chap 4 The random current expansion

In the previous chapter, we left a few inequalities to be proved:
- Griffith ineq for $h>0$ and $+$ b.c.
- GHS ineq: $\Lambda \subset\subset \mathbb{Z}^d$, $\beta>0$, $h>0$.

$$\langle \sigma_x\sigma_y\sigma_z\rangle^+_{\Lambda,\beta,h} - \langle\sigma_x\rangle\langle\sigma_y\sigma_z\rangle - \langle\sigma_y\rangle\langle\sigma_x\sigma_z\rangle - \langle\sigma_z\rangle\langle\sigma_x\sigma_y\rangle + 2\langle\sigma_x\rangle\langle\sigma_y\rangle\langle\sigma_z\rangle \leq 0$$

In the current chapter, we will define a very useful tool to prove this kind of inequalities, and in general to transform questions about the Ising model into "percolation" problems.

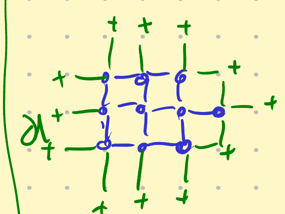
I- The expansion

Let $G=(V,E)$ be a finite graph.
We will in this chapter take $J=(J_e)_{e\in E}$ ≥ 0 and $\beta=1$, and $h=0$.
(more flexible setting).

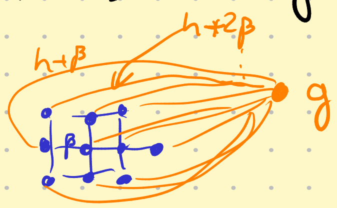
$\Omega = \{\pm 1\}^V$.

$$\forall \sigma\in\Omega, \quad \mu(\sigma) = \mu_{G,J}(\sigma) = \frac{\exp\left(\sum_{e=\{xy\}\in E} J_e\sigma_x\sigma_y\right)}{Z_{G,J}}$$

Example: For $\Lambda\subset\subset\mathbb{Z}^d$, $\beta>0$, $h>0$, we can recreate the measure $\mu^+_{\Lambda,\beta,h}$ in this setting:



$\rightsquigarrow$



G with vertices $\Lambda\cup\{g\}$
edges $E(\Lambda)\cup\{\{xg\}/x\in\Lambda\}$.

and $J_e = \beta$ if $e\in E(\Lambda)$
h $+\beta\sum_{y\in\partial\Lambda, y\sim x} 1$ if $e=\{xg\}$, $x\in\Lambda$.

Then $\mu^+_{\Lambda,\beta,h}(\cdot) = \mu_{G,J}(\cdot\mid\sigma_g=+1)$

As in the HTE, we expand $Z_{G,J}$, but this time we completely expand the exponential:

$$Z_{G,J} = \sum_{\sigma \in \{\pm 1\}^V} \prod_{\substack{e \in E \\ \{x,y\}}} \exp(J_e \sigma_x \sigma_y) \qquad \sum_{n_e \geq 0} \frac{J_e^{n_e} \sigma_x^{n_e} \sigma_y^{n_e}}{n_e!}$$

$$= \sum_{\sigma \in \{\pm 1\}^V} \sum_{\substack{n \in \mathbb{N}^E \\ (n_e)_{e \in E}}} \prod_{\substack{e \in E \\ \{x,y\}}} \frac{J_e^{n_e}}{n_e!} \sigma_x^{n_e} \sigma_y^{n_e}.$$

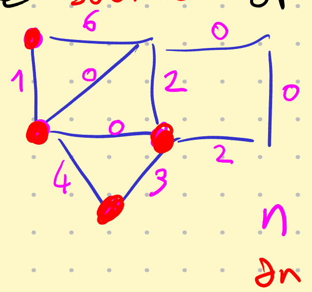
$$= \sum_{n \in \mathbb{N}^E} \left(\prod_e \frac{J_e^{n_e}}{n_e!} \right) \underbrace{\left[\sum_{\sigma \in \{\pm 1\}^V} \prod_{x \in V} \sigma_x^{\sum_{e \sim x} n_e} \right]}_{\begin{cases} 0 \text{ if } \exists x / \sum_{e \sim x} n_e \text{ odd} \\ 2^{|V|} \text{ otherwise} \end{cases}}$$

$$= 2^{|V|} \sum_{\substack{n \in \mathbb{N}^E / \\ \partial n = \emptyset}} \prod_{e \in E} \frac{J_e^{n_e}}{n_e!}$$

A configuration $n \in \mathbb{N}^E$ is called a current, and $\partial n := \{x \in V / \sum_{e \sim x} n_e = 1[2]\}$ be the sources of n.

Remark: $\mathrm{card}(\partial n)$ has to be even.

Let also $w(n) = \prod_{e \in E} \frac{J_e^{n_e}}{n_e!}$.



And just like HTE, we get:

Theo (Random current expansion)

Let $A \subset V$ be an even subset of vertices.

Then $\langle \sigma_A \rangle_{G,J} = \dfrac{\sum_{n / \partial n = A} w(n)}{\sum_{n / \partial n = \emptyset} w(n)}$.

Proof: Exercise!

Remark: By symmetry here $\langle \sigma_A \rangle = 0$ for $|A|$ odd.

Exercise: For Λ, $\beta>0$, $h>0$, +b.c., we constructed G, J before ($V = \Lambda \cup \{g\}$ ...)

- Show that $\forall A \subset \Lambda$,

$$\langle \sigma_A \rangle^+_{\Lambda,\beta,h} = \begin{cases} \langle \sigma_A \rangle_{G,J} & \text{if } A \text{ even} \\ \langle \sigma_{A \cup \{g\}} \rangle_{G,J} & \text{if } A \text{ odd} \end{cases}$$

and deduce its r.c. expansion.
random current.

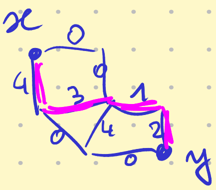
- Show the First Griffith inequality in that case.

Exercise: Show that if we apply $\mathbb{N}^E \to \mathcal{P}(E)$
$n \mapsto \{e \in E / n_e \text{ odd}\}$,
we recover the HTE from the RCE.

II- The switching lemma

The key property of the RCE, and its advantage over HTE, is called the switching lemma.

For $n \in \mathbb{N}^E$, and $x, y \in V$, we say that



$x \overset{n}{\leftrightarrow} y$ iff there exists a path γ from x to y s.t. $\forall e \in \gamma$, $n_e > 0$.

Example: If $\partial n = \{x\,y\}$, then $x \overset{n}{\leftrightarrow} y$.

Theo (switching lemma)

Let $F : \mathbb{N}^E \to \mathbb{R}$, let $A \subset V$ be even, and $x \neq y \in V$.

Then

$$\sum_{\substack{m/\partial m = A \\ n/\partial n = \{x y\}}} w(m)\, w(n)\, F(m+n) = \sum_{\substack{m/\partial m = A \Delta \{x,y\} \\ n/\partial n = \emptyset}} w(m)\, w(n)\, F(m+n)\, 1_{x \overset{m+n}{\leftrightarrow} y}$$

switching!

Application : This doesn't look like much, but it allows us to give a probabilistic interpretation of some expressions from RCE (which, remember, don't look a priori like expectations for a measure on $\mathbb{N}^E$).

For instance, $\forall x \neq y \in V$,

$$\langle \sigma_x \sigma_y \rangle^2_{G,J} = \frac{\sum\limits_{\substack{n,m / \partial n = \{xy\} \\ \partial m = \{xy\}}} w(n)\, w(m)}{\sum\limits_{\substack{n,m / \partial n = \phi, \\ \partial m = \phi}} w(n)\, w(m)}$$

$$= \frac{\sum\limits_{n,m / \partial n = \partial m = \phi} w(n)\, w(m)\, 1_{x \overset{m+n}{\longleftrightarrow} y}}{\sum\limits_{n,m / \partial n = \partial m = \phi} w(n)\, w(m)}$$

$$= \mathbb{P}\left(x \overset{m+n}{\longleftrightarrow} y \right)$$

For m, n random currents, indep with distrib. $\dfrac{w(.)}{\sum\limits_{\partial n = \phi} w(n)}$.

Proof of the switching lemma:

$$\sum_{\substack{\partial m = A \\ \partial n = \{xy\}}} w(m)\, w(n) \quad F(m+n)$$

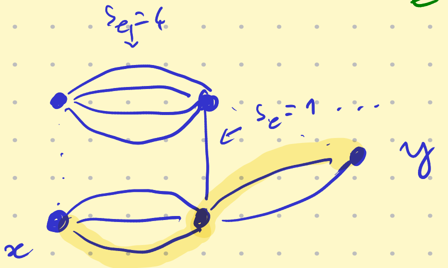
$s = m + n$

$$= \sum_{\substack{s \in \mathbb{N}^E / \\ \partial s = A \Delta \{xy\}}} F(s) \left(\sum_{\substack{n \leq s / \\ \partial n = \{xy\}}} w(n)\, w(s-n) \right)$$

$$= \sum_{\substack{s \in \mathbb{N}^E / \\ \partial s = A \Delta \{xy\}}} F(s) \cdot \left(\sum_{\substack{n \leq s \\ \partial n = \{xy\}}} \prod_{e \in E} \frac{J_e^{s_e}}{n_e!\,(s_e - n_e)!} \right)$$

$$= \sum_{\substack{s \in \mathbb{N}^E / \\ \partial s = A \Delta \{xy\}}} F(s)\, w(s) \cdot \left(\sum_{\substack{n \leq s \\ \partial n = \{xy\}}} \prod_{e \in E} \binom{s_e}{n_e} \right)$$

To compute $\sum_{n\le s} \cdots$, Consider the multigraph on vertices V with s_e edges between u, v st $\{uv\}=e$.



$\leftarrow G_s$

as subsets of edges

Then we are counting the number of subgraphs of G_s s.t. the vertices x, y have odd degree in the subgraph, & all others have even degree.

In particular, $x \leftrightarrow y$ in the subgraph, so $x \xleftrightarrow{s} y$, otherwise the sum $\sum_{n\le s} \cdots$ is 0.

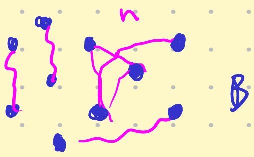
When $x \xleftrightarrow{s} y$, let us fix a path $\gamma : x \leftrightarrow y$ in G_s. Then the symmetric difference with γ induces a bijection between our subgraphs and those that are even everywhere.

So
$$\sum_{\substack{n\le s\\ \partial n=\{xy\}}} \prod_e \binom{s_e}{n_e} = \mathbb{1}_{x \xleftrightarrow{s} y} \sum_{\substack{n\le s\\ \partial n=\emptyset}} \prod_e \binom{s_e}{n_e}.$$

By making the first computation in reverse, we get the statement! $\boxtimes$

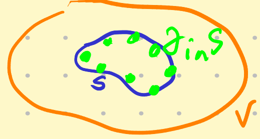
The lemma can be extended in (at least) two ways:

- Let A, B be even subsets of V.
  For $n \in \mathbb{N}^E$, we say that $n \in \mathcal{F}_B$ if the classes of B for the $\xleftrightarrow{n}$ equiv. relation all have even cardinal.



Then
$$\sum_{\substack{\partial m=A\\ \partial n=B}} w(m)\,w(n)\,F(m+n) = \sum_{\substack{\partial m=A\Delta B\\ \partial n=\emptyset}} w(m)\,w(n)\,\mathbb{1}_{m+n\in\mathcal{F}_B}.$$

("same" proof!)



- Let $S \subset V$, and $E_S = \{ \{x y\} \in E \,/\, x \in S, y \in S \}$.

For $n \in \mathbb{N}^{E_S}$, let $w_S(n) = \prod_{e \in E_S} \frac{J_e^{n_e}}{n_e!}$. Also (for later), $\partial_{in} S := \{ x \in S \,/\, \exists y \notin S \text{ st } \{x y\} \in E \}$

Then for $x, y \in S$, $A \subset E$ even,

$$\sum_{\substack{m \in \mathbb{N}^E / \partial m = A \\ n \in \mathbb{N}^{E_S} / \partial n = \{x y\}}} w(m)\, w_S(n) = \sum_{\substack{m \in \mathbb{N}^E / \partial m = A \Delta \{x y\} \\ n \in \mathbb{N}^{E_S} / \partial n = \emptyset}} w(m)\, w_S(n)\, \mathbb{1}_{x \overset{m|_{E_S} + n}{\longleftrightarrow} y}$$

Exercise: Using the 1^{st} generalization, prove the 2^{nd} Griffith ineq. for $\mu^+_{\Lambda, \beta, h}$ $(h > 0)$

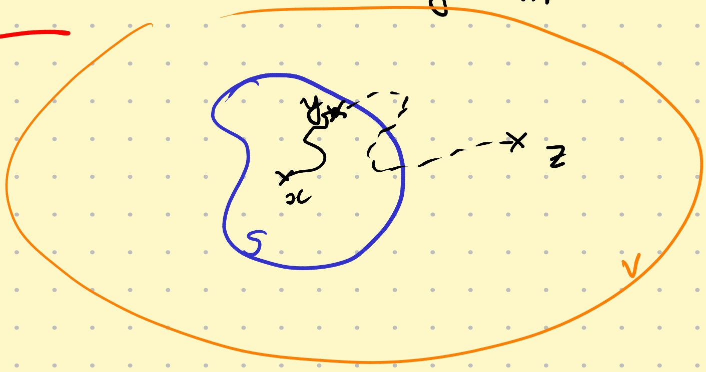
III - A factory of inequalities

As we just saw, the switching lemma is useful to prove inequalities. We will exemplify again with two ineq: First the Simons-Lieb ineq (which we will use to show exp. decay of correlations for $\beta < \beta_c$), and the GHS ineq (which we used before to show uniqueness of Gibbs measure at $h \neq 0$)

Theo: (SL inequality)

Let $S \subset V$, $x \in S$, $z \in V \setminus S$. Then

$$\langle \sigma_x \sigma_z \rangle_{G,J} \leq \sum_{y \in \partial_{in} S} \langle \sigma_x \sigma_y \rangle_{S, J|_{E_S}} \langle \sigma_y \sigma_z \rangle_{G,J}$$



Proof:

$$\sum_{y\in\partial_{in}S} \langle\sigma_x\sigma_y\rangle_S\langle\sigma_y\sigma_z\rangle$$

$$= \frac{1}{Z_S Z}\sum_{y\in\partial_{in}S}\ \sum_{\substack{m\in\mathbb{N}^E \\ \partial m=\{x z\}, \\ n\in\mathbb{N}^{E_S} \\ \partial n=\{y z\}}} w(m)\, w_S(n)$$

where
$Z=\sum_{\partial m=\emptyset} w(m)$
$Z_S=\sum_{\partial n=\emptyset} w_S(n)$

Switch!

$$= \frac{1}{Z_S Z}\sum_{y\in\partial_{in}S}\ \sum_{\substack{\partial m=\{x z\} \\ \partial n=\emptyset}} w(m)\, w_S(n)\, 1_{y \overset{n+m|_S}{\longleftrightarrow} z}$$

$$= \frac{1}{Z_S Z}\sum_{\substack{\partial m=\{x z\} \\ \partial n=\emptyset}} w(m)\, w_S(n) \underbrace{\left(\sum_{y\in\partial_{in}S} 1_{y \overset{n+m_S}{\longleftrightarrow} z}\right)}$$

≥ 1: since $\partial m=\{x z\}$,
$x \overset{m}{\longleftrightarrow} z$
so $\exists\, y\in\partial_{in}S$ st
$x \overset{m|_S}{\longleftrightarrow} y$.

$$\geq \frac{1}{Z_S Z}\sum_{\substack{\partial m=\{x z\} \\ \partial n=\emptyset}} w(m)\, w_S(n)$$

$$= \langle\sigma_x\sigma_z\rangle_{G,\beta}. \quad \boxtimes$$

Theo (GHS ineq, version 2)
Let $x,y,z,t\in V$ be all distinct.

Then $\langle xyzt\rangle - \langle xy\rangle\langle zt\rangle - \langle xz\rangle\langle yt\rangle - \langle xt\rangle\langle yz\rangle$
$+2\langle xt\rangle\langle yt\rangle\langle zt\rangle \leq 0.$

For $A\subset V$ we denote
$\langle A\rangle := \langle\sigma_A\rangle_{G,\beta}$

__Exercise__ Deduce from this GHS ineq that for $\Lambda,\beta,h>0$, t.b.c,
$\langle xyz\rangle^+ - \langle xy\rangle^+\langle z\rangle^+ - \langle xz\rangle^+\langle y\rangle^+ - \langle x\rangle^+\langle yz\rangle^+ + 2\langle x\rangle^+\langle y\rangle^+\langle z\rangle^+ \leq 0$
as we wanted before.

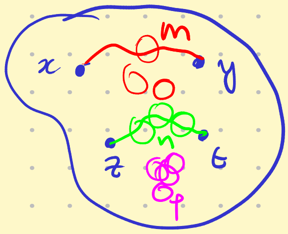
To prove this theo, we need a lemma:

Lemma: $\displaystyle\sum_{\substack{\partial m = xy\\ \partial n = zt}} w(m)w(n)1_{x \overset{m+n}{\leftrightarrow} t} \geq \langle zt\rangle \sum_{\substack{\partial m = xy\\ \partial n = \emptyset}} w(m)w(n)1_{x \overset{m+n}{\leftrightarrow} t}$

Remark: writing $\langle zt\rangle = \dfrac{\sum_{\partial p = zt} w(p)}{\sum_{\partial p = \emptyset} w(p)}$, this means

$$\sum_{\substack{\partial m = xy\\ \partial n = zt\\ \partial p = \emptyset}} w(m)w(n)w(p)\, 1_{x \overset{m+n}{\leftrightarrow} t} \geq \sum_{\substack{\partial m = xy\\ \partial n = \cancel{\emptyset}\ zt\\ \partial p = \cancel{zt}\ \emptyset}} w(m)w(n)w(p)\, 1_{x \overset{m+\cancel{n}\,p}{\leftrightarrow} t}$$

change var $n \leftrightarrow p$.



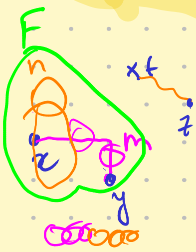
"it is easier to connect $x \leftrightarrow t$ with $m+n$ than it is with $m+p$ (because n has already a path $z \leftrightarrow t$.)

Proof of lemma:

It is enough to prove

$$\boxed{\sum_{\substack{\partial m = xy\\ \partial n = zt}} w(m)w(n)\, 1_{x \overset{m+n}{\not\leftrightarrow} t}} \leq \langle zt\rangle \sum_{\substack{\partial m = xy\\ \partial n = \emptyset}} w(m)w(n)\, 1_{x \overset{m+n}{\not\leftrightarrow} t}$$

(the quantities are $Z^{xy}Z^{zt}$ − these ↗, where $Z^A = \sum_{\partial n = A} w(n)$)



$$= \sum_{\substack{F \subset E\\ F \text{ connected}\\ x \in F\\ t \notin F}} \sum_{\substack{\partial m = xy\\ \partial n = zt}} w(m)w(n)\, \mathbb{1}_{\text{Cluster of } x \text{ in } \{m+n>0\} = F}\,;$$

i.e. $\forall e \in F$, $m_e + n_e > 0$ and $\forall e \in \partial F$ (i.e. $e \notin F$ and $\exists e' \in F / e \sim e'$) $m_e = n_e = 0$.

Given F, we split m into $m_1 = m|_F$ "$F \cup \partial F$" and $m_2 = m|_{F^c}$.
Then $w(m) = w(m_1)w(m_2)$.

$$= \sum_{F \dots} \left(\sum_{\substack{\partial m_1 = xy\\ \partial n_1 = \emptyset}} w(m_1)w(n_1)\, \mathbb{1}_{C_x(m_1+n_1) = F} \right) \underbrace{\left(\sum_{\substack{\partial m_2 = \emptyset\\ \partial n_2 = tz}} w(m_2)w(n_2) \right)}_{Z^{\emptyset}_{F^c}\, Z^{tz}_{F^c} \,=\, (Z^{\emptyset}_{F^c})^2 \langle \sigma_t \sigma_z \rangle_{F^c} \,\underset{\text{Griffith}}{\leq}\, (Z^{\emptyset}_{F^c})^2 \langle tz \rangle}$$

$$\text{So} \leq \langle tz \rangle \sum_{F\ldots} \Big(\sum_{\substack{\partial m_1 = xy \\ \partial n_1 = \emptyset}} w(m_1) w(n_1) \mathbb{1}_{C_x(m_1+n_1)=F} \Big) \Big(\sum_{\substack{\partial m_2 = \emptyset \\ \partial n_2 = \emptyset}} w(m_2) w(n_2) \Big)$$

$$= \langle tz \rangle \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} w(m) w(n) \mathbb{1}_{x \overset{m+n}{\not\leftrightarrow} t} \qquad \blacksquare$$

Proof of GHS ineq:

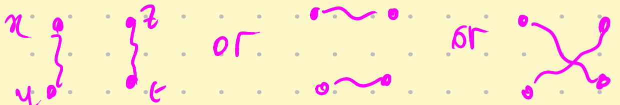
First, $\langle xy \rangle \langle zt \rangle = \frac{1}{(Z^\emptyset)^2} \sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m) w(n) \overset{\text{switch}}{=} \frac{1}{Z^2} \sum_{\substack{\partial m = xyzt \\ \partial n = \emptyset}} w(m) w(n) \mathbb{1}_{z \overset{m+n}{\leftrightarrow} t}$

By doing this on all terms, we get:

$$\langle xyzt \rangle - \langle xy \rangle \langle zt \rangle - \langle xz \rangle \langle yt \rangle - \langle xt \rangle \langle yz \rangle$$

$$= \frac{1}{Z^2} \sum_{\substack{\partial m = xyzt \\ \partial n = \emptyset}} w(m) w(n) \Big[1 - \mathbb{1}_{z \overset{m+n}{\leftrightarrow} t} - \mathbb{1}_{y \overset{m+n}{\leftrightarrow} t} - \mathbb{1}_{x \overset{m+n}{\leftrightarrow} t} \Big]$$

m looks like



so out of the 3 events, either 1 or 3 happen.

$$\text{So } 1 - \mathbb{1}_{..} - \mathbb{1}_{..} - \mathbb{1}_{..} = -2\,\mathbb{1}_{x,y,z,t \text{ all connected}}$$

$$= -2\,\mathbb{1}_{x \leftrightarrow t} \mathbb{1}_{z \leftrightarrow t}.$$

By switching back,

$$= \frac{-2}{Z^2} \sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m) w(n) \mathbb{1}_{x \overset{m+n}{\leftrightarrow} t}$$

$$\overset{\text{lemma}}{\leq} -2 \frac{\langle zt \rangle}{Z^2} \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} w(m) w(n) \mathbb{1}_{x \overset{m+n}{\leftrightarrow} t}$$

$$\overset{\text{switch}}{=} -2 \frac{\langle zt \rangle}{Z^2} \sum_{\substack{\partial m = yt \\ \partial n = xt}} w(m) w(n) = -2 \langle zt \rangle \langle yt \rangle \langle xt \rangle. \qquad \blacksquare$$

IV - Sharpness of phase ~~transition~~

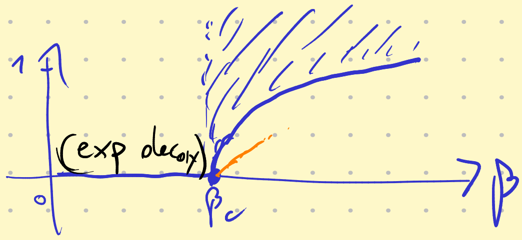
We now have all the tools to greatly improve our understanding of the Ising measures $\mu^\pm$ on $\mathbb{Z}^d$ from the last chapter. For instance, recall that $m^+(\beta) = \lim_{n\to\infty} \downarrow \langle\sigma_0\rangle^+_{\Lambda_n,\beta}$ $(h=0)$,

$$\beta_c = \{\inf \beta > 0 \,/\, m^+(\beta) > 0\}.$$

We will show:

Theo (Sharpness of phase transition) Aizenman - Barsky - Fernandez '87, Duminil-Copin - Tassion '18

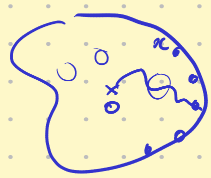
- $\forall \beta < \beta_c$, $\exists c > 0 \,/$ $\forall n$, $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \leq e^{-cn}$.
- $\forall \beta \geq \beta_c$, $m^+(\beta) \geq \dfrac{\sqrt{\beta-\beta_c}}{1+\sqrt{\beta-\beta_c}}$. "mean Field lower bound"



We will prove only $\geq \dfrac{\beta-\beta_c}{1+\beta-\beta_c}$.

In fact with the same tools but a bit more work, it is possible to show continuity of the phase transition $(m^+(\beta_c) = 0)$. See Duminil-Copin's lecture notes from PIMS 2017.

Proof of the theo: For $S \subset\subset \mathbb{Z}^d$, with $0 \in S$, let



$$\phi_\beta(S) := \sum_{x \in \partial_{in} S} \langle\sigma_0\sigma_x\rangle^{free}_S$$

(in the R.C. representation, $\phi_\beta(S) = \dfrac{1}{\sum_{\partial n = \emptyset} w(n)} \sum_{x\in\partial_{in} S} \sum_{\partial n = \{0x\}} w(n)$)

We will show:

1) If $\exists S \subset\subset \mathbb{Z}^d \,/\, \phi_\beta(S) < 1$, then $\exists c(\beta) > 0$ s.t. $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \leq e^{-cn}$
$\quad 0 \in S$

2) If $\forall S \subset\subset \mathbb{Z}^d$, $\phi_\beta(S) \geq 1$, then $\exists c(\beta)$ s.t. $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \geq c > 0$
$\quad 0 \in S$

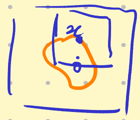
These will imply that $\tilde\beta_c := \sup\{\beta > 0 \,/\, \inf_{0\in S\subset\subset\mathbb{Z}^d} \phi_\beta(S) < 1\} = \beta_c$.

(and the first part of the theo).

1) Under this assumption, let $K = \mathrm{diam}(S)$ and $n > 2K$.

Recall that $\mu^+_{\Lambda_n,\beta} = \mu_{G,J}(\cdot \mid \sigma_g = +1)$ for a certain G, J

Then $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} = \langle\sigma_0\sigma_g\rangle_{G,J}$ as proven before



$$\leq \sum_{x\in\partial_{in}S} \langle\sigma_0\sigma_x\rangle_S \underbrace{\langle\sigma_x\sigma_g\rangle_G}$$

$$= \langle\sigma_x\rangle^+_{\Lambda_n,\beta} \leq \langle\sigma_0\rangle^+_{\Lambda_{n-K},\beta}$$ by monotonicity in domain.

$$\leq \varphi_\beta(S)\,\langle\sigma_0\rangle^+_{\Lambda_{n-K},\beta}$$

so ... $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \leq c\,(\varphi_\beta(S))^{\frac{n}{K}}$ and $\varphi_\beta(S)^{\frac{1}{K}} < 1$ ...

2) First, for G, J (corresponding to Λ_n), we want to show:

(*) $$\frac{\partial}{\partial\beta}\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \geq \frac{1}{Z^2}\sum_{\substack{\partial m=\emptyset\\ \partial n=\emptyset}} \phi_\beta(C_g^c)\, w(m)w(n)\, 1_{0 \overset{m+n}{\not\leftrightarrow} g}$$

$Z = \sum_{\partial n=\emptyset} w(n)$

where C_g is the CC of g for $\overset{m+n}{\longleftrightarrow}$.

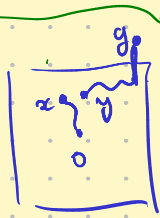
• $$\frac{\partial}{\partial\beta}\langle\sigma_0\rangle^+_{\Lambda_n,\beta} = \frac{\partial}{\partial\beta}\langle\sigma_0\sigma_g\rangle_G = \sum_{\{x,y\}\in E} \langle 0xyg\rangle - \langle 0g\rangle\langle xy\rangle$$

all edges of G (incl. touching g)

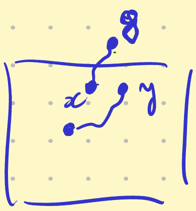
$$= \sum_{\{xy\}\in E}\frac{1}{Z^2}\left(\sum_{\substack{\partial m=0xyg\\ \partial n=\emptyset}} w(m)w(n) - \sum_{\substack{\partial m=0g\\ \partial n=xy}} w(m)w(n)\right)$$

switch

$$= \frac{1}{Z^2}\sum_{\{xy\}\in E}\underbrace{\left(\sum_{\substack{\partial m=0xyg\\ \partial n=\emptyset}} w(m)w(n)\, 1_{x \overset{m+n}{\not\leftrightarrow} y}\right)}$$



or



(connect° in m+n)

• $$\sum_{\substack{\partial m=\emptyset\\ \partial n=\emptyset}} \phi_\beta(C_g^c)\, w(m)w(n)\, 1_{0 \not\leftrightarrow g} = \sum_{\substack{S\sqcup C=V\\ 0\in S}}\ \sum_{\substack{\partial m=\emptyset\\ \partial n=\emptyset}} w(m)w(n)\, 1_{C_g=C}\, \phi_\beta(S)$$

↑

But $\phi_\beta(S) = \sum_{x \in \partial_{in} S} \frac{Z_S^{0x}}{Z_S^{\phi}} \leq \sum_{\substack{x \sim y \\ x \in S \\ y \in C}} \frac{Z_S^{0x}}{Z_S^{\phi}}$

$\sum_{\partial n_S = 0x} w(n_S)$

By decomposing m, n into the part in S and that in C (since they are 0 in between),

$$= \sum_{\substack{S \sqcup C = V \\ 0 \in S}} \Big(\sum_{\substack{\partial m_C = \phi \\ \partial n_C = \phi}} w(m_C) w(n_C) 1_{C_g = C} \Big) \underbrace{\Big(\sum_{\substack{\partial m_S = \phi \\ \partial n_S = \phi}} w(m_S)\, w(n_S) \Big)}_{= (Z_S^{\phi})^2} \phi_\beta(S)$$

$$\leq \sum_{\{x y\} \in E} \sum_{\substack{S \sqcup C = V \\ 0 \in S \\ x \in S \\ y \in C}} \Big(\sum_{\substack{\partial m_C = \phi \\ \partial n_C = \phi}} w(m_C) w(n_C) 1_{C_g = C} \Big) Z_S^{0x} Z_S^{\phi}$$

$$= \sum_{\{xy\} \in E} \sum_{\substack{S \sqcup C = V \\ 0 \in S \\ x \in S \\ y \in C}} \sum_{\substack{\partial m = 0x \\ \partial n = \phi}} w(m) w(n) 1_{C_g = C}$$

$$= \sum_{\{xy\} \in E} \sum_{\substack{\partial m = 0x \\ \partial n = \phi}} w(m) w(n)\, 1_{g \leftrightarrow y}\, 1_{x \not\leftrightarrow y}.$$

$$= \sum_{\{xy\} \in E} \sum_{\substack{\partial m = 0xyg \\ \partial n = yg}} w(m) w(n)\, 1_{x \not\leftrightarrow y}$$

almost the lemma we had for GHS...
Exercise!

$$\leq \sum_{xy} \underbrace{\langle gy \rangle}_{\leq 1} \sum_{\substack{\partial m = 0xyg \\ \partial n = \phi}} w(m) w(n)\, 1_{x \not\leftrightarrow y}$$

$$\leq Z \frac{\partial}{\partial \beta} \left(\langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \right).$$

Now that we have $(*)$, we get

$$\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \geq \inf_S \phi_\beta(S) \underbrace{\frac{1}{Z^2} \Big(\sum_{\substack{\partial m = \phi \\ \partial n = \phi}} w(m) w(n)\, 1_{0 \not\leftrightarrow g} \Big)}_{= 1 - \langle \sigma_0 \rangle^+}$$ (proven previously)

So in our regime, $f' \geq 1 - f^2$ where $f(\beta) = \langle \sigma_0 \rangle^+_{\Lambda_n, \beta}$

$\geq 1 - f$

so $\frac{f'}{1-f} \geq 1$. We integrate between β_0 and β:

$$\log\left(\frac{1-f(\beta_0)}{1-f(\beta)}\right) \geq \beta - \beta_0$$

$$> \log(1 + \beta - \beta_0)$$

so... $f(\beta) \geq \frac{\beta - \beta_0}{1 + \beta - \beta_0}$. $\boxtimes$

Exercise: • Show that for $\beta < \beta_c$, $\exists\, c(\beta)$ s.t.

$\forall x, y$, $\langle \sigma_x \sigma_y \rangle^+_\beta \leq \exp(-c\|x-y\|_\infty)$

↑ measure on $\mathbb{Z}^d$

and

$\chi(\beta) = \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle^+_\beta < \infty$ "susceptibility".

• Show that for $\beta > \beta_c$, $\exists\, c(\beta)$ s.t.

$\forall x, y$, $\langle \sigma_x \sigma_y \rangle^+_\beta \geq c$. "long-range order".