Method of cumulants and mod-Gaussian convergence of the graphon models

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When looking at a sum $S_n = \sum_{i=1}^{n} A_i$ of centered i.i.d. random variables, the fluctuations are universally predicted by the **central limit theorem**

$$\frac{S_n}{\sqrt{n \text{ Var}(A_1)}} \xrightarrow{\text{d}} \mathcal{N}(0, 1).$$

This is not the whole story:

- **large deviations** (Cramér, 1938): $\log (\mathbb{P}[S_n \geq nx]) \simeq - n I(x)$.
- **speed of convergence** (Berry, 1941; Esseen, 1945):

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P} \left[ \frac{S_n}{\sqrt{n \text{ Var}(A_1)}} \leq s \right] - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt \right| \leq \frac{3 \mathbb{E}[|A_1|^3]}{(\text{Var}(A_1))^{3/2} \sqrt{n}}.$$

- **local limit theorem** (Gnedenko, 1948; Stone, 1965): if $A_1$ is non-lattice distributed and $\text{Var}(A_1) = 1$, then

$$\sqrt{n} \ \mathbb{P} [S_n \in (\sqrt{n}x, \sqrt{n}x + h)] \simeq \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} h.$$
Many other sequences of random variables are asymptotically normal: functionals of Markov chains, martingales, etc.

**Idea:** there is a renormalisation theory of random variables that allows one to go beyond the central limit theorem, and to prove in one time the CLT and the other limiting results.

**Definition (Mod-Gaussian convergence)**

A sequence of real random variables \((X_n)_{n \in \mathbb{N}}\) is mod-Gaussian with parameters \(t_n \to +\infty\) and limit \(\psi(z)\) if, locally uniformly on a domain \(D \subset \mathbb{C}\),

\[
\mathbb{E}[e^{zX_n}] e^{-\frac{t_n z^2}{2}} = \psi_n(z) \to \psi(z)
\]

with \(\psi\) continuous on \(D\) and \(\psi(0) = 1\).

For a sum of i.i.d. \(S_n\), one looks at \(X_n = \frac{S_n}{n^{1/3}}\); \(t_n = n^{1/3} \text{Var}(A_1)\) and \(\psi(z) = \exp\left(\frac{\mathbb{E}[(A_1)^3]}{6}z^3\right)\).
Example: let \( X_n = \text{Re}(\log \det(I_n - M_n)) \), with \( M_n \sim \text{Haar}(U(n)) \). One has the mod-Gaussian convergence

\[
\mathbb{E}[e^{zX_n}] e^{-\frac{(\log n)z^2}{4}} \to \frac{G(1 + \frac{z}{2})^2}{G(1 + z)}, \quad G = \text{Barnes’ function}.
\]

Later: Markov chains, random graphs, random permutations, etc.

Remark: one can replace the exponent \( \frac{z^2}{2} \) of the Gaussian distribution by the exponent \( \eta(z) \) of any infinitely divisible distribution.

Objectives:

1. Explain the consequences of mod-Gaussian convergence.
2. Describe general conditions which ensure the mod-Gaussian convergence.
3. Prove the mod-Gaussian convergence of a large class of models of random graphs.
Mod-Gaussian convergence and bounds on cumulants
Method of cumulants

If $X$ is a random variable with convergent Laplace transform, its cumulants are:

$$
\kappa^{(r)}(X) = \frac{d^r}{dz^r} \left( \log \mathbb{E}[e^{zX}] \right) \bigg|_{z=0}.
$$

So, $\log \mathbb{E}[e^{zX}] = \sum_{r=1}^{\infty} \frac{\kappa^{(r)}(X)}{r!} z^r$. The first cumulants are

$$
\kappa^{(1)}(X) = \mathbb{E}[X] ; \quad \kappa^{(2)}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{Var}(X) ; \\
\kappa^{(3)}(X) = \mathbb{E}[X^3] - 3 \mathbb{E}[X^2] \mathbb{E}[X] + 2 (\mathbb{E}[X])^3.
$$

The Gaussian distribution $\mathcal{N}(m, \sigma^2)$ is characterized by $\kappa^{(1)}(X) = m$, $\kappa^{(2)}(X) = \sigma^2$, $\kappa^{r\geq 3}(X) = 0$.

**Idea:** characterize similarly the mod-Gaussian convergence of a sequence $(X_n)_{n \in \mathbb{N}}$. 
Definition (Method of cumulants)

A sequence of random variables \((S_n)_{n \in \mathbb{N}}\) satisfies the hypotheses of the method of cumulants with parameters \((D_n, N_n, A)\) if:

(MC1) One has \(N_n \to +\infty\) and \(\frac{D_n}{N_n} \to 0\).

(MC2) The first cumulants satisfy

\[
\kappa^{(1)}(S_n) = 0;
\kappa^{(2)}(S_n) = (\sigma_n)^2 N_n D_n;
\kappa^{(3)}(S_n) = L_n N_n (D_n)^2
\]

with \(\lim_{n \to \infty} (\sigma_n)^2 = \sigma^2 > 0\) and \(\lim_{n \to \infty} L_n = L\).

(MC3) All the cumulants satisfy

\[
|\kappa^{(r)}(S_n)| \leq N_n (2D_n)^{r-1} r^{r-2} A^r.
\]
Mod-Gaussian convergence and its consequences

If \((S_n)_{n \in \mathbb{N}}\) satisfies the hypotheses MC1-MC3, then

\[ X_n = \frac{S_n}{(N_n)^{1/3}(D_n)^{2/3}} \]

is mod-Gaussian convergent, with \(t_n = (\sigma_n)^2 \left( \frac{N_n}{D_n} \right)^{1/3} \) and \(\psi(z) = \exp \left( \frac{Lz^3}{6} \right)\).

Consequences:

1. **Central limit theorem:** if \(Y_n = \frac{S_n}{\sqrt{\operatorname{Var}(S_n)}}\), then \(Y_n \xrightarrow{d} \mathcal{N}(0, 1)\).

2. **Speed of convergence:**

\[
d_{\text{Kol}}(Y_n, \mathcal{N}(0, 1)) \leq \left( \frac{3A}{\sigma_n} \right)^3 \sqrt{\frac{D_n}{N_n}}.
\]

This inequality relies on the general estimate

\[
d_{\text{Kol}}(\mu, \nu) \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{\hat{\mu}(\xi) - \hat{\nu}(\xi)}{\xi} \right| d\xi + \frac{24}{\pi T} \left\| \frac{d\nu(x)}{dx} \right\|_{\infty}.
\]
3. Normality zone and moderate deviations: if \( y \ll \left( \frac{N_n}{D_n} \right)^{1/6} \), then

\[
P[Y_n \geq y] = P[\mathcal{N}(0, 1) \geq y] (1 + o(1)).
\]

If \( 1 \ll y \ll \left( \frac{N_n}{D_n} \right)^{1/4} \), then

\[
P[Y_n \geq y] = \frac{e^{-\frac{y^2}{2}}}{y\sqrt{2\pi}} \exp\left( \frac{Ly^3}{6\sigma^3} \sqrt{\frac{D_n}{N_n}} \right) (1 + o(1)).
\]

This estimate relies on the Berry–Esseen inequality and an argument of change of measure.

4. Local limit theorem: for any exponent \( \epsilon \in (0, \frac{1}{2}) \),

\[
\lim_{n \to \infty} \left( \frac{N_n}{D_n} \right)^{\epsilon} P \left[ Y_n - y \in \left( \frac{D_n}{N_n} \right)^{\epsilon} (a, b) \right] = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} (b - a).
\]

Thus, \( Y_n \) is normal between the two scales \( \left( \frac{N_n}{D_n} \right)^{-1/2} \) and \( \left( \frac{N_n}{D_n} \right)^{1/6} \).
Joint cumulants and dependency graphs
Dependency graphs

Let \( S = \sum_{v \in V} A_v \) be a sum of random variables, and \( G = (V, E) \) a dependency graph for \( (A_v)_{v \in V} \): if \( V_1 \) and \( V_2 \) are two disjoint subsets of \( V \) without edge \( e = \{v_1, v_2\} \) between \( v_1 \in V_1 \) and \( v_2 \in V_2 \), then \( (A_v)_{v \in V_1} \) and \( (A_v)_{v \in V_2} \) are independent.

Example:

\[
\begin{align*}
(A_1, A_2, \ldots, A_5) &\perp (A_6, A_7), \text{ but one has also } (A_1, A_2, A_3) &\perp A_5.
\end{align*}
\]

Parameters of the graph: \( D = \max_{v \in V} (\deg v + 1), \)
\( N = \text{card}(V), \)
\( A = \max_{v \in V} \|A_v\|_\infty. \)
Theorem (Bound on cumulants; Féray–M.–Nikeghbali, 2013)

If $S$ is a sum of random variables with a dependency graph of parameters $(D, N, A)$, then for any $r \geq 1$,

$$|\kappa^{(r)}(S)| \leq N (2D)^{r-1} r^{r-2} A^r.$$ 

Corollary: if $S_n = \sum_{i=1}^{N_n} A_{i,n}$ with the $A_{i,n}$’s bounded by $A$ and a sparse dependency graph of maximal degree $D_n \ll N_n$, then MC3 is satisfied.

The proof of the bound relies on the notion of joint cumulant:

$$\kappa(A_1, A_2, \ldots, A_r) = \left. \frac{d^r}{dz_1 dz_2 \cdots dz_r} \left( \log \mathbb{E}[e^{z_1A_1+z_2A_2+\cdots+z_rA_r}] \right) \right|_{z_1=\cdots=z_r=0}$$

$$= \sum_{\pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_{\ell(\pi)} = [1, r]} (-1)^{\ell(\pi)-1} (\ell(\pi) - 1)! \prod_{i=1}^{\ell(\pi)} \mathbb{E} \left[ \prod_{j \in \pi_i} A_j \right].$$
Properties of joint cumulants

1. For any random variable $X$, $\kappa^{(r)}(X) = \kappa(X, X, \ldots, X)$ ($r$ occurrences).
2. The joint cumulants are multilinear and invariant by permutation.
3. If $\{A_1, A_2, \ldots, A_r\}$ can be split in two independent families, then $\kappa(\{A_1, \ldots, A_r\}) = 0$.

Consider a sum $S = \sum_{v \in V} A_v$ with a dependency graph $G$ of parameters $(D, N, A)$.

$$\kappa^{(r)}(S) = \sum_{v_1, v_2, \ldots, v_r} \kappa(A_{v_1}, A_{v_2}, \ldots, A_{v_r})$$

and the sum can be restricted to families $\{v_1, v_2, \ldots, v_r\}$ such that the induced multigraph $H = G[v_1, v_2, \ldots, v_r]$ is connected. Actually,

$$|\kappa(A_{v_1}, A_{v_2}, \ldots, A_{v_r})| \leq A^r 2^{r-1} \text{ST}_H,$$

where $\text{ST}_H$ is the number of spanning trees of $H$. 
Sketch of proof of the bound

1. In the expansion of $\kappa(A_1, \ldots, A_r)$, many set partitions yield the same moment $M_\pi = \prod_{i=1}^{\ell(\pi)} \mathbb{E}[\prod_{j \in \pi_i} A_j]$, so

$$
\kappa(A_1, \ldots, A_r) = \sum_{\pi'} M_{\pi'} \left( \sum_{\pi \rightarrow H \pi'} \mu(\pi) \right)
$$

$$
|\kappa(A_1, \ldots, A_r)| \leq A^r \sum_{\pi'} \left| \sum_{\pi \rightarrow H \pi'} \mu(\pi) \right|.
$$

2. The functional $F_{H/\pi'} = \sum_{\pi \rightarrow H \pi'} \mu(\pi)$ depends only on the contraction $H/\pi'$ of $H$ along $\pi'$, and one can show that is up to a sign the bivariate Tutte polynomial

$$
|F_{H/\pi'}| = T_{H/\pi'}(1, 0) \leq T_{H/\pi'}(1, 1) = ST_{H/\pi'}.
$$
3. A pair \((\pi', T \in ST_{H/\pi'})\) can be associated to a bicolored spanning tree of \(H\), hence

\[
\sum_{\pi'} ST_{H/\pi'} \leq 2^{r-1} ST_H.
\]

The bound on the cumulant of the sum \(S\) follows by noticing that:

- given a vertex \(v_1\) and a Cayley tree \(T\), the number of lists \((v_2, \ldots, v_r)\) such that \(T\) is contained in \(H = G[v_1, \ldots, v_r]\) is smaller than \(D^{r-1}\);
- the number of pairs \((v_1 \in V, T \text{ Cayley tree})\) is \(N r^{r-2}\).

The proof leads to the notion of **weighted dependency graph**.
Definition (Weighted dependency graph; Féray, 2016)

A sum \( S = \sum_{v \in V} A_v \) admits a weighted dependency graph \( G = (V, E) \) of parameters \((\text{wt}: E \to \mathbb{R}_+, A)\) if, for any family \( \{v_1, v_2, \ldots, v_r\} \),

\[
|\kappa(A_{v_1}, A_{v_2}, \ldots, A_{v_r})| \leq A^r \sum_{T \in ST_{G[v_1, \ldots, v_r]}} \left( \prod_{(v_i, v_j) \text{ edge of } T} \text{wt}(v_i, v_j) \right).
\]

The same proof gives:

\[
|\kappa(S)| \leq N (2D)^{r-1} r^{r-2} A^r
\]

with \( N = \text{card}(V) \) and \( D = \frac{1}{2} \left( 1 + \max_{v \in W} \left( \sum_{w \sim v} \text{wt}(v, w) \right) \right) \).
Let $S_n = \sum_{i=1}^{N_n} A_{i,n}$ be a sum of random variables, with $|A_{i,n}| \leq A$ a.s. and a dependency graph of maximal degree $D_n$. We suppose that

$$\frac{D_n}{N_n} \to 0 \ ; \quad \frac{\text{Var}(S_n)}{N_nD_n} \to \sigma^2 > 0 \ ; \quad \frac{\kappa^{(3)}(S_n)}{N_n(D_n)^2} \to L.$$ 

Then, $S_n - \mathbb{E}[S_n]$ satisfies the hypotheses of the method of cumulants, and all its consequences. Moreover, one has the concentration inequality:

$$\mathbb{P}[|S_n - \mathbb{E}[S_n]| \geq \varepsilon] \leq 2 \exp \left(-\frac{\varepsilon^2}{9 \left(\sum_{i=1}^{N_n} \mathbb{E}[|A_i|]\right)D_n A}\right) \leq 2 \exp \left(-\frac{\varepsilon^2}{9 \, N_n D_n A^2}\right).$$
Functionals of ergodic Markov chains

Let \((X_n)_{n \in \mathbb{N}}\) be a reversible \textbf{ergodic Markov chain} on a finite state space \(\mathcal{X}\) of size \(M\), and \(f : \mathcal{X} \to \mathbb{R}\). We set \(S_n(f) = \sum_{i=1}^{n} f(X_i)\), and we denote \(\pi\) the stationary distribution, \(P\) the transition matrix, and

\[
\theta_P = \max\{|z| \mid z \neq 1, \ z \text{ eigenvalue of } P\}.
\]

The sequence \((S_n(f))_{n \in \mathbb{N}}\) has a weighted dependency graph and satisfies the hypotheses of the method of cumulants, with parameters

\[
D_n = \frac{1 + \theta_P}{2(1 - \theta_P)}, \ N_n = n, \text{ and } A = 2\|f\|_\infty \sqrt{M}.
\]

**Remarks:**

1. If \(f = 1_{X_i = a}\), then one can take \(A = 2\).

2. One can remove the hypothesis of reversibility if

\[
\lim_{n \to \infty} \frac{\text{Var}(S_n(f))}{n} = \text{Var}_\pi(f) + 2 \sum_{i=1}^{\infty} \text{cov}_\pi(f(X_0), f(X_i)) \neq 0.
\]
Consider the **Ising model** on $\Lambda \subset \mathbb{Z}^d$, which is the probability measure on spin configurations $\sigma \in \{\pm 1\}^\Lambda$ proportional to $\exp(-\mathcal{H}_\beta,h(\sigma))$, with

$$
\mathcal{H}_{\beta,h}(\sigma) = -\beta \sum_{i,j \in \Lambda} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i.
$$

If $h \neq 0$ or $\beta < \beta_c(d)$, then the Ising model has a unique limiting probability measure $\mu_{\beta,h,\mathbb{Z}^d}$ on $\mathbb{Z}^d$.

Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a growing sequence of boxes, and $M_n = \sum_{i \in \Lambda_n} \sigma_i$ be the magnetization. Under $\mu_{\beta,h,\mathbb{Z}^d}$, $(M_n - \mathbb{E}[M_n])_{n \in \mathbb{N}}$ has a weighted dependency graph and satisfies the hypotheses of the method of cumulants if

- $h \neq 0$ (non-zero ambient magnetic field);
- $h = 0$ and $\beta < \beta_1(d) < \beta_c(d)$ (very high temperature).
Subgraph counts in graphon models
If $G = (V_G, E_G)$ is a finite graph, one says that $F = (V_F, E_F)$ is a subgraph of $G$ if there is a map $\psi : V_F \rightarrow V_G$ such that

$$\forall e = \{x, y\} \in E_F, \quad \{\psi(x), \psi(y)\} \in E_G.$$ 

**Density** of $F$ in $G$: $t(F, G) = \frac{|\text{hom}(F, G)|}{|V_G|^{|V_F|}} = \frac{6}{6^3} = \frac{1}{36}.$

**Objective:** establish the mod-Gaussian convergence of $t(F, G_n)$ for some models $(G_n)_{n \in \mathbb{N}}$ of random graphs.
A graph function is a measurable function $g : [0, 1]^2 \to [0, 1]$ that is symmetric: $g(x, y) = g(y, x)$ almost everywhere. If $F$ is a graph on $k$ vertices and $g$ is a graph function, the density of $F$ in $g$ is

$$t(F, g) = \int_{[0,1]^k} \left( \prod_{\{i,j\} \in E_F} g(x_i, x_j) \right) dx_1 \, dx_2 \cdots \, dx_k.$$ 

Let $\mathcal{F}$ be the set of graph functions, and $\mathcal{G} = \mathcal{F} / \sim$ its quotient by the relation:

$$g \sim h \iff \exists \sigma \text{ Lebesgue isomorphism of } [0, 1], \text{ with } h(x, y) = g(\sigma(x), \sigma(y)).$$

**Definition (Graphon; Lovász–Szegedy, 2006)**

A graphon is an element $\gamma = [g]$ of the quotient space $\mathcal{G}$. Endowed with the topology of convergence of all the observables $t(F, \cdot)$, $\mathcal{G}$ is a compact metrisable space.
From graphons to random graphs

To any graphon $\gamma = [g]$, one can associate a random graph $G_n(\gamma)$ on $n$ vertices:

1. One chooses $n$ independent uniform variables $X_1, \ldots, X_n$ in $[0, 1]$.
2. One connects $i$ to $j$ in $G_n(\gamma)$ according to a Bernoulli variable of parameter $g(X_i, X_j)$, independently for each pair $\{i, j\}$.

Conversely, to any graph $G$ on $n$ vertices, one can associate a graph function $g$: 
Convergence of graphon models

**Theorem (Lovász–Szegedy, 2006)**

If \( \gamma \) is the graphon associated to a graph \( G \), then \( t(F, G) = t(F, \gamma) \) for any finite graph \( F \). If \( \gamma_n(\gamma) \) is the random graphon associated to the random graph \( G_n(\gamma) \), then \( E[t(F, \gamma_n(\gamma))] = t(F, \gamma) \) and

\[
\gamma_n(\gamma) \to_P \gamma.
\]

We introduce the algebra \( \mathcal{O} \) of finite graphs \( F \), endowed with the degree \( \text{deg } F = \text{card}(V_F) \) and with the product \( F_1 \times F_2 = F_1 \sqcup F_2 \). One evaluates an **observable** \( f \in \mathcal{O} \) by linear extension of the rule \( F(\gamma) = t(F, \gamma) \). The convergence of graphon models amounts to:

\[
\forall \gamma \in \mathcal{G}, \forall f \in \mathcal{O}, f(\gamma_n(\gamma)) \to_P f(\gamma).
\]
Let \( \gamma \) be a graphon, \( F \) a finite graph on \( k \) vertices, \( N_{n,k} = n^k \) and

\[
S_n(F, \gamma) = n^k t(F, G_n(\gamma)) = \sum_{\psi : [1,k] \rightarrow [1,n]} 1_{\text{\( \psi \) is a morphism from \( F \) to \( G_n(\gamma) \)}} = \sum_{\psi : [1,k] \rightarrow [1,n]} A_\psi.
\]

Given independent uniform random variables \((X_i)_{1 \leq i \leq n}\) and \((U_{i,j})_{1 \leq i < j \leq n}\), one can write:

\[
A_\psi = \prod_{\{i<j\} \in E_F} 1_{U_{\psi(i), \psi(j)} \leq g(X_{\psi(i)}, X_{\psi(j)})}.
\]

If \( \psi \) and \( \phi \) have disjoint images, then \( A_\psi \) and \( A_\phi \) are independent. Therefore, for any \( n \in \mathbb{N} \), \( \gamma \in \mathcal{G} \), \( f \in \mathcal{O}_k \), \( S_n(f, \gamma) \) is a sum of random variables with a dependency graph of parameters

\[
D_{n,k} = k^2 n^{k-1}; \quad N_{n,k} = n^k; \quad A = \|f\|_{\mathcal{O}_k}.
\]
Asymptotics of the first cumulants

The computation of the limits $\sigma^2(f, \gamma)$ and $L(f, \gamma)$ involves the operation of junction of graphs. If $F$ and $G$ are finite graphs of size $k$, $a \in V_F$ and $b \in V_G$, we denote $(F \bowtie G)(a, b)$ the graph on $2k - 1$ vertices obtained by identifying $a \in V_F$ with $b \in V_G$.

$$\lim_{n \to \infty} \frac{\text{cov}(S_n(F_1, \gamma), S_n(F_2, \gamma))}{n^{2k-1}} = \sum_{1 \leq a, b \leq k} t((F_1 \bowtie F_2)(a, b), \gamma) - t(F_1, \gamma) t(F_2, \gamma).$$
Theorem (Féray–M.–Nikeghbali, 2016)

Fix \( \gamma \in \mathcal{G} \), \( f \in \mathcal{O}_k \), and define

\[
\kappa_2(F, G) = \frac{1}{k^2} \sum_{1 \leq a, b \leq k} (F \bowtie G)(a, b) - F \cdot G;
\]

\[
\kappa_3(F, G, H) = \frac{1}{k^4} \sum_{1 \leq a, b, c \leq k} (F \bowtie G \bowtie H)(a, b, c) + 2 F \cdot G \cdot H - (F \bowtie G)(a, b) \cdot H - (G \bowtie H)(b, c) \cdot F - (F \bowtie H)(a, c) \cdot G
\]

\[
+ \frac{1}{k^4} \sum_{\mathbb{Z}/3\mathbb{Z}} \sum_{1 \leq a \neq c, d \leq k} (F \bowtie G \bowtie H)(a, b; c, d) + F \cdot G \cdot H - (F \bowtie G)(a, b) \cdot H - (G \bowtie H)(c, d) \cdot F.
\]

If \( \kappa_2(f, f)(\gamma) \neq 0 \), then \( S_n(f, \gamma) \) satisfies MC1-MC3 with parameters \( D_{n,k} = k^2 n^{k-1} \), \( N_{n,k} = n^k \) and \( A = \|f\|_{\mathcal{O}_k} \). Moreover,

\[
\sigma^2 = \kappa_2(f, f)(\gamma)
\]

\[
L = \kappa_3(f, f, f)(\gamma).
\]
So, any subgraph count of a random graph $G_n(\gamma)$ stemming from any graphon $\gamma \in \mathcal{G}$ is generically mod-Gaussian convergent.

**Example:** If $K_3 = \begin{tikzpicture}[baseline=-0.5ex]
\fill (0,0) circle (0.075cm);
\fill (1,0) circle (0.075cm);
\fill (0,1) circle (0.075cm);
\draw (0,0) -- (0,1);
\draw (0,0) -- (1,0);
\end{tikzpicture}$ and $H = \begin{tikzpicture}[baseline=-0.5ex]
\fill (0,0) circle (0.075cm);
\fill (1,0) circle (0.075cm);
\fill (0,1) circle (0.075cm);
\fill (1,1) circle (0.075cm);
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}$, then the density of triangles $t(K_3, G_n(\gamma))$ satisfies the central limit theorem:

$$Y_n = \sqrt{n} \frac{t(K_3, G_n(\gamma)) - t(K_3, \gamma)}{3 \sqrt{t(H, \gamma) - t(K_3, \gamma)^2}} \rightarrow \mathcal{N}(0,1),$$

assuming that the denominator is positive. Furthermore, one has

$$d_{\text{Kol}}(Y_n, \mathcal{N}(0,1)) \leq \frac{81}{(t(H, \gamma) - t(K_3, \gamma)^2)^{3/2} \sqrt{n}}$$

for $n$ large enough; the concentration inequality

$$\mathbb{P} \left[ |t(K_3, G_n(\gamma)) - t(K_3, \gamma)| \geq \varepsilon \right] \leq 2 \exp \left( -\frac{n\varepsilon^2}{3} \right);$$

as well as a moderate deviation result and a local limit theorem.
Mod-Gaussian moduli spaces

We consider a compact metrisable space $\mathcal{M}$, where convergence is controled by a graded algebra of observables $\mathcal{O}$.

\[ \mathcal{M} \]

\[ M_3 \times \mathcal{M}_1 \]

\[ M_4 \times \mathcal{M}_2 \]

Informal definition: each parameter $m \in \mathcal{M}$ generates its own random perturbations $(M_n(m))_{n \in \mathbb{N}}$, and for any observable $f \in \mathcal{O}$, the sequence $(f(M_n(m)))_{n \in \mathbb{N}}$ is mod-Gaussian convergent after appropriate renormalisation, assuming $\kappa_2(f,f)(m) \neq 0$. 
One can prove that:
  ▶ the space of probability measures on a compact space;
  ▶ the space of permutons;
  ▶ the Thoma simplex

are mod-Gaussian moduli spaces for the following observables and random variables:
  ▶ polynomial functionals of empirical measures of random sequences;
  ▶ counts of motives in random permutations;
  ▶ random characters values associated to random integer partitions.

**Informal conjecture:** if one approximates a continuous object by a random discrete one, the observables of the model usually have mod-Gaussian fluctuations (example: the Gromov–Hausdorff–Prohorov space).
The end