Exam 2 - Solutions

Ex 1: By translation-invariance, it is enough to show it for $x=0$.

Conditioning on $\sigma|_{\partial\Lambda_n}$, we have $(n=\|y\|_\infty)$

$$\langle\sigma_0\sigma_y\rangle = \sum_{\substack{\xi\in\{\pm1\}^{\partial\Lambda_n},\\ \xi_y=+1}} \langle\sigma_0\cancel{\sigma_y}\,|\,\sigma|_{\partial\Lambda_n}=\xi\rangle\,\langle 1_{\sigma|_{\partial\Lambda_n}=\xi}\rangle$$

$$\underline{\ast}\sum_{\substack{\xi\in\{\pm1\}^{\partial\Lambda_n}\\ \xi_y=-1}} \langle\sigma_0\cancel{\sigma_y}\,|\,\sigma|_{\partial\Lambda_n}=\xi\rangle\,\langle 1_{\sigma|_{\partial\Lambda_n}=\xi}\rangle$$

But in the 1st sum, $\langle\sigma_0|\sigma|_{\partial\Lambda_n}=\xi\rangle = \langle\sigma_0\rangle^{\xi}_{\Lambda_n} \overset{\text{monotonicity of b.c.}}{\le} \langle\sigma_0\rangle^{+}_{\Lambda_n} \le e^{-cn}$

and in the 2nd, $\langle\sigma_0|\sigma|_{\partial\Lambda_n}=\xi\rangle = \langle\sigma_0\rangle^{\xi}_{\Lambda_n} \ge \langle\sigma_0\rangle^{-}_{\Lambda_n} \overset{\text{symmetry}}{=} -\langle\sigma_0\rangle^{+}_{\Lambda_n} \ge -e^{-cn}$

so $\langle\sigma_0\sigma_y\rangle \le e^{-cn}\underbrace{\left(\sum_{\xi\in\{\pm1\}^{\partial\Lambda_n}}\langle 1_{\sigma|_{\partial\Lambda_n}=\xi}\rangle\right)}_{=1}$.

Ex 2: Let $\varepsilon>0$. Let $r>0$ be s.t.

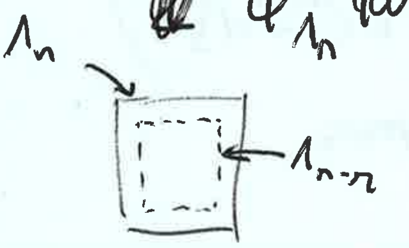
$$|\phi^1_{\Lambda_r}(1_{e_0\in\omega}) - d^1| \le \varepsilon.$$

e_0 := edge to the right of origin

(we drop the p,q everywhere)

Then for $n\ge0$ large,

$$\phi^1_{\Lambda_n}(|\omega|) = \sum_{e\in E_n}\phi^1_{\Lambda_n}(e\in\omega) \le \sum_{e\in E(\Lambda_{n-r})}\phi^1_{\Lambda_n}(e\in\omega) + \sum_{e\in E(\Lambda_n\setminus\Lambda_{n-r})} 1$$



② By monotonicity of domain, in the first sum,

$$\Phi^1_{\Lambda_n}(e \in \omega) \leqslant \Phi^1_{\Lambda_r}(e_0 \in \omega) \leqslant d^1 + \varepsilon$$

(even for e vertical by $\frac{\pi}{2}$-rotation invariance)

$$\text{So } \Phi^1_{\Lambda_n}(|\omega|) \leqslant |E(\Lambda_{n-r})|\,(d^1+\varepsilon) + \underbrace{|E(\Lambda_n) \setminus E(\Lambda_{n-r})|}_{O(n^{d-1}r)}$$

$$\text{so } \limsup_n \frac{1}{\underbrace{|E_n|}_{\substack{\leq \\ \text{cst}.\, n^d}}} \Phi^1_{\Lambda_n}(|\omega|) \leqslant d^1+\varepsilon.$$

So $\leq d^1$.

On the other hand, $\forall e \in E_n$, $\Phi^1_{\Lambda_n}(1_{e\in\omega}) \underset{n\to\infty}{\searrow} d^1$ (by def. of the ∞-vol measure & translation invariance)

so $\Phi^1_{\Lambda_n}(1_{e\in\omega}) \geqslant d^1$

and $\frac{1}{E_n}\Phi^1_{\Lambda_n}(|\omega|) \geqslant d^1$.

These give the convergence.

Ex3: By random currents expansions,

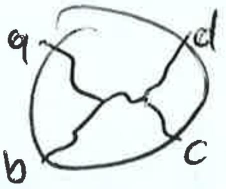
$$\langle ab\rangle\langle cd\rangle + \langle ad\rangle\langle bc\rangle - \langle abcd\rangle - \langle ac\rangle\langle bd\rangle$$

$$= \left(\frac{1}{Z_\emptyset^{\cancel{2}}}\right) \cdot \left[\sum_{\substack{m \in \mathbb{N}^E \\ \text{st } \partial m = \{ab\}, \\ n \in \mathbb{N}^E \\ \text{st } \partial n = \{cd\}}} \overset{\prod_e \frac{J_e^{m_e}}{m_e!}}{w(m)}\, w(n) + \sum_{\substack{\partial m = ad \\ \partial n = bc}} w(m)w(n) - \sum_{\substack{\partial m = abcd \\ \partial n = \emptyset}} w(m)w(n) - \sum_{\substack{\partial m = ac \\ \partial n = bd}} w(m)w(n) \right]$$

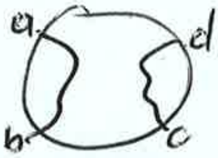
$$= \left(\frac{1}{Z_\emptyset^2}\right) \cdot \left(\sum_{\substack{\partial m = abcd \\ \partial n = \emptyset}} w(m)\, w(n) \cdot \left[1_{c \xleftrightarrow{m+n} d} + 1_{b \xleftrightarrow{m+n} c} - 1 - 1_{b \xleftrightarrow{m+n} d} \right] \right)$$

by switching lemmas.

③ As $\partial m = abcd$, ~~the connections~~ in $\{e / m_e \geq 1\}$, there has to be a pairwise connection of $\{a, b, c, d\}$. By planarity, this implies that the connections in $m+n$ are either



OR



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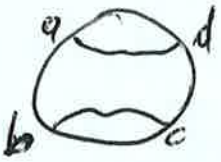


(but not  in particular)

One checks directly that in each case,

$$1_{c \leftrightarrow d} + 1_{b \leftrightarrow c} - 1 - 1_{b \leftrightarrow d} = 0.$$

This gives the equality. Then by Griffith $\langle abcd \rangle \geq \langle ac \rangle \langle bd \rangle$ which gives the inequality.

Ex4: 1) As in the course,

$$Z_I = 2^{|V|} \sum_{\substack{n \in \mathbb{N}^E \\ \text{st } \partial n = \phi}} w(n) \qquad \text{where } w(n) = \prod_e \frac{\beta^{n_e}}{n_e!} = \frac{\beta^{\sum_e n_e}}{\prod_e n_e!}.$$

2) In Edwards-Sokal, we showed that by sampling σ and then ~~[illegible]~~ taking $(\omega_e)_{e \in E}$ st

- $\forall e = \{xy\} / \sigma_x \neq \sigma_y$, $\omega_e = 0$
- $\forall e = \{xy\} / \sigma_x = \sigma_y$, $\omega_e = \begin{cases} 1 & \text{with pr. } 1 - e^{-2\beta} \\ 0 & \text{otherwise} \end{cases}$

all indep (conditionally on σ), then $\omega \sim FK(q = 2, p = 1 - e^{-2\beta})$.

④ In fact we had a joint measure ν s.t.

$$\nu(\omega,\sigma) = \frac{1}{Z_{FK}} \, p^{|\omega|} (1-p)^{|\omega^c|} \, \mathbb{1}_{(\omega,\sigma)\text{ compatible}}$$

with 2nd marginal μ the Ising measure.

We proved, For a fixed σ,

$$\mu(\sigma) = \sum_{\omega} \nu(\omega,\sigma) = \frac{1}{Z_{FK}} \exp\Big(\beta \sum_{\substack{e=\{x,y\} \\ \in E}} (\sigma_x \sigma_y - 1) \Big) \quad \leftarrow \text{see proof of E-S.}$$

$$= \frac{e^{-\beta|E|}}{Z_{FK}} \cdot Z_I \, \mu(\sigma)$$

$$\text{so } Z_I = e^{\beta|E|} Z_{FK}.$$

3) As $\partial N = \emptyset$, and $\forall e \in E$, $U_e \equiv N_e \; [2]$, we must have $\partial U = \emptyset$ as well.

Conversely, if $\partial u = \emptyset$, we see that such a u appears with positive proba (for instance with $n = u$)

If $u \in \{0,1,2\}^E$ is given and s.t. $\partial u = \emptyset$, then a given $n \in \mathbb{N}^E$ produces $U = u$ iff n_e is 0 when $u_e = 0$

$$\left.\begin{array}{ll} \text{odd} & \text{—— } 1 \\ \text{even} \geq 2 & \text{—— } 2. \end{array}\right\} (*)$$

Such an n is automatically $\partial n = \emptyset$.

So the probability of $U = u$ is

$$\frac{1}{Z_{rc}} \sum_{n \in \mathbb{N}^E \text{ st } (*)} w(n) \propto \prod_{e \in E} \begin{cases} 1 & \text{if } u_e = 0 \\ \prod_{n_e \text{ odd}} \frac{\beta^{n_e}}{n_e!} & \text{if } u_e = 1 \\ \prod_{n_e \text{ even} \geq 2} \frac{\beta^{n_e}}{n_e!} & \text{if } u_e = 2 \end{cases}$$

$$= \prod_{e \in E} \begin{cases} 1 & \text{if } u_e = 0 \\ \operatorname{sh}\beta & \text{if } u_e = 1 \\ (\operatorname{ch}\beta - 1) & \text{if } u_e = 2. \end{cases} \qquad \text{for } u \text{ st } \partial u = \emptyset.$$

⑤ 4) $\forall e \in E$, $\overline{w}_e = \begin{cases} 0 \text{ iff } & N_e = 0 \text{ and } \xi^{\varphi}_e = 0 \quad \Leftrightarrow U_e = 0 \text{ and } \xi^{\varphi}_e = 0 \\ +1 \text{ iff } & N_e \text{ even} \geq 2 \text{ OR } (N_e = 0, \xi^{\varphi}_e = 1) \quad \Leftrightarrow U_e = 2 \text{ or } U_e = 0, \xi^{\varphi}_e = 1 \\ -1 \text{ iff } & N_e \text{ odd} \quad \Leftrightarrow U_e = 1. \end{cases}$

We have the distribution of U.

By the equivalences, $\overline{w}$ amounts to doing the following:

$\forall e \in E$, if $U_e = 1$ we set $\overline{w}_e = -1$

if $U_e = 2$ we set $\overline{w}_e = +1$

if $U_e = 0$ we set $\overline{w}_e = +1$ with prob p'

0 —— $1-p'$.

Note that $\overline{w}$ is st $\forall v \in V$, there is an even number of -1 in $\overline{w}$ around v.

Therefore, for a given $\overline{w}$, the set of u that may give $\overline{w}$ are those s.t.

- when $\overline{w}_e = 0$, $u_e = 0$
- when $\overline{w}_e = -1$, $u_e = 1$
- when $\overline{w}_e = +1$, $u_e = 0$ or 2

} (**) and such a u is always st $\partial u = \emptyset$.

and the sum of their weights is

$$\sum_{\substack{u \in \{0,1,2\}^E \text{ st } \partial u = \emptyset \\ \text{and } (*)}} \prod_{e \in E} \begin{cases} 1 \text{ if } u_e = 0 \\ \operatorname{sh}\beta \text{ if } u_e = 1 \\ \operatorname{ch}\beta - 1 \text{ if } u_e = 2 \end{cases}$$

$$= \prod_{\substack{e \in E / \\ \overline{w}_e = 0}} \Big(\underset{\substack{\uparrow \\ \text{weight} \\ \text{for } u_e = 0}}{1} \cdot \underset{\substack{\uparrow \\ \text{we keep } 0 \\ (\text{Ber}(p'))}}{(1-p')} \Big) \prod_{\substack{e \in E / \\ \overline{w}_e = +1}} \Big(\underset{\substack{\uparrow \\ \text{weight} \\ \text{for } u_e = 1}}{(\operatorname{ch}\beta - 1)} + \underset{\substack{\uparrow \\ \text{weight} \\ \text{for } u_e = 0 \\ \text{and we set} \\ \overline{w}_e = 1 \\ (\text{Ber}(p'))}}{1 \cdot p'} \Big) \prod_{\substack{e \in E \\ \overline{w}_e = -1}} \underset{\substack{\uparrow \\ \text{weight} \\ \text{for } u_e = 2}}{\operatorname{sh}\beta}$$

$$= \prod_{\substack{e \in E / \\ \overline{w}_e = 0}} e^{-\beta} \prod_{\substack{e \in E / \\ \overline{w}_e = +1}} \operatorname{sh}\beta \prod_{\substack{e \in E \\ \overline{w}_e = -1}} \operatorname{sh}\beta$$

$$= \prod_{e \in E} (\operatorname{sh}\beta)^{|\overline{w}_e|} (e^{-\beta})^{1-|\overline{w}_e|}$$

⑥ 5] For a given $w \in \{0,1\}^E$, we want to count the number of $\bar{w} \in \{-1,0,1\}^E$ such that $|\bar{w}| = w$

and $\bar{w}$ has nonzero proba (that is, around every $v \in V$, even number of -1 edges in $\bar{w}$)

Clearly, this amounts to chosing the edges in the subset $H \subset E_w = \{e / w_e = 1\}$ where we put $\bar{w}_e = -1$, and this subset has to be even (all vertices have even degree in H).

we claim that there are $2^{|w| + K(w) - |V|}$ such subsets.

indeed, for any graph (possibly not connected) G', we know that by HTE,

$$Z_I(G', \beta') := \sum_{\sigma \in \{\pm 1\}^{V'}} e^{\beta \sum_{\{x,y\} \in E'} \sigma_x \sigma_y} = 2^{|V|} ch\beta'^{|E'|} \sum_{\substack{H \subset E' \\ \partial H = \emptyset}} (th\beta')^{|H|}$$

As $\beta' \to \infty$, the l.h.s. concentrates on configs st σ is constant on each cluster of G' (those maximize $\underset{|E|}{\underset{\shortparallel}{\sum \sigma_x \sigma_y}}$), and is equivalent to

$$2^{K(G')} e^{\beta' |E|}$$

while in the r.h.s, $ch\beta' \sim \frac{e^{\beta'}}{2}$ and $th\beta' \to 1$

So we have $2^{|V'|} e^{\beta' |E'|} 2^{-|E'|} \# \{H \subset E' / \partial H = \emptyset\}$

and we must have

$$2^{K(G')} = 2^{|V'|} 2^{-|E'|} \# \{H \subset E' / \partial H = \emptyset\}$$

Applying to the graph with vertices V and eges E_w gives the claim.

⑦ 6) Any $w \in \{0,1\}^E$ has positive probability for W (for instance take $\overline{w} = w$) and its probability is proportional to the sum of weights of all $2^{|w|+k(w)-|V|}$ possible $\overline{w}$:

$$2^{|w|+k(w)-|V|} \prod_{e \in E} \mathrm{sh}\beta^{w_e} \, e^{-\beta(1-w_e)}$$

$$\propto 2^{k(w)} (2\,\mathrm{sh}\beta)^{|w|} (e^{-\beta})^{|w^c|}$$

recall $p = 1 - e^{-2\beta}$

and $2\,\mathrm{sh}\beta = \lambda p$

$e^{-\beta} = \lambda(1-p)$ where $\lambda = e^{\beta}$.

$$\propto 2^{k(w)} p^{|w|} (1-p)^{|w^c|}.$$

This is the FK measure from question 2).

7) For a given $h \subset E$ st $\partial h = \emptyset$, the probability of $H=h$ is proportional to $\mathrm{th}\beta^{|h|}$.

~~On the other hand, the distribution of $\hat{N}$ can be found from U: it is the set of edges where U_e is 1 or 2. So for a given $\hat{n} \in \{0,1\}^E$ the proba that $\hat{N} = \hat{n}$ is proportional to~~

~~$\prod_{e / \hat{n}_e = 0} 1 \prod_{e/\hat{n}_e=1} \mathrm{sh}\beta + \mathrm{ch}\beta - 1$~~

This is also the distribution of $N_{odd} = \{e \in E / N_e \text{ is odd}\}$ (same constraint $\partial n_{odd} = \emptyset$ and weight $\prod_{e \in N_{odd}} \mathrm{sh}\beta \prod_{e \notin N_{odd}} 1 + (\mathrm{ch}\beta - 1)$

$\propto (\mathrm{th}\beta)^{|n_{odd}|}$)

To get the distribution of $\hat{N}$ from that of N_{odd}, it is enough to chose a (free) set of edges where N is even ≥ 2

⑧ among those not in N_{odd}, and add them to get $\hat{N}$.

The edges we add come with a weight $ch\beta - 1$ while the others have weight 1, so we may create them by ~~a~~ taking ξ' a Bernoulli perco with parameter $\frac{ch\beta - 1}{1 + ch\beta - 1} = 1 - \frac{1}{ch\beta} =: p^*$.

- On edges already in N_{odd}, adding ξ' doesn't change the positivity, so we have

$$\widehat{N_{odd} + \xi'} \overset{d}{=} \hat{N}$$

and $\widehat{H+\xi'} \overset{d}{=} \hat{N}$.

8) Sample H, ~~a~~ ξ' and ξ, and define $\hat{N} = \widehat{H+\xi'}$ and $W = \widehat{\hat{N} + \xi}$, then they have the correct marginals and

$$H \subset \hat{N} \subset W \text{ as.}$$

Let μ_{HT} be the distri of H (HTE measure on even subgraphs)

μ_{RC} be the law of $\hat{N}$ (trace of random current with $\partial n = \emptyset$)

$\phi_{2,p}$ be the FK-perco proba, then

$$\mu_{HT} \leq_{st} \mu_{RC} \leq_{st} \phi_{2,p}$$

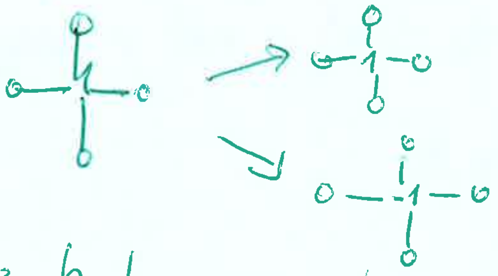
9) Sure.

⑨ Ex5: 1) We Define a Markov chain on Ω in the following way. Starting from $h_0 \in \Omega$, we sample a site $(i,j) \sim \mathcal{U}nif(\Lambda_n \setminus \{(0,0)\})$. Then we sample h_1 with distribution $\mu(h_1 | h_1|_{\Lambda_n \setminus \{i,j\}} = h_0|_{\Lambda_n \setminus \{i,j\}})$.

We need to show that it is aperiodic to apply the coupling arguments of the course.

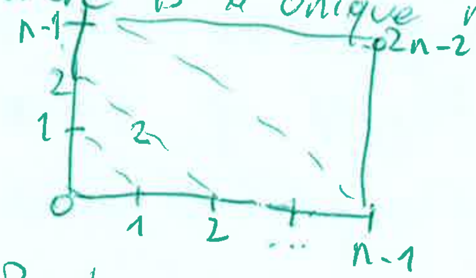
First, if all the neighbours of (i,j) have the same value in h_0, then we have two possibilities for h_1:

Ex:



Otherwise, $h_0|_{\Lambda_n \setminus \{i,j\}}$ characterizes h_0, (we cannot flip)

Notice also that there is a unique maximal height function h_{max}:



that is, $\forall h \in \Omega$, $h \leq h_{max}$.

We show that any $h \neq h_{max}$ admits an up-flip, that is, a site $(i,j) \neq (0,0)$ st around (i,j), h looks like

$$\begin{array}{ccccc} & & m+1 & & \\ & & | & & \\ m+1 & — & m & — & m+1 \\ & & | & & \\ & & m+1 & & \end{array}$$

(possibly on boundary)

so that it may flip to become $\begin{array}{ccccc} & & m+1 & & \\ & & | & & \\ m+1 & — & m+2 & — & m+1 \\ & & | & & \\ & & m+1 & & \end{array}$.

This implies that any config can reach h_{max}, and then since the transition is such that $P(h,h') = P(h',h)$, it implies irreducibility.

⑩ Let $h \in \Omega$, consider a site attaining the global minimum of h. If this site is $\neq (0,0)$, we can up-flip at this site:

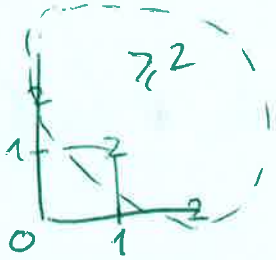
meaning: if $\exists$ such a site $\neq (0,0)$

If this site is $(0,0)$, then h is

Consider the global min among sites $\neq (0,0)$.

If there is one on a site that is not a neighbour of $(0,0)$, we can flip it up.

Otherwise, the config is



If we keep going, we see that any config has a site that can be flipped up, unless it is h_{max}.

That concludes the proof of irreducibility.

Then, as for the Ising Glauber dynamics, it is enough to prove that for $h_0 \geqslant h_0' \in \Omega$, we can couple the Markov chains starting from h_0, h_0' so that $h_1 \geqslant h_1'$ a.s.

We chose the same site for both chains. If $x \in \Lambda_n$ is such that $h_0(x) > h_0'(x)$, then any choice of flips will result in $h_1 \geqslant h_1'$. Now we consider the case $h_0(x) = h_0'(x)$.

If x is not a local minimum for h_0', then it cannot be flipped up in h_0', and again we must have $h_1 \geqslant h_1'$.

Now we suppose:

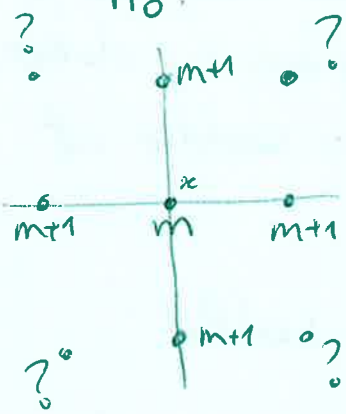
- $h_0(x) = h_0'(x)$
- x is a local min for h_0'

So x is also a local min for h_0 ($h_0 \geqslant h_0'$, in particular on neighbours of x..)

~~We want to show~~

Then the proba to flip upwards in h_0 at x is

h_0:



$$\frac{c^{n_0}}{c^{n_0} + c^{4-n_0}} = \frac{1}{1+c^{4-n_0}}$$

where n_0 is the number of "?" equal to $m+2$ in h_0.
(= number of saddles after flipping).

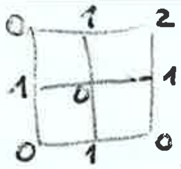
and in h_0', it is $\frac{1}{1+c^{4-n_0'}}$.

As $h_0 \geqslant h_0'$, $n_0 \geqslant n_0'$ and sine $c \geqslant 1$,

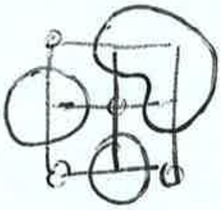
$$\frac{1}{1+c^{4-n_0}} \geqslant \frac{1}{1+c^{4-n_0'}}$$

and we can couple the walks so that every time we flip upwards in h_0', we also do it in h_0. This ensures $h_1 \geqslant h_1'$.

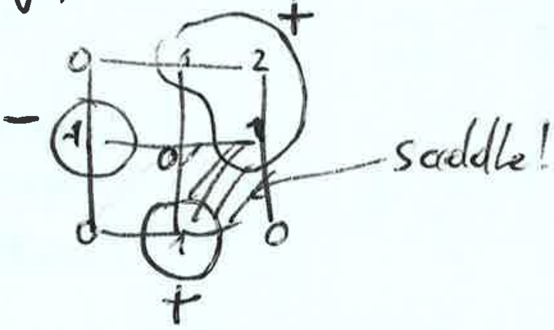
2) Example : For $|h|$ =



there are 2^3 possible h, obtained by chosing the sign in each of the 3 clusters:

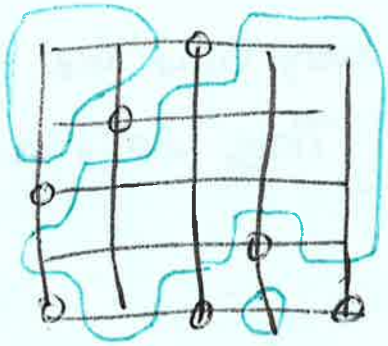


what are their relative weights? They are given by the number of saddles that differ between them. It is easy to see that those are the saddles on squares that have two "0". More precisely, when two neighbouring clusters get the same sign, it creates a saddle at their interface.



In general, let $G_{|h|}$ be the graph whose vertices are the connected components of the subgraph $\Lambda_n \setminus \{x / |h|_x = 0\}$. Between two clusters $\mathcal{C}, \mathcal{C}'$, we put $n_{\mathcal{C},\mathcal{C}'}$ edges, where $n_{\mathcal{C},\mathcal{C}'}$ is the number of squares in Λ_n that have two "0" and an element of $\mathcal{C}$, an element of $\mathcal{C}'$.

Ex:



$\rightsquigarrow$

Then, any config H on Λ_n s.t. $|H|=|n|$ is a choice of spins σ_H on $G_{|n|}$, and among possible H's we have

~~$w(H)\propto$~~

$$\mu(H \mid |H|=|n|) \propto C^{\#\{\overset{\{e,e'\}}{\overset{\shortparallel}{e}}\in E(G_{|n|}) / \sigma_H(e)=\sigma_H(e')\}}$$

$$\propto \exp\left(\frac{\ln C}{2} \sum_{e=\{e,e'\}\in E(G_{|n|})} \sigma_H(e)\,\sigma_H(e') \right)$$

which is the distribution of an Ising model on $G_{|n|}$. Notice that the temperature $\beta = \frac{\ln C}{2}$ is >0.

3) See "Delocalization of the height function of the six-vertex model", Duminil-Copin, Karrila, Manolescu, Oulamara, Appendix A. 3