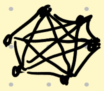
chap 1 Ising's Ising model, and Curie Weiss' Ising model.

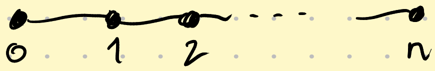
In this chapter we show the (non)existence of a phase transition in 2 simple cases: the dimension 1 (i.e. 

for n large)

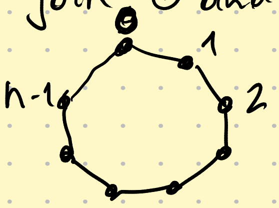
and the complete graph.  K_n for n large.

I- Dimension 1 (Ising's Ising model)

A) Partition Function (with a magnetic Field)

Consider the line graph 

$n \in \mathbb{N}^*$.

For simplicity, we consider periodic boundary conditions, that is, we join 0 and $n-1$ to get the cycle graph with n vertices: 

Let $\Omega = \{\pm 1\}^{[\![0;n-1]\!]}$ be the set of all spin configurations.

Let $\beta > 0$ and $h \in \mathbb{R}$. We consider the proba:

$$\forall \sigma \in \Omega, \quad \mu_{\beta,h,n}(\sigma) = \frac{1}{Z_{\beta,h,n}} \exp\left(\beta \sum_{i=0}^{n-1} \sigma_i \sigma_{i+1} + h \sum_{i=0}^{n-1} \sigma_i\right)$$

with the convention $\sigma_n := \sigma_0$, and

$$Z_{\beta,h,n} = \sum_{\sigma \in \Omega} \exp\left(\beta \sum_{i=0}^{n-1} \sigma_i \sigma_{i+1} + h \sum_{i=0}^{n-1} \sigma_i\right)$$

$\forall \beta > 0,\ h \in \mathbb{R},\ n \geq 2$

Theo: $Z_{\beta,h,n} = \lambda_+^n + \lambda_-^n$, where

$$\lambda_\pm = e^\beta \cosh(h) \pm \sqrt{\underbrace{e^{2\beta}\cosh(h)^2 - 2\sinh(2\beta)}_{>0}}$$

Proof: We write $Z_{\beta,h,n} = \sum_{\sigma_0,\dots,\sigma_{n-1} \in \{\pm 1\}} \prod_{i=0}^{n-1} e^{\beta \sigma_i \sigma_{i+1} + h \sigma_i}$ (*)

Consider a 2×2 matrix P with rows & columns indexed by the set $\{\pm 1\}$:

$$P = \begin{array}{c} \\ \ominus \\ \oplus \end{array} \begin{array}{c} \begin{array}{cc} \ominus & \oplus \end{array} \\ \begin{pmatrix} P_{++} & P_{+-} \\ P_{-+} & P_{--} \end{pmatrix} \end{array}$$

Take P st $\overbrace{\ }^{\forall a,b \in \{\pm 1\}}$ $P_{a,b} = \exp(\beta ab + ha)$.

That is,

$$P = \begin{pmatrix} e^{\beta + h} & e^{-\beta + h} \\ e^{-\beta - h} & e^{\beta - h} \end{pmatrix}.$$

By $(*)$,

$$Z_{\beta,h,n} = \sum_{\sigma_0,\dots,\sigma_n \in \{\pm 1\}} P_{\sigma_0\sigma_1} P_{\sigma_1\sigma_2} \cdots P_{\sigma_{n-1}\sigma_0}$$

$$= \sum_{\sigma_0 \in \{\pm 1\}} P^n_{\sigma_0,\sigma_0} \quad \leftarrow \text{matrix product!}$$

$$= \mathrm{Tr}(P^n)$$

Compute the eigenvalues of P... They are $\lambda_\pm$ and different, so P is diagonalizable, and

$$\mathrm{Tr}(P^n) = \lambda_+^n + \lambda_-^n$$

<u>Remark</u>: P is called a transfer matrix, and it is a classical technique in statistical mechanics (consider a matrix indexed by the states of the system on a given "slice"...)

We computed the partition function, which may not look like much... But in fact $Z_{\beta,h,n}$ should not be thought of as a normalizing constant, and rather as a function of the parameters (β, h, n) that encodes all the physical behaviour, as we shall see now.

B] Magnetization

We consider the magnetization $m_{\beta,h,n} = \left\langle \frac{1}{n} \sum_{i=0}^{n-1} \sigma_i \right\rangle_{\beta,h,n}$

Theo $\forall \beta > 0,\ h \in \mathbb{R},\ n \geq 2,$

$$m_{\beta,h,n} = \frac{1}{n} \frac{\partial (\ln Z_{\beta,h,n})}{\partial h} = \underbrace{\frac{e^{\beta} \sinh(h)}{\sqrt{e^{2\beta} \cosh(h)^2 - 2\sinh(2\beta)}}}_{m^*_{\beta,h} :=} + O\left(\frac{1}{n}\right)$$

Proof: $\frac{\partial}{\partial h}(Z_{\beta,h,n}) = \sum_{\sigma \in \Omega} \frac{\partial}{\partial h}\left(\exp\left(\beta \sum \sigma_i \sigma_{i+1} + h \sum \sigma_i\right)\right)$

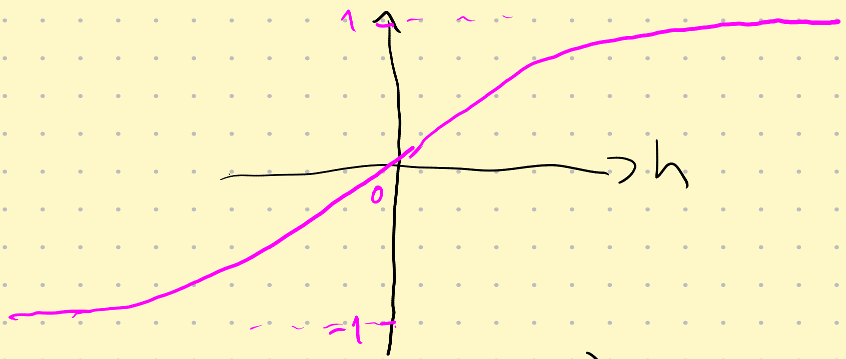
$$= \sum_{\sigma \in \Omega} \left(\sum_{i=0}^{n-1} \sigma_i\right) \underbrace{\exp\left(\beta \sum \sigma_i \sigma_{i+1} + h \sum \sigma_i\right)}_{\text{almost } \mu_{\beta,h,n}(\sigma)\,!}$$

So $\underbrace{\frac{1}{Z_{\beta,h,n}} \frac{\partial}{\partial h}(Z_{\beta,h,n})}_{= \frac{\partial}{\partial h}(\log Z_{\beta,h,n})} = \sum_{\sigma \in \Omega} \left(\sum_{i=0}^{n-1} \sigma_i\right) \mu_{\beta,h,n}(\sigma)$

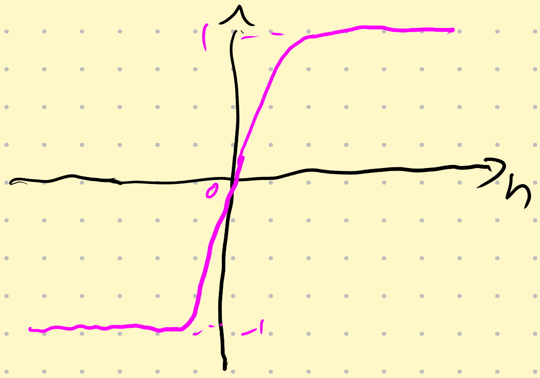
$$= \left\langle \sum_{i=0}^{n-1} \sigma_i \right\rangle.$$

This proves the first formula.
The second is just simple derivation and comparison. ☒
(we have $m^*_{\beta,h} = \frac{\partial \log \lambda_+(\beta,h)}{\partial h}$)

Portrait of $m^*_{\beta,h}$:



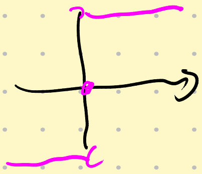
($\beta > 0$ fixed, "small")



($\beta > 0$ fixed, "big")

At any finite temperature, when we remove the magnetic field, no magnetization remains.

There is no phase transition (or rather it happens at $\beta = +\infty$, for which



← exercise)

We can actually find the limiting distribution of $M_n := \frac{1}{n}\sum_{i=0}^{n-1}\sigma_i$, for n large, via its Laplace transform:

$$F_{\beta,h,n}(t) := \langle e^{tM_n} \rangle_{\beta,h,n}$$

It satisfies $F_{\beta,h,n} \simeq \dfrac{Z_{\beta,\, h+\frac{t}{n},\, n}}{Z_{\beta,h,n}}$ ← exercise.

Prop: $\forall \beta > 0,\ h \in \mathbb{R}$,

$$\sqrt{n}\left(M_n - m^*_{\beta,h}\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2)$$

where $\sigma^2 = \dfrac{e^{-\beta}\cosh(h)}{\left(e^{2\beta}\cosh(h)^2 - 2\sinh(2\beta)\right)^{3/2}}$

Proof

The log of the Laplace transform of $\sqrt{n}\, M_n$ satisfies

$$\log \langle e^{t\sqrt{n}(M_n - m^*_{\beta,h})} \rangle = \log \frac{Z_{\beta,\, h+t/\sqrt{n},\, n}}{Z_{\beta,h,n}} - t\sqrt{n}\, m^*_{\beta,h}$$

$$= n \log \lambda_+\left(\beta, h + \tfrac{t}{\sqrt{n}}\right) - n \log \lambda_+(\beta,h) - t\sqrt{n}\, m^*_{\beta,h} + o(1)$$

We Taylor expand λ_+ in terms of its second argument around h at order 2, which gives

$$\log \langle \quad " \quad \rangle = t\sqrt{n}\, \cancel{\frac{\partial \log \lambda_+}{\partial h}(\beta,h)} + \frac{t^2}{2}\frac{\partial^2 \log\lambda_+}{\partial h^2}(\beta,h) - \cancel{t\sqrt{n}\, m^*_{\beta,h}} + o(1)$$

$$= (\text{computation})\dots$$

$$= \frac{t^2}{2}\sigma^2 + o(1)$$

We deduce that the characteristic function of $\sqrt{n}(M_n - m^*_{\beta,h})$ is

$Z_n :=$

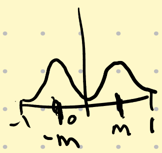
$$\Phi_{Z_n}(t) = \langle e^{itZ_n} \rangle_{\beta,h} = \exp\left(-\frac{t^2}{2}\sigma^2 + o(1)\right)$$

and we conclude with Levy's theorem. ☐

<u>Remark</u>: Even for $h=0$ (so $m^+_{\beta,h} = 0$), this is interesting.

It says that M_n has order $\frac{c}{\sqrt{n}}$ with high probability.

Previously we knew only that $\langle M_n \rangle$ was small, but it might a priori be the case that M_n has a symmetric distribution with potential magnetization,



so that the system would be typically in magnetization $+m$ or $-m$.

The previous prop shows it is not the case: with high proba, there is roughly as many + spins as − spins.

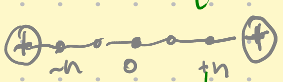
In terms of chapter 0,

$$\forall \beta > 0, \quad \left\langle \left| \frac{1}{n} \sum \sigma_i \right| \right\rangle_{\beta,0,n} \xrightarrow[n\to\infty]{} 0 .$$

There remains a 3^rd possibility we discussed for breaking the symmetry and look for magnetization:

C) + boundary condition

<u>Exercise</u>: Consider the Ising model on $[\![-n, +n]\!] \subset \mathbb{Z}$ with $J \equiv 1$, $\beta > 0$, and $h=0$. Show that



$$\langle \sigma_0 \rangle^+_{\beta,n} = 2\frac{\tanh(\beta)^{n+1}}{1+\tanh(\beta)^{2n+2}} \xrightarrow[n\to\infty]{} 0$$

using transfer matrices.

We will soon have a better tool to show this formula.

One may also try this exercise for $h \neq 0$ ...

<u>CCL</u>: In dimension 1, at $h=0$ the circle is never magnetic. This is not so surprising, as we cannot see boundary condition: a single unlucky spin breaks it all. There is an analogy with Markov chain that we do

not develop here, but notice that the process on ~~[illegible]~~ is "almost Markovian", which would help defining a limiting process on $\mathbb{Z}$,

II - The Curie - Weiss model defined by Kac in 1968

A) Motivation: mean Field

In this section, we take $J \equiv 1$.

Recall that we would like to study the (magnetization of) Ising model on big pieces of $\mathbb{Z}^d$, for which the energy has the form $\beta \sum_{e=\{i,j\}} \sigma_i \sigma_j$. For a given site i, the contribution of σ_i is

i.e. the term in the exponential

$$\beta \sigma_i \underbrace{\sum_{j \sim i} \sigma_j}_{\text{sum on the } 2d \text{ neighbours of } i} = 2 d \beta \sigma_i \cdot \underbrace{\frac{1}{2d} \sum_{j \sim i} \sigma_j}_{\text{a local magnetisation density}}$$

In the "mean field" approximation, we suppose that these local magnetisation densities can be approximated by a global one: $\frac{1}{|V|} \sum_{j \in V} \sigma_j$. This gives a new energy:

$$\frac{2d\beta}{|V|} \sum_{i,j} \sigma_i \sigma_j ,$$

where the sum is over all pairs of vertices.

This amounts to considering the Ising model on a complete graph, which we do now.

B) Definition

Let $n \in \mathbb{N}^*$ and $\beta > 0$. The Curie-Weiss model is the probability on $\Omega = \{\pm 1\}^{\{0, \dots, n-1\}}$ given by

$$\forall \sigma \in \Omega, \quad \mu_{\beta,n}(\sigma) = \frac{1}{Z_{\beta,n}} \exp\left(\frac{\beta}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sigma_i \sigma_j \right) \qquad (h=0)$$

with $Z_{\beta,n} = \dots$

Let $M_n = \frac{1}{n} \sum_{i=0}^{n-1} \sigma_i$ be the magnetization.

C] Spontaneous magnetization

Theo Let $\beta_c = \frac{1}{2}$.

- If $\beta \leq \beta_c$, $\forall \varepsilon > 0$, $\mu_{\beta,n}(|M_n| > \varepsilon) \xrightarrow[n\to\infty]{} 0$.

- If $\beta > \beta_c$, $\exists\, m^*_\beta > 0$ st

  $\forall \varepsilon > 0$, $\mu_{\beta,n}\left(|M_n - m^*_\beta| > \varepsilon \text{ and } |M_n + m^*_\beta| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$.

- At high temp, no spontaneous magnet: $M_n \simeq 0$ with high prob

- At low temp, $M_n \simeq m^*_\beta$ with prob $\simeq \frac{1}{2}$

  $-m^*_\beta$ with prob. $\simeq \frac{1}{2}$.

Finally, a phase transition!

Remark: it is still the case that $\forall \beta$, $\langle M_n \rangle_\beta = 0$.

Proof: Note that $\mu_{n,\beta}(\sigma) = \frac{1}{Z_{\beta,n}} \exp\left(\frac{\beta}{n}\left(\sum_{i=0}^{n-1} \sigma_i\right)^2\right)$

$$= \frac{1}{Z_{\beta,n}} \exp\left(\frac{\beta}{n}\left(\sum_{i=0}^{n-1} \sigma_i\right)^2\right).$$

So

$$Z_{\beta,n} = \sum_\sigma \exp\left(\frac{\beta}{n}\left(\sum_{i=0}^{n-1} \sigma_i\right)^2\right).$$

"total magnetization"

The probability of any σ depends only on $\sum_{i=0}^{n-1} \sigma_i \in [\![-n, n]\!]$.

Reciproquely, given a value of $s \in [\![-n ; n]\!]$ with $s \equiv n\,[2]$, there are $\binom{n}{\frac{n+s}{2}}$ configs σ with this total magnetization.

(indeed there has to be $\frac{n+s}{2}$ spins $\oplus$ and $\frac{n-s}{2}$ spins $\ominus$).

Therefore,

$$Z_{\beta,n} = \sum_{\substack{-n \leq s \leq n, \\ s \equiv n[2]}} \binom{n}{\frac{n+s}{2}} \exp\left(\frac{\beta}{n} s^2\right). \qquad (*)$$

(This induces a proba on the set of possible s, given by

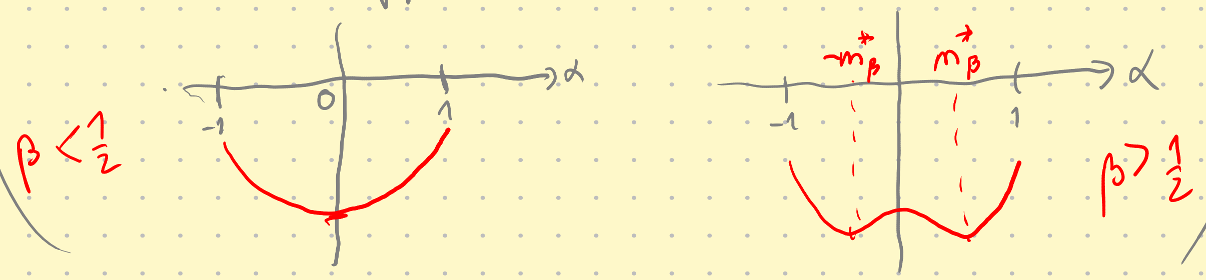
$\dfrac{\binom{n}{\frac{n+s}{2}} \exp\left(\frac{\beta}{n} s^2\right)}{Z_{\beta,n}}$, and the distribution of σ is obtained by taking s with this distribution and then σ by chosing the set of $\frac{n+s}{2}$ spins $\oplus$ uniformly).

We want to know which values of s dominate the sum $(*)$ (it is a problem of large deviation).
If we take $n \to \infty$ and $s \sim \alpha n$ $(\alpha \in [-1,1])$, we have by Stirling's Formula,

Exercise

$$-\frac{1}{n} \log \left(\binom{n}{\frac{n+s}{2}} \exp\left(\frac{\beta}{n} s^2\right)\right) \xrightarrow[n\to\infty]{} + \underbrace{\frac{1+\alpha}{2} \ln \frac{1+\alpha}{2} + \frac{1-\alpha}{2} \ln \frac{1-\alpha}{2} - \beta \alpha^2}_{=: f_\beta(\alpha)}.$$

Key property: f_β looks like



Therefore in the sum $(*)$, all terms are $\leq \exp\left(-n \min_{[-1,1]} f_\beta + o(n)\right)$
and some terms have this order.
So $\exp\left(-n \min_{[-1,1]} f_\beta + o(n)\right) \leq Z_{\beta,n} \leq 2n \exp\left(-n \min_{[-1,1]} f_\beta + o(n)\right)$

and $\quad \frac{1}{n} \log Z_{\beta,n} \xrightarrow[n\to\infty]{} -\min_{[-1,1]} f_\beta$

Similarly, for any $I =]a,b[\subset]-1,1[$, by the same reasoning,

$$\frac{1}{n} \log\left(\mu_{n,\beta}(M_n \in I)\right) = \frac{1}{n} \log\Big(\sum_{\substack{s \in]an, bn[\\ s \equiv n [2]}} \binom{n}{\frac{n+s}{2}} e^{\frac{\beta}{n} s^2} \Big) - \frac{1}{n} \log\left(Z_{\beta,n}\right)$$

$$\xrightarrow[n\to\infty]{} -\min_{\alpha \in I} f_\beta(\alpha) + \min_{\alpha \in]-1,1[} f_\beta(\alpha) \quad (\leq 0)$$

As a result, (if $\bar{I}$ contains none of the global minima of f_β, then $\mu_{n,\beta}(M_n \in I)$ goes to 0 as $n \to \infty$ (exp. fast!)) (**)

(so M_n concentrates on global minima of f_β)

So it enough to show the key property, and use (**) on sets like $]-1,-\varepsilon[\cup]-\varepsilon, 1[$, ...

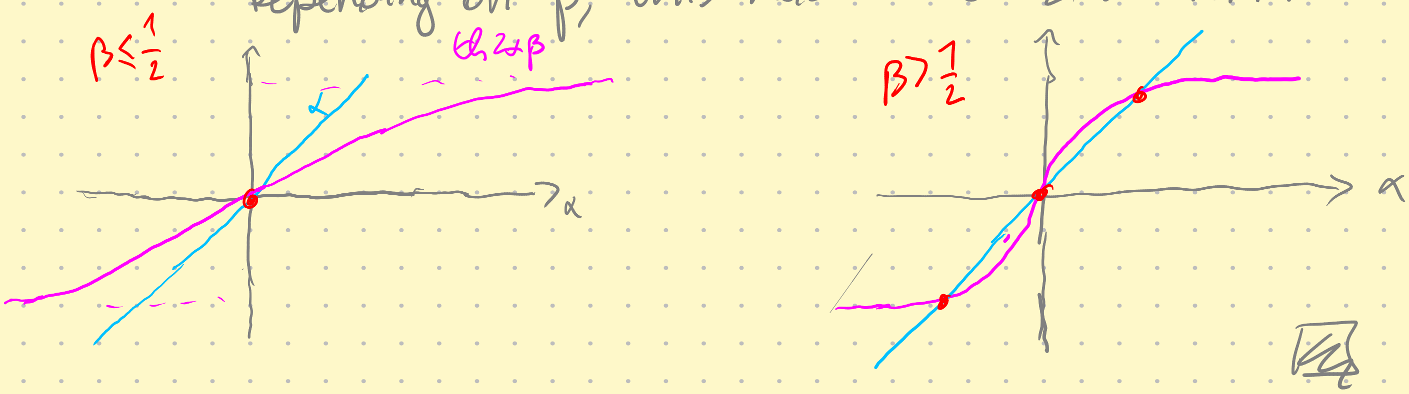
The key property is simple analysis:

$$f_\beta'(\alpha) = \cdots = \frac{1}{2} \ln \frac{1-\alpha}{1+\alpha} + 2\alpha\beta .$$

This ⊙ For ...

$$\alpha = \tanh(2\alpha\beta)$$

Depending on β, this has 0 or 2 solutions.



$\boxtimes$

Remark: We got more:

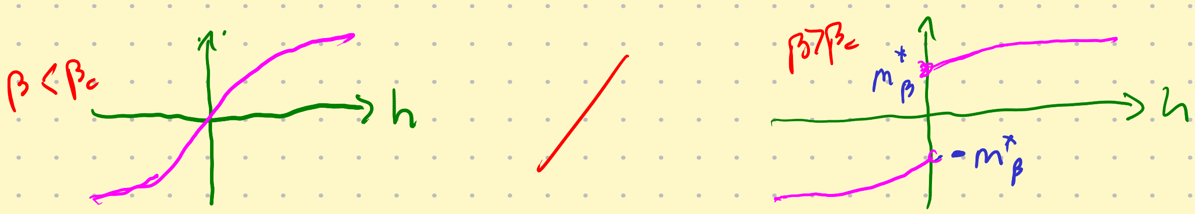
- exponential speed of convergence (with a rate depending on β and ε)
- $m^*_\beta = \tanh(2\beta\, m^*_\beta)$.
- m^*_β is ↗ in β and goes to 1 as $\beta \to \infty$.

Exercise Find the behaviour of the magnetisation in the presence of a magnetic field $h \neq 0$, that is,

$$\mu_{N,\beta,h} = \frac{1}{Z_{N,\beta,h}} \exp\left(\frac{\beta}{n} \sum_{i,j} \sigma_i \sigma_j + h \sum_i \sigma_i \right) .$$

spoiler: $m^*_{\beta,h}$ satisfies $m^*_{\beta,h} = \tanh(2\beta m^*_{\beta,h} + h)$

Show in particular that the magnetisation concentrates on a single value that behaves like:



- When $\beta > \beta_c$, the magnetization, as a function of h, is discontinuous at $h=0$.

- For $h=0$, the function $\beta \mapsto m^*_\beta$ is non analytic at β_c (otherwise it would be completely 0, as it is for $\beta < \beta_c$!).

In physics, we say that there is a phase transition when a rupture of analyticity happens.

D/ Free energy

As a corollary of the previous proof, we get:

Cor: $\forall \beta > 0$, $-\frac{1}{n} \log Z_{n,\beta} \xrightarrow[n\to\infty]{} \min_{x\in[-1,1]} F_\beta(x)$

where $F_\beta(x) = \frac{1+x}{2} \ln \frac{1+x}{2} + \frac{1-x}{2} \ln \frac{1-x}{2} - \beta x^2$.

The quantity $\lim_{n\to\infty} -\frac{1}{n} \log Z_{n,\beta}$ can usually be

in gen, number of spins

shown to exist via soft submultiplicativity arguments. It is called the free energy of the model.

It can be thought of as the "typical energy per site" (up to a factor β)

(recall that we took $\mu(\sigma) = \frac{1}{Z} \exp(-\beta \cdot \text{Energy}(\sigma))$.

Imagine n non-interacting particles, each with energy h, then $Z = e^{-\beta n h}$ and $\frac{1}{n} \log Z = \beta h$).

It is also sometimes called the pressure (as we think of it as an energy per unit of volume).

E] Critical exponents

We may try to quantify the ruptures of analyticity of the function $m^*_{\beta,h}$. This gives a classification of phase transitions.
For instance, how does $m^*_{\beta,0}$ approach 0 as $\beta \to \beta_c$?

Exercise
- Using a Taylor expansion on the implicit equation
$$m^*_{\beta,0} = \tanh(2\beta\, m^*_{\beta,0}),$$
show that
$$m^*_{\beta,0} \underset{\beta\to\beta_c^+}{\sim} \left(6(\beta-\beta_c)\right)^{1/2}.$$
- Differenting with respect to h on
$$m^*_{\beta,h} = \tanh(2\beta\, m^*_{\beta,h} + h),$$
show that $\left.\frac{\partial m^*_{\beta,h}}{\partial h}\right|_{h=0} \underset{\beta\to\beta_c^-}{\sim} \frac{\beta_c}{\beta_c-\beta}$.

In both cases, we find a behaviour of the form $(\beta-\beta_c)^\alpha$ for some power α. These powers are called critical exponents, and have names associated to physical interpredations. Here we showed that the
- susceptipility exponent is $\gamma = 1$
- order parameter exponent is $b = 1/2$.

There are others (see section 2.5.3 in Friedli-Velenik)