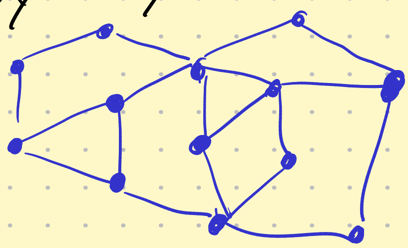
Exact solution on $\mathbb{Z}^2$

In this chapter we give a glimpse on "integrability" features of the Ising model on planar graphs. Roughly speaking, this means that several quantities like the free energy F and magnetization m can be expressed using previously known, classic functions.

I. The Kac-Ward theorem

A) On a planar graph

We start with a finite planar graph $G=(V,E)$ embedded in the plane with edges being straight lines (such an embedding exists by Fáry's theorem).



Let $(J_e)_{e\in E}$ be positive coupling constants on the edges.

We know that, by High Temperature Expansion, the Ising partition function (at $\beta=1$, $h=0$, $J=(J_e)$),

$$Z_{G,J} = 2^{|V|} \prod_{e\in E} \cosh(J_e) \cdot \sum_{\substack{H\subset E\\ \partial H=\emptyset}} \prod_{e\in H} \underbrace{\tanh(J_e)}_{=:x_e}$$

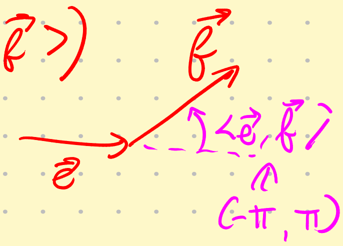
H even everywhere

Let $Z' = \sum_{\substack{H\subset E\\ \partial H=\emptyset}} \prod_{e\in E} x_e$.

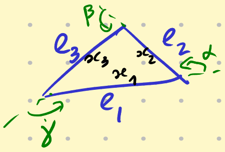
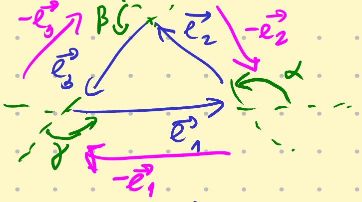
every $e\in E$ gives 2 oriented edges, oriented $\vec{e}$ and $-\vec{e}$

Let $\vec{E}$ be the oriented edges of G $(|\vec{E}|=2|E|)$, and let Λ be the square matrix indexed by $\vec{E}$ given by

$$\forall \vec{e},\vec{f}\in\vec{E},\quad \Lambda_{\vec{e},\vec{f}} = \begin{cases} x_e \exp\left(\frac{i}{2}\langle\vec{e},\vec{f}\rangle\right) & \text{if } \\ 0 & \text{otherwise} \end{cases}$$



For instance, $\Lambda_{\vec{e},-\vec{e}}=0$

Ex: $G=$  $\rightsquigarrow$ 

$$\Lambda = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc|c} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & -\vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 & \\ \end{array}$$

$$\Lambda = \left(\begin{array}{cccccc} 0 & x_1 e^{i\alpha/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 e^{i\beta/2} & 0 & 0 & 0 \\ & & \vdots & & & \end{array}\right)\begin{array}{c} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \end{array}$$

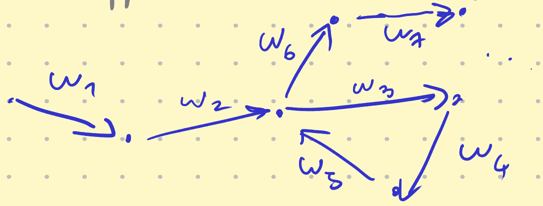
Theo (Kac-Ward, '52)

$$\det(I-\Lambda) = (Z')^2 .$$

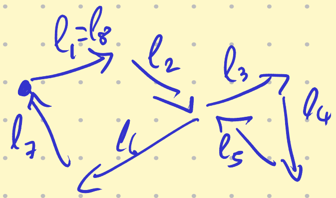
We show a clever "short proof" due to Lis ('15).

Proof: Notations: $(\omega_1, \dots, \omega_{n+1})$

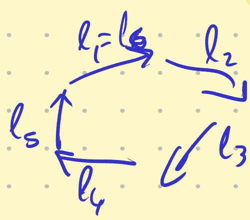
- A path ω " is a sequence of oriented edges st the end of one is the beginning of the next.
  We suppose that $\forall i$, $\omega_i \neq -\omega_{i+1}$ ("non backtracking")



- A loop $\ell = (\ell_1, \dots, \ell_{n+1})$ is a path s.t. $\ell_1 = \ell_{n+1}$



- A loop is self-avoiding if no vertex is re-used.



($n=6$ is the number of used edges)

- For a path $\omega = (\omega_1, \dots, \omega_{n+1})$, we define
  $x(\omega) = \prod_{i=1}^{n} x_{\omega_i}$

$$\lambda(\omega) = \prod_{i=1}^{n} \Lambda_{\omega_i,\omega_{i+1}} = x(\omega)\,\exp\left(\frac{i}{2}\sum_{i=1}^{n}\langle \omega_i,\omega_{i+1}\rangle\right)$$

For a loop $\ell=(\ell_1,\dots,\ell_{n+1})$

$$w(\ell)=\frac{\lambda(\ell)}{n}.$$

We extend x, λ, w into measures on the set of all paths/loops.

Lemma 1:

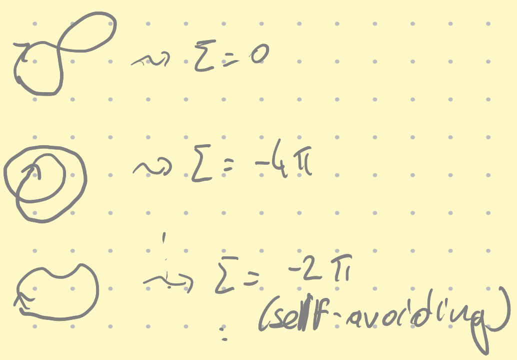
- If $\omega=(\omega_1,\dots,\omega_k,\omega_{k+1},\dots,\omega_{n+1})$, $\underbrace{\omega_1,\dots,\omega_k}_{\omega'}$, $\underbrace{\omega_k,\omega_{k+1},\dots,\omega_{n+1}}_{\omega''}$

  then $\lambda(\omega)=\lambda(\omega')\,\lambda(\omega'')$.
- For a loop ℓ, $\lambda(\ell^{-1}) = \lambda(\ell) = \pm x(\ell)$

  ↑ reverse order and all edges

  and for a self-avoiding loop ℓ, $\lambda(\ell) = -x(\ell)$
- For ω going from $\vec{e}$ to $-\vec{e}$, $\lambda(\omega) = -\lambda(\omega^{-1})$
  $= \pm i\, x(\omega)$.

[Simple checks:
for a loop, $\sum$ angles $= 0\ [2\pi]$.
and for self-av. loop, $= \pm 2\pi$.
. . .)



Let $\mathcal{L}$ be the set of all loops.

There are at most $|V|\,\Delta\,(\Delta-1)^{n-1}$ loops of length n, where $\Delta = \max_{v\in V} \deg v$

We now suppose $\|x\|_\infty < \frac{1}{\Delta-1}$, so that

$$x(\mathcal{L}),\ w(\mathcal{L}),\ \lambda(\mathcal{L}) \text{ are } < \infty.$$

Since both sides of the theorem are polynomials in $(x_e)_{e\in E}$, it is enough to prove it for $\|x\|_\infty$ small!

Lemma 2: $\det(I-\Lambda) = \exp(-w(\mathcal{L}))$.

proof: $w(\mathcal{L}) = \sum_{n\geq 1} \sum_{|\ell|=n} \frac{\lambda(\ell)}{n} = \sum_{n\geq 1} \frac{Tr(\Lambda^n)}{n} = \sum_{n\geq 1} \sum_{i=1}^{|E|} \frac{\alpha_i^n}{n}$

where (α_i) are the eigenvalues of Λ, with mult.

$$= \sum_i \sum_{n\geq 1} \frac{\alpha_i^n}{n}$$

$$= -\ln\left(\prod_i 1-\alpha_i\right)$$

$$= -\ln(\det(I-\Lambda)) \qquad \square$$

The idea is now to look at the dependency of $\det(I-\Lambda)$ in x_e for a given $e\in E$. We have

$$\det(I-\Lambda) = \exp(-w(\mathcal{L}))$$

$$= A \exp\left[-\underbrace{w(\{\ell \text{ visits } \vec{e} \text{ or } -\vec{e}\})}\right]$$

where A indep of x_e.

We work on this quantity now.

Lemma 3: $w(\{\ell \text{ visits } \vec{e} \text{ and } -\vec{e}\}) = 0$.

$\forall \vec{e}\in\vec{E}$.

proof: On the set of loops that visit $\vec{e}$ and $-\vec{e}$, we consider an involution that reverses the walk between the first $\vec{e}$ and the last $-\vec{e}$:

$$\ell = (\ell_1, \ell_2, \dots, \underbrace{\vec{e}, \dots\dots, -\vec{e}}_{\text{a path } \omega \text{ from } \vec{e} \text{ to } -\vec{e}}, \ell_{p+1}, \dots, \ell_{n+1})$$

$$\ell' = (\ell_1, \ell_2, \dots, \underbrace{\qquad \omega^{-1} \qquad}, \ell_{p+1}, \dots, \ell_{n+1}).$$

(Similarly if $-\vec{e}$ comes first).

By Lemma 1, $\lambda(\omega) = -\lambda(\omega^{-1})$

so $\lambda(\ell') = -\lambda(\ell)$ (concatenation)

so $w(\ell') = -w(\ell)$

and it is an involution ... $\square$

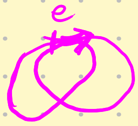
As a result, $\det(I-\Lambda) = A \exp\left[-2\, w(\{\ell \text{ visits } \vec{e} \text{ and not } -\vec{e}\})\right]$.

(by Lemma 3 and symmetry)

Lemma 4 $\exp(-w\{\ell \text{ visits } \vec{e} \text{ and not } -\vec{e}\}) = 1-\lambda(\{\ell \text{ starts from } \vec{e},\ \text{visits } \vec{e} \text{ once, doesn't visit } -\vec{e}\})$

Proof: $w(\{\ell \text{ visits } \vec{e} \text{ and not } -\vec{e}\})$

$$= \sum_{\substack{\ell \text{ vis. } \vec{e} \\ \text{not } -\vec{e}}} \frac{\lambda(\ell)}{|\ell|} = \sum_{\substack{\ell \text{ starts from } \vec{e} \\ \text{doesn't visit } -\vec{e}}} \frac{\lambda(\ell)}{|\text{visits of } \ell \text{ at } \vec{e}|}$$

ex: $\ell =$  done 3 times ⇝ 6 visits at $\vec{e}$,

$|\ell| = 3n$,

n shifted loops and 2 of them start at $\vec{e}$.

On the l.h.s, total contrib. of the n loops: $n \cdot \frac{\lambda(\ell)}{3n}$. On r.h.s, $2 \cdot \frac{\lambda(\ell)}{6}$.

$$= \sum_{k \geq 1} \sum_{\substack{\ell_1, \dots, \ell_k \\ \text{start at } \vec{e}, \\ \text{visit } e \text{ once}}} \frac{\lambda(\ell_1 \cdot \ell_2 \cdots \ell_k)}{k} \leftarrow = \lambda(\ell_1) \cdots \lambda(\ell_k)$$

visits e once so doesn't visit $-\vec{e}$

$$= -\ln\Big(1 - \sum_{\substack{\ell \text{ start at } \vec{e}, \\ \text{visit } e \text{ once}}} \lambda(\ell)\Big) \quad \square$$

Therefore, $\det(I-\Lambda) = A \exp(-2\, w(\ell \text{ visits } \vec{e} \text{ and not } -\vec{e}))$
$= A\,(1-\lambda(\ell \text{ starts at } \vec{e}, \text{ visits } e \text{ once}))^2$

is of the form

$A\,(a + b\, x_e)^2$. A, a, b indep of x_e.

This is the case for all $e \in E$.

This implies that $\det(I-\Lambda) = (P(x_e)_{e\in E})^2$ where P is a polynomial of max degree 1 in each x_e

(sthg like $P(x_e)_{e\in E} = 3 - 2x_{e_1}x_{e_2} + 7x_{e_1}x_{e_3}x_{e_4} + \dots$)

We now show that $P(x_e)_{e\in E} = Z'(x_e)_{e\in E}$.

For this, we write $\sim$ the equivalence relation on power series in the $(x_e)_{e\in E}$ of having the same coefs on monomials that have degree ≤ 1 in each x_e.

For instance $1+4x_{e_1}-6x_{e_1}x_{e_2}+7x_{e_3}^2 \sim 1+4x_{e_1}-6x_{e_1}x_{e_2}-3x_{e_2}x_{e_3}^2$

Then,

$$P(x_e)_{e\in E} = \det(I-\Lambda)^{1/2} = \exp\left(-\tfrac{1}{2}\, w(\mathcal{L})\right)$$

$$\sim \exp\Big(-\frac{1}{2}\sum_{\ell \text{ auto-ev loop}} w(\ell)\Big)$$

$$= \exp\Big(\sum_{\ell \text{ auto-ev loop}} \frac{x(\ell)}{2|\ell|}\Big)$$

$2|\ell|$ rooted loops correspond to the same (unrooted) cycle
Recall: this cycle is vertex-transitive.

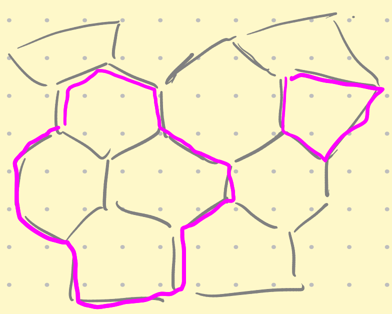
$$= \exp\Big(\sum_{C \text{ cycle of } G} \prod_{e\in C} x_e\Big)$$

$$\sim \sum_{k\ge 0}\frac{1}{k!}\sum_{C_1,\dots,C_k \text{ disjoint cycles}} \prod_{e\in C_1\cup\dots\cup C_k} x_e$$

don't use the same edges

$$= \sum_{k\ge 0}\ \sum_{\substack{\{C_1,\dots,C_k\}\\ \text{set of } k \text{ disj. cycles}}} \prod_{e\in C_1\cup\dots\cup C_k} x_e$$

Now, suppose that G is trivalent, that is, all its vertices have degree ≤ 3. Then every $H\subset E$ st $\partial H=\emptyset$ can be written uniquely as a union of disjoint cycles:

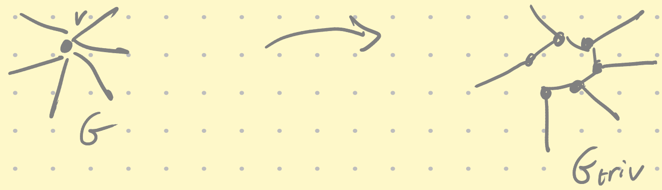


So in that case, we get $\det(I-\Lambda)^{1/2} \sim \sum_{\substack{H\subset E\\ \partial H=\emptyset}} \prod_{e\in H} x_e$

and therefore $\det(I-\Lambda) = (Z^{\emptyset})^2$.

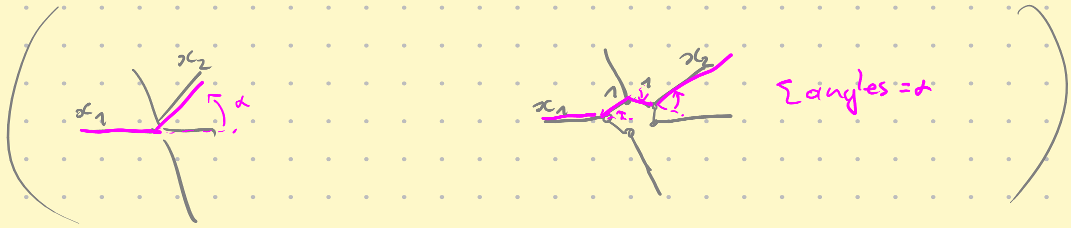
Now, if G is not trivalent, we create a trivalent decoration in the following way:

$\forall$ vertex $v \in G$,



ex:

Then G_{triv} is trivalent. Moreover, there is a one-to-one correspondence between loops on G and on G_{triv} ! (see example).
The same goes for even subgraphs H on G and G_{triv}.
If we set $x_e = 1$ on each "small" edge added in G_{triv}, we get that the bijection on loops preserves the weight λ, so w as well.



So in general,

$$\det(I - \Lambda_G) = \exp(-w(\mathcal{L}_G))$$

$$= \exp(-w(\mathcal{L}_{G_{triv}}))$$

$$= \Big(\sum_{\substack{H \subset E_{triv} \\ \partial H = \emptyset}} w(H)\Big)^2$$

$$= \Big(\sum_{\substack{H \subset E_G \\ \partial H = \emptyset}} w(H)\Big)^2 .$$

weight-preserving bij. on loops

previous case

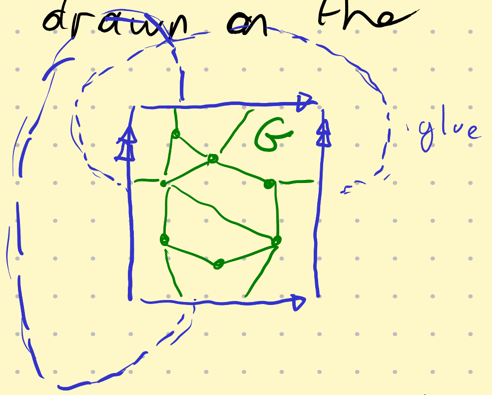
weight-preserving bij on even subgraphs

$\boxtimes$

B] On the torus

Our aim will be to compute $\det(I - \Lambda)$ for large pieces of $\mathbb{Z}^2$, and deduce the free energy. However, this big determinant is not easy to compute in general. But if we put periodic

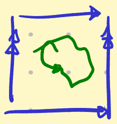
boundary conditions, we will actually be able to diagonalize Λ.
This is why we would like to adapt Kac-Ward's theo to graphs drawn on the torus $\mathbb{R}^2/\mathbb{Z}^2$. (same as periodic b.c.)

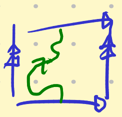


We suppose again that G is drawn with straight lines (on the torus). We can still define Λ.

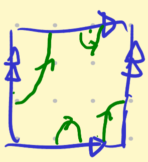
If we try to adapt the previous proof, we see that the first difference is in Lemma 1:

if ℓ is a self-avoiding loop on the torus, we don't necessarily have $\lambda(\ell) = -x(\ell)$.

Ex:  total winding $= -2\pi \rightsquigarrow \lambda(\ell) = -x(\ell)$

 total winding $= 0 \rightsquigarrow \lambda(\ell) = x(\ell)$.

Topology Facts: let ℓ be an oriented, self-avoiding loop on the torus. Let $a_\ell, b_\ell \in \mathbb{Z}$ be the number of oriented crossings of $\rightarrow$, $\uparrow$ respectively (# crossings from right - # crossings from left). Then



$a_\ell = 2-1 = 1$
$b_\ell = -1$

- either $(a_\ell, b_\ell) = (0,0)$, then total winding $= \pm 2\pi$ (so $\lambda(\ell) = -x(\ell)$)
- or a_ℓ, b_ℓ are s.t. $\gcd(a_\ell, b_\ell) = 1$, and then total winding $= 0$ (so $\lambda(\ell) = x(\ell)$)

Try examples!

In the second case, a, b cannot be both even.
So it is actually equivalent to $(a_\ell, b_\ell) \bmod 2 \in \{(1,0),(0,1),(1,1)\}$
while the first is equivalent to $(a_\ell, b_\ell) \bmod 2 = (0,0)$.

(The advantage of this remark is that we can just look at the number of crossings mod 2 instead of oriented crossings.)

The rest of the proof is unchanged, until we get that (for a trivalent graph),

$$\det(I-\Lambda)^{1/2} \sim \exp\left(-\frac{1}{2}\sum_{\ell \text{ autoev}} w(\ell)\right)$$

$$= \exp\left(\sum_{\ell \text{ autoev}} \varepsilon_\ell \frac{x(\ell)}{2|\ell|}\right), \quad \varepsilon_\ell = \begin{cases} +1 & \text{if } (a_\ell, b_\ell) \bmod 2 = (0,0) \\ -1 & \text{otherwise.} \end{cases}$$

$$\sim \sum_{k\geq 0} \sum_{\substack{C=\{C_1,\dots,C_k\} \\ \text{disj cycles}}} \varepsilon_C \cdot \left(\prod_{e\in C} x_e\right) \quad \text{where } \varepsilon_C = \prod_{i=1}^{k} \varepsilon_{C_i}$$

$$= \begin{cases} +1 & \text{if } C_1 \cup \dots \cup C_k \text{ cross both } \uparrow, \rightarrow \text{ an even number of times} \\ -1 & \text{otherwise.} \end{cases}$$

<u>Trick</u>: For $\sigma, \sigma' \in \{\pm 1\}$, let $\Lambda^{\sigma,\sigma'}$ be the matrix obtained from new weights obtained from x by changing x_e into σx_e | resp $\sigma' x_e$ when e crosses $\rightarrow$ | resp. $\uparrow$

Then $\det(I-\Lambda^{\sigma\sigma'})$ has an expression as a sum on C with new signs given by the array:

(σ,σ') \ (a_c,b_c)	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	
$(1,1)$	$+1$	-1	-1	-1	←(previous case)
$(1,-1)$	$+1$	-1	$+1$	$+1$	←(we multiply x by -1 every time we cross $\uparrow$)
$(-1,1)$	$+1$	$+1$	-1	$+1$	...
$(-1,-1)$	$+1$	$+1$	$+1$	-1	...

From this array, we get that

$$-\det(I-\Lambda^{1,1})^{1/2} + \det(I-\Lambda^{1,-1})^{1/2} + \det(I-\Lambda^{-1,1})^{1/2} + \det(I-\Lambda^{-1,-1})^{1/2}$$

$$= 2\cdot \sum_{k\geq 0} \sum_{\substack{C=\{C_1,\dots,C_k\} \\ \text{disjoint cycles}}} \prod_{e\in C} x_e .$$

The end of the proof, going from trivalent to generic, is identical.

We have proved:

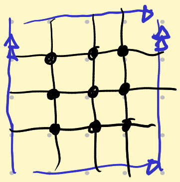
Theo Let G be a graph on the torus. Then

$$Z' = \frac{1}{2}\left| -\det(I - \Lambda^{1,1})^{1/2} + \det(I-\Lambda^{-1,1})^{1/2} + \det(I-\Lambda^{1,-1})^{1/2} + \det(I-\Lambda^{-1,-1})^{1/2} \right|$$

precisely: chose a root of $\det(I-\Lambda^{1,1})$ as a formal expression in the (x_e). Use the same root, with some of the x_e negated, for

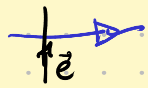
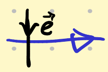
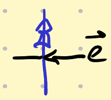
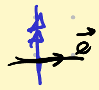
II - Free energy of Ising on $\mathbb{Z}^2$:

We saw that boundary conditions don't affect the value of the free energy. So we will work on a box of size n with periodic b.c.:

 $\leftarrow G_n$ $(n=3$ here$)$

We generalize the previous construction of $\Lambda^{\pm 1, \pm 1}$.

Let $z, w \in \mathbb{C}^*$, we define $\Lambda^{z,w}$ obtained from $\Lambda^{1,1}$ by multiplying $\Lambda_{\vec{e}, \vec{f}}$ by

- z if 
- $1/z$ if 
- w if 
- $1/w$ if 

We define $P_n(z,w) = \det(I - \Lambda^{zw}_{n})$ ← for G_n.

This is a polynomial in z, z^{-1}, w, w^{-1} (a Laurent polynomial).

We want to compute the asymptotic behavior of

$$\frac{1}{2}\left| -P_n(1,1)^{1/2} + P_n(1,-1)^{1/2} + P_n(-1,1)^{1/2} + P_n(-1,-1)^{1/2} \right|.$$

The following will be very helpful:

Theo $\forall z, w \in \mathbb{C}^*,\quad P_n(z,w) = \prod_{k=0}^{n-1} \prod_{\ell=0}^{n-1} P_1\Big(\underbrace{e^{\frac{2ik\pi}{n}} z^{1/n}}, \underbrace{e^{\frac{2i\ell\pi}{n}} w^{1/n}}\Big)$

i.e. the n solutions of $z'^n = z$ / $w'^n = w$.

Proof: Consider the operator Λ defined on the whole space, that is, on $\vec{E}(\mathbb{Z}^2)$. More precisely, if $f : \vec{E}(\mathbb{Z}^2) \to \mathbb{C}$, we define $(\Lambda f)_{\vec{e}} = \sum_{\vec{e}' \in \vec{E}(\mathbb{Z}^2)} \Lambda_{\vec{e},\vec{e}'} f_{\vec{e}'}$ where $\Lambda_{\vec{e},\vec{e}'}$ is defined geometrically as before.

As there is a finite number of $\vec{e}'$ s.t. $\Lambda_{\vec{e},\vec{e}'} \neq 0$, this is well-defined.

This linear operator on $\mathbb{C}^{\vec{E}(\mathbb{Z}^2)}$ has some finite-dimensional stable subspace. For instance, consider the subspace $V_n(z,w)$ of n-(z,w)-quasi periodic functions:

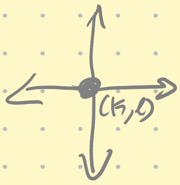
$$f \in V_n(z,w) \iff \forall \vec{e} \in \vec{E}, \begin{cases} f_{\vec{e}+\boxed{(0,n)}} = z f_{\vec{e}} \\ f_{\vec{e}+(n,0)} = \frac{1}{w} f_{\vec{e}} \end{cases}$$

translate $\vec{e}$ by $(0,n)$.

convention for later

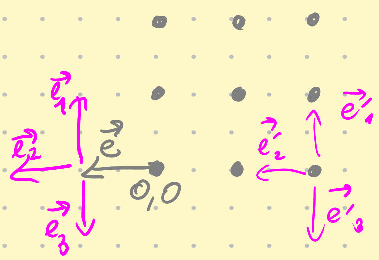
It is easy to check that $f \in V_n(z,w) \Rightarrow \begin{cases} \Lambda f \in V_n(z,w) \\ (I-\Lambda) f \end{cases}$

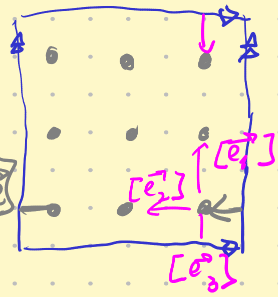
and $V_n(z,w)$ has dimension $4n^2$.

(choice of the values on  for $k, \ell \in [\![0; n-1]\!]$)

Moreover, $\Lambda\big|_{V_n(z,w)}$ on this basis is exactly $\Lambda_n^{z,w}$!

Ex: $n=3$:

on $\mathbb{Z}^2$: 

on G_3: $[\vec{e}]$ 

($f \in V_3(z,w)$ is sent to $[f]$ on G_3)

$$\Lambda f_{\vec{e}} = \sum_{i=1}^{3} \Lambda_{\vec{e},\vec{e}_i} f_{\vec{e}_i} = \sum_{i=1}^{3} \underbrace{\Lambda_{\vec{e},\vec{e}_i} \, w \, f_{\vec{e},\vec{e}_i}}$$

this is the value we put on $[f]_{[\vec{e}],[\vec{e}_i]}$

$$= \sum_{i=1}^{3} \Lambda^{z,w}_{[\vec{e}],[\vec{e}_i]} [f]_{[\vec{e}],[\vec{e}_i]}$$

Now we find eigenfunctions (eigenvectors) in $V_n(z,w)$ for Λ, which correspond to eigenfunctions of $\Lambda_n^{z,w}$

the 4×4 matrix $\Lambda_1^{z,w}$ depends on z,w, and by a direct computations, one checks that on an open set of $(\mathbb{C}^*)^2$ it has 4 distinct eigenvalues (compute the discriminant of its characteristic polynomial: it is a nonzero polynomial in $z^{\pm1}, w^{\pm1}$ so $\neq 0$ on an open set)

So for these z,w, there are 4 independent eigenfunctions for $\Lambda_1^{z,w}$, so there are 4 independent eigenfunctions of Λ in $V_1(z,w)$.

But $V_1(z,w) \subset V_n(z^n, w^n)$. (!)

so this gives 4 indep. eigenfunctions of $\Lambda_n^{z^n,w^n}$, and of $I-\Lambda_n^{z^n,w^n}$.

The product of their eigenvalues is $\det(I-\Lambda_1^{z,w}) = P_1(z,w)$.

Doing the same for $(e^{\frac{2ik\pi}{n}}z, e^{\frac{2il\pi}{n}}w)$ where $k,l \in [\![0;n-1]\!]$, we get $4n^2$ eigenfunctions of the same $\Lambda_n^{z^n,w^n}$. They are indep. as they belong to different subsets $V_1(e^{\frac{2ik\pi}{n}}z, e^{\frac{2il\pi}{n}}w)$ which are in direct sum for distincts (k,l). So we have found all eigenfunctions of $\Lambda_n^{z^n,w^n}$, and the product of eigenvalues gives

$$P_n(z^n,w^n) = \det(I-\Lambda_n^{z^n,w^n}) = \prod_{k,l} \det(I-\Lambda_1^{e^{\frac{2i\pi k}{n}}z,\, e^{\frac{2i\pi l}{n}}w})$$

$$= \prod_{k,l} P_1(e^{\frac{2ik\pi}{n}}z, e^{\frac{2il\pi}{n}}w).$$

This holds for z,w in an open set but both sides are polynomial, so they are equal for all z/w. Applying at any given n^{th} roots of z,w gives the theorem. $\square$

As a result, $P_n(1,1) = \prod_{0\le k,l\le n-1} P_1(e^{\frac{2ik\pi}{n}}, e^{\frac{2il\pi}{n}})$

so $\frac{1}{n^2}\log|P_n(1,1)| = \frac{1}{n^2}\sum_{0\le k,l\le n-1} \log|P_1(e^{\frac{2ik\pi}{n}}, e^{\frac{2il\pi}{n}})|$

$$\xrightarrow[n\to\infty]{} \underbrace{\iint_{[0,1]^2} \log|P_1(e^{2i\pi u}, e^{2i\pi v})|\,du\,dv}_{=: I}.$$

as a Riemann sum.

Similarly, $\frac{1}{n^2}\log|P_n(\pm 1, \pm 1)| \to I$.

Moreover, looking at the previous array,

$$0 \leq -P_n(+1,+1) \leq P_n(-1,+1) + P_n(+1,-1) + P_n(-1,-1)$$

so $\max_{\sigma,\sigma'} |P_n(\sigma,\sigma')| \leq |P_n(1,1)| \leq 3 \max_{\sigma,\sigma'} |P_n(\sigma,\sigma')|$

which implies that $\frac{1}{n^2}\log\left(\frac{1}{2}\left|-P_n(1,1)^{1/2} + P_n(-1,1)^{1/2} + P_n(1,-1)^{1/2} + P_n(-1,-1)^{1/2}\right|\right)$

$$\underset{n\to\infty}{\sim} \frac{1}{2n^2}\log\left(\max_{\sigma,\sigma'}|P_n(\sigma,\sigma')|\right)$$

$$\sim \frac{1}{2} I$$

Transforming I into a contour integral in the complex plane, which looks fancier, we get:

Theo We consider $G_n = \{0,..,n-1\}^2$ with periodic b.c. and constant weights x.

Then $\frac{1}{n^2}\log Z'(G_n) \xrightarrow[n\to\infty]{} \frac{1}{2\cdot(2i\pi)^2}\iint_{\mathbb{T}^2} \log\left(P_1(z,w)\right)\frac{dz}{z}\frac{dw}{w}$.

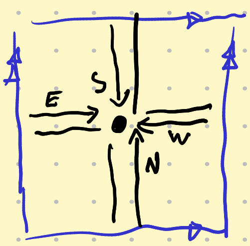
" $\{|z|=1, |w|=1\} \subset \mathbb{C}^2$.

where $P_1(z,w) = \det\left(I - \Lambda_1^{zw}\right)$

$$= 4x^4 - x^2\left(z+\frac{1}{z}\right)\left(w+\frac{1}{w}\right) + 1$$

$\Leftarrow$ For $|z|=|w|=1$, this is real and >0!

Just compute:

G_1 

$$\rightsquigarrow \Lambda_1^{zw} = \begin{array}{c} \\ N \\ S \\ E \\ W \end{array}\begin{array}{c} \begin{array}{cccc} N & S & E & W \end{array} \\ \begin{pmatrix} xz & 0 & xze^{-i\pi/4} & xze^{i\pi/4} \\ 0 & \frac{x}{z} & \frac{x}{z}e^{i\pi/4} & \frac{x}{z}e^{-i\pi/4} \\ \frac{x}{w}e^{i\pi/4} & \frac{x}{w}e^{-i\pi/4} & \frac{x}{w} & 0 \\ xwe^{-i\pi/4} & xwe^{i\pi/4} & 0 & xw \end{pmatrix}\end{array}$$

$\rightsquigarrow \det\left(I - \Lambda_1^{zw}\right) = \ldots$

Cor For the Ising model on $\mathbb{Z}^2$ with temperature $\beta > 0$ and $h=0$, the free energy is

$$f(\beta) = \log 2 + 2\log \mathrm{ch}\beta + \frac{1}{2\cdot(2i\pi)^2}\iint_{\mathbb{T}^2}\log\left(-\mathrm{th}^2\beta\left(z+\frac{1}{z}\right)\left(w+\frac{1}{w}\right) + \mathrm{th}^4\beta + 1\right)\frac{dz}{z}\frac{dw}{w}$$

Proof: By the beginning of the chapter,

$$Z_{G_n,\beta} = 2^{n^2} (\mathrm{ch}\,\beta)^{2n^2} Z'(G_n)$$

so $\frac{1}{n^2} \log Z_{G_n,\beta} \xrightarrow[n\to\infty]{} \log 2 + 2\log \mathrm{ch}\beta + \frac{1}{2} I$ For $x = \mathrm{th}\,\beta$ ...

$\boxtimes$