us prove that the actions α^t are not essentially free. By construction, the indices $[\Gamma : \Lambda_i^t]$ are pairwise coprime. Thus, LEMMA 1.4.3 implies that the number of $x \in \Gamma/\Gamma_n^t$ that are fixed by every element in Γ_A is equal to the number of $(y_1,\ldots,y_n) \in \Gamma/\Lambda_1^t \times \cdots \times \Gamma/\Lambda_n^t$ that are fixed for the diagonal action by every element in Γ_A. By construction of Λ_i^t, this number is equal to $r_1[\Gamma : \Lambda_1^t] \times \cdots \times r_n[\Gamma : \Lambda_n^t]$ which is equal to $r_1 \ldots r_n[\Gamma : \Gamma_n^t]$. Thus, LEMMA 1.2.6 implies that the μ_t-measure of the set of points whose stabilizer for α^t contains Γ_A is t. In particular, this implies that $\mathrm{IRS}(\alpha^t)$ is non-trivial. Thus α^t is allosteric. Moreover, this also implies that for all $0 < s < t < 1$, the actions α^s and α^t are not measurably isomorphic, and thus not topologically isomorphic since every α^t is uniquely ergodic by LEMMA 1.2.5, and this finally implies that the measures $\mathrm{IRS}(\alpha^s)$ and $\mathrm{IRS}(\alpha^t)$ are distinct. □

Proof of THEOREM *1.4.2.* Let Σ' be a non-orientable surface of genus $g \geq 3$. Consider the usual embedding of an orientable surface Σ of genus $g-1$ into $\mathbb{R}^3$ in such a way that the reflexions in all 3 coordinate planes map the surface to itself, and let ι to be the fixed-point free antipodal map $x \mapsto -x$. Then Σ' is homeomorphic to the quotient of Σ by ι, and the covering $\Sigma \mapsto \Sigma/\iota \approx \Sigma'$ is called the orientation covering. We decompose Σ as the union of two surfaces Σ_A and Σ_B with one boundary, of genus $|A|$ and $|B|$ respectively, with $|A| \leq |B|$, so that $\iota(\Sigma_A) \subset \Sigma_B$. Fix a point $p \in \Sigma_A \cap \Sigma_B$, then Van Kampen's Theorem implies that the fundamental group Γ of the surface Σ based at p is isomorphic to $\Gamma_A *_{\mathbb{Z}} \Gamma_B$ with $\Gamma_A = \pi_1(\Sigma_A, p), \Gamma_B = \pi_1(\Sigma_B, p)$ and $\mathbb{Z} \approx \pi_1(\Gamma_A \cap \Gamma_B, p)$. The fundamental group Γ' of Σ' based at $p' = \iota(p)$ naturally contains the subgroup Γ as an index-two subgroup. Fix a curve contained in Σ_B that joins p to $\iota(p)$. This produces an element $\gamma_0 \in \Gamma' \setminus \Gamma$, that satisfies $\gamma_0 \Gamma_A \gamma_0^{-1} \leq \Gamma_B$.

Let $(\alpha^t)_{0<t<1}$ be a continuum of allosteric Γ-actions on (X_t, μ_t) given by Theorem 1.4.1. The actions $\beta^t : \Gamma' \curvearrowright (Y_t, \nu_t)$ induced by the Γ-actions α^t are allosteric, see Proposition 1.2.10. Let us prove that the set of points in Y_t whose stabilizer for β^t contains Γ_A has ν_t-measure $t/2$. Since β^t is an induced action and $[\Gamma' : \Gamma] = 2$, the Γ'-action β^t is measurably isomorphic to a p.m.p. Γ'-action on $(X_t \times \{0,1\}, \mu_t \times \text{ unif})$, still denoted by β^t, that satisfies the following two properties:

1. For every $\gamma \in \Gamma' \setminus \Gamma$, the sets $X_t \times \{0\}$ and $X_t \times \{1\}$ are switched by $\beta^t(\gamma)$.
2. For every $\gamma \in \Gamma$, for every $x \in X_t$, $\beta^t(\gamma)(x,0) = (\alpha^t(\gamma)x, 0)$ and $\beta^t(\gamma)(x,1) = (\alpha^t(\gamma_0\gamma\gamma_0^{-1})x, 1)$.

This implies that for all $(x,\varepsilon) \in X_t \times \{0,1\}$, the subgroup Γ_A is contained in $\text{Stab}_{\beta^t}(x,\varepsilon)$ if and only if either $\varepsilon = 0$ and Γ_A is contained in $\text{Stab}_{\alpha^t}(x)$, or $\varepsilon = 1$ and $\gamma_0\Gamma_A\gamma_0^{-1}$ is contained in $\text{Stab}_{\alpha^t}(x)$. Thus, the set of points whose stabilizer for β^t contains Γ_A has ν_t-measure

$$\frac{t + \mu_t(\{x \in X_t \mid \gamma_0\Gamma_A\gamma_0^{-1} \leq \text{Stab}_{\alpha^t}(x)\})}{2}.$$

In order to finish the proof, it is enough to prove that the intersection of $\gamma_0\Gamma_A\gamma_0^{-1}$ and $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ is non-trivial, since any element in $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ acts essentially freely for α^t. The conjugation by γ_0 induces a group automorphism $\varphi : \Gamma \mapsto \Gamma$, such that $\varphi(\Gamma_A) \leq \Gamma_B$. Since Γ_A is not contained in the derived subgroup $D(\Gamma)$, so is $\varphi(\Gamma_A)$. But the amalgamated subgroup $\mathbb{Z}$ is contained in $D(\Gamma)$, thus so is $\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. This implies that the intersection $\varphi(\Gamma_A) \cap (\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B})$ is non-empty. We deduce that the set of points whose stabilizer for β^t contains Γ_A has ν_t-measure $t/2$. We conclude that the actions β^t are neither measurably nor topologically pairwise isomorphic and that their IRS are pairwise disjoint as in Theorem 1.4.1. □

Remark 1.4.4. — Let $\alpha : \Gamma \curvearrowright (C, \mu)$ be an allosteric action. Then we have

$$\text{supp}(\text{IRS}(\alpha)) \subset \overline{\{\text{Stab}_\alpha(x) \mid x \in C\}}.$$

This implies that the support of $\text{IRS}(\alpha)$ doesn't contain any non-trivial subgroup with only finitely many conjugates, because otherwise the closure of the set $\{\text{Stab}_\alpha(x) \mid x \in C\}$ would contain a closed minimal Γ-invariant set $\neq \{\{1_\Gamma\}\}$.

Carderi, Gaboriau and Le Maître proved (personal communication) that the perfect kernel of a surface group coincides with the set of its infinite index subgroups. This implies that allosteric actions of surface groups are not totipotent (a p.m.p. action is ***totipotent*** if the support of its IRS coincide with the perfect kernel of the group, see [CGLM20]).

REMARK 1.4.5. — A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is ***weakly mixing*** if for every $\varepsilon > 0$ and every finite collection Ω of measurable subsets of X, there exists a $\gamma \in \Gamma$ such that for every $A, B \in \Omega$

$$|\mu(\gamma A \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

With this definition, it is easily seen that the restriction of a weakly mixing action to a finite index subgroup remains weakly mixing. Thus PROPOSITION 1.2.8 implies that the IRS's of non-amenable surface groups we have constructed are not weakly mixing.

REMARK 1.4.6. — The proof of our main theorem applies mutatis mutandis to branched orientable surface groups, that is fundamental groups of closed orientable branched surfaces (see FIGURE 1.3). These groups can be written as amalgams. Fix an integer $g \geq 2$ as well as $2g$ letters $x_1, y_1, \ldots, x_g, y_g$. Fix a partition of $\{1, \ldots, g\}$ into n nonempty intervals $A_1, \ldots, A_n$. Let Γ_k be the free group generated by x_i and y_i for every $i \in A_k$, and let $\mathbb{Z} \to \Gamma_k$ be the injective homomorphism defined by sending the generator of $\mathbb{Z}$ to the product $\prod_{i \in A_k} [x_i, y_i]$. Then the amalgam $*_{\mathbb{Z}} \Gamma_i$ is a branched orientable surface group, and any branched orientable surface group can be obtained this way. The fundamental group of a closed orientable branched surface of genus ≥ 2 is a residually p-finite group for every prime p, see [KM93, Theorem 4.2]. Thus our method of proof applies to branched orientable surface groups, with any Γ_k in the role played by Γ_A during the proof of THEOREM 1.4.1.

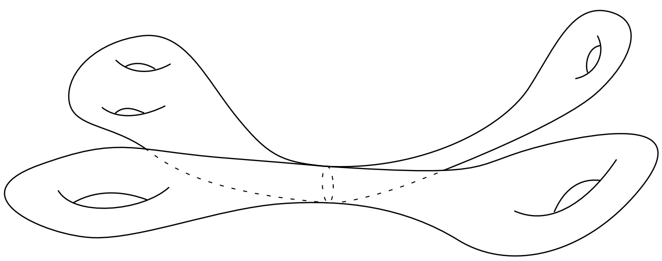

FIGURE 1.3. — A branched surface

QUESTION 1.4.7. — Is the fundamental group of a compact hyperbolic 3-manifold allosteric? More generally, is the fundamental group of a compact orientable aspherical 3-manifold allosteric?

References

[AE07] Miklós Abért and Gábor Elek. Non-abelian free groups admit non-essentially free actions on rooted trees. arXiv:0707.0970, 2007.

[AE12] Miklós Abért and Gábor Elek. Hyperfinite actions on countable sets and probability measure spaces. In *Dynamical systems and group actions. Dedicated to Anatoli Stepin on the occasion of his 70th birthday*, pages 1–16. Providence, RI: American Mathematical Society (AMS), 2012.

[AGV14] Miklós Abért, Yair Glasner, and Bálint Virág. Kesten's theorem for invariant random subgroups. *Duke Math. J.*, 163(3):465–488, 2014.

[Bau62] Gilbert Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.

[Bau67] Benjamin Baumslag. Residually free groups. *Proc. Lond. Math. Soc. (3)*, 17:402–418, 1967.

[Bek20] Bachir Bekka. Character rigidity of simple algebraic groups. arXiv:1908.06928, 2020.

[BG04] Nicolas Bergeron and Damien Gaboriau. Asymptotique des nombres de Betti, invariants ℓ^2 et laminations. *Comment. Math. Helv.*, 79(2):362–395, 2004.

[BGK17] Lewis Bowen, Rostislav I. Grigorchuk, and Rostyslav V. Kravchenko. Characteristic random subgroups of geometric groups and free abelian groups of infinite rank. *Trans. Am. Math. Soc.*, 369(2):755–781, 2017.

[BLT19] O. Becker, A. Lubotzky, and A. Thom. Stability and invariant random subgroups. *Duke Math. J.*, 168(12):2207–2234, 2019.

[CGLM20] Alessandro Carderi, Damien Gaboriau, and François Le Maître. On dense totipotent free subgroups in full groups. arXiv:2009.03080, 2020.

[CJLMT22] Alessandro Carderi, Matthieu Joseph, François Le Maître, and Romain Tessera. Belinskaya's theorem is optimal, 2022.

[CP] Darren Creutz and Jesse Peterson.

[Cre17] Darren Creutz. Stabilizers of actions of lattices in products of groups. *Ergodic Theory Dyn. Syst.*, 37(4):1133–1186, 2017.

[DM14] Artem Dudko and Konstantin Medynets. Finite factor representations of Higman-Thompson groups. *Groups Geom. Dyn.*, 8(2):375–389, 2014.

[GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitaly I. Suschanskii. Automata, dynamical systems, and groups. In *Dynamical systems, automata, and infinite groups. Transl. from the Russian*, pages 128–203. Moscow: MAIK Nauka/Interperiodica Publishing, 2000.

[Gri11] Rostislav I. Grigorchuk. Some topics in the dynamics of group actions on rooted trees. *Proc. Steklov Inst. Math.*, 273:64–175, 2011.

[Gru57] Karl W. Gruenberg. Residual properties of infinite soluble groups. *Proc. Lond. Math. Soc. (3)*, 7:29–62, 1957.

[GW15] Eli Glasner and Benjamin Weiss. Uniformly recurrent subgroups. In *Recent trends in ergodic theory and dynamical systems. International conference in honor of S. G. Dani's 65th birthday, Vadodara, India, December 26–29, 2012. Proceedings*, pages 63–75. Providence, RI: American Mathematical Society (AMS), 2015.

[Jos21] Matthieu Joseph. Continuum of allosteric actions for non-amenable surface groups, 2021.

[KM93] Goansu Kim and James McCarron. On amalgamated free products of residually p-finite groups. *J. Algebra*, 162(1):1–11, 1993.

[LBMB18] Adrien Le Boudec and Nicolás Matte Bon. Subgroup dynamics and C^*-simplicity of groups of homeomorphisms. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):557–602, 2018.

[Mag35] Wilhelm Magnus. Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring. *Math. Ann.*, 111:259–280, 1935.

[PT16] Jesse Peterson and Andreas Thom. Character rigidity for special linear groups. *J. Reine Angew. Math.*, 716:207–228, 2016.

[Ser77] Jean-Pierre Serre. *Arbres, amalgames*, SL_2, volume 46. Société Mathématique de France (SMF), Paris, 1977.

[SZ94] Garrett J. Stuck and Robert J. Zimmer. Stabilizers for ergodic actions of higher rank semisimple groups. *Ann. Math. (2)*, 139(3):723–747, 1994.

[Tak51] Mutuo Takahasi. Note on chain conditions in free groups. *Osaka Math. J.*, 3:221–225, 1951.

[TTD18] Simon Thomas and Robin Tucker-Drob. Invariant random subgroups of inductive limits of finite alternating groups. *J. Algebra*, 503:474–533, 2018.

[Ver12] Anatoly M. Vershik. Totally nonfree actions and the infinite symmetric group. *Mosc. Math. J.*, 12(1):193–212, 2012.

Part II

Quantitative orbit equivalence for $\mathbb{Z}$-actions

Chapter 2

Belinskaya's theorem is optimal

The content of this chapter is the same as that of the article [CJLMT22], which is a joint work with A. Carderi, F. Le Maître and R. Tessera.

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2.1 Introduction

Given two ergodic measure-preserving (invertible) transformations T_1, T_2 of a standard probability space (X, μ), the *conjugacy problem* asks whether there is a third measure-preserving invertible transformation S such that $ST_1 = T_2S$. Although the conjugacy problem is intractable in full generality, various invariants have been devised over the years. Two of the most important ones are the *spectrum* and the *dynamical entropy*. The first completely classifies compact transformations [HvN42], while the second completely classifies Bernoulli shifts [Sin59, HvN42].

In this paper, we are interested in natural weakenings of the conjugacy problem obtained through the notion of *orbit equivalence*. Two measure-preserving transformations T_1, T_2 are ***orbit equivalent*** if there is a measure-preserving transformation S such that ST_1S^{-1} and T_2 have the same orbits (such an S is called an orbit equivalence between T_1 and T_2). A stunning theorem of Dye states that all ergodic measure-preserving transformations of a standard probability space are orbit equivalent [Dye59], so orbit equivalence for measure-preserving ergodic transformations is a trivial weakening of conjugacy.

In order to circumvent this indistinguishability, we will compare orbit equivalences between measure-preserving transformations in a quantitative way. This fits into the emerging field of *quantitative orbit equivalence* for group actions. One of its tacit aims is to capture meaningful geometric invariants, such as Følner functions [DKLMT20], growth rates [Aus16b], etc., or ergodic theoretic invariants, such as dynamical entropy [Aus16].

In our setup of measure-preserving transformations, quantifications will be imposed on *orbit equivalence cocycles*. Given an orbit equivalence S between two ergodic measure-preserving transformations T_1 and T_2, the orbit equivalence cocycles $c_1, c_2 : X \to \mathbb{Z}$ are the maps uniquely defined by the following equation: for all $x \in X$

$$ST_1(x) = T_2^{c_2(x)}S(x) \text{ and } T_2S(x) = ST_1^{c_1(x)}(x). \tag{2.1}$$

Belinskaya's theorem is probably the first result on quantitative orbit equivalence. In the literature, it is often stated as a symmetric result on integrable orbit equivalence of ergodic measure-preserving transformations. However, her result is asymmetric and can be stated as follows.

Theorem 2.1.1 (Belinskaya [Bel68]). — *Let T_1 and T_2 be two ergodic measure-preserving transformations, let S be an orbit equivalence between them and suppose that the previously defined cocycle c_1 is integrable, i.e.*

$$\int_X |c_1(x)|\, d\mu < +\infty.$$

Then T_1 and T_2 are flip-conjugate: either T_1 is conjugate to T_2 or T_1^{-1} is conjugate to T_2.

It is natural to wonder whether Belinskaya's theorem remains valid if one weakens the integrability assumptions. For example, one could ask that one of the orbit equivalence cocycle belongs to $\mathrm{L}^p(X, \mu)$ for some $p \in (0, 1)$.

We will consider more general integrability assumptions. Given a function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, we say that a measurable integer-valued function f is ***φ-integrable***

if
$$\int_X \varphi(|f(x)|)d\mu < +\infty.$$
Our first main result concerns orbit equivalence of measure-preserving transformations for which one of the orbit equivalence cocycles is φ-integrable, for some ***sublinear*** function φ, that is satisfying $\lim_{t\to+\infty} \varphi(t)/t = 0$. This is for example the case for $\varphi(t) = t^p$. With this integrability condition, the conclusion of Belinskaya's theorem does not hold.

THEOREM 2.1.2 (see THEOREM 2.4.14). — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function and T_1 be an ergodic measure-preserving transformation. Then there is an ergodic measure-preserving transformation T_2 and an orbit equivalence S between T_1 and T_2 such that the associated cocycle c_1 is φ-integrable but the transformations T_1 and T_2 are not flip-conjugate.*

The fact that the hypotheses on φ are fairly weak gives us much freedom. For example, the above theorem even implies that Belinskaya's theorem does not hold if we assume that one of the two orbit equivalence cocycles belongs to $\mathrm{L}^p(X,\mu)$ for *all* $p \in (0,1)$. Indeed if we consider for instance the sublinear function $\varphi(t) = t/\ln(t+1)$, then φ-integrability implies being in $\mathrm{L}^p(X,\mu)$ for all $p < 1$.

A symmetric way to strengthen THEOREM 2.1.2 involves the concept of φ-integrable orbit equivalence. We say that two measure-preserving transformations are ***φ-integrable orbit equivalent*** if there is an orbit equivalence S such that *both* orbit equivalence cocycles c_1 and c_2 are φ-integrable. In this context, we obtain a similar conclusion to THEOREM 2.1.2, but we have to make one additional assumption on T_1.

THEOREM 2.1.3 (see COROLLARY 2.3.11). — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let T_1 be an ergodic measure-preserving transformation and assume that $(T_1)^n$ is ergodic for some $n \geq 2$. Then there is another ergodic measure-preserving transformation T_2 such that T_1 and T_2 are φ-integrable orbit equivalent but not flip-conjugate.*

Concrete examples of transformations to which this theorem applies are Bernoulli shifts, irrationnal rotations on the circle and the m-odometer for any integer m. One can show that the only ergodic measure-preserving transformations that are *not* covered by this theorem are the ones that factor onto the universal odometer, that is, the transformation $t \mapsto t+1$ on the profinite completion $\widehat{\mathbb{Z}}$.

Let us point out that the proof of THEOREM 2.1.2 uses THEOREM 2.1.3, so the two results are not independent. As we will explain later, THEOREM 2.1.2 also depends on the Baire category theorem.

However, THEOREM 2.1.3 is somewhat more explicit. It relies on the following simple construction, which was already used in [LM18, Thm. 4.8]. We begin with an ergodic transformation T_1 with $(T_1)^n$ ergodic. We also need a periodic transformation P all of whose orbits have cardinality n and are contained in those of

T_1. The transformation T_2 is constructed by composing P and the transformation induced by T_1 on a fundamental domain of P. Then T_1 and T_2 have the same orbits. However, $(T_2)^n$ is not ergodic and thus T_1 and T_2 are not flip-conjugate. As a byproduct, the transformations T_1 and T_2 do not have the same spectrum, as the spectrum of T_2 contains $\exp(2i\pi/n)$ whereas the spectrum of T_1 doesn't. The main task then becomes to construct P so that the orbit equivalence cocycles between T_1 and T_2 satisfy the required integrability conditions.

For many concrete measure-preserving transformations, explicit examples of such periodic transformations P with specific integrability conditions can be obtained. We will give details in the case of the Bernoulli shift, see Example 2.3.3.

Shannon orbit equivalence and dynamical entropy A remarkable consequence of Theorem 2.1.3 can be stated in the context of *Shannon orbit equivalence*, as defined by Kerr and Li [KL19]. Two measure-preserving transformations are Shannon orbit equivalent if there exists an orbit equivalence between them whose orbit equivalence cocycles c_1 and c_2 have both finite Shannon entropy. After showing that dynamical entropy is an invariant of Shannon orbit equivalence for measure-preserving actions of many groups, such as $\mathbb{Z}^n$ for every $n \geq 2$, they implicitly asked whether dynamical entropy is an invariant of Shannon orbit equivalence for measure-preserving transformations and wondered whether Shannon orbit equivalence could actually directly imply flip-conjugacy. We show that it is not the case.

Theorem 2.1.4 (see Theorem 2.3.17). — *Let $T_1 \in \mathrm{Aut}(X, \mu)$ be an ergodic transformation and assume that $(T_1)^n$ is ergodic for some $n \geq 2$. Then there exists $T_2 \in \mathrm{Aut}(X, \mu)$ such that T_1 and T_2 are Shannon orbit equivalent but not flip-conjugate.*

The above theorem is obtained by applying Theorem 2.1.3 with any sublinear function φ such that $\ln(1+t) = O(\varphi(t))$. Indeed, for any such function, φ-integrable orbit equivalence implies Shannon orbit equivalence, see Theorem 2.3.15.

We also observe that Shannon orbit equivalence preserves finiteness of dynamical entropy, see Proposition 2.3.20. This is now subsumed by a recent preprint of Kerr and Li who proved that the dynamical entropy is preserved under Shannon orbit equivalence [KL22].

Question 2.1.5. — For which unbounded sublinear metric-compatible functions φ is it true that dynamical entropy is an invariant of φ-integrable orbit equivalence?

By the above discussion, we already know that this holds for all φ such that $\ln(1+t) = O(\varphi(t))$. On the other hand, using Dye's theorem, it is not hard to see that any two ergodic measure-preserving transformations are φ-integrable orbit equivalent for *some* sublinear unbounded function φ (cf. the proof of [DKLMT20, Prop. 4.24]). So not every sublinear unbounded function satisfies the condition of the question.

φ-integrable full groups The proof of both our main results will make crucial use of the notion of *φ-integrable full group*. Whenever T is an ergodic measure preserving transformation of the probability space (X,μ), Dye defined a Polish group $[T]$, called the full group of T. This group is by definition the set of all measure-preserving transformations U of (X,μ) whose orbits are contained in T-orbits. More precisely, $U \in [T]$ if there is a function c_U, called the ***T-cocycle of** U*, such that $U(x) = T^{c_U(x)}(x)$ for all $x \in X$. The above stated theorem of Dye, that all ergodic transformations are obit equivalent, was originally stated in terms of full groups: whenever T_1 and T_2 are ergodic transformations, the full groups $[T_1]$ and $[T_2]$ are conjugate.

In our context, once φ is fixed, the reasonable analogue of the full group associated with this integrability condition, would be the set of transformations $U \in [T]$ such that the cocycle c_U is φ-integrable. However, for this set to be a subgroup of $[T]$, we will have to impose a mild restriction on φ. We say that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a ***metric-compatible function*** if

- (subadditivity) for all $s,t \in \mathbb{R}_+$, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$.
- (separation) $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$.
- (monotonicity) φ is a non-decreasing function.

The name metric-compatible comes from the observation that whenever d is a metric and φ a metric-compatible function, then $\varphi \circ d$ is also a metric. The following theorem is a combination of Lemma 2.2.14 and Theorem 2.4.1.

Theorem 2.1.6. — *Let φ be a metric-compatible function and let T be a measure preserving transformation of the probability space (X,μ). Then the set*

$$[T]_\varphi := \left\{ U \in [T] \colon \int_X \varphi(|c_u(x)|) d\mu < +\infty \right\}$$

is a group. Moreover the function

$$\boldsymbol{d}_{\varphi,T}(U,V) := \int_X \varphi(|c_U(x) - c_V(x)|) d\mu$$

is a complete, right-invariant and separable metric on $[T]_\varphi$ whose induced topology is a group-topology. In particular $([T]_\varphi, \boldsymbol{d}_{\varphi,T})$ is a Polish group.

It turns out that any sublinear function is dominated by a sublinear metric-compatible function, see Lemma 2.2.12. This will allows us to reduce the proof of Theorem 2.1.2 and Theorem 2.1.3 to the case where φ is metric-compatible and thereby to exploit the group structure of $[T]_\varphi$.

Genericity of weakly mixing Let us come back to Theorem 2.1.2. As the conclusions of Theorem 2.1.3 are stronger, we just need to show Theorem 2.1.2

whenever $(T_1)^n$ is non-ergodic for all $n \geq 2$. Observe that this condition is incompatible with the notion of *weakly mixing*, as all the powers of any weakly mixing transformation are ergodic. Therefore our strategy is to provide for every ergodic transformation T_1 a weakly mixing transformation T_2 which has the same orbits as T_1 and whose T_1-cocycle is φ-integrable. We do not have any constructive argument for this and we proceed through the Baire category theorem.

THEOREM 2.1.7 (see THEOREM 2.4.15). — *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic element. Then the set of all measure-preserving transformations in $[T]_\varphi$ which are weakly mixing and have the same orbits as T is a dense G_δ set in the Polish space of aperiodic transformations of $[T]_\varphi$.*

Besides the Baire category theorem, there are two other main ingredients in the proof of THEOREM 2.1.7. One is a result of Conze [Con72] which claims that starting from any ergodic mesure-preserving transformation, the first return map to a *generic* measurable subset gives rise to a weakly mixing transformation. The second is a sublinear ergodic theorem which may be of independent interest.

THEOREM 2.1.8 (see THEOREM 2.4.5). — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear non-decreasing function. Let $U \in \mathrm{Aut}(X,\mu)$ and $f\colon X \to \mathbb{C}$ measurable such that $\varphi(|f|) \in \mathrm{L}^1(X,\mu)$. Then for almost every $x \in X$*

$$\lim_n \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right) = 0.$$

The convergence also holds in L^1, that is

$$\lim_n \int_X \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right) d\mu = 0.$$

Outline of the paper In Section 4.1, after a few preliminaries, we present the framework and establish basic properties of φ-integrable full groups. In Section 2.3, we explain our construction of periodic transformations in φ-integrable full groups and use it to prove THEOREM 2.1.2. In Section 2.4, we first prove that φ-integrable full groups are Polish groups. We then use the Baire category theorem and prove that weakly mixing elements are generic in the set of aperiodic elements in $[T]_\varphi$. Combining this with THEOREM 2.1.2, we finally prove THEOREM 2.1.3. In the appendix, we present a proof of Belinskaya's theorem which is due to Katznelson and is not publicly available to our knowledge.

2.2 Quantitative orbit equivalence and full groups

2.2.1 Preliminaries

Throughout the paper, (X,μ) will denote a standard probability space without atoms. Recall that such spaces are measurably isomorphic to the interval $[0,1]$

equipped with the Lebesgue measure. A bimeasurable bijection $T\colon X \to X$ is a ***measure-preserving transformation*** of (X,μ) if for all measurable sets $A \subseteq X$, one has $\mu(T^{-1}(A)) = \mu(A)$. We denote by $\mathrm{Aut}(X,\mu)$ the group of all measure-preserving transformations of (X,μ), two such transformations being identified if they coincide on a conull set. The group $\mathrm{Aut}(X,\mu)$ will be equipped with the ***uniform metric*** d_u defined by

$$d_u(T_1,T_2) := \mu(\{x \in X\colon T_1(x) \neq T_2(x)\}).$$

This metric is bi-invariant and complete [Hal17, p. 73].

Remark 2.2.1. — One can always modify measure-preserving transformations on null-sets without changing its equivalence class in $\mathrm{Aut}(X,\mu)$. Indeed the saturation of any null set is still a null set. This will often be used implicitly in the sequel.

The ***support*** of a measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ is the measurable set $\mathrm{supp}(T) := \{x \in X\colon T(x) \neq x\}$.

A measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ is ***periodic*** if the T-orbit of almost every $x \in X$ is finite. A ***fundamental domain*** of a periodic transformation $T \in \mathrm{Aut}(X,\mu)$ is a measurable subset $A \subseteq X$ which intersects almost every T-orbit at exactly one point. Every periodic transformation admits such a fundamental domain, as can be seen by fixing a Borel linear order $<$ on X and taking for D the set of $<$-least points in each orbit of the transformation. A measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ is ***aperiodic*** if the T-orbit of almost every $x \in X$ is infinite. It is ***ergodic*** if every T-invariant measurable set is either null or conull.

The ***full group*** of a measure-preserving transformation T is the group

$$[T] := \{U \in \mathrm{Aut}(X,\mu)\colon \forall x \in X,\ \exists n \in \mathbb{Z} \text{ such that } U(x) = T^n(x)\}.$$

Remark 2.2.2. — Note that $U \in [T]$ if and only if the U-orbit of every point $x \in X$ is contained in the T-orbit of x. By Remark 2.2.1, we actually have that $U \in [T]$ if and only if the U-orbit of *almost* every point $x \in X$ is contained in the T-orbit of x.

Two measure-preserving transformations $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ ***have the same orbits*** if for almost every $x \in X$, the T_1-orbit of x coincides with the T_2-orbit of x. By the above remark, this is equivalent to following condition: $T_1 \in [T_2]$ and $T_2 \in [T_1]$. We say that two measure-preserving transformations $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ are ***orbit equivalent*** if there exists $S \in \mathrm{Aut}(X,\mu)$ such that ST_1S^{-1} and T_2 have the same orbits, that is $ST_1S^{-1} \in [T_2]$ and $T_2 \in [ST_1S^{-1}]$.

Fix an aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$. Any $U \in [T]$ is completely determined by its T-***cocycle***, defined as the unique function $c_U\colon X \to \mathbb{Z}$ satisfying the equation $U(x) = T^{c_U(x)}(x)$ for all $x \in X$. The T-cocycle satisfies the so-called cocycle identity: given $U, V \in [T]$, we have

$$c_{UV}(x) = c_U(V(x)) + c_V(x) \text{ for all } x \in X. \tag{2.2}$$

Let $T \in \mathrm{Aut}(X, \mu)$ and $A \subseteq X$ be a measurable subset. The ***first return time*** of T to A is the map $n_{T,A} \colon A \to \mathbb{N}^*$ defined by

$$n_{T,A}(x) := \min\{n \in \mathbb{N}^* \colon T^n(x) \in A\}.$$

This function is well-defined up to measure zero by Poincaré's recurrence theorem. For convenience, we extend $n_{T,A}$ to all X, setting it to be 0 on $X \setminus A$. Kac's lemma [Kac47] yields the following inequality

$$\int_X n_{T,A}(x) d\mu \leq 1. \tag{2.3}$$

The ***first return map*** of T with respect to A is the transformation $T_A \in [T] \leq \mathrm{Aut}(X, \mu)$ defined by

$$T_A(x) := T^{n_{T,A}(x)}(x).$$

By definition, we have $\mathrm{supp}(T_A) = A$ and $x, y \in A$ are in the same T-orbit if and only if they are in the same T_A-orbit. Whenever T is aperiodic, the first return time $n_{T,A}$ coincides with the T-cocycle c_{T_A} of T_A.

LEMMA 2.2.3. — *Let $T \in \mathrm{Aut}(X, \mu)$, let $P \in \mathrm{Aut}(X, \mu)$ be a periodic transformation and D a fundamental domain of P. Let $U := T_D P$. Then the following are true.*

(i) $U_D = T_D$.

(ii) If $x \in D$ and $n(x)$ is the cardinality of the P-orbit of x, then $n_{U,D}(x) = n(x)$.

(iii) If $P \in [T]$, then T and U have the same orbits.

Proof. We first prove (i) and (ii). Clearly $U_D(x) = T_D(x) = x$ for every $x \notin D$. Since D is a fundamental domain for P, for all $x \in D$ and $i \in \{1, \dots, n(x) - 1\}$, we have $P^i(x) \notin D$. Since $T_D(x) = x$ for all $x \notin D$, we deduce by induction that

$$U^i(x) = P^i(x) \notin D \text{ for all } x \in D \text{ and } i \in \{1, \dots, n(x) - 1\}.$$

So for all $x \in D$, one has $U^{n(x)}(x) = UU^{n(x)-1}(x) = T_D P^{n(x)}(x) = T_D(x)$. This shows Item (i) and (ii).

We now prove Item (iii). Clearly, $U \in [T]$. We need to show that $T \in [U]$. Observe that for almost every x, the U-orbit of x meets D: indeed, if $x \in P^i(D)$ for $i \in \{1, \dots, n(x) - 1\}$, then $U^{-i}(x) = P^{-i}(x) \in D$. Since being in the same orbit is an equivalence relation, it is enough to show that any two points in D, which belong to the same T-orbit, are in the same U-orbit. This follows directly from (i). □

We will also need the following lemma which can be proven with the same kind of arguments as above.

LEMMA 2.2.4. — *Let $U \in \mathrm{Aut}(X,\mu)$ and let A be a measurable subset of X which intersects every U-orbit. Then $(U_A)^{-1}U$ is periodic and A is a fundamental domain for it.*

Proof. Since A intersects every U-orbit, for almost every $x \in X \setminus A$ there exists a smallest $n \geq 1$ such that $U^{-n}(x) \in A$. Remark that $((U_A)^{-1}U)^{-n}(x) = U^{-n}(x) \in A$ and hence A intersects every $(U_A)^{-1}U$-orbit. If $x \in A$, then for every $0 \leq n < n_{U,A}(x)$ we have that

$$((U_A)^{-1}U)^n(x) = U^n(x) \notin A \text{ and } ((U_A)^{-1}U)^{n_{U,A}(x)}(x) = U_A^{-1}UU^{n_{U,A}(x)-1}(x) = x.$$

Since A intersects every $(U_A)^{-1}U$-orbit, we obtain both that every $(U_A)^{-1}U$-orbit is finite and that A is a fundamental domain for $(U_A)^{-1}U$. □

2.2.2 φ-integrable orbit equivalence and full groups

We first define the notion of *φ-integrable orbit equivalence*.

DEFINITION 2.2.5. — Fix $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$. Two aperiodic transformations $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ are ***φ-integrable orbit equivalent*** if there exists $S \in \mathrm{Aut}(X,\mu)$ such that ST_1S^{-1} and T_2 have the same orbits and their respective cocycles are φ-integrable. To be more precise, we ask that

$$\int_X \varphi(|c_{ST_1S^{-1}}(x)|)d\mu < +\infty \text{ and } \int_X \varphi(|c_{T_2}(x)|)d\mu < +\infty,$$

where $c_{ST_1S^{-1}}$ is the T_2-cocycle of ST_1S^{-1} and c_{T_2} is the ST_1S^{-1}-cocycle of T_2, defined for all $x \in X$ by the equations

$$ST_1S^{-1}(x) = T_2^{c_{ST_1S^{-1}}(x)}(x) \text{ and } T_2(x) = (ST_1S^{-1})^{c_{T_2}(x)}(x).$$

When $\varphi(t) = t^p$ for some $p \in (0,+\infty)$, we recover the notion of L^p orbit equivalence.

REMARK 2.2.6. — We warn the reader that even though the term L^p orbit equivalence is often used in the literature, this terminology may sound a bit deceptive. Indeed, since the integrability condition has no reason to be preserved under composition of orbit equivalences, we do not expect φ-integrable (even L^p) orbit equivalence to be an equivalence relation for every concave function φ, although we don't have any counterexample. The fact that it is the case for $p = 1$ seems to be a rather artificial consequence of Belinskaya's theorem.

In our work, the function φ is at most linear and for our main theorems the function is assumed to be ***sublinear***, that is $\lim_{t\to+\infty} \varphi(t)/t = 0$. For example we are interested in the case of L^p orbit equivalence for $p \leq 1$, or in the case of $\varphi(t) = \log(1+t)$.

In the context of φ-orbit equivalence, it is natural to consider the the set of measure-preserving transformations U whose cocycle c_U is φ-integrable. In order for this set to be a group, the following conditions on φ are required.

DEFINITION 2.2.7. — A function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is ***metric-compatible*** if:

- (subadditivity) for all $s, t \in \mathbb{R}_+$, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$.
- (separation) $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$.
- (monotonicity) φ is a non-decreasing function.

EXAMPLE 2.2.8. — Any concave function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ is metric-compatible. In particular for every $p \leq 1$, the function $\varphi(t) = t^p$ is metric-compatible. It is moreover sublinear whenever $p < 1$. Other examples of sublinear metric-compatible functions are given by $\varphi(t) = \log(1+t)$ or $\varphi(t) = t/\log(2+t)$.

The term "metric-compatible" was coined because of the following property: whenever d is a metric on a set X, then $\varphi \circ d$ is also a metric on X.

CONVENTION. — For all $t \in \mathbb{R}$, we use the notation

$$|t|_\varphi := \varphi(|t|).$$

The map $(s,t) \mapsto |s-t|_\varphi$ is a metric on $\mathbb{R}$.

DEFINITION 2.2.9. — Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. The ***φ-integrable full group*** of an aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$ is

$$[T]_\varphi := \left\{ U \in [T]\colon \int_X |c_U(x)|_\varphi d\mu < +\infty \right\},$$

where $c_U\colon X \to \mathbb{Z}$ denotes the T-cocycle of U.

Given a metric-compatible function φ, the φ-integrable full group $[T]_\varphi$ is indeed a group: given $U, V \in [T]_\varphi$, the cocycle identity implies that

$$c_{UV^{-1}}(x) = c_U(V^{-1}(x)) + c_{V^{-1}}(x) = c_U(V^{-1}(x)) - c_V(V^{-1}(x)).$$

We then get that

$$\begin{aligned} \int_X |c_{UV^{-1}}(x)|_\varphi d\mu &\leq \int_X |c_U(V^{-1}(x))|_\varphi d\mu + \int_X |c_V(V^{-1}(x))|_\varphi d\mu \\ &= \int_X |c_U(x)|_\varphi d\mu + \int_X |c_V(x)|_\varphi d\mu < +\infty. \end{aligned} \tag{2.4}$$

EXAMPLE 2.2.10. — If φ is any metric-compatible function which is bounded, then $[T]_\varphi = [T]$ and if φ is the identity map, then we recover the L^1 full group $[T]_1$ defined by the third named author in [LM18]. Any other such φ gives rise to new[1] examples of full groups, such as L^p full groups $[T]_p$ for $0 < p < 1$ obtained with the function $\varphi(t) = t^p$, or else $[T]_{\log}$ obtained with the function $\varphi(t) = \log(1+t)$.

Remark 2.2.11. — Given a metric-compatible function φ, it is now straightforward to check that two aperiodic transformations $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ are φ-integrable orbit equivalent if and only if there is $S \in \mathrm{Aut}(X,\mu)$ such that $ST_1S^{-1} \in [T_2]_\varphi$ and $T_2 \in [ST_1S^{-1}]_\varphi$. However, the notion of φ-orbit equivalence is a priori weaker than conjugacy of φ-integrable full groups. Indeed conjugacy of φ-integrable full groups is an equivalence relation but φ-integrable orbit equivalence may not be, see Remark 2.2.6. This is in contrast with the case of classical orbit equivalence, see [Kec10, Thm. 4.1].

In our two main results, namely Theorem 2.1.2 and Theorem 2.1.3, the sublinear function φ is not assumed to be metric-compatible. The following lemma will allow us to reduce to the case where φ is in addition metric-compatible.

Lemma 2.2.12. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Then there is a sublinear metric-compatible function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \le \psi(t)$ for all t large enough.*

Proof. Set

$$\theta\colon \mathbb{R}_+^* \to \mathbb{R}_+, \quad \theta(t) := \min\left(1, \sup_{s\ge t} \frac{\varphi(s)+1}{s}\right);$$
$$\psi\colon \mathbb{R}_+ \to \mathbb{R}_+, \quad \psi(t) := \int_0^t \theta(s)ds.$$

Noting that θ is positive-valued and non-increasing, it is straightforward to check that ψ is non-decreasing, subadditive and that $\psi(t) = 0$ if and only if $t = 0$. Moreover the fact that $\theta(t)$ tends to 0 as t approaches $+\infty$ implies that ψ is sublinear. Now remark that for every $t \in \mathbb{R}_+^*$

$$\psi(t) = \int_0^t \theta(s)ds \ge \int_0^t \theta(t)ds = t\theta(t).$$

For $t \in \mathbb{R}_+^*$ large enough so that $\sup_{s\ge t} \frac{\varphi(s)+1}{s} \le 1$ we finally have

$$t\theta(t) = t\sup_{s\ge t}\frac{\varphi(s)+1}{s} = t\sup_{s\ge t}\frac{\varphi(s)+1}{s} \ge t\frac{\varphi(t)+1}{t} = \varphi(t)+1 \ge \varphi(t)$$

so we are done. □

Remark 2.2.13. — Given a sublinear function φ, Lemma 2.2.12 grants us a sublinear metric-compatible function ψ such that $\varphi(t) \le \psi(t)$ for all t large enough.

[1]We can actually characterize when $[T]_\varphi = [T]_\psi$ and more generally when $[T]_\varphi \le [T]_\psi$, see Proposition 2.4.2.

Therefore, for any measurable function $f\colon X \to \mathbb{Z}$ we have

$$\int_X \psi(|f(x)|)d\mu < +\infty \text{ implies that } \int_X \phi(|f(x)|)d\mu < +\infty.$$

In particular, ψ-integrable orbit equivalence implies φ-orbit equivalence and any element in a ψ-integrable full group will have φ-integrable cocycle.

We will state most of our results in the comfortable context of (sublinear) metric-compatible functions. However, many of our statement could be easily generalized to the general context of sublinear functions through REMARK 2.2.13. We will explicitly do so only in our main theorems, THEOREM 2.1.2 and THEOREM 2.1.3.

2.2.3 Metric properties of φ-integrable full groups

We now introduce and study a natural extended pseudo-metric on full groups from which φ-integrable full groups naturally arise.

LEMMA 2.2.14. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Let $\boldsymbol{d}_{\varphi,T}\colon [T]\times[T] \to \mathbb{R}_+ \cup \{+\infty\}$ be the function defined by*

$$\boldsymbol{d}_{\varphi,T}(U,V) := \int_X |c_U(x) - c_V(x)|_\varphi d\mu.$$

Then the following are true.

(i) The group $[T]_\varphi$ is determined by $\boldsymbol{d}_{\varphi,T}$:

$$[T]_\varphi = \{U \in [T]\colon \boldsymbol{d}_{\varphi,T}(U,\mathrm{id}) < +\infty\}.$$

(ii) The restriction of $\boldsymbol{d}_{\varphi,T}$ to $[T]_\varphi \times [T]_\varphi$ is a metric on $[T]_\varphi$ which is right-invariant, that is, for all $U,V,W \in [T]_\varphi$,

$$\boldsymbol{d}_{\varphi,T}(UW,VW) = \boldsymbol{d}_{\varphi,T}(U,V).$$

Proof. Item (i) is an immediate consequence of the definition of $[T]_\varphi$.

Let us now prove Item (ii). The fact that $\boldsymbol{d}_{\varphi,T}$ is a metric is a straightforward consequence of the fact that $(s,t) \mapsto |s-t|_\varphi$ is a metric on $\mathbb{R}$. The right-invariance follows from the cocycle identity (2.2) and the fact that the transformation W is measure-preserving:

$$\begin{aligned} \boldsymbol{d}_{\varphi,T}(UW,VW) &= \int_X |c_{UW}(x) - c_{VW}(x)|_\varphi \, d\mu \\ &= \int_X |c_U(W(x)) - c_V(W(x))|_\varphi \, d\mu \\ &= \int_X |c_U(x) - c_V(x)| \, d\mu \\ &= \boldsymbol{d}_{\varphi,T}(U,V). \end{aligned}$$

□

Example 2.2.15. — Consider the metric-compatible function $\varphi := \min(\mathrm{id}_{\mathbb{R}_+}, 1)$. Then it is straightforward to check that $\boldsymbol{d}_{\varphi,T} = d_u$ is the uniform metric on $[T] = [T]_\varphi$.

Another example is obtained by taking $\varphi := \mathrm{id}_{\mathbb{R}_+}$; we then recover the L^1 metric on the L^1 full group $[T]_1 = [T]_\varphi$.

In order to compare φ-integrable full groups, we are led to compare asymptotically metric-compatible functions. We will use the following standard notation: given two real-valued functions f and g, we write $f(t) = O(g(t))$ as $t \to +\infty$ if there exist $t_0 > 0$ and $C > 0$ such for all $t > t_0$, we have $|f(t)| \leq C|g(t)|$. Since the functions we consider are subadditive, it is enough to compare them on the integers.

Lemma 2.2.16. — *Let φ, ψ be two metric compatible function. Then the following are equivalent.*

(i) $\varphi(t) = O(\psi(t))$ *as* $t \to +\infty$.

(ii) There exists $C > 0$ *such that* $\varphi(t) \leq C\psi(t)$ *for all* $t \geq 1$.

(iii) There exists $C > 0$ *such that* $\varphi(k) \leq C\psi(k)$ *for all integer* $k \in \mathbb{N}$.

Proof. We first prove that (i) implies (ii). Let $t_0 \geq 1$ and $D > 0$ such that for all $t > t_0$, we have $\varphi(t) \leq D\psi(t)$. Set $C := \max(D, \varphi(t_0)/\psi(1))$ and observe that since φ and ψ are non-decreasing, $\varphi(t) \leq C\psi(t)$ for all $t \geq 1$.

The implication (ii) $\Longrightarrow$ (iii) is straightforward, so we are left with proving (iii) $\Longrightarrow$ (i). Let $C > 0$ such that $\varphi(k) \leq C\psi(k)$ for all integer $k \in \mathbb{N}$. Fix a real number $t \geq 2$ and let $n \in \mathbb{N}^*$ such that $n \leq t < n+1$. Then we have

$$\varphi(t) \leq \varphi(n+1) \leq C\psi(n+1) \leq C(\psi(t) + \psi(1)) \leq C\left(1 + \frac{\psi(1)}{\psi(t)}\right)\psi(t),$$

and since $\psi(t) \geq \psi(1)$ for every $t \geq 1$, the proof is complete. □

We now compare φ-integrable full groups for different metric-compatible functions.

Lemma 2.2.17. — *Let φ and ψ be two metric-compatible functions and fix an aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$. If $\varphi(t) = O(\psi(t))$ as $t \to +\infty$, then $[T]_\psi \leq [T]_\varphi$. Moreover, the inclusion map is Lipschitz.*

Proof. By LEMMA 2.2.16, there is $C > 0$ such that $\varphi(k) \leq C\psi(k)$ for all integer $k \in \mathbb{N}$. Let $U, V \in [T]$. Then for almost every $x \in X$,

$$|c_U(x) - c_V(x)|_\varphi \leq C|c_U(x) - c_V(x)|_\psi.$$

Integrating over X, we get that $\boldsymbol{d}_{\varphi,T}(U,V) \leq C\boldsymbol{d}_{\psi,T}(U,V)$. The lemma now follows immediatly. □

COROLLARY 2.2.18. — *Whenever T is an aperiodic measure-preserving transformation, we have*

$$[T]_1 \leq [T]_\varphi \leq [T],$$

and the inclusion maps are Lipschitz,

Proof. Since φ is subadditive, we have $\varphi(t) = O(t)$ as $t \to +\infty$. Moreover $\min(1,t) = O(\varphi(t))$ as $t \to +\infty$. The conclusion now follows from LEMMA 2.2.17. □

We will show in PROPOSITION 2.4.2 that LEMMA 2.2.17 is an equivalence. For this, we will make a crucial use of the fact that the topologies induced by these metrics are Polish group topologies, see THEOREM 2.4.1.

REMARK 2.2.19. — Let $d_T \colon X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ be the extended metric on X defined by

$$d_T(x,y) := \inf\{n \in \mathbb{N} \colon T^n(x) = y \text{ or } T^n(y) = x\}.$$

Then by definition of the T-cocycle of any $U \in [T]$, we have that for all $x \in X$, $d_T(U(x), x) = |c_U(x)|$. For all $U, V \in [T]_\varphi$, the cocycle identity implies that $c_{UV^{-1}}(x) = c_U(V^{-1}(x)) - c_V(V^{-1}(x))$. Since V preserves the measure, we obtain

$$\begin{aligned}
\boldsymbol{d}_{\varphi,T}(U,V) &= \int_X |c_U(V^{-1}(x)) - c_V(V^{-1}(x))|_\varphi \\
&= \int_X |c_{UV^{-1}}(x)|_\varphi d\mu \\
&= \int_X \varphi(d_T(UV^{-1}(x), x)) d\mu \\
&= \int_X \varphi(d_T(U(x), V(x)) d\mu.
\end{aligned}$$

We won't use this formula thereafter. However, this point of view allows one to define φ-integrable full groups of non-necessarily free actions of finitely generated groups. Some of the arguments given in this paper work in this wider context; this will be examined in an upcoming work.

2.3 Flexibility of φ-integrable orbit equivalence

2.3.1 Construction of cycles in φ-integrable full groups

An ***n-cycle***, $n \geq 2$, is a periodic transformation $P \in \mathrm{Aut}(X, \mu)$ whose orbits have cardinality either 1 or n. The aim of this section is to prove the following result.

THEOREM 2.3.1. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X, \mu)$ be aperiodic. Then for all measurable $A \subseteq X$ and all integer $n \geq 2$, there exists an n-cycle $P \in [T]_\varphi$ whose support is equal to A.*

REMARK 2.3.2. — The hypothesis that φ is sublinear is necessary, as the result is false for L^1 full groups of certain aperiodic transformations. Indeed if $T \in \mathrm{Aut}(X, \mu)$ is ergodic, then there exists an n-cycle in $[T]_1$ whose support is $A \subseteq X$ if and only if $\exp(2i\pi/n)$ is in the spectrum of the restriction of T_A to A [LM18, Thm. 4.8]. In particular the L^1 full group of the Bernoulli shift contains no n-cycle with full support for any $n \geq 2$. By contrast, THEOREM 2.3.1 says that as soon as $p < 1$, its L^p full group contains an n-cycle of full support for every $n \geq 2$.

EXAMPLE 2.3.3. — In certain concrete situations, we can exhibit explicit involutions. Let T be the Bernoulli shift on $(\{0,1\}, \kappa)^{\otimes \mathbb{Z}}$, where κ is the uniform measure on $\{0,1\}$. Then for every $0 < p < 1/2$, there exists an involution in $[T]_p$ with full support and fundamental domain $X_0 := \{(x_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} \colon x_0 = 0\}$.

Indeed, for all $x \in X_0$, let $N(x)$ be the infimum of $n \geq 1$ such that 1 appears strictly more often than 0 in $\{x_1, \ldots, x_n\}$. Then the map $\pi\colon x \in X_0 \mapsto T^{N(x)}(x) \in \{0,1\}^{\mathbb{Z}} \setminus X_0$ is almost everywhere well-defined and injective. Thus it can be extended to an involution $P \in [T]$ with full support and fundamental domain X_0. Standard estimates on the simple random walk on $\mathbb{Z}$ imply that P belongs to $[T]_p$ for all $0 < p < 1/2$.

REMARK 2.3.4. — THEOREM 2.3.1 tells us that any measurable subset $A \subseteq X$ is the support of an involution. The situation is less flexible regarding fundamental domains. For example, the subset X_0 introduced in the previous example cannot be the fundamental domain of any involution in the L^p full group of the Bernoulli shift for $1/2 \leq p \leq 1$, as a consequence of a result of Liggett [Lig02]. Note that his result is more general and stated in probabilistic terms; the connection to our context and a purely ergodic-theoretic version of his proof will be presented in the second named author's PhD thesis.

A ***partial measure-preserving transformation*** of (X, μ) is a bimeasurable measure-preserving bijection π between two measurable subsets $\mathrm{dom}(\pi)$ and $\mathrm{rng}(\pi)$ of X, called respectively the *domain* and the *range* of π. The ***support*** of π is the set

$$\mathrm{supp}(\pi) := \{x \in \mathrm{dom}(\pi) \colon \pi(x) \neq x\} \cup \{x \in \mathrm{rng}(\pi) \colon \pi^{-1}(x) \neq x\}.$$

A ***pre-cycle*** of length $n \geq 2$ is a partial measure-preserving transformation $\pi\colon \mathrm{dom}(\pi) \to \mathrm{rng}(\pi)$ of (X,μ) such that if we set $B := \mathrm{dom}(\pi) \setminus \mathrm{rng}(\pi)$, then

- $\{\pi^0(B), \dots, \pi^{n-2}(B)\}$ is a partition of $\mathrm{dom}(\pi)$,
- $\{\pi^1(B), \dots, \pi^{n-1}(B)\}$ is a partition of $\mathrm{rng}(\pi)$.

The set $B = \mathrm{dom}(\pi) \setminus \mathrm{rng}(\pi)$ is called the ***basis*** of the pre-cycle π.

A pre-cycle π of length n can be extended to an n-cycle P and called the ***closing cycle*** of π, as follows:

$$P(x) := \begin{cases} \pi(x) & \text{if } x \in \mathrm{dom}(\pi), \\ \pi^{-(n-1)}(x) & \text{if } x \in \mathrm{rng}(\pi) \setminus \mathrm{dom}(\pi), \\ x & \text{else.} \end{cases}$$

Observe that the support of P coincides with the support of the pre-cycle π and that the basis B is a fundamental domain for the restriction of P to its support. A pre-cycle π is ***induced*** by $T \in \mathrm{Aut}(X,\mu)$ if for all $x \in \mathrm{dom}(\pi)$, we have $\pi(x) = T_{\mathrm{supp}(\pi)}(x)$.

Lemma 2.3.5. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation, let π be a pre-cycle induced by T and let P be its closing cycle. Then*

$$\boldsymbol{d}_{\varphi,T}(P,\mathrm{id}) \leq 2\boldsymbol{d}_{\varphi,T}(T_{\mathrm{supp}(\pi)},\mathrm{id}).$$

In particular P belongs to $[T]_\varphi$.

Proof. Let n be the length of the pre-cycle π, let $A := \mathrm{supp}(\pi)$ and let $B := \mathrm{dom}(\pi) \setminus \mathrm{rng}(\pi)$ the basis of π. Since π is induced by T, for all $x \in \mathrm{dom}(\pi)$, one has $\pi(x) = P(x) = T_A(x)$. This implies that $c_P(x) = c_{T_A}(x)$ for all $x \in \mathrm{dom}(\pi)$. Thus,

$$\begin{aligned} \boldsymbol{d}_{\varphi,T}(P,\mathrm{id}) &= \int_{\mathrm{dom}(\pi)} |c_{T_A}(x)|_\varphi d\mu + \int_{P^{n-1}(B)} |c_{P^{-(n-1)}}(x)|_\varphi d\mu \\ &\leq \boldsymbol{d}_{\varphi,T}(T_A,\mathrm{id}) + \int_B |c_{P^{n-1}}(x)|_\varphi d\mu. \end{aligned}$$

Moreover, for all $x \in B$, the cocycle identity yields

$$|c_{P^{n-1}}(x)|_\varphi \leq |c_P(x)|_\varphi + |c_P(P(x))|_\varphi + \dots |c_P(P^{n-2}(x))|_\varphi.$$

We now use the fact that P preserves the measure and that $\mathrm{dom}(\pi) = B \sqcup P(B) \sqcup \cdots \sqcup P^{n-2}(B)$ to get

$$\int_B |c_{P^{n-1}}(x)|_\varphi d\mu \leq \int_{\mathrm{dom}(\pi)} |c_P(x)|_\varphi d\mu \leq \int_X |c_{T_A}(x)|_\varphi d\mu,$$

which concludes the proof. □

Kac's lemma, that is Equation (2.3), implies that for every measurable $A \subseteq X$, the first return map T_A belongs to $[T]_1$, which is contained in $[T]_\varphi$ for every metric-compatible function φ by COROLLARY 2.2.18. We will need a more quantitative version of this fact.

LEMMA 2.3.6. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation, let $A \subseteq X$ be a measurable subset and let $C > 0$. Then*

$$\boldsymbol{d}_{\varphi,T}(T_A, \mathrm{id}) \leq C\varphi(1)\mu(A) + \sup_{t>C} \frac{\varphi(t)}{t}.$$

Proof. Recall that the T-cocycle of T_A is the return time $n_{T,A}$, which is non-negative. Set $B \coloneqq \{x \in A\colon n_{T,A}(x) \leq C\}$. We have

$$\begin{aligned}\boldsymbol{d}_{\varphi,T}(T_A, \mathrm{id}) &= \int_B \varphi(n_{T,A}(x))d\mu + \int_{A\setminus B} \varphi(n_{T,A}(x))d\mu \\ &\leq C\varphi(1)\mu(B) + \int_{A\setminus B} \frac{\varphi(n_{T,A}(x))}{n_{T,A}} n_{T,A}(x)d\mu \\ &\leq C\varphi(1)\mu(A) + \left(\sup_{t>C} \frac{\varphi(t)}{t}\right) \int_{A\setminus B} n_{T,A}(x)d\mu\end{aligned}$$

and the last integral is at most 1 by Kac's lemma, see Equation (2.3). □

COROLLARY 2.3.7. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Then $\boldsymbol{d}_{\varphi,T}(T_A, \mathrm{id})$ tends to 0 as $\mu(A)$ approaches 0.*

Proof. Fix $\varepsilon > 0$. By sublinearity, let $C > 0$ such that for all $t > C$, we have $\varphi(t)/t < \varepsilon$. For all measurable $A \subseteq X$, if $\mu(A) < \varepsilon/C\varphi(1)$, then $\boldsymbol{d}_{\varphi,T}(T_A, \mathrm{id}) < 2\varepsilon$, which concludes the proof. □

REMARK 2.3.8. — In particular, by taking φ bounded, we recover the well-known fact that $d_u(T_A, \mathrm{id})$ tends to 0 as $\mu(A)$ approaches 0 (see LEMMA 2.4.9).

The following lemma is a direct consequence of Rokhlin's lemma.

LEMMA 2.3.9. — *Let $T \in \mathrm{Aut}(X,\mu)$ be aperiodic and $A \subseteq X$ be measurable. For all $\varepsilon > 0$ and all integer $n \geq 2$, there exists a pre-cycle π of length n, induced by T, such that $\mathrm{supp}(\pi) \subseteq A$ and $\mu(A \setminus \mathrm{supp}(\pi)) \leq \varepsilon$.*

Proof. Since T is aperiodic, T_A is aperiodic on its support. We apply Rokhlin's lemma to T_A to find a measurable subset $B \subseteq A$ such that $B, T_A(B), \dots, (T_A)^{n-1}(B)$ are pairwise disjoint and

$$\mu\Big(A \setminus \big(B \sqcup \cdots \sqcup (T_A)^{n-1}(B)\big)\Big) \leq \varepsilon.$$

Then the restriction of T_A to $B \sqcup \cdots \sqcup (T_A)^{n-2}(B)$ is a pre-cycle of length n, which is induced by T_A and thus by T. Finally, its support satisfies the desired assumptions. □

We are now ready to prove the existence of n-cycles with prescribed support in φ-integrable full groups.

Proof of THEOREM *2.3.1.* Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic element, let $A \subseteq X$ be a measurable subset and let $n \geq 2$. Since φ is sublinear, we can and do fix a sequence $(C_k)_{k\geq 1}$ of strictly positive numbers such that

$$\sup_{t>C_k} \frac{\varphi(t)}{t} \leq 2^{-k} \text{ for all } k \geq 1.$$

Then, we use LEMMA 2.3.9 to construct inductively a sequence $(\pi_k)_{k\geq 0}$ of pre-cycles of length n induced by T, whose supports are pairwise disjoint subsets of A and such that for all $k \geq 1$,

$$\mu\Big(A \setminus \big(\mathrm{supp}(\pi_0) \sqcup \cdots \sqcup \mathrm{supp}(\pi_{k-1})\big)\Big) \leq \frac{1}{2^k C_k}.$$

This inequality implies in particular that for all $k \geq 1$, we have $\mu(\mathrm{supp}(\pi_k)) \leq 1/(2^k C_k)$. Let P_k be the closing cycle of π_k and let $P \in \mathrm{Aut}(X,\mu)$ be the n-cycle defined by $P(x) := P_k(x)$ for $x \in \mathrm{supp}(P_k)$ and $P(x) := x$ for $x \notin A$. The support of P is equal to A and by LEMMA 2.3.5 and LEMMA 2.3.6, we have

$$\begin{aligned}
\boldsymbol{d}_{\varphi,T}(P,\mathrm{id}) &= \sum_{k\geq 0} \boldsymbol{d}_{\varphi,T}(P_k,\mathrm{id}) \\
&\leq 2\sum_{k\geq 0} \boldsymbol{d}_{\varphi,T}(T_{\mathrm{supp}(\pi_k)},\mathrm{id}) \\
&\leq 2\boldsymbol{d}_{\varphi,T}(T_{\mathrm{supp}(\pi_0)},\mathrm{id}) + 2\sum_{k\geq 1}\left(\varphi(1)C_k\mu(\mathrm{supp}(\pi_k)) + \sup_{t>C_k}\frac{\varphi(t)}{t}\right).
\end{aligned}$$

The second term is by construction a converging series, so we are done. □

2.3.2 Construction of φ-integrable orbit equivalences

Let us now prove THEOREM 2.1.3.

THEOREM 2.3.10. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic. For all $n \geq 2$, there exists $U \in \mathrm{Aut}(X,\mu)$ such that T and U are φ-integrable orbit equivalent and U^n is not ergodic.*

Proof. By LEMMA 2.2.12, there is a sublinear metric-compatible function ψ such that $\varphi(t) \leq \psi(t)$ for all t large enough. In particular, ψ-integrable orbit equivalence implies φ-orbit equivalence (cf. REMARK 2.2.13). Hence if the theorem holds for ψ then it holds for φ. Therefore, by replacing φ by ψ, we may and do assume that φ is a metric-compatible function.

By THEOREM 2.3.1, there exists an n-cycle $P \in [T]_\varphi$ whose support is X. We fix a fundamental domain D for P and we let $U := T_D P$. By LEMMA 2.2.3 the following hold:

- the first return maps U_D and T_D coincide: $U_D = T_D$;
- for all $x \in D$, we have $U_D(x) = U^n(x)$;
- T and U have same orbits.

By the second item, the set D is U^n-invariant. So U^n is not ergodic.

We will now prove that T and U are φ-integrable orbit equivalent. Since T and U have same orbits, we are left to show that $T \in [U]_\varphi$ and $U \in [T]_\varphi$. As a direct consequence of Kac's lemma, see Equation (2.3), we have that $T_D \in [T]_1 \leq [T]_\varphi$ and therefore $U = T_D P \in [T]_\varphi$.

We now prove that $T \in [U]_\varphi$. In the sequel, if a measure-preserving transformation V belongs to $[T] = [U]$, we shall denote by c^T_V the T-cocycle of V and by c^U_V the U-cocycle of V.

Claim. — Let $V \in [T]$. Then for all $y \in D$ such that $V(y) \in D$,

$$\left|c^U_V(y)\right| \leq n \left|c^T_V(y)\right|.$$

Proof of Claim. Note that since y and $V(y)$ belong to D, any $i \in \mathbb{Z}$ such that $U^i(z) = V(z)$ must be a multiple of n. If we combine this with the fact that $U_D(z) = U^n(z)$ for all $z \in D$ and that $U_D = T_D$, we obtain:

$$\begin{aligned}\left|c^U_V(y)\right| &= \min\{|i| \colon U^i(y) = V(y)\} \\ &= n \min\{|i| \colon U^i_D(y) = V(y)\} \\ &= n \min\{|i| \colon T^i_D(y) = V(y)\} \\ &\leq n \min\{|i| \colon T^i(y) = V(y)\} \\ &\leq n \left|c^T_V(y)\right|.\end{aligned}$$

□

Let $x \in X$. By definition of U, there are two integers $0 \leq k, l \leq n-1$ such that $U^k(x) \in D$ and $U^l(T(x)) \in D$. By the cocycle identity,

$$c^U_{U^l T U^{-k}}(U^k(x)) = c^U_{U^l}(T(x)) + c^U_T(x) + c^U_{U^{-k}}(U^k(x)) = l + c^U_T(x) - k.$$

Hence

$$\left|c^U_T(x)\right| \leq \left|c^U_{U^l T U^{-k}}(U^k(x))\right| + n.$$

Using the claim for $V = U^l T U^{-k}$ and $y = U^k(x)$, we obtain

$$\left|c^U_T(x)\right| \leq n \left|c^T_{U^l T U^{-k}}(U^k(x))\right| + n.$$

Integrating over X, we get

$$\int_X \left|c^U_T(x)\right|_\varphi d\mu \leq \max_{0 \leq k, l \leq n-1} \int_X n \left|c^T_{U^l T U^{-k}}(U^k(x))\right|_\varphi d\mu + \varphi(n),$$

which is bounded since $U^l T U^{-k} \in [T]_\varphi$. Hence $T \in [U]_\varphi$ and this concludes the proof of the theorem. □

The following direct corollary says that the analogue of Belinskaya's theorem for φ-integrable orbit equivalence does not hold as soon as φ is sublinear.

Corollary 2.3.11. — *Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let $T \in \mathrm{Aut}(X, \mu)$ be an ergodic transformation and assume that T^n is ergodic for some $n \geq 2$. Then there exists $U \in \mathrm{Aut}(X, \mu)$ such that T and U are φ-integrable orbit equivalent but not flip-conjugate.*

Proof. Let $n \geq 2$ such that T^n is ergodic. By the previous theorem, we find $U \in \mathrm{Aut}(X, \mu)$ such that U is φ-integrable orbit equivalent to T and U^n is not ergodic. In particular U cannot be flip-conjugate to T because otherwise U^n would be flip-conjugate to T^n which is ergodic. □

Question 2.3.12. — Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X, \mu)$ be an ergodic transformation such that T^n is non-ergodic for all $n \geq 2$. Does there exist $U \in \mathrm{Aut}(X, \mu)$ such that T and U are φ-integrable orbit equivalent but not flip-conjugate?

As we will see in Section 2.4.5, a weaker result holds in full generality: for every ergodic $T \in \mathrm{Aut}(X, \mu)$ and every sublinear metric-compatible function, there is $U \in [T]_\varphi$ such that U and T have the same orbits, but are not flip-conjugate. This relies on the Baire category theorem, using the fact that $[T]_\varphi$ is a Polish group (see Section 2.4.1).

2.3.3 Connection to Shannon orbit equivalence

Let I be a countable set and $f \colon X \to I$ a measurable map. The Shannon entropy of f is the quantity

$$H(f) := -\sum_{i \in I} \mu(f^{-1}\{i\}) \log \mu(f^{-1}\{i\}).$$

Definition 2.3.13 (Kerr-Li). — Two aperiodic transformations $T_1, T_2 \in \mathrm{Aut}(X, \mu)$ are ***Shannon orbit equivalent*** if there exists $S \in \mathrm{Aut}(X, \mu)$ such that ST_1S^{-1} and T_2 have the same orbits and

$$H(c_{ST_1S^{-1}}) < +\infty \text{ and } H(c_{T_2}) < +\infty,$$

where $c_{ST_1S^{-1}}$ is the T_2-cocycle of ST_1S^{-1} and c_{T_2} is the ST_1S^{-1}-cocycle of T_2.

Lemma 2.3.14. — *There are two positive constants $C_1, C_2 > 0$ such that for any measurable function $f \colon X \to \mathbb{Z}$, we have*

$$H(f) \leq C_1 \int_X \log(1 + |f(x)|) d\mu + C_2.$$

The proof we propose is inspired by a classical proof that integrable functions have finite Shannon entropy, see for instance [Aus16, Lem. 2.1] or [Dow11, Fact 1.1.4].

Proof. Let $f_+ := \max(f, 0)$ and $f_- := \min(f, 0)$, so that $f = f_+ + f_-$. We have

$$\int_X \log(1 + |f|)d\mu = \int_X \log(1 + f_+)d\mu + \int_X \log(1 - f_-)d\mu.$$

By subadditivity, see for instance [Dow11, Chap. 1],

$$H(f) = H(f_+ + f_-) \leq H(f_+) + H(-f_-).$$

Hence, it is enough to prove the lemma for $f\colon X \to \mathbb{N}$. So let us fix $f : X \to \mathbb{N}$. For all $n \in \mathbb{N}$, let $p_n := \mu(f^{-1}\{n\})$. By definition of the Shannon entropy,

$$H(f) = -\sum_{n\geq 0} p_n \log p_n.$$

Studying the variations of the function $g_t(s) = st + e^{-s-1}$, one checks that for all $t > 0$ and $s \in \mathbb{R}$, $-t \log t \leq st + e^{-s-1}$. Applying this for $t = p_n$ and $s = 2\log(n + 1)$ and summing over n, we get

$$H(f) \leq 2 \sum_{n\geq 0} p_n \log(1 + n) + \sum_{n\geq 0} \frac{e^{-1}}{(n + 1)^2}.$$

To conclude, we observe that $\sum_{n\geq 0} p_n \log(1 + n) = \int_X \log(1 + f(x))d\mu$. □

We immediately deduce the following comparison between φ-integrable orbit equivalence and Shannon orbit equivalence.

Theorem 2.3.15. — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\log(1 + t) = O(\varphi(t))$ as $t \to +\infty$. Then for any aperiodic transformation $T \in \mathrm{Aut}(X, \mu)$, every $S \in [T]$ whose T-cocycle is φ-integrable has finite Shannon entropy.*

In particular, if two aperiodic transformations $S, T \in \mathrm{Aut}(X, \mu)$ are φ-integrable orbit equivalent, then they are Shannon orbit equivalent.

Remark 2.3.16. — Note that for every $p \in (0, +\infty)$, we have $\log(1 + t) = O(t^p)$ as $t \to +\infty$. Therefore L^p orbit equivalence implies Shannon orbit equivalence for measure-preserving transformations.

In [KL19], Kerr and Li asked whether Shannon orbit equivalence of ergodic transformations implies flip-conjugacy. We prove that it is not the case.

Theorem 2.3.17. — *Let $T \in \mathrm{Aut}(X, \mu)$ be an ergodic transformation, assume that T^n is ergodic for some $n \geq 2$. Then there exists $U \in \mathrm{Aut}(X, \mu)$ such that T and U are Shannon orbit equivalent but not flip-conjugate.*

Proof. Let us consider the sublinear metric-compatible function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi(t) := \log(1+t)$. By COROLLARY 2.3.11, there exists $U \in \mathrm{Aut}(X,\mu)$ such that T and U are φ-integrable orbit equivalent, but not flip-conjugate. On the other hand, the transformations T and U are Shannon orbit equivalent by THEOREM 2.3.15. □

2.3.4 Finiteness of entropy and Shannon orbit equivalence

Kerr and Li implicitly asked whether dynamical entropy is an invariant of Shannon orbit equivalence for ergodic measure-preserving transformations. Shortly after a first version of our paper appeared, they obtained a positive answer [KL22, Thm. A]. In this section, we provide a short proof that *finiteness* of dynamical entropy is an invariant of Shannon orbit equivalence. We start by recalling a definition of dynamical entropy of measure-preserving transformations which is convenient for our purposes.

DEFINITION 2.3.18. — Let $T \in \mathrm{Aut}(X,\mu)$. A measurable map $f\colon X \to I$, where I is countable, is called ***T-dynamically generating*** if there is a full measure set $X_0 \subseteq X$ such that for all distinct $x, y \in X_0$, there is $n \in \mathbb{Z}$ such that $f(T^n(x)) \neq f(T^n(y))$.

DEFINITION 2.3.19. — The ***dynamical entropy*** of a measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ is the infimum of the Shannon entropies of its T-dynamically generating functions.

The above definition is not the standard definition, however it is equivalent by a theorem of Rokhlin [Rok67, Thm. 10.8]. Also note that by definition, the dynamical entropy of $T \in \mathrm{Aut}(X,\mu)$ is finite if and only if T admits a dynamically generating function of finite entropy.

PROPOSITION 2.3.20. — *Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation with infinite dynamical entropy and let $U \in [T]$ be a transformation whose T-cocycle has finite Shannon entropy. Then U has infinite dynamical entropy.*

Proof. Let $f\colon X \to I$ be a U-dynamically generating function and denote by c_U the T-cocycle of U. We claim that the couple $(f, c_U)\colon X \to I \times \mathbb{Z}$ is T-dynamically generating. Indeed, let $x, y \in X$ such that

$$c_U(T^n(x)) = c_U(T^n(y)) \text{ and } f(T^n(x)) = f(T^n(y)) \text{ for all } n \in \mathbb{Z}.$$

The first equality and the cocycle identity imply that $c_{U^n}(x) = c_{U^n}(y)$ for all $n \in \mathbb{Z}$. So for all $n \in \mathbb{Z}$

$$f(U^n(x)) = f(T^{c_{U^n}(x)}(x)) = f(T^{c_{U^n}(y)}(y)) = f(U^n(y)).$$

Since f is U-dynamically generating, the above equation implies that $x = y$.

Assume by contradiction that U has finite entropy. This implies that there exists a U-generating function f with finite Shannon entropy. Since c_U has finite

Shannon entropy, then so does the function (f, c_U). But we have seen that (f, c_U) is a T-generating function, hence we deduce that T has finite dynamical entropy and the proof is complete. □

Corollary 2.3.21. — *Suppose T_1, T_2 are two aperiodic measure-preserving transformations which are Shannon orbit equivalent. Then T_1 has finite dynamical entropy if and only if T_2 has finite dynamical entropy.*

Remark 2.3.22. — As explained before, Kerr and Li recently obtained that dynamical entropy itself is preserved under Shannon orbit equivalence [KL22, Thm. A]. Suppose now that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function that satisfies $\log(1+t) = O(\varphi(t))$ as $t \to +\infty$, such as $\varphi(t) = \log(1+t)$ or $\varphi(t) = t^p$. By Kerr and Li's result and Theorem 2.3.15, dynamical entropy is an invariant of φ-integrable orbit equivalence. In particular it is an invariant of L^p orbit equivalence for $p \in (0, +\infty)$. When φ is moreover sublinear, this is the only invariant of φ-integrable orbit equivalence that we know for ergodic transformations, even for L^p orbit equivalence where $p \in (0, 1)$.

2.4 Weakly mixing elements are generic in $[T]_\varphi$

This last section is dedicated to the proof of Theorem 2.1.2: we are going to show that for every sublinear metric-compatible function φ and ergodic transformation T, there is an element $U \in [T]_\varphi$ which has the same orbit as T but is not flip-conjugated to T.

Note that we have shown in Corollary 2.3.11 that this is already the case if T is an ergodic transformation such that T^n is ergodic for some $n \geq 2$. Therefore we will restrict ourselves to the case when there exists $n \geq 2$ such that T^n is not ergodic. For such transformations, we will not construct any explicit $U \in [T]_\varphi$, but we will use the Baire category theorem. We will show that given an aperiodic transformation T, the possible candidates of such U are generic, see Theorem 2.4.15.

We start with three preparatory sections to introduce the required material. We believe them to be of independent interest. In the first one, we show that the metric $\boldsymbol{d}_{\varphi,T}$ is a complete separable metric inducing a Polish group topology on $[T]_\varphi$, see Theorem 2.4.1. In the second one, we prove a sublinear ergodic theorem in the context of φ-integrability which will play a crucial role later on. In the third one, we study continuity properties of the first return map.

2.4.1 Polish group topology

Recall that the full group $[T] \leq \mathrm{Aut}(X, \mu)$ is closed and separable for the topology induced by the uniform metric d_u and therefore it is a Polish group [Kec10, Prop. 3.2]. We shall see that φ-integrable full groups provide further interesting classes of Polish groups.

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. We introduced the φ-integrable full group $[T]_\varphi$ as the group of measure-preserving transformation whose cocycle is φ-integrable. In Lemma 2.2.14, we defined a metric $\boldsymbol{d}_{\varphi,T}$ on $[T]_\varphi$. The goal of this section is to prove that the topology induced by $\boldsymbol{d}_{\varphi,T}$ on $[T]_\varphi$ is a Polish group topology.

Theorem 2.4.1. — *The metric $\boldsymbol{d}_{\varphi,T}$ is complete, separable and right-invariant on $[T]_\varphi$ and the topology generated by $\boldsymbol{d}_{\varphi,T}$ is a group topology. In particular $[T]_\varphi$ is a Polish group.*

Proof. We have already shown in Lemma 2.2.14 that $\boldsymbol{d}_{\varphi,T}$ is right-invariant. Corollary 2.2.18 tells us that the inclusion $[T]_\varphi \hookrightarrow [T]$ is Lipschitz and in particular any $\boldsymbol{d}_{\varphi,T}$-Cauchy sequence is d_u-Cauchy. Since d_u is complete, any $\boldsymbol{d}_{\varphi,T}$-Cauchy sequence has a d_u-limit.

Claim. — Let $(U_n)_{n\geq 0}$ be a $\boldsymbol{d}_{\varphi,T}$-Cauchy sequence of elements of $[T]_\varphi$ and let $U \in [T]$ be its d_u-limit. Then $U \in [T]_\varphi$ and $\lim_n \boldsymbol{d}_{\varphi,T}(U_n,U) = 0$. In particular $\boldsymbol{d}_{\varphi,T}$ is complete.

Proof of the claim. Since $(U_n)_{n\geq 0}$ is $\boldsymbol{d}_{\varphi,T}$-Cauchy, there is m such that for all $n \geq m$,

$$\int_X |c_{U_n}(x)|_\varphi d\mu = \boldsymbol{d}_{\varphi,T}(U_n,\mathrm{id}) \leq \boldsymbol{d}_{\varphi,T}(U_n,U_m) + \boldsymbol{d}_{\varphi,T}(U_m,\mathrm{id}) \leq 1 + \boldsymbol{d}_{\varphi,T}(U_m,\mathrm{id}).$$

Moreover since $\lim_n d_u(U_n,U) = 0$, we have that $(c_{U_n})_{n\geq 0}$ converges in measure to c_U and thus a subsequence of $(c_{U_n})_{n\geq 0}$ converges pointwise to c_U. Fatou's lemma then implies that $U \in [T]_\varphi$. The triangle inequality for $|\cdot|_\varphi$ gives

$$\begin{aligned}\int_X \big||c_{U_n}(x) - c_U(x)|_\varphi - |c_{U_m}(x) - c_U(x)|_\varphi\big| d\mu &\leq \int_X |c_{U_n}(x) - c_{U_m}(x)|_\varphi d\mu \\ &= \boldsymbol{d}_{\varphi,T}(U_n,U_m),\end{aligned}$$

hence the sequence $(|c_{U_n} - c_U|_\varphi)_{n\geq 0}$ is Cauchy with respect to the L^1-metric. Since $(c_{U_n} - c_U)_{n\geq 0}$ converges in measure to 0, we must have that

$$\lim_n \boldsymbol{d}_{\varphi,T}(U_n,U) = \lim_n \int_X |c_{U_n}(x) - c_U(x)|_\varphi d\mu = 0$$

so the claim is proved. $\square_{\text{claim}}$

Let us now show that the topology induced by $\boldsymbol{d}_{\varphi,T}$ is a group topology. We start by proving the continuity of the inverse map. Let $(U_n)_{n\geq 0}$ be a sequence of elements of $[T]_\varphi$ converging to $U \in [T]_\varphi$. Then the cocycle identity gives us that

$0 = c_{UU^{-1}}(x) = c_U(U^{-1}(x)) + c_{U^{-1}}(x)$ and hence

$$\begin{aligned}
\boldsymbol{d}_{\varphi,T}(U_n^{-1}, U^{-1}) &= \int_X |c_{U_n^{-1}}(x) - c_{U^{-1}}(x)|_\varphi d\mu \\
&= \int_X |c_{U_n}(U_n^{-1}(x)) - c_U(U^{-1}(x))|_\varphi d\mu \\
&= \int_X \left| c_{U_n}(x) - c_U(U^{-1}U_n(x)) \right|_\varphi d\mu \\
&\leq \int_X |c_{U_n}(x) - c_U(x)|_\varphi d\mu + \int_X |c_U(x) - c_U(U^{-1}U_n(x))|_\varphi d\mu.
\end{aligned}$$

Since $(U_n)_{n\geq 0}$ converges to U for the metric $\boldsymbol{d}_{\varphi,T}$ and thus for the uniform metric, the right hand side converges to 0 and hence the inverse map is continuous.

We now prove that the multiplication map is continuous. Let $(U_n)_{n\geq 0}$ and $(V_m)_{m\geq 0}$ be two sequences which $\boldsymbol{d}_{\varphi,T}$-converge to U and V respectively. Then by the triangle inequality and right-invariance,

$$\boldsymbol{d}_{\varphi,T}(U_n V_n, UV) \leq \boldsymbol{d}_{\varphi,T}(U_n, U) + \boldsymbol{d}_{\varphi,T}(UV_n, UV).$$

Now remark that since the inverse map is continuous, UV_n converges to UV if and only if $V_n^{-1}U^{-1}$ converges to $V^{-1}U^{-1}$. By right-invariance and continuity of the inverse $\lim_n \boldsymbol{d}_{\varphi,T}(V_n^{-1}U^{-1}, V^{-1}U^{-1}) = 0$, which finishes the proof that $\boldsymbol{d}_{\varphi,T}$ induces a group topology on $[T]_\varphi$.

We are left to show that this topology is separable. Consider the following abelian group where we identify functions up to a null set

$$\mathrm{L}^\varphi(X, \mathbb{Z}) := \left\{ f\colon X \to \mathbb{Z}\colon \int_X |f(x)|_\varphi d\mu < +\infty \right\},$$

endowed with the metric $(f,g) \mapsto \int_X |f(x) - g(x)|_\varphi d\mu$. The function which takes $U \in [T]_\varphi$ to $c_U \in \mathrm{L}^\varphi(X,\mathbb{Z})$ is an isometry. So $[T]_\varphi$ is isometric to a metric subspace of $\mathrm{L}^\varphi(X,\mathbb{Z})$. We now prove that $\mathrm{L}^\varphi(X,\mathbb{Z})$ is separable: identify X with $[0,1]$ equipped with the Lesbegue measure and observe that the subgroup generated by characteristic functions of rational intervals is dense. Since subspaces of separable metric spaces are separable, we conclude that $[T]_\varphi$ is separable. □

We now exploit the Polish group topology to characterize the inclusion between φ-integrable full groups in terms of metric comparisons. In particular $[T]_\varphi \neq [T]_\psi$ as soon as φ and ψ are not bi-Lipschitz. However we do not know how to construct any explicit element in $[T]_\varphi \setminus [T]_\psi$.

Proposition 2.4.2. — *Let φ and ψ be two metric-compatible functions and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic measure-preserving transformation. Then the following are equivalent:*

(i) $\varphi(t) = O(\psi(t))$ *as* $t \to +\infty$.

(ii) $[T]_\psi \leq [T]_\varphi$.

The proof uses the following well-known lemma.

LEMMA 2.4.3. — *Let G be a Polish group, let $H_1 \leq H_2 \leq G$ be two subgroups of G. Suppose that H_1 and H_2 are endowed with a Polish topology which refines the topology induced by G. Then the topology of H_1 refines the topology induced by H_2.*

Proof. By hypothesis, the inclusions $H_1 \hookrightarrow G$ and $H_2 \hookrightarrow G$ are continuous. In particular, the Borel structure induced by each of their topology refines the Borel structure induced by the one of G. The Lusin-Souslin theorem states that given any two Polish spaces X and Y, if $f\colon X \to Y$ is Borel and injective then for every Borel $A \subseteq X$, the set $f(A)$ is Borel, see [Kec95, Thm. 15.1]. Therefore, we can apply it to the inclusions $H_1 \hookrightarrow G$ and $H_2 \hookrightarrow G$ to obtain that the Borel structures induced by the respective topologies of H_1 and H_2 coincide with the σ-algebra induced by the Borel subsets of G. This in particular tells us that the inclusion map $H_1 \hookrightarrow H_2$ is Borel, so it is automatically continuous by Pettis' lemma [Kec95, Thm. 9.9] which proves the lemma. □

Proof of PROPOSITION *2.4.2.* The implication (i)⇒(ii) follows from LEMMA 2.2.17, so we only need to prove (ii)⇒(i). We argue by contradiction: assume that $[T]_\psi \leq [T]_\varphi$ but (i) does not hold. By LEMMA 2.2.16, there exists a sequence $(k_n)_{n\geq 0}$ of positive integers such that

$$\lim_{n\to+\infty} \frac{\psi(k_n)}{\varphi(k_n)} = 0. \tag{2.5}$$

COROLLARY 2.2.18 tells us that $[T]_\varphi$ and $[T]_\psi$ embed continuously in $[T]$. Therefore LEMMA 2.4.3 yields that the inclusion map of $[T]_\psi$ into $[T]_\varphi$ is continuous. We will obtain our contradiction by constructing a sequence $(U_n)_{n\geq 0}$ of elements of $[T]_\psi$ such that

$$\boldsymbol{d}_{\varphi,T}(U_n, \mathrm{id}) \to 0 \text{ but } \boldsymbol{d}_{\psi,T}(U_n, \mathrm{id}) \not\to 0.$$

By Rokhlin's lemma, one can find for every $n \in \mathbb{N}$, a measurable subset $A_n \subseteq X$ such that $A_n, T(A_n), \dots T^{2k_n-1}(A_n)$ are pairwise disjoint and $\mu(A_n) \geq \frac{1}{4k_n}$. Note that

$$\mu\left(\bigsqcup_{i=0}^{k_n-1} T^i(A_n)\right) \geq \frac{1}{4}.$$

Hence, for all n such that $\varphi(k_n) \geq 4$, we can pick a measurable subset $B_n \subseteq \bigsqcup_{i=0}^{k_n-1} T^i(A_n)$ of measure exactly $\frac{1}{\varphi(k_n)}$. We then define $U_n \in [T]_\varphi$ by

$$U_n(x) := \begin{cases} T^{k_n}(x) & \text{if } x \in B_n; \\ T^{-k_n}(x) & \text{if } x \in T^{k_n}(B_n); \\ x & \text{otherwise.} \end{cases}$$

By construction $\boldsymbol{d}_\varphi(U_n, \mathrm{id}) = 2\mu(B_n)\varphi(k_n) = \frac{1}{2}$ but Equation (2.5) implies that $\boldsymbol{d}_\psi(U_n, \mathrm{id}) = 2\mu(B_n)\psi(k_n) \to 0$, a contradiction. □

Corollary 2.4.4. — *Let φ and ψ be two metric-compatible functions, let $T \in \mathrm{Aut}(X,\mu)$ be aperiodic. Then $[T]_\varphi = [T]_\psi$ if and only if $\varphi(t) = O(\psi(t))$ and $\psi(t) = O(\varphi(t))$ as $t \to +\infty$.*

2.4.2 A sublinear ergodic theorem for φ-integrable functions

In this section, we prove the following sublinear ergodic theorem, which will be a key tool in our analysis of the first return map. Given a measurable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, a measurable function $f : X \to \mathbb{C}$ is ***φ-integrable*** when $\int_X \varphi(|f(x)|)d\mu < +\infty$.

Theorem 2.4.5. — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear non-decreasing function. Let $U \in \mathrm{Aut}(X,\mu)$ and $f\colon X \to \mathbb{C}$ a measurable function f which is φ-integrable. Then for almost every $x \in X$*

$$\lim_n \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right) = 0.$$

The convergence also holds in L^1, that is

$$\lim_n \int_X \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right) d\mu = 0.$$

Proof. We start by restricting ourselves to the subadditive case. For this, we first use Lemma 2.2.12 to find a sublinear metric-compatible function ψ and $C > 0$ such that $\varphi(t) \leq \psi(t)$ for all $t \geq C$. We define $\tilde{\varphi}(t) := \psi(t) + \varphi(C)$, then $\tilde{\varphi}$ is still sublinear, non-decreasing and subadditive. Moreover, since ψ is non-decreasing we now have $\psi(t) \leq \tilde{\varphi}(t)$ for *all* $t \geq 0$. Up to replacing φ by $\tilde{\varphi}$, we thus may and do assume that φ is subadditive.

Given $n \geq 1$ and a φ-integrable function f, let for all $x \in X$

$$g_n(x) := \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right).$$

Using the fact that φ is subadditive, we deduce that the sequence of functions $(g_n)_{n\geq 1}$ satisfies Kingman's subadditivity property: for all $n, m \geq 1$ and all $x \in X$,

$$g_{n+m}(x) \leq g_n(x) + g_m(U^n(x)).$$

Kingman's subadditive theorem [Kin68] implies that $(g_n)_{n\geq 0}$ converges almost everywhere to some function g and our aim becomes to show that $g = 0$. Recall that a sequence that converges in L^1 admits an almost surely converging subsequence. In order to prove that $g = 0$, it is therefore enough to prove that $\|g_n\|_1$ converges to 0, namely to establish the second part of the theorem.

To this end, let f be a φ-integrable function and let $\varepsilon > 0$. Since $\varphi(|f|)$ is integrable and φ is non-decreasing, we find a measurable subset $A \subseteq X$ and

$K \geq 0$ such that $\int_{X\setminus A} \varphi(|f(x)|)d\mu \leq \varepsilon$ and $|f(x)| \leq K$ for every $x \in A$. For every measurable subset $B \subseteq X$, we denote $f_B := f\mathbb{1}_B$, where $\mathbb{1}_B$ is the indicator function of B. With this notation at hand, using first that φ is subadditive non-decreasing and then that U preserves the measure we obtain:

$$\begin{aligned}\limsup_n \int_X \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f_{X\setminus A}(U^k(x))\right|\right) d\mu &\leq \limsup_n \int_X \frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(|f_{X\setminus A}(U^k(x))|\right) d\mu \\ &= \int_X \varphi\left(|f_{X\setminus A}(x)|\right) d\mu \\ &\leq \varepsilon.\end{aligned}$$

Besides, since f_A is bounded by K, we have for all $x \in X$

$$\frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f_A(U^k(x))\right|\right) \leq \frac{\varphi(nK)}{n} = K\frac{\varphi(nK)}{nK}.$$

Integrating over X, we obtain

$$\int_X \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f_A(U^k(x))\right|\right) d\mu \leq K\frac{\varphi(nK)}{nK}.$$

Using that $f = f_{X\setminus A} + f_A$ and subadditivity, we deduce

$$\limsup_n \int_X \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right) d\mu \leq \varepsilon + \limsup_n K\frac{\varphi(nK)}{nK}$$

Since φ is sublinear, we finally obtain

$$\limsup_n \int_X \frac{1}{n}\varphi\left(\left|\sum_{k=0}^{n-1} f(U^k(x))\right|\right) d\mu \leq \varepsilon.$$

This proves that $\|g_n\|_1$ converges to 0, thus ending the proof of the theorem. □

Here is our main application, which will be a key tool in the following section.

Corollary 2.4.6. — *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Then for every $U \in [T]_\varphi$,*

$$\lim_n \frac{\boldsymbol{d}_{\varphi,T}(U^n, \mathrm{id})}{n} = 0.$$

Proof. For all integer $n \geq 0$ and all $x \in X$, by the cocycle identity and the triangular inequality we have

$$|c_{U^n}(x)| \leq \sum_{k=0}^{n-1} |c_U(U^k(x))|.$$

We apply THEOREM 2.4.5 to the function $f(x) := |c_U(x)|$ and get that

$$\frac{\boldsymbol{d}_{\varphi,T}(U^n,\mathrm{id})}{n} \leq \int_X \frac{1}{n}\left|\sum_{k=0}^{n-1} f(U^k(x))\right|_\varphi \underset{n\to+\infty}{\longrightarrow} 0. \qquad \square$$

REMARK 2.4.7. — We do not fully understand the asymptotic of the sequence $(\boldsymbol{d}_{\varphi,T}(U^n,\mathrm{id}))_{n\geq 0}$. For instance, when does the sequence $(\boldsymbol{d}_{\varphi,T}(U^n,\mathrm{id})/\varphi(n))_{n\geq 0}$ converge?

2.4.3 Continuity properties of the first return map

In the coming section we are primarily interested in continuity properties of the first return map. An important preliminary step is the following analogue of Kac's Lemma, saying that φ-integrable full groups are stable under first return maps.

LEMMA 2.4.8. — *Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation and let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. For all $U \in [T]_\varphi$ and all measurable subset $A \subseteq X$, we have $\boldsymbol{d}_{\varphi,T}(U_A,\mathrm{id}) \leq \boldsymbol{d}_{\varphi,T}(U,\mathrm{id})$. In particular, $U_A \in [T]_\varphi$.*

Proof. For every integer $j \geq 1$, set $A_j := \{x \in A\colon n_{U,A}(x) = j\}$ where $n_{U,A}$ is the first return time of U to A as defined in Section 4.1. Then,

$$\int_X |c_{U_A}(x)|_\varphi d\mu = \int_A |c_{U_A}(x)|_\varphi d\mu = \sum_{j=1}^{+\infty}\int_{A_j}|c_{U_A}(x)|_\varphi d\mu = \sum_{j=1}^{+\infty}\int_{A_j}|c_{U^j}(x)|_\varphi d\mu$$

By the cocycle identity, for every $j \geq 1$ we have $c_{U^j}(x) = \sum_{i=0}^{j-1} c_U(U^i(x))$, so by the triangle inequality we obtain

$$\begin{aligned}\int_X |c_{U_A}(x)|_\varphi d\mu &\leq \sum_{j=1}^{+\infty}\sum_{i=0}^{j-1}\int_{A_j}|c_U(U^i(x))|_\varphi d\mu\\ &\leq \sum_{j=1}^{+\infty}\sum_{i=0}^{j-1}\int_{U^i(A_j)}|c_U(x)|_\varphi d\mu\\ &\leq \int_X |c_U(x)|_\varphi d\mu,\end{aligned}$$

the last inequality being a consequence of the fact that the sets $U^i(A_j)$ are disjoint for $j \in \mathbb{N}$ and $i \in \{0,\dots,j-1\}$. $\square$

In order to state the continuity properties of the first return map, let us first observe that since we are working up to measure zero, the first return map with respect to a set A only depends on A up to a null set. It is therefore natural to introduce the measure algebra $\mathrm{MAlg}(X,\mu)$, defined as the algebra of measurable subsets modulo identifying subsets which differ on a null set. We endow $\mathrm{MAlg}(X,\mu)$ with the metric $d_\mu(A,B) := \mu(A \triangle B)$.

We can now recall a continuity property satisfied by the first return map in the full group, which was first observed by Keane.

LEMMA 2.4.9 ([Kea70, Lem. 3]). — *Let T be a measure-preserving transformation, then the map*

$$\begin{aligned}[T] \times \mathrm{MAlg}(X,\mu) &\to [T] \\ (U,A) &\mapsto U_A\end{aligned}$$

is continuous.

It is worth noting that the analogue of LEMMA 2.4.8 fails for the L^1-full group. Indeed, let $T \in \mathrm{Aut}(X,\mu)$ be ergodic and let $\varphi := \mathrm{id}_{\mathbb{R}_+}$ Then Kac's Lemma yields that for all measurable $A \subseteq X$ of positive measure, $\boldsymbol{d}_{\varphi,T}(T_A,\mathrm{id}) = d_{\varphi,T}(T,\mathrm{id}) = 1$. Since $T_\varnothing = \mathrm{id}$, this shows that the map $\mathrm{MAlg}(X,\mu) \to [T]_1$ defined by $A \mapsto T_A$ is not continuous.

However, the situation is not that clear when φ is sublinear.

QUESTION 2.4.10. — Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a *sublinear* metric-compatible function. Is the map $\mathrm{MAlg}(X,\mu) \to [T]_\varphi$ defined by $A \mapsto T_A$ continuous? More generally, is the map $[T]_\varphi \times \mathrm{MAlg}(X,\mu) \to [T]_\varphi$ given by $(U,A) \mapsto U_A$ continuous?

In this section we give two partial answers to the above questions. We first prove that the map $A \mapsto U_A$ satisfies a continuity property "from below". For this, we need the following version of Scheffé's lemma for sequences of $\mathbb{Z}$-valued φ-integrable functions.

LEMMA 2.4.11. — *Let $f\colon X \to \mathbb{Z}$ be a measurable function and let $(f_n)_{n\geq 0}$ be a sequence of measurable functions $f_n\colon X \to \mathbb{Z}$ that converge in measure to f. If*

$$\limsup \int_X |f_n|_\varphi \, d\mu \leq \int_X |f|_\varphi \, d\mu,$$

then $\lim_n \int_X |f_n - f|_\varphi \, d\mu = 0$.

Proof. It suffices to show that given $\varepsilon > 0$, there is $\delta > 0$ such that for all measurable function $g\colon X \to \mathbb{Z}$ satisfying

$$\mu(\{x \in X\colon f(x) \neq g(x)\}) \leq \delta \text{ and } \int_X |g|_\varphi \, d\mu \leq \int_X |f|_\varphi \, d\mu + \delta, \tag{2.6}$$

one has that $\int_X |f-g|_\varphi \, d\mu \leq \varepsilon$. To this end, fix $\varepsilon > 0$. Since $\int_X |f|_\varphi d\mu < +\infty$, by Lebesgue's dominated convergence theorem there exists $\delta_0 > 0$ such that for all measurable subset $A \subseteq X$, if $\mu(A) < \delta_0$ then $\int_A |f|_\varphi d\mu < \varepsilon$. Take $\delta := \min\{\delta_0, \varepsilon\}$. Let $g\colon X \to \mathbb{Z}$ be a measurable function satisfying (2.6). If we let $A := \{x \in X\colon f(x) \neq g(x)\}$, we have

$$\int_A |g|_\varphi d\mu = \int_X |g|_\varphi d\mu - \int_{X\setminus A} |g|_\varphi d\mu \leq \int_X |f|_\varphi d\mu - \int_{X\setminus A} |f|_\varphi d\mu + \delta \leq 2\varepsilon$$

and we can therefore conclude the proof

$$\int_X |f-g|_\varphi \, d\mu = \int_A |f-g|_\varphi \, d\mu \leq \int_A |f|_\varphi \, d\mu + \int_A |g|_\varphi \, d\mu \leq 3\varepsilon. \qquad \square$$

We can now prove the following proposition which is the φ-integrable analogue of [LM18, Prop. 3.9].

PROPOSITION 2.4.12. — *Let φ be a metric-compatible function and $T \in \mathrm{Aut}(X,\mu)$ an ergodic transformation. Consider $U \in [T]_\varphi$ and consider a measurable subset $A \subseteq X$. If $(A_n)_{n\geq 0}$ is a sequence of measurable subsets of A such that $\lim_n \mu(A \setminus A_n) = 0$, then $\lim_n \boldsymbol{d}_{\varphi,T}(U_{A_n}, U_A) = 0$.*

Proof. Since $\lim_n \mu(A \setminus A_n) = 0$ and since the first return map is continuous with respect to the uniform metric by LEMMA 2.4.9, we get that $\lim_n d_u(U_{A_n}, U_A) = 0$. This means that $(c_{U_{A_n}})_{n\geq 0}$ converges in measure to c_{U_A} and therefore that $(|c_{U_{A_n}}|_\varphi)_{n\geq 0}$ converges in measure to $|c_{U_A}|_\varphi$. Thanks to LEMMA 2.4.8, we have for all $n \geq 0$, $d_{\varphi,T}(U_{A_n}, \mathrm{id}) \leq d_{\varphi,T}(U_A, \mathrm{id})$. In other words

$$\int_X |c_{U_{A_n}}(x)|_\varphi d\mu \leq \int_X |c_{U_A}(x)|_\varphi d\mu.$$

Hence we can apply LEMMA 2.4.11, yielding

$$\lim_n \int_X |c_{U_{A_n}}(x) - c_{U_A}(x)|_\varphi d\mu = 0.$$

This precisely means that $\lim_n \boldsymbol{d}_{\varphi,T}(U_{A_n}, U_A) = 0$, so we are done. $\square$

Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function and $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. In COROLLARY 2.3.7, we proved that for any aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$, the quantity $\boldsymbol{d}_{\varphi,T}(T_A, \mathrm{id})$ tends to 0 as $\mu(A)$ approaches 0. It is natural to ask whether this holds for all aperiodic $U \in [T]_\varphi$, i.e. does $\boldsymbol{d}_{\varphi,T}(U_A, \mathrm{id})$ tends to 0 as $\mu(A)$ approaches 0? We were not able to answer this question, but we can prove the following much weaker statement. Its proof relies on our sublinear ergodic theorem (THEOREM 2.4.5), or rather on COROLLARY 2.4.6.

PROPOSITION 2.4.13. — *Let φ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Then for any aperiodic transformation $U \in [T]_\varphi$ and for any measurable subset $A \subseteq X$, there exists a sequence $(A_k)_{k\geq 0}$ of measurable subsets contained in A which intersect every U-orbit, such that $\lim_k \mu(A_k) = 0$ and $\lim_k \boldsymbol{d}_{\varphi,T}(U_{A_k}, \mathrm{id}) = 0$.*

Proof. Put $V := U_A$ and remark that for every measurable $B \subseteq A$, we have that $V_B = U_B$. As an immediate consequence of Alpern's multiple Rokhlin theorem [Alp79][2], for every $k \geq 0$, one can find a measurable subset $B_k \subseteq A$ which meets

[2]We actually only need Step 1 from the simpler proof given in [EP97].

every V-orbit in A and such that $n_{V,B_k}(B_k) = \{k, k+1\}$. The latter implies that the $V^i(B_k)$ are disjoint for $i \in \{0, ..., k-1\}$. Observe that for all $x \in X$ and $i \in \mathbb{Z}$ we have $n_{V,V^i(B_k)}(x) = n_{V,B_k}(V^{-i}(x))$. This implies that for all $x \in V^i(B_k)$, either $V_{V^i(B_k)}(x) = V^k(x)$ or $V_{V^i(B_k)}(x) = V^{k+1}(x)$. Therefore by integrating over the disjoint union of the $V^i(B_k)$ for $i \in \{0, ..., k-1\}$ we get that

$$\sum_{i=0}^{k-1} \boldsymbol{d}_{\varphi,T}(V_{V^i(B_k)}, \mathrm{id}) \leq \boldsymbol{d}_{\varphi,T}(V^k, \mathrm{id}) + \boldsymbol{d}_{\varphi,T}(V^{k+1}, \mathrm{id}),$$

whence there exists $0 \leq i_k \leq k-1$ such that

$$\boldsymbol{d}_{\varphi,T}(V_{V^{i_k}(B_k)}, \mathrm{id}) \leq \frac{\boldsymbol{d}_{\varphi,T}(V^k, \mathrm{id}) + \boldsymbol{d}_{\varphi,T}(V^{k+1}, \mathrm{id})}{k}.$$

The set $A_k := V^{i_k}(B_k)$ has measure less than $1/k$. Corollary 2.4.6 implies that the right hand side in the above formula tends to zero, which implies

$$\lim_k \boldsymbol{d}_{\varphi,T}(U_{A_k}, \mathrm{id}) = \lim_k \boldsymbol{d}_{\varphi,T}(V_{A_k}, \mathrm{id}) = 0. \qquad \square$$

2.4.4 Optimality of Belinskaya's theorem

We are now ready to prove Theorem 2.1.2: for any sublinear function φ, Belinskaya's theorem fails if we replace integrability by φ-integrability.

Theorem 2.4.14. — *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function and let $T_1 \in \mathrm{Aut}(X, \mu)$ be ergodic. Then there exists an ergodic transformation $T_2 \in [T_1]$ whose cocycle is φ-integrable such that T_1 and T_2 have the same orbits but are not flip-conjugate.*

The proof of the theorem depends on whether T_1 is weakly mixing. Indeed if it is the case, then we can use Corollary 2.3.11. Otherwise, we have to use the Baire category theorem. Indeed the candidate for T_2 is generic for the topology induced by $\boldsymbol{d}_{\varphi,T_1}$.

Theorem 2.4.15. — *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X, \mu)$ be an aperiodic element. Then the set of all elements of $[T]_\varphi$ which are weakly mixing and have the same orbits as T is a dense G_δ set in the Polish space of aperiodic elements of $[T]_\varphi$ with respect to the topology induced by $\boldsymbol{d}_{\varphi,T}$.*

We delay the proof of the above theorem to Section 2.4.5. Let us first explain how to deduce Theorem 2.4.14 from Theorem 2.4.15.

Proof of Theorem *2.4.14.* By Lemma 2.2.12, there is a sublinear metric-compatible function ψ such that $\varphi(t) \leq \psi(t)$ for all t large enough. In particular, ψ-integrability implies φ-integrability for $\mathbb{Z}$-valued functions (cf. Remark 2.2.13). Hence if the theorem holds for ψ then it holds for φ. Therefore, by replacing φ by ψ, we may and do assume that φ is a metric-compatible function.

If T_1 is weakly mixing, then all its nontrivial power are ergodic. Thus COROLLARY 2.3.11 implies that there exists $T_2 \in [T_1]_\varphi$ such that T_1 and T_2 have the same orbits but are not flip-conjugate.

If T_1 is not weakly mixing, then THEOREM 2.4.15 grant us the existence of some weakly mixing $T_2 \in [T_1]_\varphi$ such that T_1 and T_2 have the same orbits. Since T_2 is weakly mixing and T_1 isn't, they cannot be flip-conjugated. □

2.4.5 Weakly mixing elements form a dense G_δ set

This section is dedicated to the proof of THEOREM 2.4.15. Before starting the proof, we will need some terminology and preliminary propositions.

In this section we will consider the φ-integrable full groups both with the topology induced by the uniform metric d_u and the their natural topology induced by $\boldsymbol{d}_{\varphi,T}$. The metric $\boldsymbol{d}_{\varphi,T}$ is complete so we can apply the Baire category theorem in $([T]_\varphi, \boldsymbol{d}_{\varphi,T})$, see THEOREM 2.4.1. Moreover, the topology induced by $\boldsymbol{d}_{\varphi,T}$ refines the topology induced by d_u, see COROLLARY 2.2.18. Note that $([T]_\varphi, d_u)$ is not complete, indeed one can show that $[T]_\varphi$ is dense in the complete metric space $([T], d_u)$.

Denote by $\mathrm{APER} \subseteq \mathrm{Aut}(X, \mu)$ the set of aperiodic transformations.

LEMMA 2.4.16. — *Let φ be a metric-compatible function and let $T \in \mathrm{Aut}(X, \mu)$ be an aperiodic element. Then the set $\mathrm{APER} \cap [T]_\varphi$ is closed in the complete metric space $([T]_\varphi, \boldsymbol{d}_{\varphi,T})$ and hence it is a complete metric space itself.*

Proof. Note that T is aperiodic if and only if for all $n \geq 1$ we have $d_u(T^n, \mathrm{id}) = 1$. So the set APER is closed in $(\mathrm{Aut}(X, \mu), d_u)$. In particular, $\mathrm{APER} \cap [T]_\varphi$ is closed in $([T]_\varphi, d_u)$, so it is also closed in $([T]_\varphi, \boldsymbol{d}_{\varphi,T})$. □

PROPOSITION 2.4.17. — *Let φ be a sublinear metric-compatible function and consider an aperiodic element $T \in \mathrm{Aut}(X, \mu)$. Then the set*

$$\{U \in \mathrm{APER} \cap [T]_\varphi \colon T \text{ and } U \text{ have the same orbits}\}$$

is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$.

Proof. We first prove that this set is G_δ. For all $\varepsilon > 0$ and $n \geq 1$, let

$$O_{\varepsilon,n} := \left\{U \in [T]_\varphi \colon \mu(\{x \in X \colon T(x) \in \{U^{-n}(x), \ldots, U^n(x)\}\}) > 1 - \varepsilon\right\}.$$

Each $O_{\varepsilon,n}$ is open in $([T]_\varphi, d_u)$ and thus also in $([T]_\varphi, \boldsymbol{d}_{\varphi,T})$. Moreover, we have

$$\{U \in \mathrm{APER} \cap [T]_\varphi \colon T \text{ and } U \text{ have the same orbits}\} = \bigcap_{\varepsilon \in \mathbb{Q}_+^*} \bigcup_{n \geq 1} O_{\varepsilon,n},$$

which is a countable intersection of open sets, thus by definition a G_δ set.

We now prove the density. Let $U \in \mathrm{APER} \cap [T]_\varphi$. Fix a sequence $(A_k)_{k \geq 0}$ of measurable subset of X which intersect every S-orbits, such that $\lim_k \mu(A_k) = 0$

and $\lim_k \boldsymbol{d}_{\varphi,T}(U_{A_k}, \mathrm{id}) = 0$ as in Proposition 2.4.13. If we set $P_k := (U_{A_k})^{-1}U$, then we get that $(P_k)_{k\geq 0}$ tends to U. Moreover, Corollary 2.3.7 implies that $(T_{A_k})_{k\geq 0}$ tends to the identity, which implies that $(T_{A_k}P_k)_{k\geq 0}$ tends to U. On the other hand, Lemma 2.2.4 yields that the transformation P_k is periodic and A_k is a fundamental domain for it. Thus, the transformations $T_{A_k}P_k$ and T have the same orbits by Lemma 2.2.3 and the proof is completed. □

Let ERG denote the set of ergodic transformations in $\mathrm{Aut}(X,\mu)$.

Proposition 2.4.18. — *Let φ be a sublinear metric-compatible function and fix an ergodic transformation $T \in \mathrm{Aut}(X,\mu)$. Then $\mathrm{ERG} \cap [T]_\varphi$ is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$.*

Proof. By [Kec10, Thm. 3.6], $\mathrm{ERG} \cap [T]$ is a G_δ set in $(\mathrm{APER} \cap [T], d_u)$. Thus $\mathrm{ERG} \cap [T]_\varphi$ is a G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$. Finally, since T is ergodic,

$$\mathrm{ERG} \cap [T]_\varphi \supseteq \{S \in \mathrm{APER} \cap [T]_\varphi \colon S \text{ and } T \text{ have the same orbits}\}.$$

Thus, Proposition 2.4.17 yields that $\mathrm{ERG} \cap [T]_\varphi$ is dense in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$ and the proof is complete. □

Remark 2.4.19. — The hypothesis that φ is sublinear is necessary, as for any ergodic $T \in \mathrm{Aut}(X,\mu)$, we have that $\mathrm{ERG} \cap [T]_1$ is not dense in $\mathrm{APER} \cap [T]_1$. Indeed, one can define a continuous *index map* $I \colon [T]_1 \to \mathbb{Z}$ by integrating the cocycle (see [LM18, Cor. 4.20] for the fact that it takes values in $\mathbb{Z}$). Then note that $I(U) \neq 0$ for every ergodic $U \in [T]_1$: by [LM18, Prop. 4.13] every ergodic $U \in [T]_1$ is either almost positive or almost negative. Then combining [LM18, Prop. 4.17 and Prop. 3.4] yields that U has positive or negative index, so $I(U) \neq 0$. Finally, there are aperiodic elements with index 0: take $A \subseteq X$ measurable with $0 < \mu(A) < 1$, then the aperiodic transformation $U := T_A T_{X\setminus A}^{-1}$ has index zero (using again [LM18, Prop. 3.4]). By continuity of the discrete-valued index map, we conclude that $\mathrm{ERG} \cap [T]_1$ cannot be dense in $\mathrm{APER} \cap [T]_1$.

Definition 2.4.20. — A transformation $S \in \mathrm{Aut}(X,\mu)$ is ***weakly mixing*** if for all finite subset $\mathcal{F} \subseteq \mathrm{MAlg}(X,\mu)$ and all $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that for all $A, B \in \mathcal{F}$,

$$|\mu(V^n(A) \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

Given a measurable subset of positive measure $A \subseteq X$ we will denote by μ_A the probability measure on A defined by $\mu_A(B) := \mu(A \cap B)/\mu(A)$. We say that a transformation $T \in \mathrm{Aut}(X,\mu)$ is *weakly mixing on A* if $T(A) = A$ and the restriction of T to A is weakly mixing as an element of $\mathrm{Aut}(A,\mu_A)$. The following result will be crucial in the proof of Theorem 2.4.15.

Theorem 2.4.21 (Conze [Con72]). — *Let $T \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation. Then the set*

$$\{A \in \mathrm{MAlg}(X,\mu) \colon T_A \textit{ is weakly mixing on } A\}$$

is a dense G_δ set in $(\mathrm{MAlg}(X,\mu), d_\mu)$ *where* $d_\mu(A,B) := \mu(A \triangle B)$.

Denote by WMIX the set of weakly mixing transformations of $\mathrm{Aut}(X,\mu)$. We are finally ready to prove Theorem 2.4.15 which can be reformulated as follows.

Theorem 2.4.22. — *Let φ be a sublinear metric-compatible function and let $T \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation. Then the set*

$$\{U \in \mathrm{WMIX} \cap [T]_\varphi \colon T \text{ and } U \text{ have the same orbits}\}$$

is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$.

Proof. By the Baire category theorem, the intersection of two dense G_δ subsets is a dense G_δ subset. Hence by Proposition 2.4.17, it suffices to show that $\mathrm{WMIX} \cap [T]_\varphi$ is a dense G_δ set in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$, which will occupy the remainder of the proof.

By definition, a transformation U is weakly mixing if and only if for all finite subset $\mathcal{F} \subseteq \mathrm{MAlg}(X,\mu)$ and all $\varepsilon > 0$, it belongs to the set $O_{\mathcal{F},\varepsilon}$ defined by:

$$O_{\mathcal{F},\varepsilon} := \{V \in \mathrm{Aut}(X,\mu) \colon \exists n \in \mathbb{Z},\ \forall A,B \in \mathcal{F},\ |\mu(V^n(A) \cap B) - \mu(A)\mu(B)| < \varepsilon\}\,.$$

Observe that each $O_{\mathcal{F},\varepsilon}$ is open in $(\mathrm{Aut}(X,\mu), d_u)$. As before, denote by d_μ the metric on $\mathrm{MAlg}(X,\mu)$ defined by $d_\mu(A,B) = \mu(A \triangle B)$.

Claim 1. — Let $\mathcal{F} = \{A_1, ..., A_n\}$ and $\mathcal{F}' = \{A'_1, ..., A'_n\}$ be subsets of $\mathrm{MAlg}(X,\mu)$. Fix $\varepsilon > 0$. If for every $i \in \{1,\dots,n\}$ one has $\mu(A_i \triangle A'_i) < \varepsilon$, then

$$O_{\mathcal{F},\varepsilon} \subseteq O_{\mathcal{F}',5\varepsilon}.$$

Proof of the claim. Let $V \in O_{\mathcal{F},\varepsilon}$. Fix $n \in \mathbb{Z}$ such that for all $i,j \in \{1,\dots,n\}$ we have $|\mu(V^n(A_i) \cap A_j) - \mu(A_i)\mu(A_j)| < \varepsilon$. We first remark that for every measurable $B \subset X$ and $i \in \{1,\dots,n\}$, we have $|\mu(B)\mu(A_i) - \mu(B)\mu(A'_i)| < \varepsilon$ and $|\mu(B \cap A'_i) - \mu(B \cap A_i)| < \varepsilon$. The result now follows from the triangular inequality and the fact V preserves μ:

$$\begin{aligned}
|\mu(V^n(A'_i) \cap A'_j) - \mu(A'_i)\mu(A'_j)| &< |\mu(V^n(A'_i) \cap A'_j) - \mu(A'_i)\mu(A_j)| + \varepsilon \\
&< |\mu(V^n(A'_i) \cap A'_j) - \mu(A_i)\mu(A_j)| + 2\varepsilon \\
&< |\mu(V^n(A'_i) \cap A_j) - \mu(A_i)\mu(A_j)| + 3\varepsilon \\
&= |\mu(A'_i \cap V^{-n}(A_j)) - \mu(A_i)\mu(A_j)| + 3\varepsilon \\
&< |\mu(A_i \cap V^{-n}(A_j)) - \mu(A_i)\mu(A_j)| + 4\varepsilon \\
&= |\mu(V^n(A_i) \cap A_j) - \mu(A_i)\mu(A_j)| + 4\varepsilon \\
&< 5\varepsilon.
\end{aligned}$$

$\square_{\text{claim}}$

Since (X,μ) is standard, we can fix a countable dense subset $\mathcal{M}$ of $(\mathrm{MAlg}(X,\mu), d_\mu)$.

It follows from the Claim 1 that

$$\mathrm{WMIX} = \bigcap_{\varepsilon \in \mathbb{Q}_+^*} \bigcap_{\mathcal{F} \subseteq \mathcal{M} \text{ finite}} O_{\mathcal{F},\varepsilon}. \tag{2.7}$$

In particular WMIX is a G_δ set in $(\mathrm{Aut}(X,\mu), d_u)$ and hence $\mathrm{WMIX} \cap [T]_\varphi$ is a G_δ set in $([T]_\varphi, \boldsymbol{d}_{\varphi,T})$.

We now prove that WMIX is dense. By the Baire category theorem, it is enough to show that each $O_{\mathcal{F},\varepsilon}$ is dense in $(\mathrm{APER} \cap [T]_\varphi, \boldsymbol{d}_{\varphi,T})$. By Proposition 2.4.18 the set $\mathrm{ERG} \cap [T]_\varphi$ is dense in $\mathrm{APER} \cap [T]_\varphi$, so it is enough to prove that

$$\mathrm{ERG} \cap [T]_\varphi \subseteq \overline{O_{\mathcal{F},\varepsilon} \cap \mathrm{APER} \cap [T]_\varphi}. \tag{2.8}$$

So let us fix a finite subset $\mathcal{F} \subseteq \mathrm{MAlg}(X,\mu)$, a positive real $\varepsilon > 0$ and an ergodic transformation $U \in \mathrm{ERG} \cap [T]_\varphi$.

We let $(X_k)_{k\geq 0}$ be a sequence of measurable subsets such that $\mu(X_k) = 1 - 2^{-k}$. For all $k \geq 0$, we apply Conze's Theorem to the transformation U_{X_k}, which is ergodic on X_k, to find a measurable subset $Y_k \subseteq X_k$ such that $\mu(Y_k) > 1 - 2^{-k+1}$ and U_{Y_k} is weakly mixing on Y_k. Set $V_k := U_{Y_k} T_{X \setminus Y_k}$. We claim that $(V_k)_{k\geq 0}$ tends to U. Indeed since $\lim_k \mu(Y_k) = 1$, Proposition 2.4.12 yields that U_{Y_k} tends to U while Corollary 2.3.7 gives us that $T_{X\setminus Y_k}$ tends to the identity.

Claim 2. — For k large enough, we have that $V_k \in O_{\mathcal{F},\varepsilon}$.

Proof of the claim. For all $k \geq 0$, put $\mathcal{F}_k := \{A \cap Y_k : A \in \mathcal{F}\}$. Since U_{Y_k} is weakly mixing on Y_k, we have $U_{Y_k} \in O_{\mathcal{F}_k,\varepsilon/5}$. By construction, the transformations U_{Y_k} and V_k coincide on Y_k, so we also have $V_k \in O_{\mathcal{F}_k,\varepsilon/5}$. Since $\lim_k \mu(Y_k) = 1$, for k large enough and all $A \in \mathcal{F}$, we have $\mu(A \bigtriangleup (A \cap Y_k)) < \varepsilon/5$. We thus get that $V_k \in O_{\mathcal{F},\varepsilon}$ for k large enough by Claim 1. $\square_{\text{claim}}$

It follows immediately from Claim 2 that any ergodic element in $[T]_\varphi$ is a limit of aperiodic elements in $O_{\mathcal{F},\varepsilon} \cap [T]_\varphi$. This shows the inclusion (2.8), ending the proof of the theorem. $\square$

2.A Proof of Belinskaya's theorem

In this appendix, we present a short proof of Belinskaya's theorem due to Katznelson which is not publicly available to our knowledge. As in Belinskaya's original proof, a key step is the following theorem, of independent interest. The proof we present here is mainly due to Katznelson. To lighten notation, given a point $x \in X$, a map $T\colon X \to X$, and a subset $I \subseteq \mathbb{Z}$, we will write

$$T^I(x) := \{T^i(x) : i \in I\}.$$

Theorem 2.A.1. — *Let T be an aperiodic measure-preserving transformation, suppose $U \in \mathrm{Aut}(X,\mu)$ has the same orbits as T and that for almost every $x \in X$, the symmetric* *difference of the respective positive T and U orbits $T^{\mathbb{N}}(x) \triangle U^{\mathbb{N}}(x)$ is finite. Then T and U are conjugate.*

Proof (Katznelson). We will explicitly define an element S in $[T]$ such that $U = S^{-1}TS$. This will be done thanks to the following claim.

Claim. — For almost every $x \in X$, there exists a unique $j(x) \in \mathbb{Z}$ such that

$$\left|T^{\mathbb{N}+j(x)}(x) \setminus U^{\mathbb{N}}(x)\right| = \left|U^{\mathbb{N}}(x) \setminus T^{\mathbb{N}+j(x)}(x)\right|$$

Proof of the claim. For almost every $x \in X$, consider the function $\tau_x \colon \mathbb{Z} \to \mathbb{Z}$ defined by

$$\tau_x(j) := \left|T^{\mathbb{N}+j}(x) \setminus U^{\mathbb{N}}(x)\right| - \left|U^{\mathbb{N}}(x) \setminus T^{\mathbb{N}+j}(x)\right|.$$

Remark that by assumption for almost every x, the value $\tau_x(j)$ is finite for all $j \in \mathbb{Z}$. By considering the two cases $T^j(x) \in U^{\mathbb{N}}(x)$ and $T^j(x) \notin U^{\mathbb{N}}(x)$, we see that $\tau_x(j+1) = \tau_x(j) + 1$ for all $j \in \mathbb{Z}$. It follows that $\tau_x(j) = \tau_x(0) + j$ for all $j \in \mathbb{Z}$ so $j(x) := -\tau_x(0)$ is the unique element we seek. $\square_{\text{claim}}$

We set $S(x) := T^{j(x)}(x)$. By the above claim, $S(x)$ is the unique element of the T-orbit of x satisfying

$$\left|T^{\mathbb{N}}(S(x)) \setminus U^{\mathbb{N}}(x)\right| = \left|U^{\mathbb{N}}(x) \setminus T^{\mathbb{N}}(S(x))\right|. \tag{2.9}$$

By considering whether none, only one, or both of the points x and $S(x)$ belong to $T^{\mathbb{N}}(S(x)) \cap U^{\mathbb{N}}(x)$, we see that removing the point $S(x)$ from $T^{\mathbb{N}}(S(x))$ and the point x from $U^{\mathbb{N}}(x)$ does not perturb the above equation, so that

$$\left|T^{\mathbb{N}+1}(S(x)) \setminus U^{\mathbb{N}+1}(x)\right| = \left|U^{\mathbb{N}+1}(x) \setminus T^{\mathbb{N}+1}(S(x))\right|.$$

This can be rewritten as

$$\left|T^{\mathbb{N}}(TS(x)) \setminus U^{\mathbb{N}}(U(x))\right| = \left|U^{\mathbb{N}}(U(x)) \setminus T^{\mathbb{N}}(TS(x))\right|,$$

which by equation (2.9) yields the desired equivariance condition

$$SU(x) = TS(x).$$

We now have to check that $S \in [T]$. Using that T and U are invertible and a straightforward induction, we obtain that $SU^n(x) = T^nS(x)$ for all $n \in \mathbb{Z}$. In particular S induces a bijection from the T-orbit of x to the U-orbit of $S(x)$. Since $S(x) = T^{j(x)}(x)$ belongs to the T-orbit of x, which coincides with the U-orbit of x, we conclude that S induces a bijection on each T-orbit, in particular S is bijective. Finally we check that S is measure-preserving. The sets $A_n := \{x \in X \colon S(x) = T^n(x)\}$ for $n \in \mathbb{Z}$ form a partition of X. If $B \subseteq X$ is measurable, we write

$B = \bigsqcup_n A_n \cap B$ so that $\mu(S(B)) = \sum_n \mu(T^n(A_n \cap B)) = \sum_n \mu(A_n \cap B) = \mu(B)$. This ends the proof of THEOREM 2.A.1 □

Given $T \in \mathrm{Aut}(X, \mu)$, denote by $\mathcal{R}_T \subseteq X \times X$ the equivalence relation whose classes are the T-orbits. Before proceeding with the proof of Belinskaya's theorem, we recall the following well-known lemma. Its usefulness towards proving Belinskaya's theorem was pointed out to us by Todor Tsankov.

LEMMA 2.A.2 (Mass-transport principle). — *Let $T \in \mathrm{Aut}(X, \mu)$ and $f\colon \mathcal{R}_T \to \mathbb{N}$ be a measurable map. Then*

$$\int_X \sum_{n\in\mathbb{Z}} f(x, T^n(x))d\mu = \int_X \sum_{n\in\mathbb{Z}} f(T^n(x), x)d\mu.$$

Proof. Since f is non-negative, Tonelli's theorem tells us that

$$\begin{aligned}\int_X \sum_{n\in\mathbb{Z}} f(x, T^n(x))d\mu &= \sum_{n\in\mathbb{Z}} \int_X f(x, T^n(x))d\mu \\ &= \sum_{n\in\mathbb{Z}} \int_X f(T^{-n}(x), x)d\mu \\ &= \sum_{n\in\mathbb{Z}} \int_X f(T^n(x), x)d\mu \\ &= \int_X \sum_{n\in\mathbb{Z}} f(T^n(x), x)d\mu.\end{aligned}$$ □

THEOREM 2.A.3 (Belinskaya's theorem). — *Let $T \in \mathrm{Aut}(X, \mu)$ be ergodic, let $U \in [T]_1$. Then T and U are flip-conjugate.*

Proof. Define a T-invariant total order $\leq_T$ on each T-orbit by setting $x \leq_T y$ if there is $n \geq 0$ such that $y = T^n(x)$. We will write $x <_T y$ whenever $x \neq y$ and $x \leq_T y$. Define $f\colon \mathcal{R}_T \to \mathbb{N}$ by:

$$f(x,y) := \begin{cases} 1 & \text{if } x \leq_T y <_T U(x) \text{ or } U(x) <_T y \leq_T x, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by c_U the T-cocycle of U. By assumption, c_U is integrable. Remark that $f(x, T^n(x)) = 1$ if and only if $0 \leq n < c_U(x)$ or $c_U(x) < n \leq 0$, so $\sum_{n\in\mathbb{Z}} f(x, T^n(x)) = |c_U(x)|$. We thus have

$$\int_X \sum_{n\in\mathbb{Z}} f(x, T^n(x))d\mu = \int_X |c_U(x)|\, d\mu < +\infty.$$

Using the mass-transport principle (LEMMA 2.A.2), we deduce that

$$\int_X \sum_{n\in\mathbb{Z}} f(T^n(x), x)d\mu < +\infty,$$

in particular for almost every $x \in X$, the sum $\sum_{n\in\mathbb{Z}} f(T^n(x), x)$ is finite.

This implies that for almost every $x \in X$, there are only finitely many integers $n \in \mathbb{Z}$ such that $U^n(x) \leq_T x <_T U^{n+1}(x)$ or $U^{n+1}(x) <_T x \leq_T U^n(x)$. Since the U-orbit of almost every point is infinite, for almost every x, we must have that either $U^n(x) \leq_T x$ or $U^n(x) \geq_T x$ for all but finitely many $n \geq 0$. By ergodicity of U and up to replacing U with its inverse, we can assume that for almost all $x \in X$ we have $U^n(x) \geq_T x$ for all but finitely many $n \geq 0$.

By the symmetric argument, for all but finitely many $n \leq 0$, we have that $U^n(x) \leq_T x$ and therefore we must have that $\{T^n(x) \colon n \geq 0\} \triangle \{U^n(x) \colon n \geq 0\}$ is finite. The conclusion now follows from Theorem 2.A.1. □

References

[Alp79] Steve Alpern. Generic properties of measure preserving homeomorphisms. In Manfred Denker and Konrad Jacobs, editors, *Ergodic Theory*, volume 729, pages 16–27. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979.

[Aus16a] Tim Austin. Behaviour of Entropy Under Bounded and Integrable Orbit Equivalence. *Geom. Funct. Anal.*, 26(6):1483–1525, 2016.

[Aus16b] Tim Austin. Integrable measure equivalence for groups of polynomial growth. *Groups Geom. Dyn.*, 10(1):117–154, 2016.

[Bel68] Raisa M. Belinskaya. Partitions of Lebesgue space in trajectories defined by ergodic automorphisms. *Funkts. Anal. Prilozh.*, 2(3):190–199, 1968.

[Con72] Jean Pierre Conze. Équations fonctionnelles et systèmes induits en théorie ergodique. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 23(1):75–82, 1972.

[DKLMT20] Thiebout Delabie, Juhani Koivisto, François Le Maître, and Romain Tessera. Quantitative measure equivalence. *arXiv:2002.00719*, 2020.

[Dow11] Tomasz Downarowicz. *Entropy in Dynamical Systems*. New Mathematical Monographs. Cambridge University Press, Cambridge, 2011.

[Dye59] Henry A. Dye. On Groups of Measure Preserving Transformations. I. *Am. J. Math.*, 81(1):119–159, 1959.

[EP97] Stanley J. Eigen and Vidhu S. Prasad. Multiple Rokhlin tower theorem: A simple proof. *New York J. Math.*, 3A:11–14, 1997.

[Hal17] Paul R. Halmos. *Lectures on Ergodic Theory*. Dover Publications, Mineola, NY, 2017.

[HvN42] Paul R. Halmos and John von Neumann. Operator Methods in Classical Mechanics, II. *Ann. of Math.*, 43(2):332–350, 1942.

[Kac47] Mark Kac. On the notion of recurrence in discrete stochastic processes. *Bull. Am. Math. Soc.*, 53(10):1002–1010, 1947.

[Kea70] Michael Keane. Contractibility of the Automorphism Group of a Nonatomic Measure Space. *Proc. Am. Math. Soc.*, 26(3):420–422, 1970.

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kec10] Alexander S. Kechris. *Global Aspects of Ergodic Group Actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[Kin68] John F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. R. Stat. Soc., Ser. B*, 30:499–510, 1968.

[KL19] David Kerr and Hanfeng Li. Entropy, Shannon orbit equivalence, and sparse connectivity. *arXiv:1912.02764*, 2019.

[KL22] David Kerr and Hanfeng Li. Entropy, virtual abelianness, and shannon orbit equivalence, 2022.

[Lig02] Thomas M. Liggett. Tagged particle distributions or how to choose a head at random. In *In and out of equilibrium. Probability with a physics flavor. Papers from the 4th Brazilian school of probability, Mambucaba, Brazil, August 14–19, 2000*, pages 133–162. 2002.

[LM18] François Le Maître. On a measurable analogue of small topological full groups. *Adv. Math.*, 332:235–286, 2018.

[Rok67] Vladimir A. Rokhlin. Lectures on the entropy theory of measure-preserving transformations. *Russian Math. Surveys*, 22(5):1, 1967.

[Sin59] Yakov G. Sinai. On the Notion of Entropy of a Dynamical System. *Dokl. Akad. Nauk SSSR*, 124:768–771, 1959.

Chapter 3

Cycles in φ-integrable full groups

Table of contents of Chapter 3

This chapter is devoted to exploring consequences of the shift-coupling inequality in the context of φ-integrable full groups. This inequality is due to Thorisson [Tho95] and was used by Liggett to prove the following probabilistic theorem.

Theorem 3.0.1 ([Lig02, Thm. 1.1]). — *Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. random variables whose distribution is the Bernoulli distribution of parameter $1/2$. Let μ be the conditional law of $(X_n)_{n\in\mathbb{Z}}$ given that $X_0 = 1$. Then any $\mathbb{Z}$-valued random variable N such that the law of $(X_{n+N})_{n\in\mathbb{Z}}$ is equal to μ satisfies $\mathbb{E}[|N|^{1/2}] = \infty$.*

The proof provided by Liggett in [Lig02] is probabilistic. We propose here to follow the lines of Liggett's proof with the language of φ-integrable full groups introduced in [CJLMT22]. The aim is to state Theorem 3.0.1 in terms of cycles in φ-integrable full groups of Bernoulli shifts, see Theorem 3.2.3. Apart from the shift of point of view and the use of new objects such as φ-integrable full groups, no originality is claimed in this chapter.

3.1 Shift-coupling inequality

3.1.1 The inequality

Let X be a standard Borel space. If μ and ν are two Borel measures on X, we denote by $\|\mu-\nu\|_{\mathrm{TV}}$ the ***total variation distance*** between μ and ν, defined by

$$\|\mu-\nu\|_{\mathrm{TV}} := \sup\left\{\left|\int_X f(x)d\mu - \int_X f(x)d\nu\right|, f : X \to [-1,1] \text{ measurable}\right\}.$$

Let μ and ν be two Borel measures. Then ν is ***absolutely continuous*** with respect to μ if for all Borel subset $A \subseteq X$, $\mu(A) = 0$ implies $\nu(A) = 0$. If ν is absolutely continuous with respect to μ and μ, ν are σ-finite measures, then Radon-Nikodym Theorem states that there exists a function $d\nu/d\mu : X \to \mathbb{R}_+$ called the ***Radon-Nikodym derivative*** of ν with respect to μ, which is unique up to equality μ-almost everywhere, such that for all $A \subseteq X$,

$$\nu(A) = \int_A \frac{d\nu}{d\mu}(x)d\mu.$$

There is a useful formula to compute the total variation distance between two measures when one is absolutely continuous with respect to the other.

Lemma 3.1.1. — *Let μ and ν be two σ-finite Borel measures on X. If ν is absolutely continuous with respect to μ, then*

$$\|\mu - \nu\|_{\mathrm{TV}} = \int_X \left|1 - \frac{d\nu}{d\mu}(x)\right| d\mu.$$

Proof. For any measurable function $f : X \to [-1, 1]$,

$$\begin{aligned} \left|\int_X f(x)d\mu - \int_X f(x)d\nu\right| &= \left|\int_X f(x)\left(1 - \frac{d\nu}{d\mu}(x)\right)d\mu\right| \\ &\leq \int_X \left|1 - \frac{d\nu}{d\mu}(x)\right| d\mu. \end{aligned}$$

Since the last quantity is independent of f, we then get that

$$\|\mu - \nu\|_{\mathrm{TV}} \leq \int_X \left|1 - \frac{d\nu}{d\mu}(x)\right| d\mu.$$

For the reverse inequality, let $f : X \to [-1, 1]$ be defined by $f(x) = 1$ if $(d\nu/d\mu)(x) \leq 1$ and $f(x) = -1$ if $(d\nu/d\mu)(x) > 1$. Then we obtain

$$\begin{aligned} \|\mu - \nu\|_{\mathrm{TV}} \geq \left|\int_X f(x)d\mu - \int_X f(x)d\nu\right| &= \left|\int_X f(x)\left(1 - \frac{d\nu}{d\mu}(x)\right)d\mu\right| \\ &= \int_X \left|1 - \frac{d\nu}{d\mu}(x)\right| d\mu, \end{aligned}$$

which proves the lemma. □

In the sequel, if $T : X \to X$ is a Borel bijection and $c : X \to \mathbb{Z}$ a Borel map, we denote by $T^c : X \to X$ the map defined by $T^c(x) := T^{c(x)}(x)$.

Proposition 3.1.2 (Shift-coupling inequality). — *Let $T : X \to X$ be a Borel bijection. Let μ be a Borel measure on X and let $c : X \to \mathbb{Z}$ be a Borel map. For all $n \geq 1$, let ν_n be the measure defined by $\nu_n := T_*\mu + \cdots + (T^n)_*\mu$. Then*

$$\|\nu_n - (T^c)_*\nu_n\|_{\mathrm{TV}} \leq 2\int_X \min(|c(x)|, n)d\mu.$$

Proof. If $k \leq l$ are two integers, we denote by $[\![k,l]\!]$ the interval $\{k, k+1, \dots, l\}$ and by $\mathbf{1}_A$ the indicator function of a set A. Let $f : X \to [-1,1]$ be a measurable function. Then we have

$$\begin{aligned}
\left|\int_X (f(x) - f(T^{c(x)}(x)))d\nu_n\right| &= \left|\sum_{k=1}^{n}\int_X f(T^k(x))d\mu - \sum_{k=1}^{n}\int_X f(T^{k+c(x)}(x))d\mu\right| \\
&= \left|\sum_{k\geq 0}\int_X f(T^k(x))(\mathbf{1}_{[\![1,n]\!]}(k) - \mathbf{1}_{[\![c(x)+1,c(x)+n]\!]}(k))d\mu\right| \\
&\leq \sum_{k\geq 0}\int_X |\mathbf{1}_{[\![1,n]\!]}(k) - \mathbf{1}_{[\![c(x)+1,c(x)+n]\!]}(k)|d\mu.
\end{aligned}$$

Now, observe that if A and B are two sets, then $\mathbf{1}_{A\triangle B} = |\mathbf{1}_A - \mathbf{1}_B|$. Thus, we are left to prove that the cardinal of the set $[\![1,n]\!]\triangle[\![c(x)+1, c(x)+n]\!]$ is equal to $2\min(|c(x)|, n)$. There are two cases to check. If $n \leq |c(x)|$, then the intervals $[\![1,n]\!]$ and $[\![c(x)+1, c(x)+n]\!]$ are disjoint, thus the cardinal of their symmetric difference is $2n = 2\min(|c(x)|, n)$. If $|c(x)| < n$, then the cardinal of the intersection of $[\![1,n]\!]$ and $[\![c(x)+1, c(x)+n]\!]$ is $n - |c(x)|$ and the cardinal of their union is $n + |c(x)|$. Thus, the cardinal of their symmetric difference is $2|c(x)| = 2\min(|c(x)|, n)$, which finishes the proof. □

3.1.2 Shift-coupling inequality and φ-integrability

With the same notations as in Proposition 3.1.2, our goal in the sequel is to prove that if the function $c : X \to \mathbb{Z}$ satisfies some kind of integrability condition, then it forces the behavior of $\|\nu_n - (T^c)_*\nu_n\|_{\mathrm{TV}}$ to be slow as n goes to $+\infty$. The integrability conditions in question are obtained by metric-compatible function.

Definition 3.1.3. — A function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is ***metric-compatible*** if:

- (subadditivity) for all $s, t \in \mathbb{R}_+$, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$.
- (separation) $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$.
- (monotonicity) φ is a non-decreasing function.

A metric-compatible function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is ***sublinear*** if $\lim_{t\to+\infty} \varphi(t)/t = 0$.

Observe that if φ is a metric-compatible function and d is a metric on some set, then $\varphi \circ d$ is still a metric. Examples of metric-compatible functions are concave functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$.

The following lemma is a useful technicality on metric-compatible functions.

Lemma 3.1.4. — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. Then for all $0 < s \leq t$, we have $\varphi(t)/t \leq 2\varphi(s)/s$.*

Proof. Let $n = \lfloor t/s \rfloor$. Since $s \leq t$, we have $n \geq 1$. Since $t < (n+1)s$, we get that

$$\varphi(t) \leq \varphi((n+1)s) \leq (n+1)\varphi(s).$$

Finally, we divide by t and use the inequality $ns \leq t$ to obtain

$$\frac{\varphi(t)}{t} \leq \frac{n+1}{t}\varphi(s) \leq \frac{n+1}{n}\frac{\varphi(s)}{s} \leq 2\frac{\varphi(s)}{s}. \qquad \square$$

DEFINITION 3.1.5. — Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. A measurable map $f : X \to \mathbb{R}$ is ***φ-integrable*** if

$$\int_X \varphi(|f(x)|)d\mu < +\infty.$$

PROPOSITION 3.1.6. — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $T : X \to X$ be a Borel bijection and μ a Borel measure on X. For all $n \geq 1$, let ν_n be the measure defined by $\nu_n := T_*\mu + \cdots + (T^n)_*\mu$. If $c : X \to \mathbb{Z}$ is any φ-integrable map, then*

$$\lim_n \frac{\varphi(n)}{n}\|\nu_n - (T^c)_*\nu_n\|_{\mathrm{TV}} = 0.$$

Proof. For all $x \in X$ and $n \geq 1$, let $c_n(x) := \frac{\varphi(n)}{n}\min(|c(x)|, n)$. The sequence of positive functions $(c_n)_{n\geq 0}$ converges pointwise to 0. Let $n \geq 1$. If $x \in X$ is such that $|c(x)| > n$, then $c_n(x) = \varphi(n) \leq \varphi(|c(x)|)$. If $x \in X$ is such that $0 < |c(x)| \leq n$, then we have by LEMMA 3.1.4

$$c_n(x) = |c(x)|\frac{\varphi(n)}{n} = \varphi(|c(x)|)\frac{\varphi(n)}{n}\frac{|c(x)|}{\varphi(|c(x)|)} \leq 2\varphi(|c(x)|).$$

Thus, for all $x \in X$, we obtain that $c_n(x) \leq 2\varphi(|c(x)|)$. Lebesgue's dominated convergence theorem applied to the sequence $(c_n)_{n\geq 0}$ yields

$$\lim_n \frac{\varphi(n)}{n}\int_X \min(|c(x)|, n)d\mu = 0.$$

Finally, the shift-coupling inequality of PROPOSITION 3.1.2 leads to the desired conclusion. $\square$

3.2 Application to φ-integrable full groups

In this section, we apply the shift-coupling inequality, or more precisely its consequence obtained in PROPOSITION 3.1.6 to get constraints on the existence of cycles with prescribed properties in φ-integrable full groups. Let us first give the definition of these groups, which were introduced in [CJLMT22]. Let $T \in \mathrm{Aut}(X, \mu)$ be an aperiodic transformation. We denote by $[T]$ the full group of T, that is, the group of all measure-preserving transformations $U \in \mathrm{Aut}(X, \mu)$ such that for all $x \in X$, there exists $n \in \mathbb{Z}$ such that $U(x) = T^n(x)$. Any $U \in [T]$ is completely determined by its ***T-cocycle***, defined as the unique function $c_U : X \to \mathbb{Z}$ satisfying the equation $U(x) = T^{c_U(x)}(x)$ for all $x \in X$.

DEFINITION 3.2.1. — Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a metric-compatible function. The

φ-integrable full group of an aperiodic transformation $T \in \mathrm{Aut}(X,\mu)$ is

$$[T]_\varphi := \left\{ U \in [T] \colon \int_X \varphi(|c_U(x)|) d\mu < +\infty \right\},$$

where $c_U \colon X \to \mathbb{Z}$ denotes the T-cocycle of U.

This is in fact a group, which is one of the main objects of study of [CJLMT22].

3.2.1 Fundamental domains of cycles

The ***support*** of a measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ is the measurable set $\mathrm{supp}(T) := \{x \in X \colon T(x) \neq x\}$. A measure-preserving transformation $P \in \mathrm{Aut}(X,\mu)$ is ***periodic*** if the P-orbit of almost every $x \in X$ is finite. A ***fundamental domain*** of a periodic transformation $P \in \mathrm{Aut}(X,\mu)$ is a measurable subset $A \subseteq X$ which intersects almost every P-orbit at exactly one point. Let $m \geq 2$. A measure-preserving transformation $P \in \mathrm{Aut}(X,\mu)$ is a m-cycle if for almost every $x \in X$, the cardinality of the P-orbit of x is either 1 or m. Here we prove the following result.

Theorem 3.2.2. — *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear metric-compatible function. Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Fix an integer $m \geq 2$ and let $P \in [T]_\varphi$ be a m-cycle with full support. For any fundamental domain $D \subseteq X$ of P, we have*

$$\lim_n \frac{\varphi(n)}{n} \int_X \left| n - m \sum_{k=1}^{n} \mathbf{1}_D(T^{-k}(x)) \right| d\mu = 0.$$

Proof. Since P is a m-cycle with full support, the sets $D, P(D), \dots, P^{m-1}(D)$ form a partition of X. For all $k \in \{0,\dots,m-1\}$ and $x \in P^k(D)$, let $c(x) := c_{P^{-k}}(x)$, where we recall that $c_{P^{-k}}$ denotes the T-cocycle of $P^{-k} \in [T]$.

Claim. — For all $A \subseteq X$, we have $(T^c)_*\mu(A) = \mu(A \cap D)/\mu(D)$.

Proof. Let $A \subseteq X$ be measurable subset. We know that $D, P(D), \dots, P^{m-1}(D)$ form a partition of X. Moreover, if $x \in P^k(D)$ for some $k \in \{0,\dots,m-1\}$, then $T^{c(x)}(x) = P^{-k}(x)$. Thus,

$$\begin{aligned}
\mu(\{x \in X \colon T^{c(x)}(x) \in A\}) &= \sum_{k=0}^{m-1} \mu(\{x \in P^k(D) \colon P^{-k}(x) \in A\}) \\
&= \sum_{k=0}^{m-1} \mu(P^k(D) \cap P^k(A)) \\
&= m\mu(D \cap A).
\end{aligned}$$

Finally, D is a fundamental domain of a m-cycle with full support, thus $\mu(D) = 1/m$, which concludes the proof of the claim. □

We apply the claim to get that for all $k \geq 1$, the measure $(T^{k+c})_*\mu$ is absolutely continuous with respect to $(T^k)_*\mu$ and its Radon-Nikodym derivative is given by

$$\frac{d(T^{k+c})_*\mu}{d(T^k)_*\mu}(x) = \mu(D)^{-1}\mathbf{1}_D(T^{-k}(x)).$$

Since T preserves the measure μ, we have $(T^k)_*\mu = \mu$ for all $k \in \mathbb{Z}$, thus

$$\sum_{k=1}^{n}(T_k)_*\mu = n\mu.$$

Finally, Lemma 3.1.1 yields

$$\begin{aligned}\left\|\sum_{k=1}^{n}(T^k)_*\mu - \sum_{k=1}^{n}(T^{k+c})_*\mu\right\|_{\mathrm{TV}} &= \int_X \left|1 - \frac{1}{n}\sum_{k=1}^{n}\mu(D)^{-1}\mathbf{1}_D(T^{-k}(x))\right| n d\mu. \\ &= \int_X \left|n - m\sum_{k=1}^{n}\mathbf{1}_D(T^{-k}(x))\right| d\mu.\end{aligned}$$

To conclude, we observe that $c : X \to \mathbb{Z}$ is φ-integrable. Indeed, on each piece $P^k(D)$ of the finite partition $D, P(D), \ldots, P^{m-1}(D)$, the map c coincides with $c_{P^{-k}}$, which is φ-integrable since $P^{-k} \in [T]_\varphi$. Proposition 3.1.6 then yields to the desired conclusion. □

3.2.2 Cycles in the full group of a Bernoulli shift

In this section, we apply Theorem 3.2.2 in the concrete case of Bernoulli shifts.

Theorem 3.2.3. — *Let φ be a metric-compatible function. Let A be a finite space, κ the uniform measure on A, and $m := |A|$. Let $T \in \mathrm{Aut}(A^{\mathbb{Z}}, \kappa^{\mathbb{Z}})$ be the Bernoulli shift. Let $P \in [T]_\varphi$ be a m-cycle with full support. If there exists $a \in A$ such that $\{(x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}} \colon x_0 = a\}$ is a fundamental domain for P, then*

$$\lim_n \frac{\varphi(n)}{\sqrt{n}} = 0.$$

Proof. Let $X := A^{\mathbb{Z}}$ and $\mu := \kappa^{\mathbb{Z}}$. Let $a \in A$ such that the set $D := \{(x_n)_{n\in\mathbb{Z}} \in X \colon x_0 = a\}$ is a fundamental domain for P. For all $k \geq 0$, we let X_k be the random variable defined by $X_k(x) := \mathbf{1}_D(T^{-k}(x))$. Then X_k is a Bernoulli random variable with parameter $\mathbb{P}(X_k = 1) = \mu(T^k(D)) = \mu(D) = 1/m$. Moreover, $(X_k)_{k\geq 0}$ is a sequence of i.i.d. random variables. For all $n \geq 0$, let $S_n := X_1 + \ldots X_n$. By the central limit theorem, there is a constant $C > 0$ such that

$$\lim_n \mathbb{P}\left(\left|\frac{n - mS_n}{\sqrt{n}}\right| \leq 1\right) = C.$$

Thus, for all $n \geq 1$, we have

$$C \leq \mathbb{E}\left[\left|\frac{n - mS_n}{\sqrt{n}}\right|\right].$$

Moreover, by definition of S_n, we have

$$\mathbb{E}\left[\left|\frac{n - mS_n}{\sqrt{n}}\right|\right] = \frac{1}{\sqrt{n}} \int_X \left| n - m \sum_{k=1}^{n} \mathbf{1}_D(T^{-k}(x)) \right| d\mu.$$

Finally, Theorem 3.2.2 implies that $\lim_n \frac{\varphi(n)}{\sqrt{n}} = 0$, as claimed. □

Remark 3.2.4. — If φ is a metric-compatible function such that $\varphi(n)/\sqrt{n} \not\to 0$, Theorem 3.2.3 yields that there is no m-cycle P in the φ-integrable full group of the Bernoulli shift with base $(\{1, \ldots, m\}, \kappa)$ and with fundamental domain

$$\{(x_n)_{n \in \mathbb{Z}} \colon x_0 = a\}.$$

When $m = 2$, this result is optimal. In this case, one can construct an explicit involution which belongs to the L^p full group of the Bernoulli shift for all $p < 1/2$, as follows. Let T be the Bernoulli shift on $(\{0,1\}, \kappa)^{\otimes \mathbb{Z}}$, where κ is the uniform measure on $\{0,1\}$. Let $X_0 := \{(x_n)_{n \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} \colon x_0 = 0\}$. For all $x \in X_0$, let $N(x)$ be the infimum of $n \geq 1$ such that 1 appears strictly more often than 0 in $\{x_1, \ldots, x_n\}$. Then the map $\pi \colon x \in X_0 \mapsto T^{N(x)}(x) \in \{0,1\}^{\mathbb{Z}} \setminus X_0$ is well-defined and injective almost everywhere. Thus it can be extended to an involution $P \in [T]$ with full support and fundamental domain X_0. Standard estimates on the simple random walk on $\mathbb{Z}$ imply that P belongs to $[T]_p$ for all $0 < p < 1/2$.

References

[CJLMT22] Alessandro Carderi, Matthieu Joseph, François Le Maître, and Romain Tessera. Belinskaya's theorem is optimal, 2022.

[Lig02] Thomas M. Liggett. Tagged particle distributions or how to choose a head at random. In *In and out of equilibrium. Probability with a physics flavor. Papers from the 4th Brazilian school of probability, Mambucaba, Brazil, August 14–19, 2000*, pages 133–162. 2002.

[Tho95] Hermann Thorisson. Coupling methods in probability theory. *Scand. J. Stat.*, 22(2):159–182, 1995.

Chapter 4

Quantitative orbit equivalence between $\mathbb{Z}$ and D_∞

Table of contents of Chapter 4

In this chapter, we focus on quantitative aspects of orbit equivalences between p.m.p. essentially free actions of two particular groups: the group of integers $\mathbb{Z}$ and the infinite dihedral group D_∞ , which admits the presentation

$$D_\infty := \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

These groups are both infinite and amenable, so the theorem of Ornstein and Weiss [OW80] implies that any p.m.p. ergodic action of $\mathbb{Z}$ is orbit equivalent to any p.m.p. ergodic action of D_∞. We prove the following quantitative result.

THEOREM 4.0.1. — *Let $\mathbb{Z} \curvearrowright^{\alpha} (X, \mu)$ be an ergodic action. Then the following statements holds.*

(i) There exists an ergodic action $D_\infty \curvearrowright^{\beta} (Y, \nu)$ such that α and β are $L^{<1}$ orbit equivalent.

(ii) By contrast, the action α is L^1 orbit equivalent to some ergodic action of D_∞ if and only if the subgroup $2\mathbb{Z}$ doesn't act ergodically on (X, μ).

We refer to Section 4.1 for the definitions of L^1 and $\mathrm{L}^{<1}$ orbit equivalence. We prove item (ii) of THEOREM 4.0.1 in Section 4.2. For this we use a result due to Bowen, which says that any L^1 orbit equivalence extends to the end compactifications of the groups [Bow17]. In Section 4.3, we prove a combinatorial result on a special kind of involutions defined on $\mathbb{Z}$ that we call non-crossing. In Section

4.4, we use this combinatorial result and the existence of non-crossing involution in $L^{<1}$ full-groups of ergodic transformations to prove item (i) of THEOREM 4.0.1.

4.1 Preliminaries

Let Γ and Λ be two countable groups. Let $\Gamma \curvearrowright (X,\mu)$ be a p.m.p. action. A ***cocycle*** into Λ is a map $\sigma : \Gamma \times X \to \Lambda$ such that for all $\gamma, \delta \in \Gamma$, $x \in X$,

$$\sigma(\gamma\delta, x) = \sigma(\gamma, \delta \cdot x)\sigma(\delta, x).$$

Cocycles naturally appear with orbit equivalent actions. Two p.m.p. *essentially free* actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are ***orbit equivalent*** if there exists an ***orbit equivalence*** between them, that is, a bimeasurable bijection $\Phi : X \to Y$ such that $\Phi_*\mu = \nu$ and for μ-almost every $x \in X$, we have $\Phi(\Gamma \cdot x) = \Lambda \cdot \Phi(x)$. The map Φ is called an orbit equivalence between the actions. By freeness of the actions, there are two measurable maps $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$, which are called the ***orbit equivalence cocycles*** associated with Φ and are defined for all $\gamma \in \Gamma, \lambda \in \Lambda$ and μ-almost every $x \in X$ by

$$\Phi(\gamma \cdot x) = \sigma(\gamma, x) \cdot \Phi(x),$$
$$\Phi(\tau(\lambda, \Phi(x)) \cdot x) = \lambda \cdot \Phi(x).$$

Assume that Γ and Λ are finitely generated groups and fix $|\cdot|_\Gamma$ and $|\cdot|_\Lambda$ word lengths associated with finite generating systems for Γ and Λ respectively. We say that an orbit equivalence $\Phi : (X,\mu) \to (Y,\nu)$ is an $\mathbf{L}^p$ ***orbit equivalence*** ($p \in]0,+\infty[$) if the following two conditions are satisfied:

(i) $\int_X |\sigma(\gamma, x)|_\Gamma^p d\mu < +\infty$ for all $\gamma \in \Gamma$,

(ii) $\int_X |\tau(\lambda, x)|_\Lambda^p d\mu < +\infty$ for all $\lambda \in \Lambda$.

We say that Φ is an $\mathbf{L}^\infty$ ***orbit equivalence*** if for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$, the maps $|\sigma(\gamma, -)|_\Gamma$ et $|\tau(\lambda, -)|_\Lambda$ are essentially bounded. In both cases, the conditions are independent of the choice of the word length $|\cdot|_\Gamma$ and $|\cdot|_\Lambda$ associated with finite generating systems. Moreover, the cocycle identity implies that it is enough to check these integrability conditions for γ and λ belonging to any finite generating system of Γ and Λ respectively.

We say that two p.m.p. actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are $\mathbf{L}^p$ ***orbit equivalent*** ($p \in]0,+\infty]$) if there exists an L^p orbit equivalence between them. Finally, we say that the p.m.p. actions are $\mathbf{L}^{<p}$ ***orbit equivalent*** if there exists an orbit equivalence which is an L^q orbit equivalence for all $q < p$.

4.2 L^1 orbit equivalence between $\mathbb{Z}$ and D_∞

Let D_∞ be the ***infinite dihedral group*** which is defined as the free product of two copies of the cyclic group of order two. A presentation of D_∞ is given by

$$D_\infty := \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

Our aim in this section is to understand when an ergodic action of $\mathbb{Z}$ can be L^1 orbit equivalent to an ergodic action of D_∞.

Proposition 4.2.1. — *Let $\mathbb{Z} \curvearrowright^{\alpha} (Y, \nu)$ be an ergodic action. If there exists a p.m.p. essentially free action $D_\infty \curvearrowright^{\beta} (X, \mu)$ such that α and β are L^1 orbit equivalent, then the subgroup $2\mathbb{Z}$ doesn't act ergodically on (X, μ).*

Proof. Let $\Phi : (X, \mu) \to (Y, \nu)$ be an L^1 orbit equivalence between α and β. Let $\sigma : \mathbb{Z} \times X \to D_\infty$ and $\tau : D_\infty \times Y \to \mathbb{Z}$ be the orbit equivalence cocycles associated with Φ. Let us define $\widetilde{\tau} : D_\infty \times Y \to \mathbb{Z}$ by $\widetilde{\tau}(\gamma, y) = -\tau(\gamma^{-1}, y)$. By Bowen's theorem on extension of L^1 orbit equivalence cocycles on the space of ends [Bow17, Thm. 3.1], $\widetilde{\tau}$ extends to a continuous map

$$\widetilde{\tau} : D_\infty \cup \{\xi_{ab}, \xi_{ba}\} \times Y \to \mathbb{Z} \cup \{-\infty, +\infty\}$$

where $\xi_{ab} = (ab)^\infty, \xi_{ba} = (ba)^\infty$ are the two ends of D_∞ and $-\infty, +\infty$ the two ends of $\mathbb{Z}$. The map $\widetilde{\tau}$ satisfies the extended cocycle identity: for all $\gamma \in D_\infty$, $\xi \in D_\infty \cup \{\xi_{ab}, \xi_{ba}\}$ and for ν-almost every $y \in Y$

$$\widetilde{\tau}(\gamma\xi, y) = \widetilde{\tau}(\gamma, y) + \widetilde{\tau}(\xi, \beta(\gamma^{-1})y).$$

Moreover, for ν-almost every $y \in Y$, we have $\{\widetilde{\tau}(\xi_{ab}, y), \widetilde{\tau}(\xi_{ba}, y)\} = \{-\infty, +\infty\}$. In our case, the action of $\mathbb{Z}$ on its space of ends $\{-\infty, +\infty\}$ is trivial, thus for ν-almost every $y \in Y$ and all $\gamma \in D_\infty$, we have

$$\widetilde{\tau}(\gamma\xi_{ab}, y) = \widetilde{\tau}(\xi_{ab}, \beta(\gamma^{-1})y) \text{ and } \widetilde{\tau}(\gamma\xi_{ba}, y) = \widetilde{\tau}(\xi_{ba}, \beta(\gamma^{-1})y).$$

Therefore, for ν-almost every $y \in Y$,

$$\widetilde{\tau}(\xi_{ab}, \beta((ab)^{-1})y) = \widetilde{\tau}(\xi_{ab}, y).$$

This proves that the measurable sets

$$Y_{+\infty} := \{y \in Y \colon \widetilde{\tau}(\xi_{ab}, y) = +\infty\} \text{ and } Y_{-\infty} := \{y \in Y \colon \widetilde{\tau}(\xi_{ab}, y) = -\infty\}$$

are both invariant under the action of ab. Moreover, if $\widetilde{\tau}(\xi_{ab}, y) = +\infty$, then $\widetilde{\tau}(\xi_{ba}, y) = -\infty$. This implies that $\beta(a)Y_{+\infty} = Y_{-\infty}$. Thus, the measures of Y_a and of Y_b are non null. This shows that the subgroup of D_∞ generated by ab doesn't act ergodically. We construct a new p.m.p. action of $\mathbb{Z}$ on (Y, ν) that we call α'. It is enough to define $\alpha'(1)$: for all $y \in Y$, let

$$\alpha'(1)y := \begin{cases} \beta(a)y & \text{if } y \in Y_{+\infty}, \\ \beta(b)y & \text{if } y \in Y_{-\infty}. \end{cases}$$

The action α' is L^∞ orbit equivalent to β (actually, α' and β' are isometric orbit equivalent, see Chapter 5 for the definition of this notion). Since α and β are L^1 orbit equivalent, we obtain that α and α' are L^1 orbit equivalent. By Belinskaya's theorem [Bel68], the ergodic actions α and α' are flip conjugate. Since the subgroup $2\mathbb{Z}$ doesn't act ergodically for α', we deduce that it doesn't act ergodically for α, which concludes the proof. □

Remark 4.2.2. — Another way to prove Proposition 4.2.1 would be to use a result due to Le Maître. He proved that given any ergodic measure-preserving transformation $T \in \mathrm{Aut}(X, \mu)$, if there exists an involution P in the L^1 full group of T with full support, then T^2 is not ergodic [LM18, Thm. 4.8]. Proposition 4.2.1 is then a direct consequence of this result.

Proposition 4.2.3. — *Let* $\mathbb{Z} \curvearrowright^{\alpha} (X, \mu)$ *be an ergodic action. If* $2\mathbb{Z}$ *doesn't act ergodically on* (X, μ)*, then* α *is* L^∞ *orbit equivalent to an ergodic action of* D_∞.

Proof. Since $2\mathbb{Z}$ does not act ergodically, there is a partition $X = A \sqcup B$ into measurable sets of measure $1/2$ that are both invariant under $2\mathbb{Z}$. We define a p.m.p. action $D_\infty \curvearrowright^{\beta} (X, \mu)$ by the actions of its generators a and b. For all $x \in X$, we let

$$\beta(a)x = \begin{cases} \alpha(1)x & \text{if } x \in A, \\ \alpha(-1)x & \text{if } x \in B, \end{cases}$$

$$\beta(b)x = \begin{cases} \alpha(-1)x & \text{if } x \in A, \\ \alpha(1)x & \text{if } x \in B. \end{cases}$$

By construction, the actions α and β are L^∞ orbit equivalent (they are even isometric orbit equivalent), which concludes the proof. □

4.3 Non-crossing involutions

4.3.1 Definition and first properties

In this section, we denote by T the addition by 1 in $\mathbb{Z}$, that is $T(x) := x + 1$ for all $x \in \mathbb{Z}$. For all $k, l \in \mathbb{Z}$, we denote by $[\![k, l]\!]$ the interval of $\mathbb{Z}$ whose extremities are k and l. In other words, if $k \leq l$, then $[\![k, l]\!] := \{k, k+1, \ldots, l\}$ and if $l \leq k$, then $[\![k, l]\!] := \{l, l+1, \ldots, k\}$.

Definition 4.3.1. — An involution $P : \mathbb{Z} \to \mathbb{Z}$ is ***non-crossing*** if for all $x \in \mathbb{Z}$, $P([\![x, P(x)]\!]) = [\![x, P(x)]\!]$.

Remark 4.3.2. — This is equivalent to saying that for all $x, y \in \mathbb{Z}$, the intervals $[\![x, P(x)]\!]$ and $[\![y, P(y)]\!]$ are either disjoint or one is contained in the other. The non-crossing property has the following pictorial interpretation. For all $x \in \mathbb{Z}$ such that $P(x) \neq x$, we draw the circular arc in the upper half-plane whose extremities are x and $P(x)$, which is perpendicular to the horizontal axis. Then the involution P is non-crossing if and only if none of these circular arcs intersect. We refer to these arcs as the ***P upper arcs*** in the sequel.

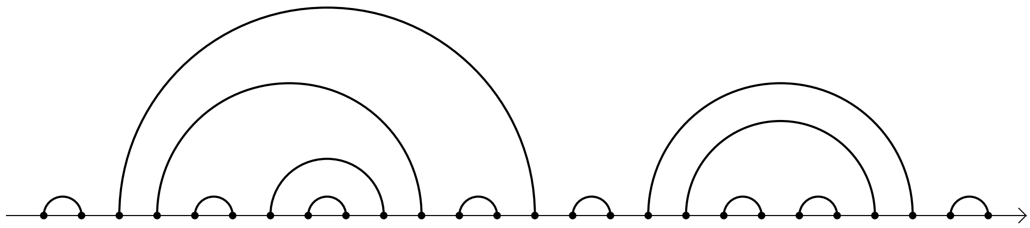

Figure 4.1. — A part of a non-crossing involution P.

We will focus on ***fixed-point free*** involutions $P : \mathbb{Z} \to \mathbb{Z}$, that is, satisfying $P(x) \neq x$ for all $x \in \mathbb{Z}$. We explain below some properties of fixed-point free, non-crossing involutions that will be useful in the sequel.

Lemma 4.3.3. — *Let $P : \mathbb{Z} \to \mathbb{Z}$ be a fixed-point free, non-crossing involution. Then TPT^{-1} is a fixed-point free, non-crossing involution.*

Proof. The map TPT^{-1} is an involution. Let us prove that it is non-crossing. For all $x \in \mathbb{Z}$, we have

$$\begin{aligned} y \in [\![x, TPT^{-1}(x)]\!] &\Leftrightarrow T^{-1}(y) \in [\![T^{-1}(x), PT^{-1}(x)]\!] = P([\![T^{-1}(x), PT^{-1}(x)]\!]) \\ &\Leftrightarrow PT^{-1}(y) \in [\![T^{-1}(x), PT^{-1}(x)]\!] \\ &\Leftrightarrow TPT^{-1}(y) \in [\![x, TPT^{-1}(x)]\!]. \end{aligned}$$

This proves that TPT^{-1} is non-crossing. Finally $TPT^{-1}(x) = x$ if and only if $PT^{-1}(x) = T^{-1}(x)$. Since P has no fixed point, this proves that TPT^{-1} has no fixed point. □

Lemma 4.3.4. — *Let $P : \mathbb{Z} \to \mathbb{Z}$ be a fixed-point free, non-crossing involution. Then for all $x \in \mathbb{Z}$, the number $P(x) - x$ is odd.*

Proof. Let $x \in \mathbb{Z}$. Since P is non-crossing, we have $P([\![x, P(x)]\!]) = [\![x, P(x)]\!]$. This implies that P induces an involution of the interval $[\![x, P(x)]\!]$. This implies that the cardinal of $[\![x, P(x)]\!]$ is even, because P has no fixed point. In other words, $P(x) - x$ is odd. □

Lemma 4.3.5. — *Let $P : \mathbb{Z} \to \mathbb{Z}$ be a fixed-point free, non-crossing involution. Let $x, y \in \mathbb{Z}$ such that $x < y$ and $P([\![x, y]\!]) = [\![x, y]\!]$. Then there exist $x_1, \dots, x_k \in \mathbb{Z}$ such that $x = x_1 < P(x_1) < \cdots < x_k < P(x_k) = y$ and*

$$[\![x, y]\!] = [\![x_1, P(x_1)]\!] \sqcup \cdots \sqcup [\![x_k, P(x_k)]\!].$$

Proof. Let $z \in [\![x,y]\!]$. Since $P([\![x,y]\!]) = [\![x,y]\!]$, we get that $P(z) \in [\![x,y]\!]$ and thus $[\![z,P(z)]\!] \subseteq [\![x,y]\!]$ because P is non-crossing. This implies that for all $z \in [\![x,y]\!]$, there exists a maximal interval of the form $[\![t,P(t)]\!]$ which contains z and is contained in $[\![x,y]\!]$. Since P is non-crossing, intervals of this form are either disjoint, or equal by maximality. Thus, there are $x_1,\dots,x_k \in [\![x,y]\!]$ such that

$$[\![x,y]\!] = [\![x_1,P(x_1)]\!] \sqcup \cdots \sqcup [\![x_k,P(x_k)]\!].$$

By permuting the elements $x_1,\dots,x_k$ and changing x_i to $P(x_i)$ if necessary, we may assume that $x = x_1 < P(x_1) < \cdots < x_k < P(x_k) = y$. □

4.3.2 A combinatorial result

Let $P : \mathbb{Z} \to \mathbb{Z}$ be a fixed-point free, non-crossing involution. By Lemma 4.3.3, the map TPT^{-1} is a fixed-point free, non-crossing involution. Thus we can represent it with non-crossing circular arcs but this time drawn in the lower half-plane. We refer to these arcs as the ***TPT*$^{-1}$ *lower arcs***. Observe that the TPT^{-1} lower arcs are obtained as the image of the P upper arcs by a reflexion across the horizontal line, followed by the translation $T : x \mapsto x+1$.

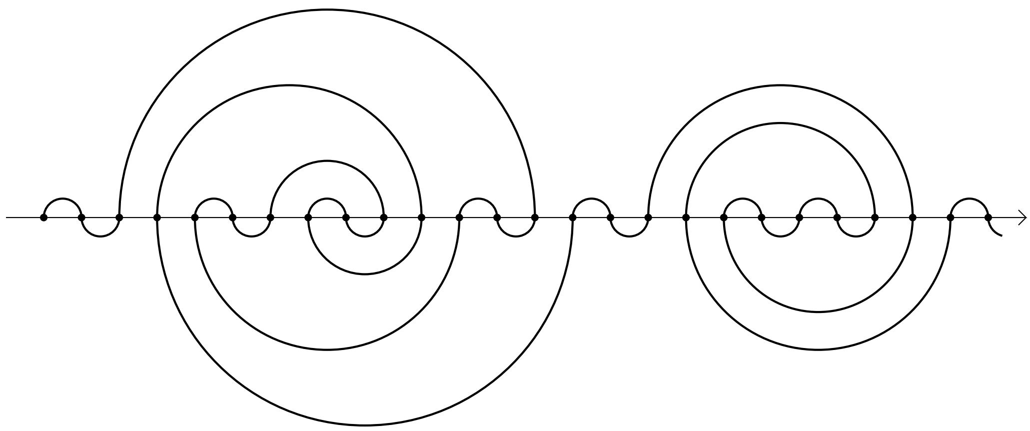

Figure 4.2. — The P upper arcs and the TPT^{-1} lower arcs

In the following result, we prove that the graph with vertex set $\mathbb{Z}$ and edge set $\{\text{P upper arcs}\} \cup \{TPT^{-1} \text{ lower arcs}\}$ is connected.

Theorem 4.3.6. — *Let $P : \mathbb{Z} \to \mathbb{Z}$ be a fixed-point free, non-crossing involution. Then the action of the infinite dihedral group $D_\infty = \langle a,b \mid a^2 = b^2 = 1\rangle$ on $\mathbb{Z}$ defined by $a \cdot x = P(x)$ and $b \cdot x = TPT^{-1}(x)$ is free and transitive.*

Proof. Let G be the graph with vertex set $\mathbb{Z}$ and egde set $\{\text{P upper arcs}\} \cup \{TPT^{-1} \text{ lower arcs}\}$. We define the length of an arc to be the distance between its extremities.

Claim. — For $x \in \mathbb{Z}$ such that $a \cdot x - x \geq 1$, there exists an integer $n \geq 1$ such that $(ba)^n \cdot x = a \cdot x + 1$ and

$$\{x, a\cdot x, (ba)\cdot x, (aba)\cdot x, \dots, (ba)^n \cdot x\} = [\![x, a\cdot x + 1]\!].$$

This claim says that for such a point x, the path in the graph G which starts at x, then takes a P upper arc, then a TPT^{-1} lower arc, etc., will visit exactly every point in the interval $[\![x, a \cdot x]\!]$ and then reach $a \cdot x + 1$.

Proof of the claim. We prove this claim by induction on the length l of the P upper arc with extremities x and $a \cdot x$, which is odd by LEMMA 4.3.4. If $l = 1$, then $n = 1$ works. Assume that the result of the claim holds for all P upper arcs of length $< 2d + 1$. Let us prove that the result holds for any P upper arc of length $l := 2d + 1$. Let $x \in \mathbb{Z}$ such that $a \cdot x - x = l$. We apply LEMMA 4.3.5 to the interval $[\![x + 1, a \cdot x - 1]\!]$, which is P-invariant. There is $x_1, \dots, x_k \in \mathbb{Z}$ such that $x + 1 = x_1 < P(x_1) < \cdots < x_k < P(x_k)$ and

$$[\![x + 1, a \cdot x - 1]\!] = [\![x_1, P(x_1)]\!] \sqcup \cdots \sqcup [\![x_k, P(x_k)]\!].$$

The P upper arcs with extremities x_i and $P(x_i)$ are the P upper arcs located immediately below the P upper arc with extremities x and $a \cdot x$. The length of any of these arcs is $< l$, thus one can use the induction hypothesis. There exists integers $n_1, \dots, n_k \geq 1$ such that for all $1 \leq i \leq k$, we have $(ba)^{n_i} \cdot x_i = a \cdot x_i + 1$ and

$$\{x_i, a \cdot x_i, (ba) \cdot x_i, \dots, (ba)^{n_i} \cdot x\} = [\![x_i, a \cdot x_i + 1]\!].$$

By concatenating these results, we obtain the claim with $n := n_1 + \cdots + n_k + 1$. $\square_{\text{claim}}$

In order to prove that the graph G is connected, it is enough to prove that for all $x \in \mathbb{Z}$, the points x and $x + 1$ are connected. Let $x \in \mathbb{Z}$. In both of the cases $a \cdot x - x \geq 1$ and $x - a \cdot x \geq 1$, the fact that x and $x + 1$ are connected is a direct consequence of the claim. Thus, G is connected, which proves the theorem. $\square$

4.4 $\mathrm{L}^{<1}$ orbit equivalence between $\mathbb{Z}$ and D_∞

Let (X, μ) be a probability space. Let $\mathrm{Aut}(X, \mu)$ be the group of all measure-preserving transformations of (X, μ). The support of an element $T \in \mathrm{Aut}(X, \mu)$ is the set

$$\mathrm{supp}(T) := \{x \in X \colon T(x) \neq x\}.$$

Let $T \in \mathrm{Aut}(X, \mu)$ be ergodic. The ***full group*** of T, denoted by $[T]$, is the group of all bimeasurable bijections $U : X \to X$ such that $U_*\mu = \mu$ and for all $x \in X$, there exists $n_x \in \mathbb{Z}$ such that $U(x) = T^{n_x}x$. By ergodicity of T, every T-orbit is infinite and thus the integer n_x is unique. The map $x \mapsto n_x$ is called the T-cocycle of U and is denoted by c_U. The $\mathbf{L^1}$ ***full group*** of T, denoted by $[T]_1$ is the group of all $U \in [T]$ such that

$$\int_X |c_U(x)| d\mu < +\infty.$$

This group was introduced by Le Maître who proved that this is a complete invariant of flip-conjugacy for ergodic actions of $\mathbb{Z}$ [LM18].

The $\mathbf{L}^{<1}$ ***full group*** of T, denoted by $[T]_{<1}$, is the group of all $U \in [T]$ such that for all $p < 1$,

$$\int_X |c_U(x)|^p d\mu < +\infty.$$

Let $x, y \in X$ be two points in the same T-orbit. Then there is a unique $n \in \mathbb{Z}$ such that $T^n(x) = y$. We denote by $[\![x,y]\!]_T$ the interval with extremities x and y, which is defined by

$$[\![x,y]\!]_T := \begin{cases} \{x, T(x), \dots, T^{n-1}(x), y\} & \text{if } n \geq 0, \\ \{x, T^{-1}(x), \dots, T^{-n+1}(x), y\} & \text{otherwise.} \end{cases}$$

The length of the interval $[\![x,y]\!]_T$ is equal to $|n|$. We say that a measurable subset $B \subseteq X$ is l-separated for T if for all $x, y \in B$ distinct, the length of the interval $[\![x,y]\!]_T$ is $\geq l$. The following result is an easy consequence of Rokhlin's lemma [Rok67].

Lemma 4.4.1. — *Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic. Then for all $l \geq 1$, there exists $B \subseteq X$ of positive measure which is l-separated for T.*

The measure of such a set B satisfies $\mu(B) \leq 1/l$. We will use l-separated sets to construct involutions in full groups which are non-crossing.

Definition 4.4.2. — Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic. An involution $P \in [T]$ is ***non-crossing*** if for μ-almost every $x \in X$, we have $P([\![x,P(x)]\!]_T) = [\![x,P(x)]\!]_T$.

Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic and let $A \subseteq X$ be of positive measure. The ***first return time*** of T to A is the map $n_{T,A} : A \to \mathbb{N} \cup \{+\infty\}$ defined for all $x \in A$ by

$$n_{T,A}(x) := \inf\{n \in \mathbb{N} \colon T^n(x) \in A\}.$$

By Poincaré recurrence theorem, $n_{T,A}(x)$ is finite for μ-almost every $x \in A$. The ***first return map*** of T to A is the map $T_A : A \to A$ given for μ-almost every $x \in A$ by

$$T_A(x) := T^{n_{T,A}(x)}(x).$$

If we denote by μ_A the measure on A induced by μ, that is, the measure defined for all $B \subseteq A$ by $\mu_A(B) := \mu(B)/\mu(A)$, then $T_A \in \mathrm{Aut}(X_A, \mu_A)$ is ergodic. We recall that by Kac's Lemma [Kac47], we have

$$\int_A n_{T,A}(x) d\mu = 1.$$

We now prove that there exist non-crossing involutions with full support in $\mathrm{L}^{<1}$ full groups. This result is implicitly contained in the proof of [CJLMT22, Thm. 3.1]. We provide here a complete proof.

Theorem 4.4.3. — *Let $T \in \mathrm{Aut}(X,\mu)$ be ergodic. Then there exists a non-crossing involution $P \in [T]_{<1}$ with full support.*

Proof. Let $T_1 := T$ and $A_1 = X$. By LEMMA 4.4.1, we fix an 2-separated subset $B_1 \subseteq A_1$ of positive measure. For all $x \in A_1$, let $n_1(x)$ be the smallest integer $n \geq 0$ such that $x \in T_1^n(B_1)$. This is finite for μ-almost every $x \in X$ by the Poincaré recurrence theorem. Let $A_2 \subseteq A_1$ be the measurable subset given by

$$A_2 := \{x \in A_1 \colon n_1(x) \text{ is even and } n_1(T_1(x)) = 0\}.$$

If $\mu(A_2) > 0$, then we construct n_2. We denote by $T_2 : A_2 \to A_2$ the first return map of T_1 with respect to A_2. By LEMMA 4.4.1, one gets $B_2 \subseteq A_2$ a 2^2-separated subset for T_2 of positive measure. Then for all $x \in A_2$, we define $n_2(x)$ as the smallest integer $n \geq 0$ such that $x \in T_2^n(B_2)$. Let $A_3 \subseteq A_2$ be the measurable subset given by

$$A_3 := \{x \in A_2 \colon n_2(x) \text{ is even and } n_2(T_2(x)) = 0\}.$$

If $\mu(A_3) > 0$, we next construct n_3. We let T_3 be the first return map of T_2 with respect to A_3 and we consider $B_3 \subseteq A_3$ a 2^3-separating subset for T_3 of positive measure. Then for all $x \in A_3$ we define $n_3(x)$ as the smallest integer $n \geq 0$ such that $x \in T_3^n(B_3)$.

Assume that this construction can be run indefinitely. We then obtain a sequence of decreasing subsets of positive measure $X = A_1 \supseteq A_2 \supseteq \dots$, a sequence of measure preserving transformations $T_k : A_k \to A_k$, a sequence $B_k \subseteq A_k$ of 2^k-separated sets for T_k of positive measure, and a sequence of maps $n_k : A_k \to \mathbb{N}$. For all $k \geq 1$, the map T_{k+1} is the first return time of T_k on A_{k+1}, where

$$A_{k+1} := \{x \in A_k \colon n_k(x) \text{ is even and } n_k(T_k(x)) = 0\}.$$

Thus we deduce that $T_k(A_{k+1}) \subseteq B_k$. Since B_k is 2^k-separated for T_k, we obtain that $\mu(A_{k+1}) \leq \mu(B_k) \leq \mu_{A_k}(B_k) \leq 1/2^k$. This implies that the sequence of sets $(A_k \setminus A_{k+1})_{k\geq 1}$ forms a partition of X. We now define the non-crossing involution P. For all $k \geq 1$ and all $x \in A_k \setminus A_{k+1}$, we let

$$P(x) := \begin{cases} T_k(x) & \text{if } n_k(x) \text{ is even and } n_k(T_k(x)) \neq 0, \\ (T_k)^{-1}(x) & \text{if } n_k(x) \text{ is odd.} \end{cases}$$

By construction, P is a non-crossing involution with full support contained in the full group $[T]$. We need to prove that $P \in [T]_{<1}$. Let c_P be the T-cocycle of P. For all $x \in A_1 \setminus A_2$, we have $|c_P(x)| = 1$. Let $k \geq 2$ and let $x \in A_k \setminus A_{k+1}$. The facts that B_{k-1} is 2^{k-1}-separated for T_{k-1} and that $T_{k-1}(A_k) \subset B_{k-1}$ yields

$$2^{k-1} \leq |c_P(x)|.$$

Thus, for all $0 < p < 1$, we have

$$\begin{aligned}\int_X |c_P(x)|^p d\mu &= \mu(A_1 \setminus A_2) + \sum_{k\geq 2} \int_{A_k\setminus A_{k+1}} |c_P(x)|^{p-1}|c_P(x)|d\mu \\ &\leq \mu(A_1 \setminus A_2) + \sum_{k\geq 2} 2^{(k-1)(p-1)} \int_{A_k\setminus A_{k+1}} |c_P(x)|d\mu.\end{aligned}$$

Claim. — For all $k \geq 1$, we have $\int_{A_k\setminus A_{k+1}} |c_P(x)|d\mu \leq 2$.

Proof of the claim. For all $k \geq 1$, we define

$$L_k := \{x \in A_k \colon n_k(x) \text{ is even and } n_k(T_k(x)) \neq 0\}.$$

Set $R_k := P(L_k)$. Then $A_k \setminus A_{k+1} = L_k \sqcup R_k$. Moreover, for all $x \in L_k$ we have $c_P(x) = n_{T,A_k}(x)$ and for all $x \in R_k$ we have $c_P(x) = -n_{T,A_k}(P(x))$. Thus

$$\begin{aligned}\int_{A_k\setminus A_{k+1}} |c_P(x)|d\mu &= 2\int_{L_k} n_{T,A_k}(x)d\mu \\ &\leq 2\int_{A_k} n_{T,A_k}(x)d\mu \\ &= 2,\end{aligned}$$

where the last equality is given by Kac's lemma. $\square_{\text{claim}}$

Therefore, we get that $\int_X |c_P(x)|^p d\mu$ is finite for every $0 < p < 1$, which proves that $P \in [T]_{<1}$.

If along the above construction we get $\mu(A_n) = 0$ for some $n \geq 1$, then we define $P := P_1 \sqcup \cdots \sqcup P_n$. By the same computation, this yields a non-crossing involution with full support in $[T]_{<1}$. $\square$

Theorem 4.4.4. — *Let $\mathbb{Z} \curvearrowright^\alpha (X, \mu)$ be an ergodic action. Then there exists an ergodic action $D_\infty \curvearrowright^\beta (Y, \nu)$ such that α and β are $\mathrm{L}^{<1}$ orbit equivalent.*

Proof. Let $T := \alpha(1)$. By Theorem 4.4.3, let $P \in [T]_{<1}$ be a non-crossing involution with full support. Then TPT^{-1} is also a non-crossing involution with full support, which belongs to $[T]_{<1}$. Let $D_\infty \curvearrowright^\beta (X, \mu)$ be the p.m.p. action defined by $\beta(a)x := P(x)$ and $\beta(b)x := TPT^{-1}(x)$. By Theorem 4.3.6, for μ-almost every $x \in X$, the restriction of β to the T-orbit of x is transitive. This proves that id_X is an orbit equivalence between α and β. Moreover, the fact that P and TPT^{-1} belong to $[T]_{<1}$ implies right away that α and β are $\mathrm{L}^{<1}$ orbit equivalent. $\square$

It is unclear how to prove a "dual" result to Theorem 4.4.4. That is, given an ergodic action $D_\infty \curvearrowright^\alpha (X, \mu)$, is there a way to construct an ergodic action $\mathbb{Z} \curvearrowright^\beta (Y, \nu)$ such that α and β are $\mathrm{L}^{<1}$ orbit equivalent?

References

[Bel68] Raisa M. Belinskaya. Partitions of Lebesgue space in trajectories defined by ergodic automorphisms. *Funkts. Anal. Prilozh.*, 2(3):190–199, 1968.

[Bow17] Lewis Bowen. Integrable orbit equivalence rigidity for free groups. *Isr. J. Math.*, 221(1):471–480, 2017.

[CJLMT22] Alessandro Carderi, Matthieu Joseph, François Le Maître, and Romain Tessera. Belinskaya's theorem is optimal, 2022.

[Kac47] Mark Kac. On the notion of recurrence in discrete stochastic processes. *Bull. Am. Math. Soc.*, 53(10):1002–1010, 1947.

[LM18] François Le Maître. On a measurable analogue of small topological full groups. *Adv. Math.*, 332:235–286, 2018.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I: The Rohlin lemma. *Bull. Am. Math. Soc., New Ser.*, 2:161–164, 1980.

[Rok67] Vladimir A. Rokhlin. Lectures on the entropy theory of measure-preserving transformations. *Russian Math. Surveys*, 22(5):1, 1967.

Part III

Quantitative orbit equivalence and graphings

Chapter 5

Isometric orbit equivalence for probability-measure preserving actions

The content of this chapter is the same as that of the article [Jos22]

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5.1 Introduction

The purpose of this chapter is to compare probability-measure preserving actions of finitely generated groups via the graphings they define. A ***graphing*** on a probability space (X, μ) is a graph $\mathcal{G}$ whose vertex set is X and whose edge set is a measurable, symmetric subset of $X \times X$, such that the equivalence relation $\mathcal{R}$ "belonging to the same connected component of $\mathcal{G}$" satisfies the following two conditions: (i) each $\mathcal{R}$-class is countable, (ii) any bimeasurable bijection $T : A \to B$ between measurable subsets $A, B \subseteq X$, such that $(x, T(x)) \in \mathcal{R}$ for all $x \in X$, preserves the measure μ.

Graphings have seen recently a sharp rise in interest as they play an important role in graph limit theory. Indeed, they serve as limit objects for sequences of bounded degree graphs, see [Lov12, Part 4] for an introduction to this theory. They are also one of the main objects in the cost theory of p.m.p. actions, which was extensively studied by Gaboriau [Gab00].

Graphings are closely related to probability measure preserving actions of countable groups. A probability measure preserving action (p.m.p. action for short) $\Gamma \curvearrowright^{\alpha} (X, \mu)$ of a countable group Γ on a standard probability space (X, μ) is a collection $(\alpha(\gamma))_{\gamma \in \Gamma}$ of bimeasurable bijections $\alpha(\gamma) : X \to X$ which preserve the probability measure μ, such that for all $\gamma, \delta \in \Gamma$, we have $\alpha(\gamma\delta) = \alpha(\gamma)\alpha(\delta)$. A p.m.p. action is ***essentially free*** if the set of points with trivial stabilizer has full measure. A p.m.p. action is ***ergodic*** if any measurable set, which is invariant under the action, has measure 0 or 1. Assume that Γ is finitely generated and fix S a ***finite generating system*** for Γ, that is, a finite symmetric set which generates the group and which does not contain the identity element 1_Γ. To any p.m.p. action $\Gamma \curvearrowright^{\alpha} (X, \mu)$, one can associate a graphing, denoted by $\alpha(S_\Gamma)$, whose vertex set is X, and whose edge set is the symmetric set $\{(x, x') \in X \times X \colon \exists s \in S, \alpha(s)x = x'\}$.

The spirit of this article fits into the framework of ***quantitative orbit equivalence*** and more generally ***quantitative measure equivalence***, [DKLMT20]. These nascent areas aim to understand how metric structures on p.m.p. actions are distorted under orbit and measure equivalences.

Bounded orbit equivalence appears to be an important notion of quantitative orbit equivalence. For instance, in the entropy theory, Austin proved that any two p.m.p. essentially free actions of amenable groups that are bounded orbit equivalent (and even integrable orbit equivalent) have the same Kolmogorov-Sinai entropy [Aus16]. Bowen and Lin proved that any two p.m.p. essentially free actions of a free group of finite rank, which are bounded orbit equivalent, have the same f-invariant [BL22]. Let Γ and Λ be two finitely generated groups and fix two finite generated systems S_Γ and S_Λ for Γ and Λ respectively. We say that two p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X, \mu)$ and $\Lambda \curvearrowright^{\beta} (Y, \nu)$ are ***bounded orbit equivalent*** if there exists a constant $C > 0$ and a bimeasurable bijection $\Phi : X \to Y$ such that $\Phi_*\mu = \nu$ and for μ-almost every $x \in X$, the map Φ is a C-biLipschitz map between the connected component of $x \in X$ in the graphing $\alpha(S_\Gamma)$ and the connected component of $\Phi(x) \in Y$ in the graphing $\beta(S_\Lambda)$. It is straightforward to check that bounded orbit equivalence does not depend on the choice of finite generating systems for the groups.

This notion retains the geometry of the group. For instance, the following result can be extracted from Shalom's work [Sha04]: two finitely generated amenable groups Γ and Λ admit p.m.p. essentially free actions that are bounded orbit equivalent if and only if Γ and Λ are biLipschitz equivalent. This result is very specific to the amenable realm, as for instance free groups of different finite ranks are biLipschitz equivalent [Pap95] but not bounded orbit equivalent because they are not even orbit equivalent [Gab00]. While bounded orbit equivalence is related to the study of graphings which are uniformly biLipschitz, we propose here to study graphings associated with p.m.p. actions up to isometry.

DEFINITION 5.1.1. — Let Γ and Λ be two finitely generated groups and fix two finite generating systems S_Γ and S_Λ respectively. Two p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X, \mu)$ and $\Lambda \curvearrowright^{\beta} (Y, \nu)$ are ***isometric orbit equivalent*** if the graphings $\alpha(S_\Gamma)$ and $\beta(S_\Lambda)$ are isometric in a measurable sense, that is, if there exists a bimeasurable bijection $\Phi : X \to Y$ such that $\Phi_* \mu = \nu$ and for μ-almost every $x \in X$, the map Φ is an isometry between the connected component of $x \in X$ in the graphing $\alpha(S_\Gamma)$ and the connected component of $\Phi(x) \in Y$ in the graphing $\beta(S_\Lambda)$.

Beware that contrary to bounded orbit equivalence, the notion of isometric orbit equivalence heavily depends on the choice of generating systems. In each of the statements in this article, when we say that two actions are isometric orbit equivalent, it refers to the generating systems of the groups that are fixed and clear in the context.

Consider Γ a finitely generated group and fix S_Γ a finite generating system of it. The ***Cayley graph*** (Γ, S_Γ) is the graph whose vertex set is Γ and whose edge set is the symmetric set $\{(\gamma, \delta) \in \Gamma^2 \colon \exists s \in S_\Gamma, \delta = \gamma s\}$. We endow the Cayley graph with its graph distance. Two Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric if there exists a bijective isometry between them. Our first result shows that the existence of isometric orbit equivalent actions that are essentially free is connected to the isometry class of the Cayley graph.

THEOREM 5.1.2 (see THEOREM 5.3.3). — *Let Γ and Λ be two finitely generated groups and fix two finite generating systems S_Γ and S_Λ respectively. Then Γ and Λ admit p.m.p. essentially free actions that are isometric orbit equivalent if and only if the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric.*

One direction of this result is immediate, because in the graphing associated with any p.m.p. essentially free action, almost every connected component is isometric to the Cayley graph of the acting group. Thus, an isometric orbit equivalence between p.m.p. essentially free actions leads to an isometry between the Cayley graphs on almost every connected components of the associated graphings. The converse is proved using the space $\mathrm{Iso}(\Gamma, \Lambda)$ of bijective isometries from (Γ, S_Γ) to (Λ, S_Λ). The group Λ acts by postcomposition on $\mathrm{Iso}(\Gamma, \Lambda)$. Similarly, the group Γ acts by postcomposition on the space $\mathrm{Iso}(\Lambda, \Gamma)$ of bijective isometries from Λ to Γ. We prove that the quotient actions $\Gamma \curvearrowright \mathrm{Iso}(\Gamma, \Lambda)/\Lambda$ and $\Lambda \curvearrowright \mathrm{Iso}(\Lambda, \Gamma)/\Gamma$, endowed with their respective Haar probability measures, are isometric orbit equivalent, see COROLLARY 5.3.2.

This connection between Cayley graph and isometric orbit equivalence leads to interesting rigidity results. Given a finitely generated group Γ with a generating system S_Γ, we denote by $\mathrm{Iso}(\Gamma)$ the group of bijective isometries of the Cayley graph (Γ, S_Γ). Equivalently, $\mathrm{Iso}(\Gamma)$ is isomorphic to the group of all graph automorphisms of the Cayley graph. We prove the following result using techniques introduced by Furman in [Fur99].

THEOREM 5.1.3 (see THEOREM 5.4.1). — *Let Γ and Λ be two finitely generated groups and fix two finite generating systems S_Γ and S_Λ respectively. Assume that the Cayley* *graphs* (Γ, S_Γ) *and* (Λ, S_Λ) *are isometric. Let* $\Gamma \curvearrowright^\alpha (X,\mu)$ *and* $\Lambda \curvearrowright^\beta (Y,\nu)$ *be two p.m.p. essentially free actions that are isometric orbit equivalent. If* $\mathrm{Iso}(\Gamma)$ *(equivalently* $\mathrm{Iso}(\Lambda)$*) is countable, then*

(i) There exists finite index subgroups $\Gamma_0 \leq \Gamma$ *and* $\Lambda_0 \leq \Lambda$ *which are isomorphic.*

(ii) There exists a Γ_0*-invariant subset* $X_0 \leq X$ *of positive measure and a* Λ_0*-invariant subset* $Y_0 \leq Y$ *of positive measure such that the p.m.p. actions* $\Gamma_0 \curvearrowright (X_0, \mu_{X_0})$ *and* $\Lambda_0 \curvearrowright (Y_0, \nu_{Y_0})$ *are measurably isomorphic.*

If in addition, every finite index subgroup of Γ *acts ergodically on* (X,μ)*, then* Γ *and* Λ *are isomorphic and* α *and* β *are measurably isomorphic.*

We discuss concrete examples of finitely generated groups $\Gamma = \langle S_\Gamma \rangle$ such that $\mathrm{Iso}(\Gamma)$ is countable in Example 5.4.3. For instance, if $\mathbb{Z}^d$ is equipped with any finite generating set $S_{\mathbb{Z}^d}$, then the Cayley graph $(\mathbb{Z}^d, S_{\mathbb{Z}^d})$ has only countably many bijective isometries. The ergodicity constraint on any finite index subgroup is fulfilled for instance by Bernoulli shifts, or more generally by any mixing action. A p.m.p. action $\Gamma \curvearrowright^\alpha (X,\mu)$ is ***mixing*** if for any measurable subset $A, B \subseteq X$,

$$\mu(\alpha(\gamma)A \cap B) \underset{\gamma\to+\infty}{\longrightarrow} \mu(A)\mu(B).$$

The isometric orbit equivalence rigidity result obtained in Theorem 5.1.3 is false in the context of bounded orbit equivalence. Indeed, Fieldsteel and Friedman proved that for any $d \geq 2$ and any ergodic action $\mathbb{Z}^d \curvearrowright^\alpha (X,\mu)$, there exists a mixing p.m.p. action $\mathbb{Z}^d \curvearrowright^\beta (Y,\nu)$ such that α and β are bounded orbit equivalent [FF86, Thm. 3].

Let $\mathbf{F}_d$ be the free group on $d \geq 2$ generators $x_1, \dots, x_d$ with its generating set $S_{\mathbf{F}_d} = \{x_1^{\pm 1}, \dots, x_d^{\pm 1}\}$. Then the Cayley graph $(\mathbf{F}_d, S_{\mathbf{F}_d})$ has uncountably many bijective isometries. We prove the following statement, which shows in particular that Theorem 5.1.3 is false in general if $\mathrm{Iso}(\Gamma)$ is uncountable.

Theorem 5.1.4. — *There exists ergodic actions* $\mathbf{F}_d \curvearrowright^\alpha (X,\mu)$ *and* $\mathbf{F}_d \curvearrowright^\beta (Y,\nu)$ *that are isometric orbit equivalent, such that* α *is not mixing but* β *is.*

Therefore, mixing is not invariant under isometric orbit equivalence. Theorem 5.1.4 is false for $\mathbf{F}_1 \simeq \mathbb{Z}$. In fact, for ergodic actions of $\mathbb{Z}$, Belinskaya proved in [Bel68] that any two p.m.p. ergodic actions of $\mathbb{Z}$ that are L^1 orbit equivalent are flip-conjugate, that is, measurably isomorphic up to an automorphism of the group $\mathbb{Z}$.

Theorem 5.1.4 is proved using a general construction of isometric orbit equivalent actions that we explain now. For a concrete illustration of Theorem 5.1.4, we refer to Section 5.5.3.

Let Γ be a finitely generated group and fix S_Γ a finite generating system. Let $\Lambda \leq \Gamma$ be a finite index subgroup. The groups Λ and Γ both act by post-composition on $\mathrm{Iso}(\Gamma)$. Then we prove in Section 5.5.1 that the quotient action $\Gamma \curvearrowright \mathrm{Iso}(\Gamma)/\Lambda$, equipped with its Haar probability measure, is isometric orbit equivalent to the diagonal action $\Gamma \curvearrowright \mathrm{Iso}(\Gamma)/\Gamma \times \Gamma/\Lambda$, equipped with the product of the Haar probability measure and the uniform probability measure. The aim in order to prove THEOREM 5.1.4 becomes to show that for some subgroups $\Lambda \leq \mathbf{F}_d$, the action $\mathbf{F}_d \curvearrowright \mathrm{Iso}(\mathbf{F}_d)/\Lambda$ is mixing. We actually give a complete characterization of such subgroups Λ in Section 5.5.2. Along the proof, we use the Howe-Moore property satisfied by the group $\mathrm{Iso}(\mathbf{F}_d)$, a result due independently to Lubotzky and Mozes [LM92] and to Pemantle [Pem92].

5.2 Preliminaries

5.2.1 Cayley graph

In the text, if we write $\Gamma = \langle S \rangle$, we always mean that Γ is a countable group and S a ***generating system*** for Γ, that is, a symmetric set which generates Γ, and which does not contain the identity element 1_Γ of Γ. In the case where the generating system S is finite, then we say that Γ is ***finitely generated***. When we say that $\Gamma = \langle S \rangle$ is finitely generated, we implicitly mean that S is a finite generating system.

Let $\Gamma = \langle S \rangle$ be a countable group. We denote by $|\cdot|_S$ the word length on Γ with respect to the generating system S. It is defined by $|1_\Gamma|_S = 1$ and for all $\gamma \in \Gamma \setminus \{1_\Gamma\}$,

$$|\gamma|_S := \min\{n \in \mathbb{N} \colon \exists s_1, \dots, s_n \in S, \gamma = s_1 \dots s_n\}.$$

The ***Cayley graph*** of $\Gamma = \langle S \rangle$, denoted (Γ, S), is the simplicial graph whose vertex set is Γ endowed with the metric $(\gamma, \delta) \mapsto |\gamma^{-1}\delta|_S$, called the word metric. In practice, this means that (Γ, S) is a graph without multiple edges or loops and there is an edge between two vertices $\gamma, \delta \in \Gamma$ if and only if there is $s \in S$ such that $\gamma = \delta s$. A map $f : \Gamma \to \Gamma$ is an isometry if for all $\gamma, \delta \in \Gamma$ we have

$$|\gamma^{-1}\delta|_S = |f(\gamma)^{-1} f(\delta)|_S.$$

The ***group of bijective isometries*** of a Cayley graph (Γ, S) is the group denoted by $\mathrm{Iso}(\Gamma, S)$ of all bijections $f : \Gamma \to \Gamma$ which are isometries. The action of Γ by left multiplication on itself yields a canonical injective group homomorphism $\Gamma \hookrightarrow \mathrm{Iso}(\Gamma, S)$. Thus, Γ identifies naturally as a subgroup of $\mathrm{Iso}(\Gamma, S)$. Each time the generating set is clear in the context, we will write $\mathrm{Iso}(\Gamma)$ instead of $\mathrm{Iso}(\Gamma, S)$. In the literature, the group $\mathrm{Iso}(\Gamma)$ is often referred to as the automorphism group of the Cayley graph (Γ, S_Γ) and denoted by $\mathrm{Aut}(\Gamma, S_\Gamma)$.

Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two countable groups. A map $f : \Gamma \to \Lambda$ is an isometry if for all $\gamma, \delta \in \Gamma$ we have

$$|\gamma^{-1}\delta|_{S_\Gamma} = |f(\gamma)^{-1} f(\delta)|_{S_\Lambda}.$$

Let $\mathrm{Iso}((\Gamma, S_\Gamma), (\Lambda, S_\Lambda))$ be the space of bijective isometries between the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ). We say that the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are ***isometric*** if $\mathrm{Iso}((\Gamma, S_\Gamma), (\Lambda, S_\Lambda))$ is nonempty. Again, if the generating systems are clear in the context, we will write $\mathrm{Iso}(\Gamma, \Lambda)$ instead of $\mathrm{Iso}((\Gamma, S_\Gamma), (\Lambda, S_\Lambda))$.

Example 5.2.1. — Here are some examples of finitely generated groups with isometric Cayley graphs.

1. Let $\Gamma = \mathbb{Z}$ be equipped with its usual generating system $S_\Gamma = \{-1, +1\}$. Let Λ be the infinite dihedral group D_∞, that is, the free product of two copies of the cyclic group C_2 of order two. A presentation of D_∞ is given by
$$D_\infty := \langle a, b \mid a^2 = b^2 = 1 \rangle.$$
Let S_Λ be the usual generating system $\{a, b\}$. Then the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric.

2. More generally, let $\Gamma = \mathbf{F}_d$ be the free group on $d \geq 2$ generators $x_1, \ldots, x_d$, and let S_Γ be the generating system $\{x_1^{\pm 1}, \ldots, x_d^{\pm 1}\}$. Let Λ be the free product of $2d$ copies of C_2, whose presentation is given by
$$\Lambda = \langle a_1, \ldots, a_{2d} \mid a_1^2 = \cdots = a_{2d}^2 = 1 \rangle,$$
and S_Λ be the usual generating system $\{a_1, \ldots, a_{2d}\}$. Then the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric.

3. Let $m, n \geq 2$ be two integers. Let $\mathbf{F}_m$ and $\mathbf{F}_n$ be the free group on m generators $x_1, \ldots, x_m$ and on n generators $y_1, \ldots, y_n$ respectively. Let $\Gamma_{m,n}$ be the direct product $\mathbf{F}_m \times \mathbf{F}_n$ and $S_{\Gamma_{m,n}}$ be the generating system
$$S_{\Gamma_{m,n}} := \{(x_1^{\pm 1}, 1_{\mathbf{F}_n}), \ldots, (x_m^{\pm 1}, 1_{\mathbf{F}_n}), (1_{\mathbf{F}_m}, y_1^{\pm 1}), \ldots, (1_{\mathbf{F}_m}, y_n^{\pm 1})\}.$$
It follows from the work of Burger and Mozes [BM00a, BM00b] that for appropriate values of m and n, the Cayley graph $(\Gamma_{m,n}, S_{\Gamma_{m,n}})$ is isometric to the Cayley graph of some virtually simple group.

4. Dyubina observed in [Dyu00] that there exist finitely generated groups $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ with isometric Cayley graphs, such that Γ is solvable but Λ not virtually solvable. This illustrates the fact that several algebraic properties are not preserved under the property of having isometric Cayley graphs.

5.2.2 Definition of isometric orbit equivalence

Let (X, μ) be a standard probability space. A bimeasurable bijection $T : X \to X$ is a ***p.m.p. automorphism*** of (X, μ) if for all measurable set $A \subseteq X$, one has $\mu(T^{-1}(A)) = \mu(A)$. We denote by $\mathrm{Aut}(X, \mu)$ the group of all p.m.p. automorphisms of (X, μ), two such automorphisms being identified if they coincide on a conull set. A bimeasurable bijection $T : (X,\mu) \to (Y,\nu)$ between two standard probability spaces is a ***p.m.p. isomorphism*** if $T_*\mu = \nu$. A p.m.p. automorphism $T \in \mathrm{Aut}(X,\mu)$ is ***aperiodic*** if the T-orbit of μ-almost every $x \in X$ is infinite. A p.m.p. action of a countable group Γ on (X,μ) is a homomorphism $\alpha : \Gamma \to \mathrm{Aut}(X,\mu)$. It is ***essentially free*** if for each $\gamma \in \Gamma \setminus \{1_\Gamma\}$, the p.m.p. automorphism $\alpha(\gamma)$ is aperiodic. If there is no need to give a name to the p.m.p. action α, we simply write γx instead of $\alpha(\gamma)x$.

A ***p.m.p. partial automorphism*** on (X,μ) is a bimeasurable bijection $T : A \to B$ between measurable subsets $A, B \subseteq X$, which preserves the measure, that is, for all measurable subset $C \subseteq A$, one has $\mu(T(C)) = \mu(C)$. A ***graphing*** on (X,μ) is a countable set $\Theta := \{T_i : A_i \to B_i \mid i \in I\}$ of p.m.p. partial automorphisms. The shortest path distance $d_\Theta(x,y)$ between two distinct points $x, y \in X$ is the smallest integer $n \in \mathbb{N} \cup \{+\infty\}$ such that there exists $i_1, \ldots, i_n \in I$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ such that $y = T_{i_1}^{\varepsilon_1} \ldots T_{i_n}^{\varepsilon_n}(x)$. This defines an (extended) metric $d_\Theta : X \times X \to \mathbb{N} \cup \{+\infty\}$.

Definition 5.2.2 (Isometric graphings). — Two graphings Θ on (X,μ) and Ξ on (Y,ν) are ***measurably isometric*** if there exists a p.m.p. isomorphism $\Phi : (X,\mu) \to (Y,\nu)$ such that for μ-a.e. $x, x' \in X$ we have

$$d_\Xi(\Phi(x), \Phi(x')) = d_\Theta(x, x').$$

Definition 5.2.3 (Isometric orbit equivalence). — Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two countable groups. Two p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X,\mu)$ and $\Lambda \curvearrowright^{\beta} (Y,\nu)$ are ***isometric orbit equivalent*** if the graphings $\alpha(S_\Gamma)$ on (X,μ) and $\beta(S_\Lambda)$ on (Y,ν) are measurably isometric.

In the definition of isometric orbit equivalence, both groups Γ and Λ come with their fixed generating system. It is thus noteworthy that the notion of isometric orbit equivalence heavily depends on the generating systems of the groups.

Example 5.2.4. — Let $T \in \mathrm{Aut}(X,\mu)$ be an aperiodic transformation. Let C_2 be the cyclic group of order two, and ν be the uniform probability measure on C_2. Let α be the p.m.p. $\mathbb{Z}$-action on $(X \times C_2, \mu \otimes \nu)$ defined by

$$\alpha(n)(x,\varepsilon) := \begin{cases} (T^n(x), \varepsilon) & \text{if } n \text{ is even,} \\ (T^n(x), 1-\varepsilon) & \text{if } n \text{ is odd.} \end{cases}$$

Let β be the p.m.p. action of the infinite dihedral group $D_\infty := \langle a, b \mid a^2 = b^2 = 1 \rangle$ on $(X \times C_2, \mu \otimes u)$ defined by the action of the two generators

$$\beta(a)(x,0) := (T(x),1), \quad \beta(a)(x,1) := (T^{-1}(x),0),$$
$$\beta(b)(x,1) := (T(x),0), \quad \beta(b)(x,0) := (T^{-1}(x),1).$$

Then the graphings $\alpha(\{\pm 1\})$ and $\beta(\{a,b\})$ are measurably isometric. Thus, if $\mathbb{Z}$

is equipped with the generating system $S_{\mathbb{Z}} := \{\pm 1\}$ and D_∞ with the generating system S_{D_∞}, then the p.m.p. actions α and β are isometric orbit equivalent.

Example 5.2.5. — Let $\Gamma = \langle S_\Gamma \rangle$ be a finitely generated group. Let $\Gamma \curvearrowright^\alpha (X, \mu)$ be a p.m.p. action and define

$$E := \{(x, x') \in X \times X \colon \exists s \in S_\Gamma, \alpha(s)x = x'\}.$$

Fix a proper d-coloring on E, that is, a measurable map $c : E \to \{1, \ldots, d\}$ such that for μ-almost every $x, x', x'' \in X$, if (x, x') and (x, x'') are distinct elements of E, then $c(x, x') \neq c(x, x'')$. This always exists when d is large enough (see [CLP16] for a precise statement) and the smallest such d is called the measurable edge chromatic number of the graphing $\alpha(S_\Gamma)$. Let Λ be the group given by the presentation

$$\Lambda := \langle a_1, \ldots, a_d \mid a_1^2 = \cdots = a_d^2 = 1 \rangle$$

and let S_Λ be the generating system $\{a_1, \ldots, a_d\}$. Let β be the p.m.p. action of Λ defined by the action of its generators

$$\beta(a_i)(x) := \begin{cases} x' & \text{if } (x, x') \in E \text{ is such that } c(x, x') = i, \\ x & \text{else.} \end{cases}$$

Then the p.m.p. actions α and β are isometric orbit equivalent.

We now explain a way to prove that p.m.p. actions are isometric orbit equivalent. For this, we need to introduce the notion of length-preserving cocycle. Let Γ and Λ be two countable groups. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action. A measurable function $\sigma : \Gamma \times X \to \Lambda$ is a ***cocycle*** if for all $\gamma, \delta \in \Gamma$,

$$\sigma(\gamma\delta, x) = \sigma(\gamma, \delta x)\sigma(\delta, x) \text{ for } \mu\text{-almost every } x \in X.$$

If $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$, we say that a cocycle $\sigma : \Gamma \times X \to \Lambda$ is ***length-preserving*** if for all $\gamma \in \Gamma$,

$$|\sigma(\gamma, x)|_{S_\Lambda} = |\gamma|_{S_\Gamma} \text{ for } \mu\text{-almost every } x \in X.$$

Lemma 5.2.6. — *Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two countable groups. Let $\Gamma \curvearrowright^\alpha (X, \mu)$ and $\Lambda \curvearrowright^\beta (Y, \nu)$ be two p.m.p. actions. Assume that there is a p.m.p. isomorphism $\Phi : (X, \mu) \to (Y, \nu)$ and two length-preserving cocycles $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$ such that for all $\gamma \in \Gamma$, $\lambda \in \Lambda$,*

$$\Phi(\alpha(\gamma)x) = \beta(\sigma(\gamma, x))\Phi(x), \quad \text{for } \mu\text{-almost every } x \in X,$$
$$\Phi(\alpha(\tau(\lambda, y))\Phi^{-1}(y)) = \beta(\lambda)y, \quad \text{for } \nu\text{-almost every } y \in Y.$$

Then α and β are isometric orbit equivalent.

Proof. Let us prove that for μ-almost every $x, x' \in X$, we have $d_{\alpha(S_\Gamma)}(x, x') = d_{\beta(S_\Lambda)}(\Phi(x), \Phi(x'))$. Let $x, x' \in X$. First, we have $d_{\alpha(S_\Gamma)}(x, x') = +\infty$ if and only if $d_{\beta(S_\Lambda)}(\Phi(x), \Phi(x')) = +\infty$. Thus, we can assume that $d_{\alpha(S_\Gamma)}(x, x') < +\infty$. Let $\gamma \in \Gamma$ such that $d_{\alpha(S_\Gamma)}(x, x') = |\gamma|_{S_\Gamma}$ and $\alpha(\gamma)x = x'$. We have

$$\begin{aligned} d_{\beta(S_\Lambda)}(\Phi(x), \Phi(x')) &= d_{\beta(S_\Lambda)}(\Phi(x), \beta(\sigma(\gamma, x))\Phi(x)) \\ &\leq |\sigma(\gamma, x)|_{S_\Lambda} \\ &= |\gamma|_{S_\Gamma} \\ &= d_{\alpha(S_\Gamma)}(x, x'). \end{aligned}$$

The reverse inequality is proved in a similar way, using the fact that τ is length-preserving. We thus obtain that

$$d_{\beta(S_\Lambda)}(\Phi(x), \Phi(x')) = d_{\alpha(S_\Gamma)}(x, x'),$$

which proves that α and β are isometric orbit equivalent. □

5.2.3 Isometric orbit equivalence for essentially free actions

Two p.m.p. actions of two countable groups $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are ***orbit equivalent*** if there exists an ***orbit equivalence*** between them, that is, a p.m.p. isomorphism $\Phi : (X, \mu) \to (Y, \nu)$ such that for μ-almost every $x \in X$,

$$\Phi(\Gamma x) = \Lambda \Phi(x).$$

Consider now two p.m.p. *essentially free* actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ and let $\Phi : (X, \mu) \to (Y, \nu)$ be an orbit equivalence between them. We therefore have two maps, which are uniquely defined by freeness of the actions:

$$\begin{aligned} \sigma : \Gamma \times X \to \Lambda, &\quad \Phi(\gamma x) = \sigma(\gamma, x)\Phi(x), \\ \tau : \Lambda \times Y \to \Gamma, &\quad \Phi(\tau(\lambda, y)\Phi^{-1}(y)) = \lambda y. \end{aligned}$$

They are called the ***orbit equivalence cocycles*** associated with Φ. Moreover, they satisfy the following properties.

Lemma 5.2.7 (Properties of the orbit equivalence cocycles). — *For μ-almost every $x \in X$ and ν-almost every $y \in Y$, the following are true.*

(i) (Cocycles) The maps $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times X \to \Gamma$ are cocycles.

(ii) (Bijections) The maps $\sigma(-, x) : \Gamma \to \Lambda$ and $\tau(-, y) : \Lambda \to \Gamma$ are bijections which are inverse one of each other: for all $\gamma \in \Gamma$, $\lambda \in \Lambda$,

$$\sigma(\tau(\lambda, y), \Phi^{-1}(y)) = \lambda \text{ and } \tau(\sigma(\gamma, x), \Phi(x)) = \gamma.$$

(iii) (Fixing the identity) We have $\sigma(1_\Gamma, x) = 1_\Lambda$ and $\tau(1_\Lambda, y) = 1_\Gamma$.

REMARK 5.2.8. — Let $[\Gamma, \Lambda]$ be the set of maps $f : \Gamma \to \Lambda$ such that $f(1_\Gamma) = f(1_\Lambda)$. There is a natural action $\Gamma \curvearrowright [\Gamma, \Lambda]$ defined by

$$(\gamma \cdot f)(\delta) := f(\gamma^{-1})^{-1} f(\gamma^{-1}\delta).$$

A ***randomorphism*** from Γ to Λ is a Γ-invariant probability measure on $[\Gamma, \Lambda]$, see [Mon06, Def. 5.2]. Monod observed that randomorphisms supported on bijections can be obtained via orbit equivalences as follows. Let Φ be an orbit equivalence between p.m.p. essentially free actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$. Let $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$ be the orbit equivalence cocycles associated with Φ. For $x \in X$ and $y \in Y$, let $\sigma_x : \Gamma \to \Lambda$ and $\tau_y : \Lambda \to \Gamma$ be defined by

$$\sigma_x(\gamma) := \sigma(\gamma^{-1}, x)^{-1},$$
$$\tau_y(\lambda) := \tau(\lambda^{-1}, y)^{-1}.$$

We deduce by LEMMA 5.2.7 that σ_x is a bijection such that $\sigma_x(1_\Gamma) = 1_\Lambda$. Similarly, we get that τ_y is a bijection such that $\tau_y(1_\Lambda) = 1_\Gamma$. Moreover, the cocycle identity for σ and τ implies that $x \in X \mapsto \sigma_x \in [\Gamma, \Lambda]$ is Γ-equivariant and that $y \in Y \mapsto \tau_y \in [\Lambda, \Gamma]$ is Λ-equivariant. Thus, the pushforward of μ by $x \mapsto \sigma_x$ is a randomorphism from Γ to Λ supported on bijections. Similarly, the pushforward of ν by $y \mapsto \tau_y$ is a randomorphism from Λ to Γ supported on bijections.

Isometric orbit equivalent actions was defined in DEFINITION 5.2.3 thanks to the graphings associated with the actions. We now characterize isometric orbit equivalent actions in terms of cocycles. In LEMMA 5.2.11, we will prove that two essentially free actions are isometric orbit equivalent if and only if there exists an orbit equivalence whose associated cocycles are length-preserving.

DEFINITION 5.2.9. — Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two countable groups. Let $\Phi : (X, \mu) \to (Y, \nu)$ be an orbit equivalence between two p.m.p. essentially free actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$. We say that Φ is a ***length-preserving orbit equivalence*** if the orbit equivalence cocycles $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times X \to \Gamma$ associated with Φ are length-preserving.

REMARK 5.2.10. — Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two countable groups. Let $\mathrm{Iso}_1(\Gamma, \Lambda)$ be the set of $f \in \mathrm{Iso}(\Gamma, \Lambda)$ such that $f(1_\Gamma) = 1_\Lambda$. This is a subspace of $[\Gamma, \Lambda]$ which is invariant under the action $\Gamma \curvearrowright \mathrm{Iso}_1(\Gamma, \Lambda)$ defined by

$$(\gamma \cdot f)(\delta) := f(\gamma^{-1})^{-1} f(\gamma^{-1}\delta).$$

Inspired by the language of randomorphism proposed by Monod [Mon06], we say that a Γ-invariant probability measure on $\mathrm{Iso}_1(\Gamma, \Lambda)$ is a ***randisometry*** from Γ to Λ. Randisometries can be obtained via length-preserving orbit equivalence as follows. Let Φ be a length-preserving orbit equivalence between two p.m.p. essentially free actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$. Let $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$ be the orbit equivalence cocycles associated with Φ. Let $\sigma_x \in [\Gamma, \Lambda]$

and $\tau_y \in [\Lambda, \Gamma]$ be defined as in REMARK 5.2.8. Since Φ is length-preserving, we get by the cocycle identity that for all $\gamma, \delta \in \Gamma$,

$$\begin{aligned}|\sigma_x(\gamma)^{-1}\sigma_x(\delta)|_{S_\Lambda} &= |\sigma(\gamma^{-1}\delta, \delta^{-1}x)|_{S_\Lambda} \\ &= |\gamma^{-1}\delta|_{S_\Gamma}.\end{aligned}$$

This means that $\sigma_x \in \mathrm{Iso}_1(\Gamma, \Lambda)$. Similarly, we have $\tau_y \in \mathrm{Iso}_1(\Gamma, \Lambda)$. Moreover, the map $x \in X \mapsto \sigma_x \in \mathrm{Iso}_1(\Gamma, \Lambda)$ is Γ-equivariant and the map $y \in Y \mapsto \tau_y \in \mathrm{Iso}_1(\Lambda, \Gamma)$ is Λ-equivariant. Thus, the pushforward of μ by $x \mapsto \sigma_x$ is a randisometry from Γ to Λ. Similarly, the pushforward of ν by $y \mapsto \tau_y$ is a randisometry from Λ to Γ.

LEMMA 5.2.11. — *Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two countable groups. Two p.m.p. essentially free actions $\Gamma \curvearrowright^{\alpha} (X, \mu)$ and $\Lambda \curvearrowright^{\beta} (Y, \nu)$ are isometric orbit equivalent if and only if there exists a length-preserving orbit equivalence between them.*

Proof. If there exists a length-preserving orbit equivalence Φ between α and β, then we conclude by LEMMA 5.2.6 that α and β are isometric orbit equivalent. Let us prove the converse. Assume that α and β are isometric orbit equivalent. Let $\Phi : (X, \mu) \to (Y, \nu)$ be a pmp isomorphism such that for μ-almost every $x, x' \in X$,

$$d_{\alpha(S_\Gamma)}(x, x') = d_{\beta(S_\Lambda)}(\Phi(x), \Phi(x')).$$

This implies that $d_{\alpha(S_\Gamma)}(x, x')$ is finite if and only if x and x' are in the same Γ-orbit. Similarly, the distance $d_{\beta(S_\Lambda)}(\Phi(x), \Phi(x'))$ is finite if and only if $\Phi(x)$ and $\Phi(x')$ are in the same Λ-orbit. Thus, we deduce that Φ is an orbit equivalence. Let $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$ be the orbit equivalence cocycles associated with Φ. By definition, for all $\gamma \in \Gamma$ and μ-almost all $x \in X$, we have

$$d_{\alpha(S_\Gamma)}(x, \alpha(\gamma)x) = d_{\beta(S_\Lambda)}(\Phi(x), \beta(\sigma(\gamma, x))\Phi(x)).$$

Since the actions are essentially free, then the left hand side is equal to $|\gamma|_{S_\Gamma}$ while the right hand side is equal to $|\sigma(\gamma, x)|_{S_\Lambda}$. We thus get that $|\sigma(\gamma, x)|_{S_\Lambda} = |\gamma|_{S_\Gamma}$. Similarly, we get that $|\tau(\lambda, y)|_{S_\Gamma} = |\lambda|_{S_\Lambda}$ for all $\lambda \in \Lambda$ and ν-almost every $y \in Y$. This proves that Φ is a length-preserving orbit equivalence. □

5.3 A canonical isometric orbit equivalence

Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two finitely generated groups. The space $\mathrm{Iso}(\Gamma, \Lambda)$ of bijective isometries between the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) is a locally compact, totally disconnected space when equipped with the topology of pointwise convergence. The space $\mathrm{Iso}(\Gamma)$ of bijective isometries of the Cayley graph (Γ, S_Γ) is a totally disconnected, locally compact group when endowed with the topology of pointwise convergence. Moreover, it contains naturally Γ as a lattice, and thus is unimodular.

Assume that $\mathrm{Iso}(\Gamma,\Lambda)$ is nonempty. Then the group $\mathrm{Iso}(\Gamma)$ acts simply transitively on $\mathrm{Iso}(\Gamma,\Lambda)$ by precomposition by the inverse. The group $\mathrm{Iso}(\Lambda)$ also acts simply transitively on $\mathrm{Iso}(\Gamma,\Lambda)$ by postcomposition. This implies that there exists a unique measure (up to a multiplicative constant) on $\mathrm{Iso}(\Gamma,\Lambda)$ which is invariant by the actions $\mathrm{Iso}(\Gamma) \curvearrowright \mathrm{Iso}(\Gamma,\Lambda)$ and $\mathrm{Iso}(\Lambda) \curvearrowright \mathrm{Iso}(\Gamma,\Lambda)$. We call it the Haar measure on $\mathrm{Iso}(\Gamma,\Lambda)$. The inverse map $\mathrm{Iso}(\Gamma,\Lambda) \to \mathrm{Iso}(\Lambda,\Gamma)$ is a bimeasurable bijection, which sends the Haar measure to the Haar measure. The pushforward of the Haar measure on $\mathrm{Iso}(\Gamma,\Lambda)$ by the quotient map $\mathrm{Iso}(\Gamma,\Lambda) \to \mathrm{Iso}(\Gamma,\Lambda)/\Lambda$, rescaled to have mass 1, is called the Haar probability measure on $\mathrm{Iso}(\Gamma,\Lambda)/\Lambda$. Let $\mathrm{Iso}_1(\Gamma,\Lambda)$ be the compact open subspace of $\mathrm{Iso}(\Gamma,\Lambda)$ defined by

$$\mathrm{Iso}_1(\Gamma,\Lambda) := \{f \in \mathrm{Iso}(\Gamma,\Lambda) \colon f(1_\Gamma) = 1_\Lambda\}.$$

This is a fundamental domain for the action $\Lambda \curvearrowright \mathrm{Iso}(\Gamma,\Lambda)$. Let m be the Haar measure on $\mathrm{Iso}(\Gamma,\Lambda)$ such that $m(\mathrm{Iso}_1(\Gamma,\Lambda)) = 1$. The restriction of m to $\mathrm{Iso}_1(\Gamma,\Lambda)$ is called the Haar probability measure on $\mathrm{Iso}_1(\Gamma,\Lambda)$. Let m_Λ be the Haar probability measure on $\mathrm{Iso}(\Gamma,\Lambda)/\Lambda$. We then obtain that the p.m.p. action $\mathrm{Iso}(\Gamma,\Lambda) \curvearrowright (\mathrm{Iso}(\Gamma,\Lambda)/\Lambda, m_\Lambda)$ is measurably isomorphic to the p.m.p. action $\mathrm{Iso}(\Gamma,\Lambda) \curvearrowright (\mathrm{Iso}_1(\Gamma,\Lambda), m)$ defined by

$$(g \cdot f) := f(g^{-1}(1_\Gamma))^{-1} f \circ g^{-1}.$$

The restriction of this action to $\Gamma \leq \mathrm{Iso}(\Gamma,\Lambda)$ boils down to the p.m.p. action $\Gamma \curvearrowright (\mathrm{Iso}_1(\Gamma,\Lambda), m)$ encountered in Remark 5.2.10 and given by

$$(\gamma \cdot f)(\delta) := f(\gamma^{-1})^{-1} f(\gamma^{-1}\delta).$$

Let m_Γ denote the Haar probability measure on $\mathrm{Iso}(\Lambda,\Gamma)/\Gamma$. One of the aim of this section is to prove that the actions $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma,\Lambda)/\Lambda, m_\Lambda)$ and $\Lambda \curvearrowright (\mathrm{Iso}(\Lambda,\Gamma)/\Gamma, m_\Gamma)$ are isometric orbit equivalent. In order to prove this, we will work with the Γ-action on $\mathrm{Iso}_1(\Gamma,\Lambda)$ and the Λ-action on $\mathrm{Iso}_1(\Lambda,\Gamma)$ instead.

Lemma 5.3.1. — *Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two finitely generated groups, such that the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric. Let μ be the Haar probability measure on $\mathrm{Iso}_1(\Gamma,\Lambda)$ and ν the Haar probability measure on $\mathrm{Iso}_1(\Lambda,\Gamma)$. Then the following are true.*

(i) The map $\sigma : \Gamma \times \mathrm{Iso}_1(\Gamma,\Lambda) \to \Lambda$ defined by $\sigma(\gamma, f) := f(\gamma^{-1})^{-1}$ is a length-preserving cocycle.

(ii) The map $\tau : \Lambda \times \mathrm{Iso}_1(\Lambda,\Gamma) \to \Gamma$ defined by $\tau(\lambda, f) := f(\lambda^{-1})^{-1}$ is a length-preserving cocycle.

(iii) The inverse map $\Phi : (\mathrm{Iso}_1(\Gamma,\Lambda), \mu) \to (\mathrm{Iso}_1(\Lambda,\Gamma), \nu)$ is a p.m.p. isomorphism,

such that for all $\gamma \in \Gamma, \lambda \in \Lambda$, we have

$$\Phi(\gamma \cdot f) = \sigma(\gamma, f) \cdot \Phi(f), \quad \textit{for all } f \in \mathrm{Iso}_1(\Gamma, \Lambda),$$
$$\Phi(\tau(\lambda, f) \cdot \Phi^{-1}(f)) = \lambda \cdot f, \quad \textit{for all } f \in \mathrm{Iso}_1(\Lambda, \Gamma).$$

Proof. We start by proving (i). The fact that σ is a cocycle is a straightforward computation:

$$\sigma(\gamma\delta, f) = f(\delta^{-1}\gamma^{-1})^{-1} = f(\delta^{-1}\gamma^{-1})^{-1} f(\gamma^{-1}) f(\gamma^{-1})^{-1} = \sigma(\gamma, \delta \cdot f)\sigma(\gamma, f).$$

Moreover, since $f \in \mathrm{Iso}_1(\Gamma, \Lambda)$, we get that

$$|\sigma(\gamma, f)|_{S_\Lambda} = |f(\gamma^{-1})^{-1} f(1_\Gamma)|_{S_\Lambda} = |\gamma|_{S_\Gamma}.$$

This proves that σ is length-preserving. The proof of (ii) is identical. For the proof of (iii), it is clear that Φ is a bimeasurable map. Moreover, since the inverse map sends Haar measure to Haar measure, we obtain that $\Phi_*\mu = \nu$. Finally, the two formulas left to prove are straightforward computations. □

We obtain the following result as a corollary.

Corollary 5.3.2. — *Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two finitely generated groups, such that the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric. Let m_Λ be the Haar probability measure on $\mathrm{Iso}(\Gamma, \Lambda)/\Lambda$ and m_Γ the Haar probability measure on $\mathrm{Iso}(\Lambda, \Gamma)/\Gamma$. Then the p.m.p. actions $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma, \Lambda)/\Lambda, m_\Lambda)$ and $\Lambda \curvearrowright (\mathrm{Iso}(\Lambda, \Gamma)/\Gamma, m_\Gamma)$ are isometric orbit equivalent.*

Proof. Let $\sigma : \Gamma \times \mathrm{Iso}_1(\Gamma, \Lambda) \to \Lambda$ be defined by $\sigma(\gamma, f) := f(\gamma^{-1})^{-1}$ and $\tau : \Lambda \times \mathrm{Iso}_1(\Lambda, \Gamma) \to \Gamma$ be defined by $\tau(\lambda, f) := f(\lambda^{-1})^{-1}$. By Lemma 5.3.1, these are length-preserving cocycles. Let μ and ν be the Haar probability measures on $\mathrm{Iso}_1(\Gamma, \Lambda)$ and $\mathrm{Iso}_1(\Lambda, \Gamma)$ respectively. Then by Lemma 5.2.6 we get that the actions $\Gamma \curvearrowright (\mathrm{Iso}_1(\Gamma, \Lambda), \mu)$ and $\Lambda \curvearrowright (\mathrm{Iso}_1(\Lambda, \Gamma), \nu)$ are isometric orbit equivalent. Thus, the p.m.p. actions $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma, \Lambda)/\Lambda, m_\Lambda)$ and $\Lambda \curvearrowright (\mathrm{Iso}(\Lambda, \Gamma)/\Gamma, m_\Gamma)$ are isometric orbit equivalent. □

In general, the action $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma, \Lambda)/\Lambda, m_\Lambda)$ is not essentially free. However, a standard trick can be used to obtain p.m.p. essentially free actions that are isometric orbit equivalent.

Theorem 5.3.3. — *Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two finitely generated groups. Then Γ and Λ admit p.m.p. essentially free actions that are isometric orbit equivalent if and only if the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric.*

Proof. Assume that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are p.m.p. essentially free actions that are isometric orbit equivalent. By Lemma 5.2.11, there exists a length-preserving orbit equivalence $\Phi : (X, \mu) \to (Y, \nu)$. Let $\sigma : \Gamma \times X \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$ be the orbit equivalence cocycles associated with Φ. For $x \in X$, let $\sigma_x : \Gamma \to \Lambda$ be defined by $\sigma_x(\gamma) := \sigma(\gamma^{-1}, x)^{-1}$. By REMARK 5.2.10, we obtain that $\sigma_x \in \mathrm{Iso}_1(\Gamma, \Lambda)$ for μ-almost every $x \in X$. Thus, there exists a bijective isometry between the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ).

Conversely, assume that the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric. Let μ and ν be the Haar probability measures on $\mathrm{Iso}_1(\Gamma, \Lambda)$ and $\mathrm{Iso}_1(\Lambda, \Gamma)$ respectively. By COROLLARY 5.3.2, the p.m.p. actions $\Gamma \curvearrowright (\mathrm{Iso}_1(\Gamma, \Lambda), m_\Gamma)$ and $\Lambda \curvearrowright (\mathrm{Iso}_1(\Lambda, \Gamma), \nu)$ are isometric orbit equivalent. If these actions are essentially free, then the proof is complete. Else, we fix two p.m.p. essentially free actions $\Gamma \curvearrowright (X, \mu_X)$ and $\Lambda \curvearrowright (Y, \mu_Y)$ and consider the p.m.p. actions.

$$\Gamma \curvearrowright (\mathrm{Iso}_1(\Gamma, \Lambda) \times X \times Y, \mu \otimes \mu_X \otimes \mu_Y), \quad \gamma(f, x, y) := (\gamma \cdot f, \gamma x, f(\gamma^{-1})^{-1} y),$$
$$\Lambda \curvearrowright (\mathrm{Iso}_1(\Lambda, \Gamma) \times X \times Y, \nu \otimes \mu_X \otimes \mu_Y), \quad \lambda(f, x, y) := (\lambda \cdot f, f(\lambda^{-1})^{-1} x, \lambda y).$$

These actions are essentially free. Moreover, as a direct consequence of LEMMA 5.3.1, there is a length-preserving orbit equivalence between them, which implies by LEMMA 5.2.11 that they are isometric orbit equivalent. □

REMARK 5.3.4. — The trick used at the end of the proof for getting essentially free actions while staying (isometric) orbit equivalent is due to Gaboriau [Gab02, Thm. 2.3].

5.4 Rigidity of isometric orbit equivalence

The aim of this section is to understand isometric orbit equivalence when the space of bijective isometries $\mathrm{Iso}(\Gamma, \Lambda)$ between $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ is countable. Observe that the cardinality of $\mathrm{Iso}(\Gamma)$, $\mathrm{Iso}(\Lambda)$ and $\mathrm{Iso}(\Gamma, \Lambda)$ coincide, because the groups $\mathrm{Iso}(\Gamma)$ and $\mathrm{Iso}(\Lambda)$ acts simply transitively on $\mathrm{Iso}(\Gamma, \Lambda)$.

We say that two countable groups Γ and Λ are ***virtually isomorphic*** if there exists finite index subgroups $\Gamma_0 \leq \Gamma$ and $\Lambda_0 \leq \Lambda$ which are isomorphic. We say that two p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are ***virtually measurably isomorphic*** if there exist finite index subgroups $\Gamma_0 \leq \Gamma$ and $\Lambda_0 \leq \Lambda$, as well as a Γ_0-invariant subset $X_0 \subseteq X$ of positive measure and a Λ_0-invariant subset $Y_0 \subseteq Y$ of positive measure, such that the p.m.p. actions $\Gamma_0 \curvearrowright (X_0, \mu_{X_0})$ and $\Lambda_0 \curvearrowright (Y_0, \nu_{Y_0})$ are measurably isomorphic.

We prove a rigidity result for isometric orbit equivalence. The strategy of the proof is modeled on the proof of orbit equivalence rigidity phenomena due to Furman [Fur99].

THEOREM 5.4.1. — *Let $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ be two finitely generated groups, such that the Cayley graphs (Γ, S_Γ) and (Λ, S_Λ) are isometric. Let $\Gamma \curvearrowright^\alpha (X, \mu)$ and $\Lambda \curvearrowright^\beta (Y, \nu)$ be two p.m.p. essentially free actions that are isometric orbit equivalent. Assume that* $\mathrm{Iso}(\Gamma)$*, equivalently* $\mathrm{Iso}(\Lambda)$*, is countable. Then Γ and Λ are virtually isomorphic groups and the p.m.p. actions α and β are virtually measurably isomorphic. If in addition, every finite index subgroup of Γ acts ergodically on (X, μ), then Γ and Λ are isomorphic and α and β are measurably isomorphic.*

Proof. Thanks to Lemma 5.2.11, we fix a length-preserving orbit equivalence $\Phi : (X,\mu) \to (Y,\nu)$ between α and β. Let $\sigma : X \times \Gamma \to \Lambda$ and $\tau : \Lambda \times Y \to \Gamma$ be the orbit equivalence cocycles associated with Φ. For $x \in X$ and $y \in Y$, let $\sigma_x : \Gamma \to \Lambda$ and $\tau_y : \Lambda \to \Gamma$ be the maps defined by $\sigma_x(\gamma) := \sigma(\gamma^{-1}, x)^{-1}$ and $\tau_y(\lambda) := \tau(\lambda^{-1}, y)^{-1}$. By Remark 5.2.10, we know that $\sigma_x \in \mathrm{Iso}_1(\Gamma,\Lambda)$ and $\tau_y \in \mathrm{Iso}_1(\Lambda,\Gamma)$ for μ-almost every $x \in X$ and ν-almost every $y \in Y$. Moreover, the map $x \mapsto \sigma_x$ is Γ-invariant and the map $y \mapsto \tau_y$ is Λ-invariant. Since $\mathrm{Iso}(\Gamma)$ is countable, the space $\mathrm{Iso}(\Gamma,\Lambda)$ is countable and thus the compact subset $\mathrm{Iso}_1(\Gamma,\Lambda)$ is finite. Thus there is $f_0 \in \mathrm{Iso}_1(\Gamma,\Lambda)$ such that the set

$$X_0 := \{x \in X \colon \sigma_x = f_0\}$$

satisfies $\mu(X_0) > 0$. We define

$$\Gamma_0 := \{\gamma \in \Gamma \colon \gamma \cdot f_0 = f_0\}.$$

This is a finite index subgroup of Γ, which leaves X_0 invariant. Indeed, for all $\gamma \in \Gamma_0$ and $x \in X_0$, we have $\sigma_{\alpha(\gamma)x} = \gamma \cdot \sigma_x = \gamma \cdot f_0 = f_0$. Let $g_0 \in \mathrm{Iso}_1(\Lambda,\Gamma)$ be the inverse of f_0 and let

$$Y_0 := \{y \in Y \colon \tau_y = g_0\}.$$

We define

$$\Lambda_0 := \{\lambda \in \Lambda \colon \lambda \cdot g_0 = g_0\}.$$

This is a finite index subgroup of Λ, which leaves Y_0 invariant. We know by Lemma 5.2.7, that for μ-almost every $x \in X$, for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$,

$$\tau(\sigma(\gamma,x),\Phi(x)) = \gamma \text{ and } \sigma(\tau(\lambda,\Phi(x)),x) = \lambda.$$

Thus, the maps σ_x and $\tau_{\Phi(x)}$ are inverses of one another, that is $\sigma_x \circ \tau_{\Phi(x)} = \mathrm{id}_\Lambda$ and $\tau_{\Phi(x)} \circ \sigma_x = \mathrm{id}_\Gamma$. Therefore, we have $\Phi(X_0) = Y_0$. Thus the map Φ induces an orbit equivalence (still denoted by) $\Phi : (X_0, \mu_{X_0}) \to (Y_0, \mu_{Y_0})$ between the p.m.p. actions $\Gamma_0 \curvearrowright (X_0, \mu_{X_0})$ and $\Lambda_0 \curvearrowright (Y_0, \nu_{Y_0})$. The orbit equivalence cocycles $\sigma_0 : \Gamma_0 \times X_0 \to Y_0$ and $\tau_0 : \Lambda_0 \times Y_0 \to \Gamma_0$ associated with this orbit equivalence are independent of the space variable. Indeed for all $\gamma \in \Gamma_0$, $\lambda \in \Lambda_0$,

$$\sigma_0(\gamma,x) = \sigma(\gamma,x) = f_0(\gamma^{-1})^{-1} \text{ for } \mu\text{-almost every } x \in X_0,$$
$$\tau_0(\lambda,y) = \tau(\lambda,y) = g_0(\lambda^{-1})^{-1} \text{ for } \nu\text{-almost every } y \in Y_0.$$

Thus, the groups Γ_0 and Λ_0 are isomorphic and the p.m.p. actions $\Gamma_0 \curvearrowright (X_0, \mu_{X_0})$ and $\Lambda_0 \curvearrowright (Y_0, \nu_{Y_0})$ are measurably isomorphic. This proves that the groups Γ and Λ are virtually isomorphic and that the actions α and β are virtually isomorphic.

If in addition, every finite index subgroup of Γ acts ergodically on (X,μ), then $\mu(X_0) = 1$. Since Φ is a p.m.p. isomorphism, we deduce that $\nu(Y_0) = 1$. In order to prove that α and β are measurably isomorphic, it remains to show that $\Gamma_0 = \Gamma$

and $\Lambda_0 = \Lambda$. Up to null set, one can assume that X_0 is a Γ-invariant full measure set. Thus, for all $\gamma \in \Gamma$ and $x \in X_0$, we have $\gamma \cdot f_0 = \gamma \cdot \sigma_x = \sigma_{\alpha(\gamma)x} = f_0$. Thus $\Gamma_0 = \Gamma$. One proves similarly that $\Lambda_0 = \Lambda$. We therefore conclude that Γ and Λ are isomorphic and that the p.m.p. actions α and β are measurably isomorphic. □

Weakly mixing actions are examples of p.m.p. actions for which every finite index subgroup acts ergodically. Concrete examples of weakly mixing actions are Bernoulli shifts. Therefore, we have the following result.

COROLLARY 5.4.2. — *Let $\Gamma = \langle S_\Gamma \rangle$ be a finitely generated group. Assume that* $\mathrm{Iso}_1(\Gamma)$ *is finite. Let (A, κ) be a probability space. Any p.m.p. action $\Lambda \curvearrowright^{\beta} (Y, \nu)$ of some finitely generated group $\Lambda = \langle S_\Lambda \rangle$ which is isometric orbit equivalent to the Bernoulli shift $\Gamma \curvearrowright (A, \kappa)^\Gamma$ is actually measurably isomorphic to it and Λ is isomorphic to Γ.*

EXAMPLE 5.4.3. — Here are examples of finitely generated groups $\Gamma = \langle S \rangle$ such that $\mathrm{Iso}_1(\Gamma, S)$ is finite. Leemann and de la Salle proved that any finitely generated group Γ admits a finite generating system S such that $\mathrm{Iso}_1(\Gamma, S)$ is finite [LdlS21]. For some finitely generated groups Γ, the set $\mathrm{Iso}_1(\Gamma, S)$ is finite for *all* finite generating systems S. For instance, let Γ be a finitely generated, torsion free group which is either of polynomial growth, or a lattice in a simple Lie group G (in case $G \simeq \mathrm{SL}_2(\mathbb{R})$, assume that Γ is uniform in G). Then for any finite generating system S of Γ, the space $\mathrm{Iso}_1(\Gamma, S)$ is finite. These facts are due to Trofimov for groups with polynomial growth [Tro85] and to Furman for lattice in simple Lie groups [Fur01]. We refer to [dlST19, Sec. 6] for a discussion about these results. Other examples of such groups are obtained by Guirardel and Horbez. They proved the following result: if Γ is a torsion-free finite index subgroup of the group of outer automorphisms of the free group $\mathbf{F}_d$ on $d \geq 3$ generators, then for any finite generating systems S of Γ, the space $\mathrm{Iso}_1(\Gamma, S)$ is finite [GH21].

REMARK 5.4.4. — The result of COROLLARY 5.4.2 is false if $\mathrm{Iso}_1(\Gamma, \Lambda)$ is infinite. For instance, let Λ_1 and Λ_2 be two non-isomorphic finite groups. Let $\Gamma = \langle S_\Gamma \rangle$ be an infinite, finitely generated group and let $\Gamma_i := \Lambda_i * \Gamma$ for $i \in \{1, 2\}$, equipped with the finite generating set $\Lambda_i \cup S_\Gamma$. By a co-induction argument, one can show that the Bernoulli shifts $\Gamma_1 \curvearrowright ([0,1], \mathrm{Leb})^{\Gamma_1}$ and $\Gamma_2 \curvearrowright ([0,1], \mathrm{Leb})^{\Gamma_2}$ are isometric orbit equivalent, see for instance [Bow11, Thm. 1.1]. However, one can choose Λ_1, Λ_2 and Γ so that Γ_1 and Γ_2 are not isomorphic.

QUESTION 5.4.5. — Let $\mathbf{F}_d$ be the free group on $d \geq 2$ generators $x_1, \ldots, x_d$ and let $S_\Gamma := \{x_1^{\pm 1}, \ldots, x_d^{\pm 1}\}$. Let $\Lambda = \langle S_\Lambda \rangle$ be a finitely generated group. Consider a p.m.p. essentially free action $\Lambda \curvearrowright (Y, \nu)$ which is isometric orbit equivalent to the Bernoulli shift $\mathbf{F}_d \curvearrowright ([0,1], \mathrm{Leb})^{\mathbf{F}_d}$. Does this imply that Λ is isomorphic to $\mathbf{F}_d$ and that the p.m.p. actions are measurably isomorphic?

This question is related to the following problem. The measurable edge chromatic number of the graphing given by the Bernoulli shift $\mathbf{F}_d \curvearrowright ([0,1], \mathrm{Leb})^{\mathbf{F}_d}$ is known to be either $2d$ or $2d+1$ [CLP16]. However, its exact value is unknown [KM16, Prob. 5.39]. As explained in Example 5.2.5, the measurable edge chromatic number is equal to $2d$ if and only if the Bernoulli shift $\mathbf{F}_d \curvearrowright ([0,1], \mathrm{Leb})^{\mathbf{F}_d}$ is isometric orbit equivalent to some p.m.p. essentially free action of the group

$$\langle a_1, \ldots, a_{2d} \mid a_1^2 = \cdots = a_{2d}^2 = 1 \rangle.$$

Remark 5.4.6. — Let $\Gamma = \langle S_\Gamma \rangle$ be either a finitely generated amenable group, or the free group $\mathbf{F}_d$ on $d \geq 2$ generators with S_Γ any free generating system. Then nontrivial Bernoulli shifts over Γ are all orbit equivalent. This is a consequence of Ornstein and Weiss' theorem [OW80] if Γ is amenable and a consequence of Bowen's theorem [Bow11] if Γ is a free group. The picture is very different when it comes to isometric orbit equivalence. Let (A, κ_A) and (B, κ_B) be two nontrivial probability spaces. Then the Bernoulli shifts $\Gamma \curvearrowright (A, \kappa_A)^\Gamma$ and $\Gamma \curvearrowright (B, \kappa_B)^\Gamma$ are isometric orbit equivalent if and only if they are measurably isomorphic. This is a consequence of the fact that there is a notion of entropy which distinguishes Bernoulli shifts up to measure-isomorphism and which is preserved under bounded orbit equivalence. For amenable groups, Kolmogorov-Sinai entropy is preserved under bounded orbit equivalence [Aus16], whereas for free groups, the f-invariant is preserved under bounded orbit equivalence [BL22].

5.5 Construction of isometric orbit equivalent actions

5.5.1 The general construction

Given a finitely generated group $\Gamma = \langle S_\Gamma \rangle$ and a finite index subgroup $\Lambda \leq \Gamma$, we explain in this section a construction of p.m.p. isometric orbit equivalent actions of Γ. In the sequel, we will use the following notation. For all $g \in \mathrm{Iso}(\Gamma)$ and all $f \in \mathrm{Iso}_1(\Gamma)$, we denote by $g \cdot f$ the element of $\mathrm{Iso}_1(\Gamma)$ defined by

$$g \cdot f : \delta \mapsto f(g^{-1}(1_\Gamma))^{-1} f(g^{-1}(\delta)).$$

We explained in Section 5.3 why the action $(g, f) \mapsto g \cdot f$ is measurably isomorphic to the action $\mathrm{Iso}(\Gamma) \curvearrowright \mathrm{Iso}(\Gamma)/\Gamma$. Beware here, that $\mathrm{Iso}(\Gamma)/\Gamma$ means the quotient of $\mathrm{Iso}(\Gamma)$ under the Γ-action on $\mathrm{Iso}(\Gamma)$ by postcomposition.

Lemma 5.5.1. — *Let $\Gamma = \langle S_\Gamma \rangle$ be a finitely generated group. Let $\Lambda \leq \Gamma$ be a finite index subgroup. Let m_Λ be the Haar probability measure on $\mathrm{Iso}(\Gamma)/\Lambda$. Let μ be the Haar probability measure on $\mathrm{Iso}_1(\Gamma)$ and u be the uniform probability measure on Γ/Λ. Then the p.m.p. action $\mathrm{Iso}(\Gamma) \curvearrowright (\mathrm{Iso}(\Gamma)/\Lambda, m_\Lambda)$ is measurably isomorphic to the p.m.p. action $\mathrm{Iso}(\Gamma) \curvearrowright (\mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda, \mu \otimes u)$ defined by*

$$g(f, q) := (g \cdot f, f(g^{-1}(1_\Gamma))^{-1} q).$$

Proof. We fix a section $s : \Gamma/\Lambda \to \Gamma$. For all $f \in \mathrm{Iso}_1(\Gamma)$ and $q \in \Gamma/\Lambda$, we let

$\psi_{f,q} \in \mathrm{Iso}(\Gamma)$ be the map defined by $\psi_{f,q}(\gamma) := s(q)f(\gamma)$. We define the subset $D \subseteq \mathrm{Iso}(\Gamma)$ by

$$D := \{\psi_{f,q} \colon f \in \mathrm{Iso}_1(\Gamma), q \in \Gamma/\Lambda\}.$$

Claim. — The set D is a fundamental domain for $\mathrm{Iso}(\Gamma)/\Lambda$, that is, for all $f \in \mathrm{Iso}(\Gamma)$, there exists a unique $\lambda \in \Lambda$ such that the map $\gamma \mapsto \lambda^{-1}f(\gamma)$ belongs to D.

Proof of the claim. Let $f \in \mathrm{Iso}(\Gamma)$. Let $\delta = f(1_\Gamma)$. Then there exists $\lambda \in \Lambda$ and $q \in \Gamma/\Lambda$ such that $\delta = \lambda s(q)$. Observe that $\gamma \mapsto \delta^{-1}f(\gamma)$ belongs to $\mathrm{Iso}_1(\Gamma)$. Then the map $\gamma \mapsto \lambda^{-1}f(\gamma)$ belongs to D, because it coincides with $\psi_{\delta^{-1}f,q}$. $\square_{\text{claim}}$

This yields an action $\mathrm{Iso}(\Gamma) \curvearrowright D$, defined for all $g \in \mathrm{Iso}(\Gamma)$ and $\psi \in D$ by

$$(g, \psi) \mapsto (\gamma \mapsto \lambda^{-1}\psi(g^{-1}(\gamma))),$$

where λ is the unique element of Λ such that the map $\gamma \mapsto \lambda^{-1}\psi(g^{-1}(\gamma))$ belongs to D. If we denote by μ_D the Haar measure on $\mathrm{Iso}(\Gamma)$ which satisfies $\mu_D(D) = 1$, then the action $\mathrm{Iso}(\Gamma) \curvearrowright D$ preserves μ_D and is measurably isomorphic to the p.m.p. action $\mathrm{Iso}(\Gamma) \curvearrowright (\mathrm{Iso}(\Gamma)/\Lambda, m_\Lambda)$.

Let us prove that the action $\mathrm{Iso}(\Gamma) \curvearrowright (D, \mu_D)$ is measurably isomorphic to the action $\mathrm{Iso}(\Gamma) \curvearrowright (\mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda, \mu \otimes u)$ defined for $g \in \mathrm{Iso}(\Gamma)$ and $(f, q) \in \mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda$ by

$$g(f, q) := (g \cdot f, f(g^{-1}(1_\Gamma))^{-1}q).$$

Let us define a map $\Phi : \mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda \to D$ by the formula

$$\Phi(f, q) := \gamma \mapsto \psi_{f,q}(\gamma).$$

This is a bijection, as it is a surjective map by definition of D and it is straightforward to check that it is an injective map. Moreover, by definition of μ and μ_D, we get $\Phi_*(\mu \otimes u) = \mu_D$. It is a straightforward computation to check that Φ intertwines the actions $\mathrm{Iso}(\Gamma) \curvearrowright \mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda$ and $\mathrm{Iso}(\Gamma) \curvearrowright D$, which finishes the proof of the lemma. $\square$

Theorem 5.5.2. — *Let $\Gamma = \langle S_\Gamma \rangle$ be a finitely generated group. Let $\Lambda \leq \Gamma$ be a finite index subgroup. Let μ_Γ be the Haar probability measure on $\mathrm{Iso}(\Gamma)/\Gamma$ and u be the uniform probability measure on Γ/Λ. Then the p.m.p. action $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma)/\Lambda, m_\Lambda)$ is isometric orbit equivalent to the diagonal action $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma)/\Gamma \times \Gamma/\Lambda, m_\Gamma \otimes u)$.*

Proof. In this proof, we will denote by $*$ the p.m.p. action $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma)/\Lambda, m_\Lambda)$ and by $\star$ the diagonal action $\Gamma \curvearrowright (\mathrm{Iso}(\Gamma)/\Gamma \times \Gamma/\Lambda, m_\Gamma \otimes u)$.

Let u be the uniform probability measure on Γ/Λ. We know by Lemma 5.5.1 that $*$ is measurably isomorphic to the action $\Gamma \curvearrowright (\mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda, \mu \otimes u)$, still denoted by $*$ and given by

$$\gamma * (f, q) := (\gamma \cdot f, f(\gamma^{-1})^{-1}q).$$

Let $\Phi : \mathrm{Iso}_1(\Gamma) \to \mathrm{Iso}_1(\Gamma)$ be the inverse map, defined by $f \circ \Phi(f) = \Phi(f) \circ f = \mathrm{id}_\Gamma$. This is a p.m.p. isomorphism by LEMMA 5.3.1. Morever, we have $\Phi(\gamma \cdot f) = f(\gamma^{-1})^{-1}\Phi(f)$. Therefore, if $\Psi : \mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda \to \mathrm{Iso}_1(\Gamma) \times \Gamma/\Lambda$ is defined by $\Psi(f,q) = (\Phi(f),q)$, then Ψ is a p.m.p. isomorphism which satisfies

$$\Psi(\gamma * (f,q)) = f(\gamma^{-1})^{-1} \star \Psi(f,q),$$
$$\Psi(f(\gamma^{-1})^{-1} * \Psi(f,q)) = \gamma \star (f,q).$$

By LEMMA 5.3.1, the map $(\gamma, f) \mapsto f(\gamma^{-1})^{-1}$ is a length-preserving cocycle, thus we obtain by LEMMA 5.2.6 that the p.m.p. actions $*$ and $\star$ are isometric orbit equivalent, which concludes the proof. □

5.5.2 The case of the free group

In this section, we characterize the subgroups $\Lambda \leq \mathbf{F}_d$ for which the p.m.p. action $\mathbf{F}_d \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ is mixing. Here, m_Λ denotes the Haar probability measure. Before this, we need to give some properties of mixing actions of locally compact groups.

Let G be a locally compact, non-compact, second countable group. A function $f : G \to \mathbb{C}$ vanishes at infinity if for all $\varepsilon > 0$, the set $\{g \in G \colon |f(g)| \geq \varepsilon\}$ is compact. A p.m.p. action $G \curvearrowright (X,\mu)$ is ***mixing*** if for all measurable subsets $A, B \subseteq X$, the function

$$g \mapsto |\mu(gA \cap B) - \mu(A)\mu(B)|$$

vanishes at infinity. With this definition, the proof of the following two lemmas is straightforward.

LEMMA 5.5.3. — *Let $G \curvearrowright (X,\mu)$ be a p.m.p. mixing action. Then for any closed subgroup $H \leq G$, the action $H \curvearrowright (X,\mu)$ is mixing.*

LEMMA 5.5.4. — *Let $H \leq G$ be a finite index closed subgroup. Let $G \curvearrowright (X,\mu)$ be a p.m.p. action. If $H \curvearrowright (X,\mu)$ is mixing, then so is $G \curvearrowright (X,\mu)$.*

In this section, we fix an integer $d \geq 2$ and we let $\mathbf{F}_d$ be the free group on d generators $x_1, \dots, x_d$ with the generating system $S := \{a_1^{\pm 1}, \dots, a_d^{\pm 1}\}$. The ***even subgroup*** of $\mathbf{F}_d$ is the normal subgroup of index two, denoted by $\mathbf{F}_d^{ev}$, consisting of all elements $\gamma \in \mathbf{F}_d$ such that $|\gamma|_S$ is even. The ***even subgroup*** of $\mathrm{Iso}(\mathbf{F}_d)$ is the closed, normal subgroup of index two of $\mathrm{Iso}(\mathbf{F}_d)$, denoted by $\mathrm{Iso}^{ev}(\mathbf{F}_d)$ and defined by

$$\mathrm{Iso}^{ev}(\mathbf{F}_d) := \{f \in \mathrm{Iso}(\mathbf{F}_d) \colon f(\mathbf{F}_d^{ev}) = \mathbf{F}_d^{ev}\}.$$

This group satisfies the Howe-Moore property.

THEOREM 5.5.5 (Lubotzky-Mozes [LM92], Pemantle [Pem92]). — *Any p.m.p. ergodic action of* $\mathrm{Iso}^{ev}(\mathbf{F}_d)$ *on a standard probability space is mixing.*

The following lemma characterizes the finite index subgroups of the even subgroup $\mathbf{F}_d^{ev}$.

LEMMA 5.5.6. — *Let $\Lambda \leq \mathbf{F}_d$ be a finite index subgroup. Then the following are equivalent.*

(i) Λ is not contained in $\mathbf{F}_d^{ev}$.

(ii) For all $\gamma \in \mathbf{F}_d$, the index $[\mathbf{F}_d^{ev} : \gamma\Lambda\gamma^{-1} \cap \mathbf{F}_d^{ev}]$ is equal to $[\mathbf{F}_d : \Lambda]$.

(iii) The action $\mathbf{F}_d^{ev} \curvearrowright \mathbf{F}_d/\Lambda$ is transitive.

(iv) There is no bipartition $\mathbf{F}_d/\Lambda = U \sqcup V$ such that for all $s \in S$, $sU = V$.

Proof. Let us prove (i)⇒(ii). Since Λ is not contained in $\mathbf{F}_d^{ev}$, there is $\lambda \in \Lambda \setminus \mathbf{F}_d^{ev}$. Let $\gamma \in \mathbf{F}_d$. Since $\mathbf{F}_d^{ev}$ is normal in $\mathbf{F}_d$, the element $\gamma\lambda\gamma^{-1}$ is not in $\mathbf{F}_d^{ev}$. Since $\mathbf{F}_d^{ev}$ has index two in $\mathbf{F}_d$, we deduce that $\gamma\Lambda\gamma^{-1}\mathbf{F}_d^{ev} = \mathbf{F}_d$. Thus, we obtain

$$[\mathbf{F}_d : \gamma\Lambda\gamma^{-1} \cap \mathbf{F}_d^{ev}] = [\mathbf{F}_d : \gamma\Lambda\gamma^{-1}][\mathbf{F}_d : \mathbf{F}_d^{ev}].$$

We obtain (ii) by dividing both sides of the equality by $[\mathbf{F}_d : \mathbf{F}_d^{ev}]$.

We now prove (ii)⇒(iii). Observe that for all $\gamma \in \mathbf{F}_d$, the group $\gamma\Lambda\gamma^{-1} \cap \mathbf{F}_d^{ev}$ is equal to the stabilizer of the coset $\gamma\Lambda$ under the action $\mathbf{F}_d^{ev} \curvearrowright \mathbf{F}_d/\Lambda$. Thus, the index $[\mathbf{F}_d^{ev} : \gamma\Lambda\gamma^{-1} \cap \mathbf{F}_d^{ev}]$ is equal to the cardinal of the orbit of the coset $\gamma\Lambda$ under the action $\mathbf{F}_d^{ev} \curvearrowright \mathbf{F}_d/\Lambda$. If we assume (ii), then we get that the cardinal of each orbit of the action $\mathbf{F}_d^{ev} \curvearrowright \mathbf{F}_d/\Lambda$ is equal to $[\mathbf{F}_d : \Lambda]$. This exactly means that the action is transitive.

Let us prove the contrapositive of (iii)⇒(iv). Assume that there exists a partition $\mathbf{F}_d/\Lambda = U \sqcup V$ such that for all $s \in S$, $sU = V$. Then we also have $sV = U$ for all $s \in S$. By induction, we get that $\gamma U = U$ and $\gamma V = V$ for all $\gamma \in \mathbf{F}_d^{ev}$. Thus, the action $\mathbf{F}_d^{ev} \curvearrowright \mathbf{F}_d/\Lambda$ is not transitive.

Finally, let us prove the contrapositive of (iv)⇒(i). Assume that $\Lambda \leq \mathbf{F}_d^{ev}$. Let $\gamma \in \mathbf{F}_d \setminus \mathbf{F}_d^{ev}$. Let $n := [\mathbf{F}_d^{ev} : \Lambda]$. Then there are $\gamma_1, \dots, \gamma_{2n} \in \mathbf{F}_d$ such that

$$\mathbf{F}_d^{ev} = \bigsqcup_{i=1}^{n} \gamma_i\Lambda \text{ and } \mathbf{F}_d \setminus \mathbf{F}_d^{ev} = \bigsqcup_{i=n+1}^{2n} \gamma_i\Lambda.$$

But for all $s \in S$, we have $s\mathbf{F}_d^{ev} = \mathbf{F}_d \setminus \mathbf{F}_d^{ev}$. Thus, this decomposition of $\mathbf{F}_d^{ev}$ and $\mathbf{F}_d \setminus \mathbf{F}_d^{ev}$ into a disjoint union of Λ-coset yields a bipartition $\mathbf{F}_d/\Lambda = U \sqcup V$ such that for all $s \in S, sU = V$. □

We can now characterize the subgroups $\Lambda \leq \mathbf{F}_d$ for which the p.m.p. action $\mathbf{F}_d \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ is mixing.

THEOREM 5.5.7. — *Let $\Lambda \leq \mathbf{F}_d$ be a finite index subgroup. Let m_Λ be the Haar probability measure on* $\mathrm{Iso}(\mathbf{F}_d)/\Lambda$. *Then the following are equivalent.*

(i) The p.m.p. action $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ *is mixing.*

(ii) The p.m.p. action $\mathbf{F}_d \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ *is mixing.*

(iii) Λ *is not contained in the even subgroup* $\mathbf{F}_d^{ev}$*.*

Proof. In this proof, we denote by μ the Haar probability measure on $\mathrm{Iso}_1(\mathbf{F}_d)$. Let $\cdot$ be the p.m.p. action $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}_1(\mathbf{F}_d), \mu)$ given by

$$(g \cdot f) \coloneqq f(g^{-1}(1_{\mathbf{F}_2}))^{-1} f \circ g^{-1}.$$

If u denotes the uniform probability measure on $\mathbf{F}_d/\Lambda$, then by Lemma 5.5.1, the p.m.p. action $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ is measurably isomorphic to the p.m.p. action $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda, \mu \otimes u)$ given by

$$g(f, q) \coloneqq (g \cdot f, f(g^{-1}(1_\Gamma))^{-1} q).$$

The proof of (i)⇒(ii) is a direct consequence of Lemma 5.5.3.

Let us prove (ii)⇒(iii). We prove the contrapositive. Assume that Λ is contained in $\mathbf{F}_d^{ev}$. Then by Lemma 5.5.6, there is a bipartition $\mathbf{F}_d/\Lambda = U \sqcup V$ such that for all $s \in S$, we have $sU = V$. By induction, we get that $\gamma U = U$ and $\gamma V = V$ for all $\gamma \in \mathbf{F}_d^{ev}$. For all $f \in \mathrm{Iso}_1(\mathbf{F}_d)$ and all $\gamma \in \mathbf{F}_d$, we have $|f(\gamma^{-1})^{-1}|_S = |\gamma|_S$. Thus we obtain that the sets $\mathrm{Iso}_1(\mathbf{F}_d) \times U$ and $\mathrm{Iso}_1(\mathbf{F}_d) \times V$ are invariant by $\mathbf{F}_d^{ev}$. Thus, the p.m.p. action $\mathbf{F}_d \curvearrowright (\mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda, \mu \otimes u)$ is not mixing, which is equivalent to saying that $\mathbf{F}_d \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ is not mixing.

We now prove (iii)⇒(i). We prove the contrapositive. Assume that $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ is not mixing. That is, the action $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda, \mu \otimes u)$ is not mixing. By Lemma 5.5.4, the p.m.p. action $\mathrm{Iso}^{ev}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda, \mu \otimes u)$ is not mixing and thus not ergodic by Theorem 5.5.5. Let $A \subseteq \mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda$ be a measurable subset of measure $1/2$ which is $\mathrm{Iso}^{ev}(\mathbf{F}_d)$-invariant. We define the following set

$$U \coloneqq \{q \in \mathbf{F}_2/\Lambda \colon \mu \otimes u(A \cap (\mathrm{Iso}_1(\mathbf{F}_d) \times \{q\}) > 0\}.$$

Observe that the subgroup $\mathrm{Iso}_1(\mathbf{F}_d)$ is contained in $\mathrm{Iso}^{ev}(\mathbf{F}_d)$. Moreover, for all $g \in \mathrm{Iso}_1(\mathbf{F}_d)$ and for all $(f, q) \in \mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda$, we have

$$g(f, q) = (f \circ g^{-1}, q).$$

Thus, the group $\mathrm{Iso}_1(\mathbf{F}_d)$ acts transitively on each $\mathrm{Iso}_1(\mathbf{F}_d) \times \{q\}$. Since A is $\mathrm{Iso}^{ev}(\mathbf{F}_d)$-invariant, we obtain that $A = \mathrm{Iso}_1(\mathbf{F}_d) \times U$ up to a conull set. We claim that the set U and its complement V form a partition of $\mathbf{F}_d/\Lambda$ such that for all $s \in S, sU = V$. Indeed, for $s \in S$, the facts that $s \notin \mathrm{Iso}^{ev}(\mathbf{F}_d)$ and that $\mathrm{Iso}^{ev}(\mathbf{F}_d)$ is normal in $\mathrm{Iso}(\Gamma)$ imply that sA is $\mathrm{Iso}^{ev}(\mathbf{F}_d)$-invariant. But sA cannot be equal to A, because otherwise A would be a $\mathrm{Iso}(\mathbf{F}_d)$-invariant set of measure $1/2$, contradicting the ergodicity of $\mathrm{Iso}(\mathbf{F}_d) \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$. Moreover, the intersection $sA \cap A$ cannot be of positive measure, otherwise it would be a $\mathrm{Iso}^{ev}(\mathbf{F}_d)$-invariant set of measure $< 1/2$, which is impossible since $\mathrm{Iso}^{ev}(\mathbf{F}_d)$ has index two in $\mathrm{Iso}(\mathbf{F}_d)$. Thus, up to conull set, we have $sA = (\mathrm{Iso}_1(\mathbf{F}_d) \times \mathbf{F}_d/\Lambda) \setminus A$. Therefore $sU = V$. By LEMMA 5.5.6, we obtain that Λ is not contained in $\mathbf{F}_d^{ev}$, which concludes the proof. □

COROLLARY 5.5.8. — *Let $\Lambda \leq \mathbf{F}_d$ be a finite index subgroup which is not included in the even subgroup $\mathbf{F}_d^{ev}$. Then the p.m.p. action $\mathbf{F}_d \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\Lambda, m_\Lambda)$ and the diagonal action $\mathbf{F}_d \curvearrowright (\mathrm{Iso}(\mathbf{F}_d)/\mathbf{F}_d \times \mathbf{F}_d/\Lambda, m_{\mathbf{F}_d} \otimes u)$ are isometric orbit equivalent but the former is mixing, whereas the latter is not.*

5.5.3 A concrete isometric orbit equivalence for $\mathbf{F}_2$

We finish this section with a concrete example of two p.m.p. ergodic actions of $\mathbf{F}_2$ that are isometric orbit equivalent but not measurably isomorphic.

Let $\mathbf{F}_2$ be the free group on two generators a and b and let $S = \{a^{\pm 1}, b^{\pm 1}\}$ be the standard generating system. Let $|\cdot|_S$ be the word length associated with S. The Cayley graph $(\mathbf{F}_2, S)$ is isomorphic to the 4-regular tree. Let $\mathrm{Iso}(\mathbf{F}_2)$ be the group of bijective isometries of $(\mathbf{F}_2, S)$, that is, the group of all bijections $f : \mathbf{F}_2 \to \mathbf{F}_2$ such that for all $\gamma, \delta \in \mathbf{F}_2$,

$$|f(\gamma)^{-1} f(\delta)|_S = |\gamma^{-1}\delta|_S.$$

Let $\mathcal{C}$ be the space of proper colorings with five colors on the vertex set of the Cayley graph $(\mathbf{F}_2, S)$. That is, an element of $\mathcal{C}$ is a map $col : \mathbf{F}_2 \to \{1,2,3,4,5\}$ such that for all $\gamma \in \mathbf{F}_2$ and for all $s \in S$, we have $col(\gamma) \neq col(\gamma s)$. The set $\mathcal{C}$ is a closed, thus compact, subspace of $\{1,2,3,4,5\}^{\mathbf{F}_2}$. The group $\mathrm{Iso}(\mathbf{F}_2)$ acts on $\mathcal{C}$ and we denote by $*$ the action, which is defined as follows: for all $f \in \mathrm{Iso}(\mathbf{F}_2)$ and $c \in \mathcal{C}$, the coloring $f * col$ is given by $\gamma \mapsto col(f^{-1}(\gamma))$. This action is simply transitive and it admits a unique invariant probability measure μ, which can be constructed as follows. First, choose uniformly at random the color of the identity element $1_{\mathbf{F}_2}$. Then, by moving radially outwards $1_{\mathbf{F}_2}$ in the Cayley graph, extend the coloring at each vertex by choosing the color uniformly at random among the admissible ones, independently at each vertex. Thus we get a p.m.p. action $*$ of $\mathbf{F}_2$ on $(\mathcal{C}, \mu)$. Let us explain briefly why this action is mixing. Fix two distinct 5-cycles $A, B \in \mathrm{Sym}(\{1,\dots,5\})$ such that for all $i \in \{1,\dots,5\}$, $A(i) \neq B(i)$. For instance, take

$$A := (1\ 2\ 3\ 4\ 5) \text{ and } B := (1\ 3\ 5\ 2\ 4).$$

This yields a transitive action $\mathbf{F}_2 \curvearrowright \{1,2,3,4,5\}$ by letting the generator a act like A and b act like B. Let Λ be the stabilizer of the point 1. If m_Λ denotes the Haar probability measure on the quotient $\mathrm{Iso}(\mathbf{F}_2)/\Lambda$, then it can be proved that the p.m.p. action $*$ of $\mathbf{F}_2$ on (C_2, μ) is measurably isomorphic to $\mathbf{F}_2 \curvearrowright (\mathrm{Iso}(\mathbf{F}_2)/\Lambda, m_\Lambda)$. By definition, $\mathbf{F}_2 \curvearrowright \{1,\dots,5\}$ is isomorphic to the action $\mathbf{F}_2 \curvearrowright \mathbf{F}_2/\Lambda$. As there is no bipartition of $\{1,\dots,5\}$ whose pieces are exchanged by any element $s \in S$, we get by LEMMA 5.5.6 and THEOREM 5.5.7 that the p.m.p. action $*$ of $\mathbf{F}_2$ on $(\mathcal{C}, \mu)$ is mixing.

Let us construct another p.m.p. action of $\mathbf{F}_2$ on $(\mathcal{C},\mu)$, that we denote by $\star$, which is isometric orbit equivalent to $*$, but which is not mixing. We define $\star$ by the action of the generators a and b of $\mathbf{F}_2$. For $col \in \mathcal{C}$, we let

$$a \star col := s * col,$$

where s is the unique element in $\{a^{\pm 1}, b^{\pm 1}\}$ such that $(s * col)(1_{\mathbf{F}_2}) = A(col(1_{\mathbf{F}_2}))$. Such an element s exists because c is a proper vertex coloring. Similarly, we define

$$b \star col := t * col,$$

where t is the unique element in $\{a^{\pm 1}, b^{\pm 1}\}$ such that $(t * col)(1_{\mathbf{F}_2}) = B(col(1_{\mathbf{F}_2}))$.

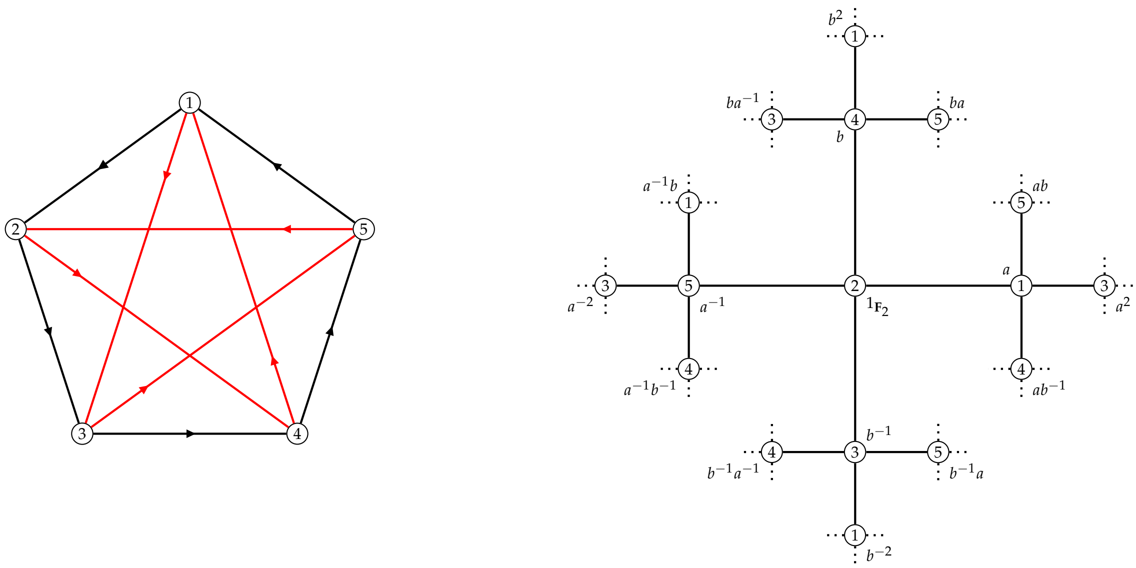


FIGURE 5.1. — On the left, black arrows represent the permutation A, red arrows B. On the right, the portion of an element $c \in \mathcal{C}$, for which $a \star col = b * col$, $a^{-1} \star col = a^{-1} * col$, $b \star col = b^{-1} * col$ and $b^{-1} \star col = a * col$.

This defines a p.m.p. action $\star$ of $\mathbf{F}_2$ on (C,μ). Observe that for each $i \in \{1,\dots,5\}$, the set

$$\{col \in \mathcal{C} \colon col(1_\Gamma) = i\}$$

is invariant by Λ. Thus, the p.m.p. action $\star$ is not mixing. Moreover, by construction, the actions $*$ and $\star$ are isometric orbit equivalent. This yields a concrete illustration of THEOREM 5.5.2. Indeed, it can be showed that:

- the p.m.p. action $*$ is measurably isomorphic to $\mathbf{F}_2 \curvearrowright (\mathrm{Iso}(\mathbf{F}_2)/\Lambda, m_\Lambda)$, where m_Λ is the Haar probability measure on $\mathrm{Iso}(\mathbf{F}_2)/\Lambda$,
- the p.m.p. action $\star$ is measurably isomorphic to the diagonal action $\mathbf{F}_2 \curvearrowright (\mathrm{Iso}(\mathbf{F}_2)/\mathbf{F}_2 \times \mathbf{F}_2/\Lambda, m_{\mathbf{F}_2} \otimes u)$ where $m_{\mathbf{F}_2}$ is the Haar probability measure on $\mathrm{Iso}(\mathbf{F}_2)/\mathbf{F}_2$ and u is the uniform probability measure on $\mathbf{F}_2/\Lambda$.

References

[Aus16] Tim Austin. Behaviour of Entropy Under Bounded and Integrable Orbit Equivalence. *Geom. Funct. Anal.*, 26(6):1483–1525, 2016.

[Bel68] Raisa M. Belinskaya. Partitions of Lebesgue space in trajectories defined by ergodic automorphisms. *Funkts. Anal. Prilozh.*, 2(3):190–199, 1968.

[BL22] Lewis Bowen and Yuqing Frank Lin. Entropy for actions of free groups under bounded orbit-equivalence, 2022.

[BM00a] Marc Burger and Shahar Mozes. Groups acting on trees: From local to global structure. *Publ. Math., Inst. Hautes Étud. Sci.*, 92:113–150, 2000.

[BM00b] Marc Burger and Shahar Mozes. Lattices in product of trees. *Publ. Math., Inst. Hautes Étud. Sci.*, 92:151–194, 2000.

[Bow11] Lewis Bowen. Orbit equivalence, coinduced actions and free products. *Groups Geom. Dyn.*, 5(1):1–15, 2011.

[CLP16] Endre Csóka, Gábor Lippner, and Oleg Pikhurko. Kőnig's line coloring and Vizing's theorems for graphings. *Forum Math. Sigma*, 4:40, 2016. Id/No e27.

[DKLMT20] Thiebout Delabie, Juhani Koivisto, François Le Maître, and Romain Tessera. Quantitative measure equivalence. *arXiv:2002.00719*, 2020.

[dlST19] Mikael de la Salle and Romain Tessera. Characterizing a vertex-transitive graph by a large ball. *J. Topol.*, 12(3):705–743, 2019.

[Dyu00] Anna Dyubina. Instability of the virtual solvability and the property of being virtually torsion-free for quasi-isometric groups. *Int. Math. Res. Not.*, 2000(21):1097–1101, 2000.

[FF86] Adam Fieldsteel and Nathaniel A. Friedman. Restricted orbit changes of ergodic $\mathbb{Z}^d$-actions to achieve mixing and completely positive entropy. *Ergodic Theory Dyn. Syst.*, 6:505–528, 1986.

[Fur99] Alex Furman. Orbit equivalence rigidity. *Ann. Math. (2)*, 150(3):1083–1108, 1999.

[Fur01] Alex Furman. Mostow-Margulis rigidity with locally compact targets. *Geom. Funct. Anal.*, 11(1):30–59, 2001.

[Gab00] Damien Gaboriau. Coût des relations d'équivalence et des groupes. *Invent. Math.*, 139(1):41–98, 2000.

[Gab02] Damien Gaboriau. On orbit equivalence of measure preserving actions. In *Rigidity in dynamics and geometry. Contributions from the programme Ergodic theory, geometric rigidity and number theory, Isaac Newton Institute for the Mathematical Sciences, Cambridge, UK, January 5–July 7, 2000*, pages 167–186. 2002.

[GH21] Vincent Guirardel and Camille Horbez. Measure equivalence rigidity of $\mathrm{Out}(F_N)$, 2021.

[Jos22] Matthieu Joseph. Isometric orbit equivalence for probability-measure preserving actions, 2022.

[KM16] Alexander S. Kechris and Andrew S. Marks. Descriptive graph combinatorics, September 2016. Preliminary version available at `http://www.math.caltech.edu/~kechris/papers/combinatorics16.pdf`.

[LdlS21] Paul-Henry Leemann and Mikael de la Salle. Cayley graphs with few automorphisms: the case of infinite groups, 2021.

[LM92] Alexander Lubotzky and Shahar Mozes. *Asymptotic Properties of Unitary Representations of Tree Automorphisms*, pages 289–298. Springer US, Boston, MA, 1992.

[Lov12] László Lovász. *Large networks and graph limits*, volume 60. Providence, RI: American Mathematical Society (AMS), 2012.

[Mon06] Nicolas Monod. An invitation to bounded cohomology. In *Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume II: Invited lectures*, pages 1183–1211. Zürich: European Mathematical Society (EMS), 2006.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I: The Rohlin lemma. *Bull. Am. Math. Soc., New Ser.*, 2:161–164, 1980.

[Pap95] P. Papasoglu. Homogeneous trees are bilipschitz equivalent. *Geom. Dedicata*, 54(3):301–306, 1995.

[Pem92] Robin Pemantle. Automorphism invariant measures on trees. *Ann. Probab.*, 20(3):1549–1566, 1992.

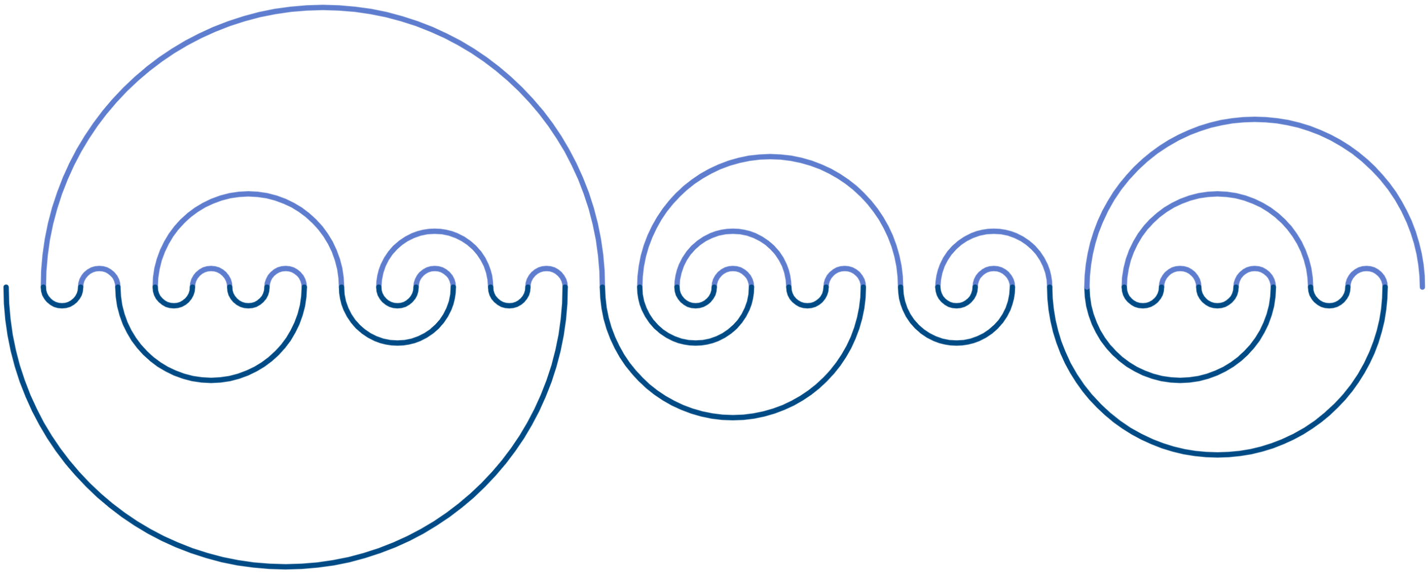
[Sha04] Yehuda Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Mathematica*, 192(2):119–185, 2004.

[Tro85] Vladimir I. Trofimov. Graphs with polynomial growth. *Math. USSR, Sb.*, 51:405–417, 1985.

Index

Résumé : Cette thèse se situe à l'interface entre dynamique topologique et dynamique mesurée. Premièrement, j'y étudie la notion d'action allostérique. Ce sont des actions génériquement libres au sens topologique mais pas génériquement libres au sens de la mesure. Ce comportement étonnant met en valeur les nuances entre sous-groupes aléatoires invariants et sous-groupes uniformément récurrents. Un second sujet d'étude est l'équivalence orbitale quantitative, qui renforce l'équivalence orbitale. Il s'agit de comprendre comment les structures métriques sur les orbites des actions peuvent être distordues par équivalence orbitale. Une grande partie des travaux de cette thèse gravite autour d'un des théorèmes fondateurs de cette théorie : le théorème de Belinskaya.

Mots clés : Dynamique topologique, dynamique mesurée, sous-groupes aléatoires invariants, allostérie, équivalence orbitale quantitative, théorème de Belinskaya.



Abstract : This PhD thesis lies at the interface between topological dynamics and measurable dynamics. First, I study the notion of allosteric actions. These actions are generically free in the sense of the topology but not generically free in the sense of the measure. This surprising behavior highlights the differences between invariant random subgroups and uniformly recurrent subgroups. The nascent theory of quantitative orbit equivalence is the second topic of this thesis. This is a strengthening of orbit equivalence, which aims to understand how metric structures on the orbits of the actions can be distorted. A large part of my work gravitates around one of the founding result of this theory: Belinskaya's theorem.

Keywords : Topological dynamics, measurable dynamics, invariant random subgroups, allostery, quantitative orbit equivalence, Belinskaya's theorem