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Mould calculus, polyhedral cones, and characters of combinatorial Hopf algebras

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ABSTRACT

We describe a method for constructing characters of combinatorial Hopf algebras by means of integrals over certain polyhedral cones. This is based on ideas from resurgence theory, in particular on the construction of well-behaved averages induced by diffusion processes on the real line. We give several interpretations and proofs of the main result in terms of noncommutative symmetric and quasi-symmetric functions, as well as generalizations involving matrix quasi-symmetric functions. The interpretation of noncommutative symmetric functions as alien operators in resurgence theory is also discussed, and a new family of Lie idempotents of descent algebras is derived from this interpretation.

1. Introduction

The algebra **Sym** of noncommutative symmetric functions is a universal object. Apart from providing noncommutative versions of various identities among classical symmetric functions [22], it can be interpreted in representation theory as the Grothendieck ring of the tower of 0-Hecke algebras [31], or as the direct sum of the descent algebras of symmetric groups (noncommutative analogues of the character rings) [22,35]. In algebraic topology, it is the homology of the loop space of the suspension

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of the infinite dimensional complex projective space [2]. It is also related to the geometry of polytopes [6]. In this context, it is also called the universal Leibniz Hopf algebra [24]. Actually, if one chooses as in [1] to define a combinatorial Hopf algebra as a graded connected Hopf algebra endowed with a character, then it becomes an initial object in the corresponding category, and its dual *QSym* (quasi-symmetric functions [23]) is a terminal object.

The most interesting applications of noncommutative symmetric functions are perhaps those related to the descent algebras, and particularly to the so-called Lie idempotents. These are idempotents of the symmetric group algebras which act on tensor algebras $T(V)$ as projectors onto the free Lie algebra $L(V)$. There is an extensive literature on the subject (see, *e.g.*, [4,30]), often related to the Hausdorff series [13,45] or various formal expansions for the solutions of differential systems [40,49].

However, before the paper [30], only three examples of Lie idempotents were known. These were the Dynkin/Specht/Wever idempotent [13,47,48], the Solomon/Eulerian idempotent [45] (both related to the Hausdorff series), and the Klyachko idempotent [29], which arose as an intertwining operator between two representations of $GL(V)$. The contribution of [30] was to show that all these examples were specializations of natural families of noncommutative symmetric functions with one or more parameters, and that their properties could be derived from noncommutative analogues of classical manipulations with ordinary symmetric functions. In particular, the properties of the Klyachko idempotent appeared as the simplest analogue of a specialization property of Hall–Littlewood functions at roots of unity. Also, Lie idempotents of descent algebras were characterized as the (suitably normalized) primitive elements with a nonzero commutative image.

This provided a potentially infinite supply of Lie idempotents, and it was tempting to assume that any new example that could arise would be readily interpreted in terms of the constructions of [30].

This belief was to be turned down by a recently discovered interpretation of **Sym**. On the occasion of a one-year seminar on Ecalle's mould calculus during the academic year 2007–2008, it was realized that the algebra of noncommutative symmetric functions (and some of its generalizations) played an important role in the theory of resurgent functions. It arises in the guise of a Hopf algebra of alien operators, which act on certain function spaces by means of intricate combinations of analytic continuations. Among those are the alien derivations, which turn out to correspond to Lie idempotents in **Sym**.

The point of view of resurgence and mould calculus leads to insights different from those from the theory of noncommutative symmetric functions. For example, a technique for constructing moulds with prescribed symmetries from random walks on the real line leads to remarkable examples of Lie idempotents given by explicit formulas.

The aim of the present paper is to explain these connections, and to try to unify both points of view. We shall see in particular that the construction of alien automorphisms (grouplike series in **Sym**) by means of random walks can be traced back to a geometric property of certain polyhedral cones, which allows to interpret the calculation in other combinatorial Hopf algebras like **WQSym** or **MQSym**.

After recalling the relevant properties of noncommutative symmetric functions (Section 2), we present in Section 3 the first version of our main result: grouplike and primitive series of noncommutative symmetric functions can be constructed by means of certain iterated integrals. To understand this property, we have to embed **Sym** in a larger algebra, **WQSym**, whose main properties are recalled in Section 4. This algebra can be regarded as based on set compositions. In Section 5, we associate two polyhedral cones with a set composition, and obtain our main result as a consequence of a geometric property: the characteristic function of a Cartesian product of cones is an alternating sum of characteristic functions of similar cones. Our first proof relies upon certain multivariate Laurent series, the so-called integer point transforms of the cones. These series represent rational functions in their domain of convergence, and given such a function together with the corresponding domain, the cone can be reconstructed unambiguously. This representation of elements of a combinatorial Hopf algebra by rational functions is reminiscent of that used in [11], although of a different nature. The exact analogue of the constructions of [11] are given in Section 6, where nonlinear operators associated with elements of **WQSym** are constructed by means of discrete iterated integrals. As in [11], an operadic interpretation is provided, and the tridendriform operad is related to **WQSym** in a simple and explicit way. This interpretation can also give rise to characters of **WQSym**, and we indicate briefly how to recover some familiar examples. Next, the constructions of Section 7 are extended to the Hopf algebra **MQSym**, which may be interpreted as based on multiset compositions. This allows to define more general polyhedral cones, to which the main result is extended in Section 8. In this context, it can be proved by a short (but tricky) algebraic calculation, whose meaning is eventually interpreted in terms of Rota–Baxter algebras (Section 9). In Section 10, we present a class of iterated integrals which can be evaluated in closed form, and obtain a new family of Lie idempotents, the Catalan family. Finally, we review the connections between noncommutative symmetric functions and alien calculus (Section 11), and sketch the proof of the isomorphism of Hopf algebras between noncommutative symmetric functions and alien operators (Section 12).

2. Noncommutative symmetric functions

2.1. *The Hopf algebra* **Sym**

By definition, **Sym** is the free associative algebra $\mathbb{K}\langle S_1, S_2, \ldots\rangle$ over an infinite sequence S_n, endowed with the grading $\deg S_n = n$ and the coproduct

$$\Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \quad (S_0 := 1), \tag{1}$$

where $\mathbb{K}$ is a field of characteristic 0.

It can be realized in terms of polynomials[1] over an auxiliary set $A = \{a_1 < a_2 < \cdots\}$ of noncommuting variables endowed with a total order. If t is another indeterminate, commuting with the a_i, we set

$$\sigma_t(A) := \overrightarrow{\prod_{i\geqslant 1}}(1-ta_i)^{-1} = (1-ta_1)^{-1}(1-ta_2)^{-1}\cdots = \sum_{n\geqslant 0} S_n(A)t^n \tag{2}$$

(the arrow means that the product should be taken from left to right) so that

$$\lambda_{-t}(A) := \overleftarrow{\prod_{1\leqslant i}}(1-ta_i) = \cdots(1-ta_2)(1-ta_1) = \sum_{n\geqslant 0} \Lambda_n(A)(-t)^n = \sigma_t(A)^{-1}. \tag{3}$$

Then the coproduct can be expressed as

$$\Delta F = F(A+B) \tag{4}$$

where $B = \{b_i \mid i \geqslant 1\}$ is another ordered alphabet isomorphic to A and where $A + B$ is interpreted as the *ordinal sum* of A and B (a noncommutative sum!), and A commutes with B for the multiplication.

As a Hopf algebra, **Sym** is not self-dual. This is clear from the definition of the coproduct, which is obviously cocommutative. Its (graded) dual is the commutative algebra *QSym* of *quasi-symmetric functions* [23,22].

2.2. *Bases*

From the generators S_n, we can form a linear basis

$$S^I = S_{i_1} S_{i_2} \ldots S_{i_r} \tag{5}$$

[1] By "polynomials", we mean formal series of finite degree.

of the homogeneous component $\mathbf{Sym}_n$, parametrized by *compositions* of n, that is, finite ordered sequences $I = (i_1, \ldots, i_r)$ of positive integers summing to n. One often writes $I \vDash n$ to mean that I is a composition of n. The dimension of $\mathbf{Sym}_n$ is 2^{n-1} for $n \geqslant 1$.

Similarly, from the Λ_n we can build a basis

$$\Lambda^I = \Lambda_{i_1} \Lambda_{i_2} \ldots \Lambda_{i_r}. \tag{6}$$

We can also look for analogues of the power-sum symmetric functions. It is here that the nontrivial questions arise. Indeed, in the commutative case, the power-sum p_n is, up to a scalar factor, the unique primitive element of degree n. Here, the primitive elements form a free Lie algebra, and it is not immediately obvious to identify the ones which should deserve the name "noncommutative power-sums".

However, we can at least give one example: the series σ_t being grouplike ($\Delta\sigma_t = \sigma_t \otimes \sigma_t$), its logarithm is primitive, and writing

$$\log \sigma_t = \sum_{n \geqslant 1} \Phi_n \frac{t^n}{n}, \tag{7}$$

we can reasonably interpret Φ_n as a noncommutative power-sum.

Finally, let us present a last basis of **Sym** which is the analog of the Schur functions in this context. In terms of the auxiliary ordered alphabet A

$$S_n(A) = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} a_{i_1} a_{i_2} \ldots a_{i_n}, \tag{8}$$

is the sum of nondecreasing words. Let us say that a word $w = a_{i_1} a_{i_2} \ldots a_{i_n}$ has a *descent* at k if $i_k > i_{k+1}$, and denote by $\mathrm{Des}(w)$ the set of such k (the *descent set* of w).

Thus, $S_n(A)$ is the sum of words of length n with no descent. Now, obviously,

$$S^I = S_{i_1} S_{i_2} \ldots S_{i_r} \tag{9}$$

is the sum of words whose descent set is contained in

$$\{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{r-1}\}. \tag{10}$$

We denote this set by $\mathrm{Des}(I)$ and call it the *descent set of the composition* I. Symmetrically, we call I the *descent composition*, and write $I = C(w)$, of any word of length n having $\mathrm{Des}(I)$ as descent set.

The *noncommutative ribbon Schur functions* are defined by

$$R_I(A) = \sum_{C(w) = I} w \tag{11}$$

so that we have

$$S^I = \sum_{J \leqslant I} R_J \tag{12}$$

where $J \leqslant I$ is the *reverse refinement order*, which means that $\mathrm{Des}(J) \subseteq \mathrm{Des}(I)$.

The product in the ribbon basis is given by

$$R_I \cdot R_J = R_{I \cdot J} + R_{I \triangleright J} \tag{13}$$

where

$$I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s) \quad \text{and} \quad I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s) \tag{14}$$

(note that this formula is obvious from the interpretation in terms of words). For example,

$$R_{132}R_2 = R_{1322} + R_{134}. \tag{15}$$

2.3. Descent algebras

Let (W, S) be a Coxeter system. One says that $w \in W$ has a descent at $s \in S$ if w has a reduced word ending by s. For $W = \mathfrak{S}_n$ and $s_i = (i, i+1)$, this means that $w(i) > w(i+1)$, whence the terminology. In this case, we rather say that i is a descent of w. Let $\mathrm{Des}(w)$ denote the descent set of w, and for a subset $E \subseteq S$, set

$$D_E = \sum_{\mathrm{Des}(w)=E} w \in \mathbb{Z}W. \tag{16}$$

Solomon has shown [46] that the D_E span a $\mathbb{Z}$-subalgebra $\Sigma(W)$ of $\mathbb{Z}W$. Moreover

$$D_{E'}D_{E''} = \sum_E c^E_{E'E''} D_E \tag{17}$$

where the coefficients $c^E_{E'E''}$ are nonnegative integers.

In the case of $W = \mathfrak{S}_n$, we encode descent sets by compositions of n as explained above. If $E = \{d_1, \ldots, d_{r-1}\}$, we set $d_0 = 0$, $d_r = n$ and $I = C(E) = (i_1, \ldots, i_r)$, where $i_k = d_k - d_{k-1}$. From now on, we shall write D_I instead of D_E, and denote by Σ_n the descent algebra of $\mathfrak{S}_n$ (with coefficients in our ground field $\mathbb{K}$).

Thus, Σ_n has the same dimension as $\mathbf{Sym}_n$, and both have natural bases labeled by compositions of n.

The natural correspondence $\Sigma_n \to \mathbf{Sym}_n$ is defined by the linear map $\alpha : D_I \to R_I$. It allows to transport the product of the descent algebra to **Sym**. The resulting operation is the internal product $*$. We set $F * G = 0$ if F and G are homogeneous of different degrees, and for technical reasons, we want our correspondence to be an *anti-isomorphism*:

$$R_I * R_J = \alpha(D_J D_I). \tag{18}$$

If we map our letters a_i to commuting variables x_i, noncommutative symmetric functions are mapped to ordinary symmetric functions. For example, the Λ_n go to the familiar elementary symmetric functions e_n and the Φ_n to the power-sums $p_n = \sum_i x_i^n$.

It is proved in [22,30] that a homogeneous element $\Pi_n \in \mathbf{Sym}_n$ is the image under α of a Lie idempotent of the descent algebra Σ_n if and only if it is primitive and has as commutative image $\frac{1}{n}p_n$.

For example, $\frac{1}{n}\Phi_n$ corresponds to the Solomon (Eulerian) idempotent [45], whilst Ψ_n defined by the recurrence

$$nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + \Psi_n \tag{19}$$

corresponds to the Dynkin operator

$$a_1 a_2 \ldots a_n \mapsto \big[\ldots\big[[a_1, a_2], a_3\big], \ldots a_n\big] \tag{20}$$

and is given by

$$\Psi_n = R_n - R_{1,n-1} + R_{1,1,n-2} - \cdots = \sum_{k=0}^{n-1} (-1)^k R_{1^k,n-k}. \tag{21}$$

A nontrivial example is the one-parameter family of Lie idempotents

$$\varphi_n(q) = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} q^{\mathrm{maj}(\sigma) - \binom{d(\sigma)+1}{2}} \sigma, \tag{22}$$

where the major index $\mathrm{maj}(\sigma)$ of a permutation is the sum of its descents. One can prove that this family interpolates between the Dynkin and Solomon idempotents and gives back the Klyachko idempotent when q is a primitive nth root of unity (see [30]). Further properties of this idempotent, including a preLie expression, can be found in [9].

3. Iterated integrals and characters

We define in this section some special elements of **Sym** corresponding to operators originally introduced by Ecalle in the framework of real resummation and alien calculus. For the sake of simplicity, we will stay within the formalism of noncommutative symmetric functions and the link with resummation will be explained at the end of this paper.

For a permutation $\sigma \in \mathfrak{S}_n$, consider the sequence of $\pm$ signs

$$\boldsymbol{\varepsilon}\bullet = (\varepsilon_1, \ldots, \varepsilon_{n-1}, \bullet), \tag{23}$$

where ε_i is the sign of $\sigma(i+1) - \sigma(i)$. Note that the symbol $\bullet$ is added so that the length of the sequence is n. With this notation, it is clear that

$$\mathrm{Des}(\sigma) = \{1 \leqslant i \leqslant n-1;\ \varepsilon_i = -\}. \tag{24}$$

Let us now introduce the *signed ribbon basis* of **Sym**, which is a slight modification of the noncommutative ribbon Schur functions:

$$R_{\boldsymbol{\varepsilon}\bullet} = (-1)^{l(I)-1} R_I \quad (R_\emptyset = 1,\ R_\bullet = R_1), \tag{25}$$

where I is the composition such that

$$\mathrm{Des}(I) = \{1 \leqslant i \leqslant n-1;\ \varepsilon_i = -\}. \tag{26}$$

So, Eq. (13) reads

$$R_{\boldsymbol{a}\bullet} R_{\boldsymbol{b}\bullet} = R_{\boldsymbol{a}+\boldsymbol{b}\bullet} - R_{\boldsymbol{a}-\boldsymbol{b}\bullet}. \tag{27}$$

For example (compare (15)),

$$R_{-++-+\bullet} R_{+\bullet} = R_{-++-+++\bullet} - R_{-++-+-+\bullet}. \tag{28}$$

Let us also define

$$\mathcal{E} = \bigcup_{n\geqslant 0} \mathcal{E}_n = \bigcup_{n\geqslant 0} \{\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n);\ \varepsilon_i = \pm\}. \tag{29}$$

Our main goal here is to study the elements of **Sym** whose coefficients in the basis $R_{\boldsymbol{\varepsilon}\bullet}$ are defined as probabilities.

Let $X_1, \ldots, X_n$ be real, independent, identically distributed random variables of density f, and define

$$m_f^{\varepsilon_1,\ldots,\varepsilon_n} = P(\varepsilon_1 S_1 > 0, \ldots, \varepsilon_n S_n > 0), \tag{30}$$

with $S_k = X_1 + \cdots + X_k$. These "weights" (whose collection is called an average induced by diffusion in [19]) will allow us to find new grouplike and primitive elements in **Sym**. Their structure does not depend on the fact that f is a probability density, since these probabilities can be defined as "iterated integrals" of f on some subsets of $\mathbb{R}^n$, and make sense for any bounded integrable function.

Definition 3.1. We denote by $\mathbf{1}_X$ the characteristic function of a subset X of $\mathbb{R}^n$, and set $\sigma_+ = \mathbf{1}_{\mathbb{R}^+}$, and $\sigma_- = \mathbf{1}_{\mathbb{R}^{-*}}$.

Let $f \in \mathcal{A} = L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a bounded integrable function. We set $m^\emptyset = 1$, $d^\emptyset = 0$ and, for a nonempty sequence of signs $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{E}$,

$$m_f^{\boldsymbol{\varepsilon}} = \int_{\mathbb{R}^n} f(x_1)\ldots f(x_n)\sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1+x_2)\ldots\sigma_{\varepsilon_n}(x_1+\cdots+x_n)\,dx_1\ldots dx_n. \tag{31}$$

In the same way, for any sequence of signs $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{E}$ of length at least 2, we set

$$\begin{aligned} d_f^{\boldsymbol{\varepsilon}} = \varepsilon_n \int_{\mathbb{R}^n} & f(x_1)\ldots f(x_n) \\ & \times \sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1+x_2)\ldots\sigma_{\varepsilon_{n-1}}(x_1+\cdots+x_{n-1})\delta(x_1+\cdots+x_n)\,dx_1\ldots dx_n, \end{aligned} \tag{32}$$

where δ is the Dirac distribution concentrated at 0 (*i.e.*, $\int_X \delta(x)\,dx$ is 1 or 0 according to whether X contains 0 or not).

Thanks to the regularity of f, these integrals are well-defined and when f is continuous at 0, Eq. (32) still makes sense for sequences of length 1 and we set $d^+ = -d^- = f(0)$. Otherwise, we can give any arbitrary value to $d^+ = -d^-$. We then have

Theorem 3.2. *Define formal series R_f, L_f and D_f of noncommutative symmetric functions by*

$$\begin{aligned} R_f &= 1 + \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} R_f^{\boldsymbol{\varepsilon}\bullet} R_{\boldsymbol{\varepsilon}\bullet}, \\ L_f &= 1 + \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} L_f^{\boldsymbol{\varepsilon}\bullet} R_{\boldsymbol{\varepsilon}\bullet}, \\ D_f &= \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} D_f^{\boldsymbol{\varepsilon}\bullet} R_{\boldsymbol{\varepsilon}\bullet}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} R_f^{\boldsymbol{\varepsilon}\bullet} &= m_f^{\boldsymbol{\varepsilon}+} \quad \big(R^{\emptyset}=1\big), \\ L_f^{\boldsymbol{\varepsilon}\bullet} &= -m_f^{\boldsymbol{\varepsilon}-} \quad \big(L^{\emptyset}=1\big), \\ D_f^{\boldsymbol{\varepsilon}\bullet} &= d_f^{\boldsymbol{\varepsilon}+} = -d_f^{\boldsymbol{\varepsilon}-} \quad \big(D^{\emptyset}=0\big). \end{aligned} \tag{34}$$

Then R_f and L_f are grouplike and D_f is primitive.

Proof. The only difficulty is to prove that R_f is grouplike. Then, the corresponding property of L_f is implied by Proposition 3.3, and the primitivity of D_f follows from Lemma 3.6.

Different proofs and generalizations of the fact that R_f is grouplike are given in the sequel. We shall first obtain it as a corollary of Proposition 5.4, which is itself equivalent to Theorem 5.2. This result provides in fact a general method of constructing characters of a certain Hopf algebra **WQSym**, whose definition is recalled in Section 4. Then a different method of constructing characters of **WQSym** is presented in Section 6. Finally, Theorem 8.1 generalizes Theorem 5.2 to the algebra **MQSym**. In this context, the result can be proved by a compact algebraic calculation, whose meaning is then traced back to the theory of Rota–Baxter algebras (Section 9).

Proposition 3.3. *If $I(f) = \int f(x)\,dx$, then*

$$R_f = L_f \cdot \sigma_{I(f)}, \tag{35}$$

where $\sigma_z = \sum_{n\geqslant 0} z^n S_n$ is the generating series of complete noncommutative symmetric functions.

Proof. This result is a consequence of the simple equality

$$m_f^{\boldsymbol{\varepsilon}+} + m_f^{\boldsymbol{\varepsilon}-} = m_f^{\boldsymbol{\varepsilon}} I(f) \quad \text{for all } \boldsymbol{\varepsilon} \in \mathcal{E}. \tag{36}$$

Let $\mathcal{E}_-$ be the subset of sequences of $\mathcal{E}$ that do not end with a $+$ sign (including the empty sequence $\emptyset$). Thanks to Eq. (36), for $\boldsymbol{\eta} \in \mathcal{E}_-$ and $k \geqslant 0$,

$$m_f^{\boldsymbol{\eta}+^{k+1}} = m_f^{\boldsymbol{\eta}} I(f)^{k+1} - \sum_{i=0}^{k} m_f^{\boldsymbol{\eta}+^{k-i}-} I(f)^i \tag{37}$$

and

$$\begin{aligned} R_f &= 1 + \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} m_f^{\boldsymbol{\varepsilon}+} R_{\boldsymbol{\varepsilon}\bullet} \\ &= 1 + \sum_{\boldsymbol{\eta}\in\mathcal{E}_-} \sum_{k\geqslant 0} m_f^{\boldsymbol{\eta}+^{k+1}} R_{\boldsymbol{\eta}+^k\bullet} \\ &= 1 + \sum_{\boldsymbol{\eta}\in\mathcal{E}_-} \sum_{k\geqslant 0} m_f^{\boldsymbol{\eta}} I(f)^{k+1} R_{\boldsymbol{\eta}+^k\bullet} - \sum_{\boldsymbol{\eta}\in\mathcal{E}_-} \sum_{k\geqslant 0} \sum_{i=0}^{k} m_f^{\boldsymbol{\eta}+^{k-i}-} I(f)^i R_{\boldsymbol{\eta}+^k\bullet} \\ &= 1 + \sum_{\boldsymbol{\eta}\in\mathcal{E}_-} \sum_{k\geqslant 0} m_f^{\boldsymbol{\eta}} I(f)^{k+1} R_{\boldsymbol{\eta}+^k\bullet} - \sum_{\boldsymbol{\eta}\in\mathcal{E}_-} \sum_{i\geqslant 0} \sum_{j\geqslant 0} m_f^{\boldsymbol{\eta}+^j-} I(f)^i R_{\boldsymbol{\eta}+^{i+j}\bullet} \\ &= 1 + \sum_{\boldsymbol{\eta}\in\mathcal{E}_-} \sum_{k\geqslant 0} m_f^{\boldsymbol{\eta}} I(f)^{k+1} R_{\boldsymbol{\eta}+^k\bullet} - \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} \sum_{i\geqslant 0} m_f^{\boldsymbol{\varepsilon}-} I(f)^i R_{\boldsymbol{\varepsilon}+^i\bullet}. \end{aligned} \tag{38}$$

Splitting the sum over $\boldsymbol{\eta}$ into two parts, according to whether $\boldsymbol{\eta}=\emptyset$ or $\boldsymbol{\eta}\in\mathcal{E}_-\backslash\{\emptyset\}$, in which case we write $\boldsymbol{\eta}=\boldsymbol{\varepsilon}-$, we have

$$\begin{aligned} R_f &= 1+\sum_{k\geqslant 0} I(f)^{k+1}R_{+^k\bullet}+\sum_{\boldsymbol{\varepsilon}\in\mathcal{E}}\sum_{k\geqslant 0} m_f^{\boldsymbol{\varepsilon}-}I(f)^{k+1}R_{\boldsymbol{\varepsilon}-+^k\bullet}-\sum_{\boldsymbol{\varepsilon}\in\mathcal{E}}\sum_{i\geqslant 0} m_f^{\boldsymbol{\varepsilon}-}I(f)^i R_{\boldsymbol{\varepsilon}+^i\bullet} \\ &= 1+\sum_{k\geqslant 1} I(f)^k R_{+^{k-1}\bullet}+\sum_{\boldsymbol{\varepsilon}\in\mathcal{E}}\sum_{k\geqslant 1} -m_f^{\boldsymbol{\varepsilon}-}I(f)^k\left(R_{\boldsymbol{\varepsilon}++^{k-1}\bullet}-R_{\boldsymbol{\varepsilon}-+^{k-1}\bullet}\right)+\sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} -m_f^{\boldsymbol{\varepsilon}-}R_{\boldsymbol{\varepsilon}\bullet}. \end{aligned} \tag{39}$$

But $S_k=R_k=R_{+^{k-1}\bullet}$ and, applying (27),

$$R_{\boldsymbol{\varepsilon}++^{k-1}\bullet}-R_{\boldsymbol{\varepsilon}-+^{k-1}\bullet}=R_{\boldsymbol{\varepsilon}\bullet}R_{+^{k-1}\bullet}, \tag{40}$$

so that

$$R_f=1+\sum_{k\geqslant 1} I(f)^k S_k+\sum_{\boldsymbol{\varepsilon}\in\mathcal{E}} -m_f^{\boldsymbol{\varepsilon}-}R_{\boldsymbol{\varepsilon}\bullet}\left(1+\sum_{k\geqslant 1} I(f)^k S_k\right), \tag{41}$$

and finally $R_f=L_f\cdot\sigma_{I(f)}$. □

Since σ_z is grouplike, we can state:

Lemma 3.4. *R_f is grouplike if and only if L_f is grouplike.* □

The case of D_f is less obvious. Given a function $f\in\mathcal{A}$, we define a one-parameter family of functions by

$$f_t(x)=f(x-t)\quad (t\in\mathbb{R}). \tag{42}$$

When $f\in\mathcal{C}_0^\infty\subset\mathcal{A}$ is an infinitely differentiable function with compact support, the weights $m_{f_t}^{\boldsymbol{\varepsilon}}$ are differentiable with respect to t, and

Proposition 3.5. *If $f\in\mathcal{C}_0^\infty$,*

$$\partial_t R_{f_t}=(Y D_{f_t})R_{f_t} \tag{43}$$

where Y is the Euler operator on **Sym** *(that is, $YR_I=|I|R_I$).*

Proof. For given functions $f_1,\dots,f_n,\dots$ in $\mathcal{A}$, we define by induction

$$\begin{bmatrix}\emptyset\\ \emptyset\end{bmatrix}(x)=\delta, \tag{44}$$

$$\begin{bmatrix}\varepsilon_1 & \cdots & \varepsilon_n\\ f_1 & \cdots & f_n\end{bmatrix}(x)=\left(\begin{bmatrix}\varepsilon_1 & \cdots & \varepsilon_{n-1}\\ f_1 & \cdots & f_{n-1}\end{bmatrix}*f_n\right)(x)\sigma_{\varepsilon_n}(x), \tag{45}$$

where $*$ is the convolution on $\mathbb{R}$

$$(f*g)(x)=\int_{\mathbb{R}} f(x-y)g(y)\,dy. \tag{46}$$

Definition 3.1 reads now

$$m_f^{\varepsilon_1,\dots,\varepsilon_n} = \int_{\mathbb{R}} \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f & \cdots & f \end{bmatrix}(x)\,dx, \tag{47}$$

and, since $f \in \mathcal{C}_0^\infty$ and $n \geqslant 1$,

$$\begin{aligned} d_f^{\varepsilon_1,\dots,\varepsilon_n} &= \varepsilon_n\left(\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f & \cdots & f \end{bmatrix} * f\right)(0) \\ &= \varepsilon_n \int_{\mathbb{R}} \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f & \cdots & f \end{bmatrix}(y) f(-y)\,dy \\ &= -\int_{\mathbb{R}} \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f & \cdots & f \end{bmatrix}(y)\left(\varepsilon_n \int_0^{\varepsilon_n\infty} f'(x-y)\,dx\right) dy \\ &= -\int_{\mathbb{R}} \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f & \cdots & f \end{bmatrix}(y)\left(\int_{\mathbb{R}} f'(x-y)\sigma_{\varepsilon_n}(x)\,dx\right) dy \\ &= -\int_{\mathbb{R}} \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} & \varepsilon_n \\ f & \cdots & f & f' \end{bmatrix}(x)\,dx, \end{aligned} \tag{48}$$

where the last identity comes from the theorem of Fubini.

Using the recursive definition of $\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f_t & \cdots & f_t \end{bmatrix}$ and integration by parts, we get

$$\partial_t \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f_t & \cdots & f_t \end{bmatrix} = \sum_{k=1}^{n-1} k d_{f_t}^{\varepsilon_1,\dots,\varepsilon_k} \begin{bmatrix} \varepsilon_{k+1} & \cdots & \varepsilon_n \\ f_t & \cdots & f_t \end{bmatrix} + n \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} & \varepsilon_n \\ f_t & \cdots & f_t & \partial_t f_t \end{bmatrix}. \tag{49}$$

Indeed, this equation is obvious for $n = 1$ and, recursively, if (49) holds for a given $n \geqslant 1$, let

$$g(t,x) = \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f_t & \cdots & f_t \end{bmatrix}, \tag{50}$$

then

$$\begin{aligned} \partial_t \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n & \varepsilon_{n+1} \\ f_t & \cdots & f_t & f_t \end{bmatrix} &= \sigma_{\varepsilon_{n+1}}(x) \int_{\mathbb{R}} \partial_t\big(g(t,y) \cdot f_t(x-y)\big)\,dy \\ &= \sigma_{\varepsilon_{n+1}}(x) \int_{\mathbb{R}} \big(\partial_t g(t,y)\big) \cdot f_t(x-y)\,dy \\ &\quad + \sigma_{\varepsilon_{n+1}}(x) \int_{\mathbb{R}} g(t,y)\big(\partial_t f_t(x-y)\big)\,dy \end{aligned} \tag{51}$$

but

$$\sigma_{\varepsilon_{n+1}}(x)\int_{\mathbb{R}} g(t,y)\big(\partial_t f_t(x-y)\big)\,dy = \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n & \varepsilon_{n+1} \\ f_t & \cdots & f_t & \partial_t f_t \end{bmatrix}, \tag{52}$$

and if we use (49) to expand $\partial_t g(t,y)$, then

$$\begin{aligned}\sigma_{\varepsilon_{n+1}}(x)\int_{\mathbb{R}} \big(\partial_t g(t,y)\big)\cdot f_t(x-y)\,dy &= \sum_{k=1}^{n-1} kd_{f_t}^{\varepsilon_1,\dots,\varepsilon_k}\begin{bmatrix} \varepsilon_{k+1} & \cdots & \varepsilon_n & \varepsilon_{n+1} \\ f_t & \cdots & f_t & f_t \end{bmatrix} \\ &\quad + n\sigma_{\varepsilon_{n+1}}(x)\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} & \varepsilon_n \\ f_t & \cdots & f_t & \partial_t f_t \end{bmatrix} * f_t(x).\end{aligned} \tag{53}$$

The last term reads

$$n\sigma_{\varepsilon_{n+1}}(x)\left(\left(\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f_t & \cdots & f_t \end{bmatrix} * \partial_t f_t\right)\sigma_{\varepsilon_n}\right) * f_t(x) \tag{54}$$

and if

$$h(x) = \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f_t & \cdots & f_t \end{bmatrix}(x), \tag{55}$$

then,

$$\begin{aligned}\big((h*\partial_t f_t)\sigma_{\varepsilon_n}\big) * f_t(x) &= -\int_{\mathbb{R}}\int_{\mathbb{R}} h(z) f'(y-z-t)\sigma_{\varepsilon_n}(y) f(x-y-t)\,dz\,dy \\ &= -\int_{\mathbb{R}} \varepsilon_n \int_0^{\varepsilon_n\infty} f'(y-z-t) f(x-y-t)\,dy\,h(z)\,dz \\ &= \varepsilon_n \int_{\mathbb{R}} \left(f(-z-t) f(x-t) - \int_0^{\varepsilon_n\infty} f(y-z-t) f'(x-y-t)\,dy\right) h(z)\,dz \\ &= \varepsilon_n (h*f_t)(0) f_t(x) + \big((h*f_t)\sigma_{\varepsilon_n}\big) * \partial_t f_t(x).\end{aligned}$$

This yields the required equality at order $n+1$.

Integrating these equations (with $\partial_t f_t(x) = -f'(x-t)$), we obtain

$$\partial_t m_{f_t}^{\varepsilon_1\dots\varepsilon_n} = \sum_{k=1}^{n} kd_{f_t}^{\varepsilon_1\dots\varepsilon_k} m_{f_t}^{\varepsilon_{k+1}\dots\varepsilon_n} \tag{56}$$

and finally

$$\partial_t R_{f_t} = (Y D_{f_t}) R_{f_t}. \qquad \square \tag{57}$$

This identity implies the primitivity of D_f. Indeed, if R_f is grouplike for all f, in particular, for a given f, all the R_{f_t} are also grouplike, and we have:

Lemma 3.6. *If R_{f_t} is grouplike for all t, then D_{f_t} is primitive for all t.*

Proof. Let us set for short $R(t) = R_{f_t}$ and $D(t) = D_{f_t}$ and first assume that $f \in \mathcal{C}_0^\infty \subset \mathcal{A}$ is an infinitely differentiable function with compact support. Since the coproduct of **Sym** has the form $\Delta F = F(A + B)$, Eq. (57) holds for the coproducts, and we have

$$\Delta\big(Y D(t)\big) = \big(\Delta R(t)\big)' \Delta\big(R(t)^{-1}\big). \tag{58}$$

Assuming that $R(t)$ is grouplike, we have

$$\big(\Delta R(t)\big)' = R'(t) \otimes R(t) + R(t) \otimes R'(t). \tag{59}$$

Substituting this expression into (58), we obtain

$$\Delta\big(Y D(t)\big) = Y D(t) \otimes 1 + 1 \otimes Y D(t). \tag{60}$$

Thus $Y D(t)$ is primitive, and so are its homogeneous components, so that $D(t)$ is primitive as well. In the general case $f \in \mathcal{A}$, since $\mathcal{C}_0^\infty$ is dense in $L^1(\mathbb{R})$ (60) still holds for $f \in \mathcal{A}$. □

Let us finally remark that we also have integral expressions of R_f and D_f in the basis Λ^I (defined in Section 2.2). Let us denote, for any element F in **Sym**, its coefficients in a basis B by $\langle F, B\rangle_I$, so that in particular

$$F = \sum \langle F, \Lambda\rangle_I \Lambda^I. \tag{61}$$

Proposition 3.7. *For $I = (i_1, \dots, i_r) \vDash n$,*

$$\langle R_f, \Lambda\rangle_I = (-1)^{r+n} \int_{\mathbb{R}} \begin{bmatrix} + & \cdots & + \\ f^{*i_1} & \cdots & f^{*i_r} \end{bmatrix}(x)\, dx = (-1)^{r+n} \int_{K_I} f(x_1) \dots f(x_n)\, dx_1 \dots dx_n, \tag{62}$$

where f^{*i} is the ith convolution power (on $\mathbb{R}$) of f and K_I is the polyhedral cone (see Section 5) in $\mathbb{R}^n$ defined by the inequalities

$$\sum_{k=1}^{i_1+\dots+i_q} x_k \geqslant 0 \quad \text{for } 1 \leqslant q \leqslant r. \tag{63}$$

Similarly,

$$\langle D_f, \Lambda\rangle_I = (-1)^{r+n} \left(\begin{bmatrix} + & \cdots & + \\ f^{*i_1} & \cdots & f^{*i_{r-1}} \end{bmatrix} * f^{*i_r} \right)(0). \tag{64}$$

Proof. The right-hand side of Eq. (62) can be rewritten as

$$(-1)^{r+n} \sum_{\substack{\varepsilon_{i_1}=+, \\ \varepsilon_{i_1+i_2}=+,\dots,\varepsilon_n=+}} \int_{\mathbb{R}} \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f & \cdots & f \end{bmatrix}(x)\, dx = (-1)^{\ell(I)+n} \sum_{J \leqslant \bar{I}^\sim} (-1)^{\ell(J)-1} \langle R_f, R\rangle_J. \tag{65}$$

Thus, the result follows from the relation [22, Eq. (63)]

$$R_I = \sum_{J \leqslant \bar{I}^{\sim}} (-1)^{\ell(J)-\ell(\bar{I}^{\sim})} \Lambda^J, \tag{66}$$

since $\ell(\bar{I}^{\sim}) = n + 1 - \ell(I)$.

Similarly, the right-hand side of (64) is

$$(-1)^{\ell(I)+n} \sum_{J \leqslant \bar{I}^{\sim}} (-1)^{\ell(J)-1} \langle D_f, R \rangle_J. \qquad \square \tag{67}$$

It remains to prove that R_f is grouplike. This will be done in the following sections and the different proofs give rise to some remarkable interpretations and developments based on these "iterated integrals". We will also emphasize the case of the Catalan average, that leads to an explicit Lie idempotent. To go ahead, we first need to introduce a few more algebraic structures.

4. More combinatorial Hopf algebras

4.1. Quasi-symmetric functions

As already mentioned, noncommutative symmetric functions and quasi-symmetric functions can help to understand certain properties of ordinary symmetric functions. In the same way, certain features of **Sym** or *QSym* can be properly understood by introducing bigger Hopf algebras, of which they are subalgebras or quotients.

The shortest way to introduce *QSym* and its duality with **Sym** is via the noncommutative Cauchy identity. Let $X = \{x_1 < x_2 < \cdots\}$ be an infinite totally ordered set of mutually commuting variables, also commuting with the a_i of **Sym**. The Cauchy kernel is the formal series

$$K(X, A) = \prod_{i \geqslant 1}^{\rightarrow} \prod_{j \geqslant 1}^{\rightarrow} (1 - x_i a_j)^{-1}, \tag{68}$$

where the arrow means that the products are taken in increasing order from left to right. It is easy to expand the product on the basis $S^I(A)$ of **Sym**:

$$K(X, A) = \prod_{i \geqslant 1}^{\rightarrow} \sigma_{x_i}(A) = \prod_{i \geqslant 1}^{\rightarrow} \sum_{j_i \geqslant 0} x_i^{j_i} S^{j_i}(A) = \sum_I M_J(X) S^J(A), \tag{69}$$

where

$$M_J(X) = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1}^{j_1} x_{i_2}^{j_2} \ldots x_{i_r}^{j_r}. \tag{70}$$

The polynomials $M_J(X)$ are called monomial quasi-symmetric functions (or quasi-monomial functions). They span a subalgebra of $\mathbb{K}[X]$, which is precisely *QSym* [23]. The bilinear map from *QSym* $\times$ **Sym** to $\mathbb{K}$ defined by

$$\langle M_I, S^J \rangle = \delta_{IJ} \tag{71}$$

realizes *QSym* as the (graded) dual of **Sym**, and any pair of bases such that

$$K(X, A) = \sum_I U_I(X) V_J(A) \tag{72}$$

are dual to each other. The dual of the ribbon basis R_I is the fundamental basis F_I, which has the explicit expression

$$F_J(X) = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ i_k < i_{k+1} \text{ if } k \in \mathrm{Des}(J)}} x_{i_1} x_{i_2} \ldots x_{i_n} = \sum_{I \geqslant J} M_I. \tag{73}$$

4.2. Moulds and their fundamental symmetries

The series R_f is grouplike if and only if the coefficients $\langle R_f, S\rangle_I$ define a character, that is, a linear form χ on *QSym* satisfying $\chi(xy) = \chi(x)\chi(y)$, with $\langle R_f, S\rangle_I = \chi(M_I)$. Similarly, the series D_f is primitive if and only if the coefficients $\langle D_f, S\rangle_I$ define an infinitesimal character, that is, a linear form ψ on *QSym* satisfying $\psi(xy) = \psi(x)\varepsilon(y) + \varepsilon(x)\psi(y)$, with $\langle D_f, S\rangle_I = \psi(M_I)$, where ε is the counit.

The product rule for the monomial basis is easily derived directly by duality. It is given by the quasi-shuffle of compositions, regarded as words over the alphabet of positive integers:

$$M_I M_J = \sum_K (K|I \uplus J) M_K \tag{74}$$

where $(K|I \uplus J)$ means the coefficient of the word K in the quasi-shuffle of the words I and J. The quasi-shuffle makes sense for words over an arbitrary additive semigroup Σ. It is recursively defined by

$$au \uplus bv = a(u \uplus bv) + b(au \uplus v) + (a+b)(u \uplus v) \tag{75}$$

where u, v are arbitrary words over Σ, and $a, b \in \Sigma$, and the condition that the empty word is neutral.

For example,

$$\begin{aligned} 13 \uplus 32 = {} & 1332 + 1332 + 1323 + 3132 + 3123 + 3213 \\ & + 162 + 432 + 423 + 135 + 315 + 333 + 45. \end{aligned} \tag{76}$$

The product formula for the basis M_I allows to identify *QSym* with the quasi-shuffle Hopf algebra over the additive semigroup of positive integers $\mathbb{K}\langle\mathbb{N}^*\rangle$ (see [28]). We identify M_I, $I = (i_1, \ldots, i_r)$ with the basis $i_1 \ldots i_r$ of $\mathbb{K}\langle\mathbb{N}^*\rangle$ (words over the alphabet of positive integers), equipped with the quasi-shuffle product and the deconcatenation coproduct.

Families of coefficients such as $\langle R_f, S\rangle_I$ or $\langle D_f, S\rangle_I$, which define linear maps from $\mathbb{K}\langle\mathbb{N}^*\rangle$ to the base field $\mathbb{K}$, appear in Ecalle's work and are called *moulds*. A mould is said to be *symmetrel* (resp. *alternel*) if and only if it defines a character (resp. an infinitesimal character) of the quasi-shuffle Hopf algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$.

Equivalently, moulds can be interpreted as nonlinear operators on **Sym**. By definition, **Sym** is a graded free associative algebra, with exactly one generator for each degree. Several sequences of generators are of common use, some of which being composed of primitive elements (such as Ψ_n or Φ_n), whilst other are sequences of divided powers (such as S_n or Λ_n), so that their generating series is grouplike. Each pair of such sequences (U_n), (V_n) defines two moulds, whose coefficients express the expansions of the V_n on the U^I, and vice-versa. These moulds can be interpreted as the automorphisms sending U_n to V_n or conversely.

Ecalle's four fundamental symmetries reflect the four possible combinations of the primitive or grouplike characteristics.

If we denote by $\mathfrak{g}$ the (completed) primitive Lie algebra of **Sym** and by $\mathcal{G} = \exp \mathfrak{g}$ the associated multiplicative group, we have the following table:

$\mathfrak{g} \to \mathfrak{g}$	Alternal
$\mathfrak{g} \to \mathcal{G}$	Symmetral
$\mathcal{G} \to \mathfrak{g}$	Alternel
$\mathcal{G} \to \mathcal{G}$	Symmetrel

4.3. *Noncommutative quasi-symmetric functions:* **FQSym** *and* **WQSym**

4.3.1. *Free Quasi-Symmetric functions*

The multiplicative structure of *QSym* in the bases F_I and M_I can be understood by lifting these to two different combinatorial Hopf algebras, which could both equally deserve the name "noncommutative quasi-symmetric functions". For this reason, the first one is called "Free Quasi-Symmetric functions" (**FQSym**) and the second one "Word Quasi-Symmetric functions" (**WQSym**).

To understand the origin of the first one, recall that the noncommutative ribbon Schur function R_I has two interpretations:

(i) as the sum of words of shape I in the free associative algebra, and
(ii) as the sum of permutations of shape I in the group algebra of the symmetric group.

So, one may ask whether it is possible to associate with each word of shape I a permutation of shape I, so as to reconcile both approaches, and interpret each permutation as a sum of words.

This is indeed possible, and the solution is given by the classical *standardization process*, familiar in combinatorics and in computer science.

The *standardized word* $\mathrm{std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labeling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma = \mathrm{std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w\sigma$ is a nondecreasing word. For example, $\mathrm{std}(bbacab) = 341625$.

Obviously, $\mathrm{std}(w)$ has the same descents as w. We can now define polynomials

$$\mathbf{G}_\sigma(A) := \sum_{\mathrm{std}(w)=\sigma} w. \tag{77}$$

It is not hard to check that the linear span of these polynomials is a subalgebra of $\mathbb{K}\langle A\rangle$, denoted by **FQSym**$(A)$, an acronym for *Free Quasi-Symmetric functions* [12].

Since the definition of the $\mathbf{G}_\sigma(A)$ involves only a totally ordered alphabet A, we can apply it to an ordinal sum $A + B$, and as in the case of **Sym**, this defines a coproduct if we assume that A commutes with B. Clearly, this coproduct is coassociative and multiplicative, so that we have a graded (and connected) bialgebra, hence again a Hopf algebra. It is isomorphic (as a Hopf algebra) to the convolution algebra of permutations of Malvenuto and Reutenauer [35] (see [12]).

By definition,

$$R_I(A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(A) \tag{78}$$

so that **Sym** is embedded in **FQSym** as a Hopf subalgebra.

It is also easy to check that **FQSym** is self-dual. If we set $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$ and $\langle \mathbf{F}_\sigma, \mathbf{G}_\tau\rangle = \delta_{\sigma,\tau}$, then $\langle FG, H\rangle = \langle F \otimes G, \Delta H\rangle$.

Since the graded dual of **Sym** is the commutative algebra *QSym*, we have a surjective homomorphism $\mathbf{FQSym}^* \twoheadrightarrow QSym$. Its description is particularly simple: it consists in replacing our noncommuting variables a_i by commuting ones x_i. Then, $\mathbf{F}_\sigma(X)$ depends only on the descent composition $I = C(\sigma)$, and is equal to the quasi-symmetric function $F_I(X)$.

Hence, the multiplication rule for the $\mathbf{F}_\sigma$ describes in particular that of the F_I. To state it, we need the following notation.

For a word w on the alphabet $\{1, 2, \ldots\}$, we denote by $w[k]$ the word obtained by replacing each letter i by the integer $i + k$. If u and v are two words, with u of length k, one defines the *shifted concatenation* $u \bullet v = u \cdot (v[k])$ and the *shifted shuffle* $u \Cup v = u \sqcup\!\sqcup (v[k])$, where $\sqcup\!\sqcup$ is the usual shuffle product, defined for words over an arbitrary alphabet A by

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v), \tag{79}$$

where u, v are arbitrary words over Σ, and $a, b \in \Sigma$, and the condition that the empty word is neutral (compare (75)).

Then, the product rule is

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \Cup \beta} \mathbf{F}_\gamma. \tag{80}$$

4.3.2. Word Quasi-Symmetric functions

Although it is possible to lift the monomial basis to **FQSym**, the resulting polynomials are not positive sums of monomials. To lift the product formula to a multiplicity-free product of nonnegative polynomials, one has to introduce the larger algebra **WQSym** [25] (which contains **FQSym**, see, *e.g.*, [42,41]). Its definition is similar to that of **FQSym**. The only difference is that standardization is replaced by a finer invariant, the *packed word*.

The *packed word* $u = \mathrm{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \cdots < b_r$ are the letters occurring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. For example, $\mathrm{pack}(64661812) = 43441512$. A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the polynomial

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w. \tag{81}$$

Under the abelianization $\chi : \mathbb{K}\langle A\rangle \to \mathbb{K}[X]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions M_I, $I = \mathrm{ev}(u) = (|u|_a)_{a\in A}$ being the evaluation vector of u, that is, the sequence whose ith term is the number of times the letter a_i occurs in w.

These polynomials span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions [25]. It is a Hopf algebra for the usual coproduct $A \mapsto A + B$.

Proposition 4.1. *The product on* **WQSym** *is given by*

$$\mathbf{M}_{u'} \mathbf{M}_{u''} = \sum_{u \in u' *_W u''} \mathbf{M}_u, \tag{82}$$

where the convolution $u' *_W u''$ *of two packed words is defined as*

$$u' *_W u'' = \sum_{v, w;\, u = v\cdot w \in \mathrm{PW},\, \mathrm{pack}(v)=u',\, \mathrm{pack}(w)=u''} u. \tag{83}$$

For example,

$$\mathbf{M}_{11}\mathbf{M}_{21} = \mathbf{M}_{1121} + \mathbf{M}_{1132} + \mathbf{M}_{2221} + \mathbf{M}_{2231} + \mathbf{M}_{3321}. \tag{84}$$

There is also a basis Φ_u of **WQSym** which is a lift of the fundamental basis F_I of *QSym*, in the sense that under abelianization, $\Phi_u(X) = F_I(X)$, where $I = \mathrm{ev}(u)$. It is defined as follows.[2]

The refinement order can be extended to packed words [5,42]. We say that w is finer than w', and write $w \succcurlyeq w'$, iff w and w' have the same standardized word and the evaluation of w is finer than the evaluation of w'. Then,

$$\Phi_u := \sum_{v;\, v \succcurlyeq u} \mathbf{M}_v. \tag{85}$$

Packed words can be naturally identified with *ordered set partitions*, also called *set compositions*, the letter a_i at the jth position meaning that j belongs to block i. For example,

$$u = 313144132 \quad \leftrightarrow \quad \Pi = \big(\{2,4,7\}, \{9\}, \{1,3,8\}, \{5,6\}\big). \tag{86}$$

As set composition Π can be represented by *segmented permutation*, that is, a permutation obtained by reading the blocks of Π in increasing order and inserting bars | between blocks. To avoid confusion, we shall always write segmented permutations between parentheses.

For example,

$$\Pi = \big(\{2,4,7\}, \{9\}, \{1,3,8\}, \{5,6\}\big) \quad \leftrightarrow \quad (247|9|138|56), \tag{87}$$

and we have, in both notations,

$$\Phi_{133142} = \mathbf{M}_{133142} + \mathbf{M}_{134152} + \mathbf{M}_{144253} + \mathbf{M}_{145263}, \tag{88}$$

$$\Phi_{(14|6|23|5)} = \mathbf{M}_{(14|6|23|5)} + \mathbf{M}_{(14|6|2|3|5)} + \mathbf{M}_{(1|4|6|23|5)} + \mathbf{M}_{(1|4|6|2|3|5)}. \tag{89}$$

Since (Φ_u) is triangular over $(\mathbf{M}_u)$, it is a basis of **WQSym**. Note that the order used for summation is a restriction of the refinement order on compositions, so is a boolean lattice. Hence, denoting by $\max(w)$ the greatest letter of a word w,

$$\mathbf{M}_u = \sum_{v;\, v \succcurlyeq u} (-1)^{\max(v)-\max(u)} \Phi_v. \tag{90}$$

For example,

$$\mathbf{M}_{133142} = \Phi_{133142} - \Phi_{134152} - \Phi_{144253} + \Phi_{145263}. \tag{91}$$

By construction, the basis Φ satisfies a product formula similar to that of Gessel's basis F_I of *QSym* (whence the choice of notation). To state it, we need an analogue of the shifted shuffle, defined on the special class of segmented permutations encoding set compositions.

The *shifted shuffle* $\alpha \Cup \beta$ of two such segmented permutations is obtained from the usual shifted shuffle $\sigma \Cup \tau$ of the underlying permutations σ and τ by inserting bars

[2] This basis is different from the basis **Q** of [5].

- between each pairs of letters coming from the same word if they were separated by a bar in this word,
- after each element of β followed by an element of α.

For example,

$$(2|1) \Cup (12) = (2|134) + (23|14) + (234|1) + (3|2|14) + (3|24|1) + (34|2|1), \tag{92}$$

$$(1|2) \Cup (12) = (1|234) + (13|24) + (134|2) + (3|1|24) + (3|14|2) + (34|1|2). \tag{93}$$

We then have [42]:

Theorem 4.2. *The product and coproduct in the basis Φ are given by*

$$\Phi_{\sigma'}\Phi_{\sigma''} = \sum_{\sigma \in \sigma' \Cup \sigma''} \Phi_\sigma, \tag{94}$$

$$\Delta\Phi_\sigma = \sum_{\sigma'|\sigma''=\sigma \text{ or } \sigma'\cdot\sigma''=\sigma} \Phi_{\mathrm{std}(\sigma')} \otimes \Phi_{\mathrm{std}(\sigma'')}. \tag{95}$$

For example, in both encodings, we have

$$\Phi_1\Phi_{121} = \Phi_{1121} + \Phi_{2132} + \Phi_{2121} + \Phi_{3121},$$

$$\Phi_{(1)}\Phi_{(13|2)} = \Phi_{(124|3)} + \Phi_{(2|14|3)} + \Phi_{(24|13)} + \Phi_{(24|3|1)}, \tag{96}$$

$$\begin{aligned} \Phi_{1312}\Phi_{21} = {} & \Phi_{131221} + \Phi_{131231} + \Phi_{131232} + \Phi_{131243} + \Phi_{141232} \\ & + \Phi_{141321} + \Phi_{142321} + \Phi_{142331} + \Phi_{142341} + \Phi_{153421} \\ & + \Phi_{242321} + \Phi_{242331} + \Phi_{242341} + \Phi_{253421} + \Phi_{353421}, \end{aligned} \tag{97}$$

$$\begin{aligned} \Delta\Phi_{23121} = {} & 1 \otimes \Phi_{23121} + \Phi_1 \otimes \Phi_{2321} + \Phi_{11} \otimes \Phi_{121} + \Phi_{211} \otimes \Phi_{21} \\ & + \Phi_{2121} \otimes \Phi_1 + \Phi_{23121} \otimes 1. \end{aligned} \tag{98}$$

Finally, the Hopf epimorphism **WQSym** $\twoheadrightarrow$ *QSym* (commutative image) gives rise by duality to a Hopf embedding **Sym** $\hookrightarrow$ **WQSym***, which is given by

$$S^I \mapsto \sum_{\mathrm{ev}(u)=I} \mathbf{N}_u \tag{99}$$

where $\mathbf{N}_u$ is the dual basis of $\mathbf{M}_u$.

5. Polyhedral cones associated with packed words

We are now in a position to explain the geometric significance of the averages induced by diffusion. Identifying the sign sequences $\boldsymbol{\varepsilon}$ with compositions I and the basis $R_{\varepsilon_1,\dots,\varepsilon_{n-1}\bullet}$ with $\pm R_I$, we have to understand why the coefficients $R_f^{\boldsymbol{\varepsilon}\bullet}$ defined in Theorem 3.2 are of the form $\pm\chi(M_I)$, for a character χ of *QSym*.

We shall see that this is the reflect of a geometric property of certain polyhedral cones associated with packed words.

5.1. Cones associated with set compositions

Let u be a packed word of length n and

$$\Pi(u) = (B_1, \dots, B_r), \quad B_k = \{b_{k,1}, \dots, b_{k,i_1}\} \tag{100}$$

be the set composition of $[n]$ encoded by u and let σ be the corresponding segmented permutation, so that for $u = 322123$, $\Pi(u) = (\{4\}, \{2, 3, 5\}, \{1, 6\})$, and $\sigma = (4|235|16)$.

Definition 5.1. The polyhedral cone K_u in $\mathbb{R}^n$ is defined by the inequalities

$$\sum_{j=1}^{k} \sum_{i \in B_j} x_i \geqslant 0 \quad \text{for } k = 1, \dots, r. \tag{101}$$

For example, with $u = 322123$ as above, K_u is defined by the system

$$\begin{cases} x_4 \geqslant 0, \\ x_4 + x_2 + x_3 + x_5 \geqslant 0, \\ x_4 + x_2 + x_3 + x_5 + x_1 + x_6 \geqslant 0. \end{cases} \tag{102}$$

We denote by $\mathbf{1}_S$ the characteristic function of a subset S of $\mathbb{R}^n$.

Let $\mathcal{F}_n$ be the space of classes of measurable functions $\mathbb{R}^n \to \mathbb{R}$, where two functions differing on a set of measure zero are identified. Define an associative product on

$$\mathcal{F} = \bigoplus_{n \geqslant 0} \mathcal{F}_n \tag{103}$$

by

$$(f \star g)(x_1, \dots, x_{m+n}) = f(x_1, \dots, x_m) g(x_{m+1}, \dots, x_{m+n}), \tag{104}$$

for $f \in \mathcal{F}_m$ and $g \in \mathcal{F}_n$.

Let $\mathcal{P}$ be the subalgebra of $\mathcal{F}$ generated by the characteristic functions $\mathbf{1}_{K_u}$ of the cones K_u.

Theorem 5.2. *The map $\alpha : \mathcal{P} \to$ **WQSym** defined by*

$$\mathbf{1}_{K_u} \mapsto (-1)^{\max(u)} \mathbf{M}_u \tag{105}$$

*is an isomorphism of algebras. That is, if the product $\mathbf{M}_u \mathbf{M}_v$ in **WQSym** is given by*

$$\mathbf{M}_u \mathbf{M}_v = \sum_w c_{uv}^w \mathbf{M}_w, \tag{106}$$

the characteristic function of the Cartesian product $K_u \times K_v$ is

$$\mathbf{1}_{K_u \times K_v} = \sum_w (-1)^{\max(u)+\max(v)-\max(w)} c_{uv}^w \mathbf{1}_{K_w}. \tag{107}$$

The rest of this section is devoted to the proof of this result. We shall first translate it into an identity involving characteristic functions of Cartesian products of different cones, and relate it to the product of the basis Φ_u of **WQSym**. We shall then represent a cone by a certain Laurent series and prove that the series corresponding to both sides of the identity coincide over a nonempty open set.

The minimal example of Theorem 5.2 is

$$\mathbf{M}_1\mathbf{M}_1 = \mathbf{M}_{12} + \mathbf{M}_{21} + \mathbf{M}_{11}, \tag{108}$$

whose counterpart is

$$\mathbf{1}_{K_1\times K_1} = \mathbf{1}_{K_{12}} + \mathbf{1}_{K_{21}} - \mathbf{1}_{K_{11}} \tag{109}$$

where

$$\begin{aligned} K_1 \times K_1 &= (x_1 \geqslant 0, x_2 \geqslant 0),\\ K_{12} &= (x_1 \geqslant 0, x_1 + x_2 \geqslant 0),\\ K_{21} &= (x_2 \geqslant 0, x_1 + x_2 \geqslant 0),\\ K_{11} &= (x_1 + x_2 \geqslant 0). \end{aligned} \tag{110}$$

This is to be compared with the product rule (80) of **FQSym**, whose minimal example is

$$\mathbf{F}_1\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{21}. \tag{111}$$

Although (80) can be derived from the embedding of **FQSym** into **WQSym**, its geometric interpretation is of a different nature. Indeed, on the one hand, $\mathbf{F}_\alpha$ can be interpreted as the characteristic function of a simplex, and the product rule reflects then the classical decomposition of a product of simplexes. On the other hand, (107) is purely a linear relation between characteristic functions, and does not follow from a dissection of the product $K_u \times K_v$, but rather from an argument of inclusion–exclusion.

Corollary 5.3. *Let f be a probability distribution over $\mathbb{R}$, and set, for u of length n*

$$m_u(f) = (-1)^{\max(u)} \int\limits_{K_u} f(x_1)\dots f(x_n)\,dx_1\dots dx_n. \tag{112}$$

Then m_u depends only on the integer composition $I = \mathrm{ev}(u)$, so that we can denote it by m_I as well. Then the formal series

$$S(f) := \sum_u m_u \mathbf{N}_u \tag{113}$$

is grouplike for the coproduct of **WQSym***. *If one embeds* **Sym** *in* **WQSym*** *by (99), then,*

$$S(f) := \sum_I m_I S^I \tag{114}$$

is grouplike in **Sym** *(that is, $m_I = \chi(M_I)$ for some character of QSym).*

5.2. Changing bases and cones

To prove Theorem 5.2, it will be easier to work with the fundamental basis Φ_u of **WQSym**.

Proposition 5.4. *If we identify as above* $(-1)^{\max(u)}\mathbf{M}_u$ *with the characteristic function of* K_u*, then* $(-1)^{\max(u)}\Phi_u$ *gets identified with the characteristic function of the cone* C_u*, defined by the conditions*

$$\sum_{i=1}^{k} x_{\sigma_i} \begin{cases} <0 & \text{if } \sigma_k \text{ is not the end of a block of } \sigma, \\ \geqslant 0 & \text{otherwise} \end{cases} \tag{115}$$

for $k = 1, \dots, n$*.*

Proof. The change of basis from the M_u to the Φ_u is given by (90). It can be rewritten as

$$(-1)^{\max(u)}\mathbf{M}_u = \sum_{v;\, v \succcurlyeq u} (-1)^{\max(v)}\Phi_v, \tag{116}$$

and the proposition follows from the fact that K_u is the union of the C_v, for $v \succcurlyeq u$, which is clear from their definitions. □

Now, Theorem 5.2 rewrites in the basis Φ_u as

Theorem 5.5. *The map* $\alpha : \mathcal{P} \to$ **WQSym** *defined by*

$$\mathbf{1}_{C_u} \mapsto (-1)^{\max(u)}\Phi_u \tag{117}$$

is an isomorphism of algebras.

In particular, if u is a nondecreasing packed word, such as $u = 111233$ (so that $\sigma = (123|4|56)$), then the characteristic function of C_u has the form

$$\mathbf{1}_{C_u} = \sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1+x_2)\dots\sigma_{\varepsilon_n}(x_1+\dots+x_n) \tag{118}$$

so that integrals of $f(x_1)\dots f(x_n)$ over C_u have the form (31). Thus, (85) implies that the series $S(f)$ coincides with R_f as defined in Theorem 3.2, so that this is actually a special case of Theorem 5.5.

The minimal example of Theorem 5.5 is

$$\Phi_1\Phi_1 = \Phi_{11} + \Phi_{21}, \tag{119}$$

whose counterpart is

$$\mathbf{1}_{C_1\times C_1} = \mathbf{1}_{C_{21}} - \mathbf{1}_{C_{11}} \tag{120}$$

where

$$\begin{aligned} C_1 \times C_1 &= (x_1 \geqslant 0, x_2 \geqslant 0), \\ C_{21} &= (x_2 \geqslant 0, x_1 + x_2 \geqslant 0), \\ C_{11} &= (x_1 < 0, x_1 + x_2 \geqslant 0). \end{aligned} \tag{121}$$

In particular, since Φ_u is a lift of F_I, we have:

Corollary 5.6. *If χ is the character of QSym defined from a function f as in Corollary 5.3, then*

$$\chi(F_I) = (-1)^{\max(u)} \int_{C_u} f(x_1)\dots f(x_n)\,dx_1\dots dx_n \tag{122}$$

for any u such that $I = \mathrm{ev}(u)$.

Example 5.7 *(The Sparre Andersen formula).* Let f be a probability distribution on $\mathbb{R}$ and $(X_n)_{n\geqslant 1}$ be a sequence of independent random variables of distribution f. Define $S_n = X_1 + \dots + X_n$ and

$$\tau_n = \mathbf{P}(S_1 < 0, S_2 < 0, \dots, S_{n-1} < 0, S_n \geqslant 0), \qquad \tau(s) = \sum_{n\geqslant 1} \tau_n s^n, \tag{123}$$

and

$$q_n = \mathbf{P}(S_n \geqslant 0). \tag{124}$$

The celebrated formula of E. Sparre Andersen (*cf.* [20, p. 413]) states that

$$\log \frac{1}{1-\tau(s)} = \sum_{n\geqslant 1} q_n \frac{s^n}{n}. \tag{125}$$

This is immediate from Corollary 5.6, since

$$\tau_n = -\chi(F_n) \quad \text{and} \quad q_n = -\chi(M_n). \tag{126}$$

But F_n is the complete homogeneous symmetric function h_n and M_n is the power-sum p_n. So (125) follows by applying χ to the well-known (Newton) identity

$$\sum_{n\geqslant 0} h_n s^n = \exp \sum_{n\geqslant 1} \frac{p_n}{n} s^n. \tag{127}$$

Another explanation of the Sparre Andersen formula relying on different Hopf algebras [37] will appear in the doctoral thesis of A. Mansuy.

Theorems 5.2 and 5.5 being equivalent, we shall prove the latter.

5.3. *The integer point transform*

Since a polyhedral cone is characterized by the set of its integral points, we shall encode C_u by the Laurent series

$$F_u := (-1)^{\max(u)} \sum_{\alpha \in C_u \cap \mathbb{Z}^n} z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}. \tag{128}$$

This identification endows the vector space spanned by the series F_u with the $\star$ product.

Such a series has a nonempty domain of convergence D_u in $\mathbb{C}^n$, and inside it, represents a rational function $f_u(x)$. The pair (f_u, D_u) allows to reconstruct C_u unambiguously. It is sometimes called the integer point transform of the polyhedral cone C_u [3].

Note that we could equivalently work with the Laplace transform of the characteristic function, and identify C_u with

$$(-1)^{\max(u)} \int\limits_{C_u} e^{-\langle p,x\rangle}\, dx. \tag{129}$$

This is again a rational function, which, together with the domain of convergence of the integral, allows the reconstruction of C_u.

For example,

$$\begin{aligned} F_{11} &= -\sum_{\alpha_1 \leqslant -1} z_1^{\alpha_1} \sum_{\alpha_2 \geqslant -\alpha_1} z_2^{\alpha_2} \\ &= -\sum_{\alpha_1 \leqslant -1} z_1^{\alpha_1} z_2^{-\alpha_1} \sum_{\alpha_2' \geqslant 0} z_2^{\alpha_2'} \end{aligned} \tag{130}$$

so that

$$F_{11} = \left(-\frac{1}{1-z_2}\frac{z_2/z_1}{1-z_2/z_1}, |z_2| < |z_1|, |z_2| < 1 \right). \tag{131}$$

We also have

$$F_{12} := \left(\frac{1}{1-z_2}\frac{1}{1-z_1/z_2}, |z_1| < |z_2| < 1 \right), \tag{132}$$

$$F_{21} := \left(\frac{1}{1-z_1}\frac{1}{1-z_2/z_1}, |z_2| < |z_1| < 1 \right). \tag{133}$$

More generally, we have

Proposition 5.8. *Let u be a packed word, and σ be the corresponding segmented permutation. Then, the rational function associated with F_u is*

$$f_u = (-1)^{\max(u)} \frac{1}{1 - z_{\sigma_n}} \prod_{i=1}^{n-1} g(\sigma_i, \sigma_{i+1}), \tag{134}$$

where

$$g(\sigma_i, \sigma_{i+1}) = \begin{cases} \frac{z_{\sigma_{i+1}}/z_{\sigma_i}}{1 - z_{\sigma_{i+1}}/z_{\sigma_i}} & \text{if } \sigma_i \text{ and } \sigma_{i+1} \text{ are not separated by a bar}, \\ \frac{1}{1 - z_{\sigma_i}/z_{\sigma_{i+1}}} & \text{otherwise}. \end{cases} \tag{135}$$

Proof. Let us write the expansion of $(-1)^{\max(u)} f_\sigma$ as

$$\sum_{i_1 \diamond_1 0} z_{\sigma_1}^{i_1} \sum_{i_1+i_2 \diamond_2 0} z_{\sigma_2}^{i_2} \sum_{i_1+i_2+i_3 \diamond_3 0} z_{\sigma_3}^{i_3} \cdots \tag{136}$$

where $\diamond_i$ is either $\geqslant$ or $<$ depending on whether there is a bar after i in σ or not.

Let $i'_2 = i_1 + i_2$. Then,

$$(-1)^{\max(u)} f_\sigma = \sum_{i_1 \diamond_1 0} z_{\sigma_1}^{i_1} z_{\sigma_2}^{-i_1} \sum_{i'_2 \diamond_2 0} z_{\sigma_2}^{i'_2} \sum_{i'_2 + i_3 \diamond_3 0} z_{\sigma_3}^{i_3} \ldots. \tag{137}$$

This sum splits therefore into the product of two partial sums, the first one being equal to $g(\sigma_1, \sigma_2)$, the remaining part being $\pm f_{(\sigma_2,\ldots)}$. □

This expression of f_u can be simplified: indeed, both choices of (135) are equal up to sign when one does not consider the domain of convergence. So, if one writes the denominators in the form $z_{\sigma_i} - z_{\sigma_{i+1}}$, the terms with a -1 match the bars, so that we have

Corollary 5.9. *The function $f_u = f_\sigma$ simplifies as*

$$f_\sigma = \frac{1}{z_{\sigma_n} - 1} \prod_{i=1}^{n-1} \frac{z_{\sigma_{i+1}}}{z_{\sigma_i} - z_{\sigma_{i+1}}}. \tag{138}$$

For example,

$$f_{211} = f_{(23|1)} = +\frac{1}{1-z_1} \frac{1}{1-z_3/z_1} \frac{z_3/z_2}{1-z_3/z_2} = \frac{z_3 z_1}{(z_2 - z_3)(z_3 - z_1)(z_1 - 1)}, \tag{139}$$

$$f_{1223} = f_{(1|23|4)} = -\frac{1}{1-z_4} \frac{1}{1-z_3/z_4} \frac{z_3/z_2}{1-z_3/z_2} \frac{1}{1-z_1/z_2}$$
$$= \frac{z_2 z_3 z_4}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)(z_4 - 1)}. \tag{140}$$

Theorem 5.10. *The $\star$ product of the Laurent series F_σ is given by*

$$F_\sigma \star F_{\sigma'} = \sum_{\sigma'' \in \sigma \Cup \sigma'} F_{\sigma''}, \tag{141}$$

that is, by the same formula as the Φ_σ of **WQSym**.

Proof. Consider the product $F_\sigma \star F_{\sigma'}$ with σ of length n and σ' of length p. The domain of convergence of this product is a nonempty open set $\mathcal{O}$, which is the Cartesian product of an open set of $\mathbb{C}^n$ and an open set of $\mathbb{C}^p$.

Then, the intersection of $\mathcal{O}$ with the domains $|z_j| < |z_i| < 1$ for all $j \in [n+1, n+p]$ and all $i \in [1, n]$ is again a nonempty open set $\mathcal{O}'$.

All the $F_{\sigma''}$ occurring in the r.h.s. of (141) converge in $\mathcal{O}'$, by definition of the segmented shifted shuffle, so that both sides of (141) define rational functions in $\mathcal{O}'$. Thus, we only have to prove equality of these rational functions.

Consider the right-hand side h as a function of z_{σ_1}. Since for each fraction, the numerator has a degree strictly lower than the denominator, this is also true for their sum. Now consider h as a reduced fraction. Apart from z_{σ_2}, the possible poles of z_{σ_1} are the $z_{\sigma'_j}$ for any j. But these poles do not occur: consider the residue. Multiplying h by $(z_{\sigma_1} - z_{\sigma'_j})$ and then putting $(z_{\sigma_1} = z_{\sigma'_j})$ yields zero in all permutations where σ_1 and σ'_j are not neighbors, and the remaining permutations can be paired as $\alpha\sigma_1\sigma'_j\beta$, $\alpha\sigma'_j|\sigma_1\beta$ whose common residue is also zero. So as a reduced fraction in z_{σ_1}, h has a denominator of degree 1 and a numerator of degree 0, hence is equal to a constant divided by $z_{\sigma_1} - z_{\sigma_2}$.

Now, putting $z_{\sigma_1} = 0$, the term corresponding to a segmented permutation τ either gives zero if $z_{\tau_1} \neq z_{\sigma_1}$ or gives $-f_{\tau_2\dots}$ otherwise. By induction, this gives the desired formula. □

Let us illustrate the proof on the example of the product $F_{11}F_{21}$. We have

$$f_{11} \star f_{21} = f_{(12)} \star f_{(2|1)} = \frac{z_2}{(z_1 - z_2)(z_2 - 1)} \frac{z_3}{(z_4 - z_3)(z_3 - 1)} \tag{142}$$

and the sum $f_{1121} + f_{1221} + f_{1321} + f_{2221} + f_{2321} + f_{3321}$ which is also $f_{(124|3)} + f_{(14|23)} + f_{(14|3|2)} + f_{(4|123)} + f_{(4|13|2)} + f_{(4|3|12)}$ in terms of segmented permutations translates into fractions as

$$\begin{aligned} &\frac{z_2 z_4 z_3}{(z_1 - z_2)(z_2 - z_4)(z_4 - z_3)(z_3 - 1)} + \frac{z_4 z_2 z_3}{(z_1 - z_4)(z_4 - z_2)(z_2 - z_3)(z_3 - 1)} \\ &\quad + \frac{z_4 z_3 z_2}{(z_1 - z_4)(z_4 - z_3)(z_3 - z_2)(z_2 - 1)} + \frac{z_1 z_2 z_3}{(z_4 - z_1)(z_1 - z_2)(z_2 - z_3)(z_3 - 1)} \\ &\quad + \frac{z_1 z_3 z_2}{(z_4 - z_1)(z_1 - z_3)(z_3 - z_2)(z_2 - 1)} + \frac{z_3 z_1 z_2}{(z_4 - z_3)(z_3 - z_1)(z_1 - z_2)(z_2 - 1)}. \end{aligned} \tag{143}$$

Now, the permutations having a pole $z_1 - z_2$ are (124|3), (4|123), and (4|3|12), that is, if one forgets their 1, (24|3), (4|23), and (4|3|2), which are the permutations up to rescaling belonging to $(1) \Cup (2|1)$. As for the poles of the r.h.s., let us consider for example the pole $z_2 - z_4$. In that case, it appears in the words (124|3) and (14|23). Then one easily checks that starting with $f_{(124|3)} + f_{(14|23)}$, multiplying by $z_2 - z_4$ and then putting $z_2 = z_4$, one finds 0. The same holds for the pole $z_1 - z_4$, where one regroups (14|23) with (4|123) and (14|3|2) with (4|13|2).

Note 5.11. Theorem 5.10 proves in particular that the $m_u(f)$ satisfy the same product formula as the $\mathbf{M}_u$ of **WQSym**, hence that the m_I are the images of the M_I by a character of *QSym*, which concludes the proof of Theorem 5.5 and of its equivalent form Theorem 5.2.

Another proof, exploiting the natural splitting of the shuffle structure, is provided in Section 8 in a broader context: set partitions each part are replaced by multisets partitions, allowing to consider integrals on a more general class of cones.

6. Rational moulds for WQSym

6.1. Set compositions as rational functions

There is another way to encode elements of **WQSym** by rational functions, which in turn define nonlinear operators on certain function spaces. Evaluating these operators on a fixed function gives then rise to a character of **WQSym**, which factors through *QSym* when the target space is a commutative algebra.

Let z_i, $i \geqslant 1$ be indeterminates. For a set of integers I, let $z^I = \prod_{i \in I} z_i$, and for a packed word u encoding a set composition $\Pi(u) = (B_1, \dots, B_m)$, define the rational function

$$M_u(Z) = \prod_{k=1}^{m} \left(\prod_{i=1}^{k} z^{B_i} - 1 \right)^{-1}. \tag{144}$$

For example, with $u = 2131231$, $\Pi(u) = (\{2,4,7\}, \{1,5\}, \{3,6\})$, and

$$M_{2131231}(Z) = \frac{1}{(z_2 z_4 z_7 - 1)(z_2 z_4 z_7 z_1 z_5 - 1)(z_2 z_4 z_7 z_1 z_5 z_3 z_6 - 1)}. \tag{145}$$

We endow the algebra of rational functions $\mathbb{C}(Z)$ with the shifted product

$$f(z_1,\dots,z_p)\star g(z_1,\dots,z_q)=f(z_1,\dots,z_p)g(z_{p+1},\dots,z_{p+q}). \tag{146}$$

The resulting structure is called the rational mould algebra [8,11].

Lemma 6.1. *The linear map ϕ : **WQSym** $\to\mathbb{C}(Z)$ defined by $\phi(\mathbf{M}_u)=M_u(Z)$ is an injective homomorphism of algebras for the $\star$ product on $\mathbb{C}(Z)$.*

Proof. The Laurent expansion of $M_u(Z)$ in the domain ($|z_k|>1$ for all k) is

$$M_u(Z)=\sum_{\alpha\in\mathbb{Z}^n_{<0};\ \mathrm{pack}(\alpha)=u} z^\alpha \tag{147}$$

where the packing of words over the negative integers is defined w.r.t. the natural order (*e.g.*, $\mathrm{pack}(-3,-5,-3,-8)=3231$). Moreover, the expansions of different M_u have no monomial in common, so that they are linearly independent. □

This embedding is the perfect analog of the embedding of **FQSym** in the rational mould algebra defined in [11]. In this reference, rational functions encode operators on formal integrals. There is an analogous situation here, which can be first understood in terms of the ordinary Fourier transform.

6.2. Associated operators

Let us consider functions f from the real line to some associative algebra, represented in terms of their Fourier transforms as

$$f(t)=\int_{-\infty}^{\infty}\hat f(\nu)e^{2i\pi\nu t}\,d\nu. \tag{148}$$

Introduce new variables ν_k and set $z_k=e^{2i\pi\nu_k}$. Assuming convergence of the integrals, we can now interpret the mould $M_u(z_1,\dots,z_m)$ as the m-linear operator

$$f_1\otimes\cdots\otimes f_m\mapsto g=M_u[f_1,\dots,f_m] \tag{149}$$

where

$$g(t)=\int_{\mathbb{R}^m}\hat f_1(\nu_1)\dots\hat f_m(\nu_m)M_u\big(e^{2i\pi\nu_1},\dots,e^{2i\pi\nu_m}\big)e^{2i\pi\nu_1 t}\dots e^{2i\pi\nu_m t}\,d\nu_1\dots d\nu_m. \tag{150}$$

Expanding M_u as a Laurent series near infinity and regarding it as the Fourier series of a distribution, we see that the result is

$$g(t)=\sum_{\alpha\in\mathbb{Z}^m_{<0};\ \mathrm{pack}(\alpha)=u} f_1(t+\alpha_1)f_2(t+\alpha_2)\dots f_m(t+\alpha_m). \tag{151}$$

For $f_1=f_2=\cdots=f_m=f$, we set simply $g(t)=M_u[f](t)$. For example,

$$M_1[f](t) = \sum_{n\geqslant 1} f(t-n), \tag{152}$$

$$M_{11}[f_1, f_2](t) = \sum_{n\geqslant 1} f_1(t-n) f_2(t-n), \tag{153}$$

$$M_{12}[f_1, f_2](t) = \sum_{n_1,n_2\geqslant 1} f_1(t-n_1-n_2) f_2(t-n_2) = M_{21}[f_2, f_1](t). \tag{154}$$

We can now rephrase Lemma 6.1 as follows:

Proposition 6.2. *If we set, for $|u| = p$ and $|v| = q$,*

$$(M_u \star M_v)[f_1, \dots f_{p+q}] = M_u[f_1 \dots, f_p] M_v[f_{p+1}, \dots, f_{p+q}] \tag{155}$$

then, the map $\mathbf{M}_u \to M_u$ is an injective homomorphism.

Observing that

$$M_{11}[f_1, f_2] = M_1[f_1 f_2] \quad \text{and} \quad M_{12}[f_1, f_2] = M_1\big[M_1[f_1] f_2\big] \tag{156}$$

and that

$$\mathbf{M}_1^2 = \mathbf{M}_{12} + \mathbf{M}_{21} + \mathbf{M}_{11}, \tag{157}$$

we have

$$M_1\big[f_1 M_1[f_2] + M_1[f_1] f_2 + f_1 f_2\big] = M_1[f_1] M_1[f_2]. \tag{158}$$

Let us set for short $M = M_1$. Then, (158) means that M is a Rota–Baxter operator (see Section 9), and moreover, we have:

Proposition 6.3. *Any operator M_u can be written as a composition of operators M and products of functions. More precisely, if $\Pi(u) = (B_1, \dots, B_m)$ is the associated set composition, set*

$$b_i = \prod_{j\in B_i} f_j \quad \textit{and} \quad [b_i] = M[b_i]. \tag{159}$$

Then,

$$\mathbf{M}_u[f_1, \dots, f_n] = \big[\dots\big[[b_1]b_2\big]\dots b_m\big]. \tag{160}$$

Proof. If v is a packed word of length n and $v = u\sigma$ for a permutation $\sigma \in \mathfrak{S}_n$, then

$$M_v[f_1, \dots, f_n] = M_u[f_{\sigma(1)}, \dots f_{\sigma(n)}], \tag{161}$$

so that it is sufficient to prove the property for nondecreasing packed words. Now,

$$M_{1^k}[f_1, \dots, f_k] = M[f_1 f_2 \dots f_k], \tag{162}$$

and if m is the maximum letter of a packed word u of length n, then

$$M\big[M_u[f_1,\ldots,f_n]f_{n+1}\ldots f_{n+k}\big] = M_{u(m+1)^k}[f_1,\ldots,f_{n+k}]. \qquad \square \tag{163}$$

For example,

$$M_{21}[f_1,f_2] = M\big[M[f_2]f_1\big], \tag{164}$$

$$M_{132}[f_1,f_2,f_3] = M\big[f_2M\big[f_3M[f_1]\big]\big], \tag{165}$$

$$M_{3121}[f_1,f_2,f_3,f_4] = M\big[f_1M\big[f_3M[f_2f_4]\big]\big]. \tag{166}$$

6.3. Operadic considerations

6.3.1. Partial compositions

The embedding of **FQSym** in the operad of rational moulds allowed the identification of various suboperads [11], in particular the dendriform operad. We shall see that the embedding of **WQSym** yields similar results for the tridendriform operad.

To define an operad structure, we need partial compositions $\circ_k$. Their definition is transparent on the operators M_u. Let Δ be the finite difference operator

$$\Delta f(t) = f(t+1) - f(t). \tag{167}$$

Note that Δ is a left inverse for M:

$$\Delta M[f](t) = f(t). \tag{168}$$

Then, the kth partial composition $M_u \circ_k M_v$, with u of length m and v of length n, is defined by

$$M_u \circ_k M_v[f_1,\ldots,f_{m+n-1}] = M_u\big[f_1,\ldots,f_{k-1},\Delta M_v[f_k,\ldots,f_{k+n-1}],f_{k+n},\ldots,f_{m+n-1}\big]. \tag{169}$$

In terms of the associated rational functions, this reads

$$\begin{aligned} M_u \circ_k M_v(Z) = {} & (z_kz_{k+1}\ldots z_{k+n-1}-1) \\ & \times M_u(z_1,\ldots,z_{k-1},z_kz_{k+1}\ldots z_{k+n-1},z_{k+n},\ldots z_{m+n-1}) \\ & \times M_v(z_k,\ldots,z_{k+n-1}). \end{aligned} \tag{170}$$

Indeed,

$$\begin{aligned} \Delta M_v[f_k,\ldots,f_{k+n-1}](t) = {} & \int_{\mathbb{R}^n} \hat{f}_k(\nu_k)\ldots\hat{f}_{k+n-1}(\nu_{k+n-1})\big(e^{2\pi i(\nu_k+\cdots+\nu_{k+n-1})}-1\big) \\ & \times e^{2\pi i\nu_k t}\ldots e^{2\pi i\nu_{k+n-1}t}\,d\nu_k\ldots d\nu_{k+n-1} \end{aligned} \tag{171}$$

so that its Fourier transform is

$$\begin{aligned}&\left(e^{2\pi i\nu}-1\right)\int\limits_{\mathbb{R}^n}\delta(\nu-\nu_k-\cdots-\nu_{k+n-1})\hat{f}_k(\nu_k)\ldots\hat{f}_{k+n-1}(\nu_{k+n-1})\\&\quad\times e^{2\pi i\nu_k t}\ldots e^{2\pi i\nu_{k+n-1}t}\,d\nu_k\ldots d\nu_{k+n-1}.\end{aligned}\tag{172}$$

Plugging this expression into (169), we obtain (170).

On this version, it is clear that M_u and M_v can be replaced by arbitrary rational functions, and it is easy to check that the axioms of an operad are satisfied. This is an analogue of the operad **Mould** of [8,11] which will be denoted here by $\mathbf{Mould}^0$.

Theorem 6.4.

(i) *The space of rational functions in Z endowed with the partial compositions $\circ_k$ above acquires the structure of an operad, which will be denoted by* $\mathbf{Mould}^1$.
(ii) *It is isomorphic to the operad λ-**RatFct** defined by Loday [32], for $\lambda = 1$.*
(iii) *The M_u span a suboperad of* $\mathbf{Mould}^1$.

Proof. (i) This can be checked directly. However, it follows from (ii).

(ii) Loday's rule for the composition in 1-**RatFct** is

$$\begin{aligned}P\circ'_k Q(x_1,\ldots,x_{m+n-1})&=P\left(x_1,\ldots,x_{k-1},\theta^1(x_k,\ldots,x_{k+n-1}),x_{k+n},\ldots,x_{m+n-1}\right)\\&\quad\times Q(x_k,\ldots,x_{k+n-1})\end{aligned}\tag{173}$$

with

$$\theta^1(x_1,\ldots,x_n)=\prod_{i=1}^{n}(x_i+1)-1.\tag{174}$$

Set

$$P'(x_1,\ldots,x_n):=\left(\prod_{i=1}^{n}n(x_i+1)-1\right)P(x_1+1,\ldots,x_n+1).\tag{175}$$

Then

$$P\circ_i Q=\sum_R R\quad\Longleftrightarrow\quad P'\circ'_i Q'=\sum_R R'.\tag{176}$$

(iii) By Proposition 6.3,

$$M_v[f_k,\ldots,f_{k+n-1}]=M[F]\tag{177}$$

where F is the product of a term $M_{v'}[f_{i_1},\ldots]$ with some f_j. Precisely, v' is obtained from v by erasing its maximal letter m, the f_j are the elements of the last block B_m of $\Pi(v)$, and the arguments of $M_{v'}$ are the remaining f_i, in their natural order. Hence,

$$\Delta M_v[f_k,\ldots,f_{k+n-1}]=F=M_{v'}[f_{i_1},\ldots,f_{i_r}]f_{j_1}\ldots f_{j_s}.\tag{178}$$

Plugging this expression into the decomposition (160) of M_u, and applying (163), we obtain $M_u\circ_k M_v$ as a multiplicity-free sum of terms M_w. $\square$

For example,

$$\begin{aligned}M_{12}\circ_2 M_{12}[f_1,f_2,f_3] &= M_{12}\big[f_1,\Delta M_{12}[f_2,f_3]\big]=M\big[M[f_1]M[f_2]f_3\big]\\ &\quad\big(\text{since } M_{12}[f_2,f_3]=M\big[M[f_2]f_3\big]\big)\\ &= M\big[M_{12}[f_1,f_2]f_3+M_{21}[f_1,f_2]f_3+M_{11}[f_1,f_2]f_3\big]\\ &= (M_{123}+M_{213}+M_{112})[f_1,f_2,f_3]. \qquad (179)\end{aligned}$$

Similarly, one can check that

$$M_{121}\circ_1 M_{12}=M_{1232}, \qquad (180)$$

$$M_{121}\circ_2 M_{12}=M_{1121}+M_{1231}+M_{2132}, \qquad (181)$$

$$M_{121}\circ_3 M_{12}=M_{2312}, \qquad (182)$$

and

$$M_{123}\circ_1 M_{12}=M_{1234}, \qquad (183)$$

$$M_{123}\circ_2 M_{12}=M_{1123}+M_{1234}+M_{2134}, \qquad (184)$$

$$M_{123}\circ_3 M_{12}=M_{1213}+M_{1223}+M_{1234}+M_{1324}+M_{2314}. \qquad (185)$$

6.3.2. Suboperads of **Mould**[1]

A *dendriform trialgebra* [34] (or tridendriform algebra) is an associative algebra whose multiplication $\odot$ splits into three pieces

$$x\odot y=x\prec y+x\circ y+x\succ y, \qquad (186)$$

where $\circ$ is associative, and

$$(x\prec y)\prec z=x\prec(y\odot z), \qquad (187)$$

$$(x\succ y)\prec z=x\succ(y\prec z), \qquad (188)$$

$$(x\odot y)\succ z=x\succ(y\succ z), \qquad (189)$$

$$(x\succ y)\circ z=x\succ(y\circ z), \qquad (190)$$

$$(x\prec y)\circ z=x\circ(y\succ z), \qquad (191)$$

$$(x\circ y)\prec z=x\circ(y\prec z). \qquad (192)$$

The free dendriform trialgebra on one generator is known to be based on *reduced plane trees*, *i.e.*, plane rooted trees in which each vertex which is not a leaf has at least two children. These trees are counted by the little Schröder numbers [34]. This algebra is naturally embedded in **WQSym**. Indeed, **WQSym** is tridendriform, the partial products being given by

$$\mathbf{M}_{w'}\prec\mathbf{M}_{w''}=\sum_{w=u.v\in w'*_W w'',\ |u|=|w'|;\ \max(v)<\max(u)}\mathbf{M}_a, \qquad (193)$$

$$\mathbf{M}_{w'} \circ \mathbf{M}_{w''} = \sum_{w=u.v \in w' *_W w'',\ |u|=|w'|;\ \max(v)=\max(u)} \mathbf{M}_a, \tag{194}$$

$$\mathbf{M}_{w'} \succ \mathbf{M}_{w''} = \sum_{w=u.v \in w' *_W w'',\ |u|=|w'|;\ \max(v)>\max(u)} \mathbf{M}_a. \tag{195}$$

Lemma 6.5. *On the operators M_u, these operations translate as*

$$(M_u \succ M_v)[f_1, \ldots, f_{n+m}] = M\big[M_u[f_1, \ldots, f_n] \Delta M_v[f_{n+1}, \ldots, f_{m+n}]\big], \tag{196}$$

$$(M_u \prec M_v)[f_1, \ldots, f_{n+m}] = M\big[\Delta M_u[f_1, \ldots, f_n] M_v[f_{n+1}, \ldots, f_{m+n}]\big], \tag{197}$$

$$(M_u \circ M_v)[f_1, \ldots, f_{n+m}] = M\big[\Delta M_u[f_1, \ldots, f_n] \Delta M_v[f_{n+1}, \ldots, f_{m+n}]\big]. \tag{198}$$

Proof. Direct verification. □

With each reduced plane tree T, one can associate an element $\mathcal{M}_T$ of **WQSym**, defined by means of a map $\mathcal{T}$ from words to trees [42]. From each word w of length n, we build a plane tree $\mathcal{T}(w)$ recursively defined as follows. If $m = \max(w)$ and w has exactly k occurrences of m, $\mathcal{T}(w)$ is obtained from the factorization

$$w = v_0 m v_1 m v_2 \ldots v_{k-1} m v_k \tag{199}$$

by grafting $\mathcal{T}(v_0), \mathcal{T}(v_1), \ldots, \mathcal{T}(v_k)$ (in this order) on a common root. We then set

$$\mathcal{M}_T = \sum_{\mathcal{T}(w)=T} w = \sum_{\mathcal{T}(u)=T} \mathbf{M}_u. \tag{200}$$

One can show [42] that these polynomials span a Hopf subalgebra of **WQSym**, which is precisely the free algebra on one generator for the tridendriform operad. This generalizes the embedding in **FQSym** of the free algebra on one generator for the dendriform operad, which is based on planar binary trees [33,26].

As in the case of binary trees and **FQSym**, the sum of the M_u for all u having a given Schröder tree $\mathcal{T}(u) = T$ is extremely simple:

Theorem 6.6. *The mould associated with a plane tree is*

$$\mathcal{M}_T(Z) = \sum_{\mathcal{T}(u)=T} M_u(Z) = \prod_{\bullet \in T} \frac{1}{H_\bullet(Z)} \tag{201}$$

where $\bullet$ runs over the internal nodes of T, and

$$H_\bullet(z) = \left(\prod_{z \in V(T_\bullet)} z - 1\right) \tag{202}$$

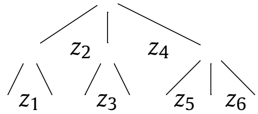
where $T_\bullet$ is the subtree with root $\bullet$ and $V(T_\bullet)$ the set of the variables labeling its sectors.

Proof. In terms of the tridendriform operations, $\mathcal{M}_T$ is given in [43] as

$$\mathcal{M}_T = (\mathcal{M}_{T_1} \succ \mathbf{M}_1) \circ (\mathcal{M}_{T_2} \succ \mathbf{M}_1) \circ \cdots \circ (\mathcal{M}_{T_{k-1}} \succ \mathbf{M}_1) \prec \mathbf{M}_{T_k} \tag{203}$$

if T has as subtrees of its root $T_1, \ldots, T_k$ in this order. The result follows then from Lemma 6.5. □

For example, the mould associated with the tree



(204)

is

$$\frac{1}{(z_1-1)(z_3-1)(z_5z_6-1)(z_1z_2z_3z_4z_5z_6-1)}. \tag{205}$$

Indeed, the tridendriform expression for $\mathcal{M}_T$ is

$$(M_1 \succ M_1) \circ (M_1 \succ M_1) \prec (M_1 \circ M_1) \tag{206}$$

and the associated operator is therefore

$$\mathcal{M}_T[f_1, f_2, f_3, f_4, f_5, f_6] = M\big[M[f_1]f_2M[f_3]f_4M[f_5f_6]\big] \tag{207}$$

whose rational mould is indeed given by (205).

6.4. Characters

Evaluating the M_u on a fixed function f yields a character of **WQSym**. For example, the natural character of **WQSym** [27]

$$\chi(\mathbf{M}_u) = \binom{t}{\max(u)} \tag{208}$$

is obtained by choosing for f the characteristic function of $[0, \infty)$:

$$M_u[\mathbf{1}_{\mathbb{R}_+}](t) = \binom{t}{\max(u)}. \tag{209}$$

Another more trivial character is obtained by choosing $f(t) = q^t$. The result is simply $q^{(n+1)t}M_u(z_1 = q, \ldots, z_m = q)$. Combining both gives

$$\mathbf{M}_u \mapsto M_I\left(\frac{1-q^t}{1-q}\right), \tag{210}$$

the specialization of the monomial quasi-symmetric function to the q-integer alphabet $[t]_q$ (when t is an integer) as defined in [30, Eq. (169)].

7. Matrix quasi-symmetric functions (MQSym)

7.1. Multiwords and packed matrices

Almost all known combinatorial Hopf algebras arise as quotients or subalgebras of an algebra based on *packed integer matrices*, *i.e.*, matrices of nonnegative integers (of arbitrary size) without null rows or columns. This is **MQSym**, the Hopf algebra of matrix quasi-symmetric functions, introduced in [12]. We shall present here only the required background and refer the reader to [12] for more details.

Matrices of nonnegative integers are in bijection with ordered partitions of multisets, that is, sequences of sets which may contain several occurrences of the same element. Indeed, given a matrix M, the coefficient M_{ij} is the number of occurrences of the letter j in the ith part P_i of an ordered partition P. The condition that no row is empty translates as no multiset of the sequence is empty, and the condition that no column is empty that the union of all multisets is packed (compare with the definition of packed words). For example,

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \quad \leftrightarrow \quad \{1,1,2\}\{1\}\{3,3,3,4\}. \tag{211}$$

Let $\boldsymbol{A} = \boldsymbol{A}' \cdot A$ and $\boldsymbol{B} = \boldsymbol{B}' \cdot B$ be two multiset partitions with last parts A and B. In order to define their product, as in the case of **WQSym**, we shall first shift the values in $\boldsymbol{B}$ by the maximum value m of $\boldsymbol{A}$. Let us denote this shifted set by $\boldsymbol{B}[m]$. Then their product $\pi(\boldsymbol{A}, \boldsymbol{B}) = \rho(\boldsymbol{A}, \boldsymbol{B}[m])$ is defined by

$$\rho(\boldsymbol{A}, \boldsymbol{B}) = \rho\big(\boldsymbol{A}, \boldsymbol{B}'\big) \cdot B + \rho\big(\boldsymbol{A}', \boldsymbol{B}\big) \cdot A + \rho\big(\boldsymbol{A}', \boldsymbol{B}'\big) \cdot (A \cup B). \tag{212}$$

For example,

$$\begin{aligned} \pi\big(\{1,1,2\}\{1\}, \{1,1,1,2\}\big) &= \rho\big(\{1,1,2\}\{1\}, \{3,3,3,4\}\big) \\ &= \{1,1,2\}\{1\}\{3,3,3,4\} + \{1,1,2\}\{3,3,3,4\}\{1\} \\ &\quad + \{3,3,3,4\}\{1,1,2\}\{1\} + \{1,1,2,3,3,3,4\}\{1\} \\ &\quad + \{1,1,2\}\{1,3,3,3,4\}. \end{aligned} \tag{213}$$

This is not the original definition of **MQSym** but this version is better suited for our purpose. Moreover it is easy to see the embedding of **WQSym** into **MQSym**: it is the linear span of the elements $\boldsymbol{P}$ such that the standard partition $\boldsymbol{P}$ is an ordered partition of the **set** (not multiset!) $[\max(\boldsymbol{P})]$. It is obviously a subalgebra of **MQSym**.

8. Polyhedral cones associated with multiset compositions

8.1. Main results

An alphabet $X = \{x_1 < \cdots < x_p\}$ can be identified with the tuple of coordinate functions $(x_1, \ldots, x_p)$ of $\mathbb{R}^p$. Then, one can associate with a multiset A on X the sum of its elements, *e.g.*,

$$s_{\{x_1,x_1,x_3,x_4,x_4,x_4\}} = 2x_1 + x_3 + 3x_4. \tag{214}$$

With this identification, one can associate with an ordered multiset partition $\boldsymbol{A}(X) = A_1(X) \ldots A_r(X)$ ($r = l(\boldsymbol{A})$) a subset $K_{\boldsymbol{A}(X)}$ of $\mathbb{R}^p$ defined by the inequalities

$$s_{A_1(X)\ldots A_i(X)} = \sum_{j=1}^{i} s_{A_j(X)} = s_{A_1(X)\cup\cdots\cup A_i(X)} \geqslant 0 \quad \text{for } i = 1, \ldots, r, \tag{215}$$

whose characteristic function is

$$\mathbf{1}_{\boldsymbol{A}(X)} = \prod_{i=1}^{r} \sigma_+(s_{A_1(X)\cup\cdots\cup A_i(X)}), \tag{216}$$

where $\sigma_+ = \mathbf{1}_{\mathbb{R}^+}$.

If we consider

$$\mathcal{F} = \bigoplus_{n\geqslant 0} \mathcal{F}\big(\mathbb{R}^n, \mathbb{R}\big) = \bigoplus_{n\geqslant 0} \mathcal{F}_n \tag{217}$$

as a graded algebra for the product $\pi(f, g) = f \star g$, *i.e.*, on $\mathcal{F}_p \times \mathcal{F}_q$

$$\pi(f, g)(x_1, \ldots, x_{p+q}) = f(x_1, \ldots, x_p)g(x_{p+1}, \ldots, x_{p+q}), \tag{218}$$

we have the following generalization of Theorem 5.2.

Theorem 8.1. *The linear map α defined from multiset compositions (of an alphabet $X = \{x_1 < \cdots < x_p\}$) to $\mathcal{F}(\mathbb{R}^p, \mathbb{R})$ by*

$$\alpha\big(\boldsymbol{A}(X)\big) = (-1)^{l(\boldsymbol{A})}\mathbf{1}_{\boldsymbol{A}(X)} \tag{219}$$

is such that

$$\pi\big(\alpha\big(\boldsymbol{A}(X)\big), \alpha\big(\boldsymbol{B}(Y)\big)\big) = \alpha\big(\pi\big(\boldsymbol{A}(X), \boldsymbol{B}(Y)\big)\big). \tag{220}$$

Proof. The proof proceeds by induction and relies upon the following simple identity (compare with (109)). For two real numbers a and b,

$$\sigma_+(a)\sigma_+(b) = \sigma_+(a)\sigma_+(a+b) + \sigma_+(b)\sigma_+(a+b) - \sigma_+(a+b). \tag{221}$$

If $l(\boldsymbol{A}) = l(\boldsymbol{B}) = 1$, then $\pi(\alpha(\boldsymbol{A}(X)), \alpha(\boldsymbol{B}(Y))) = \sigma_+(s_{\boldsymbol{A}(X)})\sigma_+(s_{\boldsymbol{B}(Y)})$ and (221) with $a = s_{\boldsymbol{A}(X)}$ and $b = s_{\boldsymbol{B}(Y)}$ gives (220) in the case $l(\boldsymbol{A}) = l(\boldsymbol{B}) = 1$. Let us assume now that Eq. (220) holds for $l(\boldsymbol{A}) + l(\boldsymbol{B}) \leqslant n$, for a given $n \geqslant 2$. If $\boldsymbol{A}(X) = \boldsymbol{A}'(X)A(X)$ (resp. $\boldsymbol{B}(Y) = \boldsymbol{B}'(Y)B(Y)$) with $l(\boldsymbol{A}) + l(\boldsymbol{B}) = n + 1$, then,

$$\alpha\big(\boldsymbol{A}(X)\big) = \alpha\big(\boldsymbol{A}'(X)A(X)\big) = -\alpha\big(\boldsymbol{A}'(X)\big)\sigma_+(s_{\boldsymbol{A}(X)}) \tag{222}$$

and

$$\alpha\big(\boldsymbol{B}(Y)\big) = \alpha\big(\boldsymbol{B}'(Y)B(Y)\big) = -\alpha\big(\boldsymbol{B}'(Y)\big)\sigma_+(s_{\boldsymbol{B}(Y)}), \tag{223}$$

with the convention $\alpha(\boldsymbol{A}'(X)) = 1$ (resp. $\alpha(\boldsymbol{B}'(Y)) = 1$) if $\boldsymbol{A}'$ is empty (resp. $\boldsymbol{B}'$ is empty). We have

$$\begin{aligned}\pi\big(\alpha\big(\boldsymbol{A}(X)\big), \alpha\big(\boldsymbol{B}(Y)\big)\big) &= \pi\big(\alpha\big(\boldsymbol{A}'(X)\big)\sigma_+(s_{\boldsymbol{A}(X)}), \alpha\big(\boldsymbol{B}'(Y)\big)\sigma_+(s_{\boldsymbol{B}(Y)})\big) \\ &= \pi\big(\alpha\big(\boldsymbol{A}'(X)\big), \alpha\big(\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)})\sigma_+(s_{\boldsymbol{B}(Y)})\end{aligned} \tag{224}$$

and using once again formula (221) with $a = s_{\boldsymbol{A}(X)}$ and $b = s_{\boldsymbol{B}(Y)}$, the product $\pi(\alpha(\boldsymbol{A}(X)), \alpha(\boldsymbol{B}(Y)))$ splits into three terms:

$$\begin{aligned}\pi\big(\alpha\big(\boldsymbol{A}(X)\big),\alpha\big(\boldsymbol{B}(Y)\big)\big) &= \pi\big(\alpha\big(\boldsymbol{A}'(X)\big),\alpha\big(\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)})\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}) \\ &\quad + \pi\big(\alpha\big(\boldsymbol{A}'(X)\big),\alpha\big(\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{B}(Y)})\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}) \\ &\quad - \pi\big(\alpha\big(\boldsymbol{A}'(X)\big),\alpha\big(\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}). \end{aligned} \tag{225}$$

Thanks to the definition of α and of the product on $\mathcal{F}$,

$$\begin{aligned}\pi\big(\alpha\big(\boldsymbol{A}(X)\big),\alpha\big(\boldsymbol{B}(Y)\big)\big) &= -\pi\big(\alpha\big(\boldsymbol{A}(X)\big),\alpha\big(\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}) \\ &\quad - \pi\big(\alpha\big(\boldsymbol{A}'(X)\big),\alpha\big(\boldsymbol{B}(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(y)}) \\ &\quad - \pi\big(\alpha\big(\boldsymbol{A}'(X)\big),\alpha\big(\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}). \end{aligned} \tag{226}$$

Recursively, we can use (220) for the three terms:

$$\begin{aligned}\pi\big(\alpha\big(\boldsymbol{A}(X)\big),\alpha\big(\boldsymbol{B}(Y)\big)\big) &= -\alpha\big(\pi\big(\boldsymbol{A}(X),\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}) \\ &\quad - \alpha\big(\pi\big(\boldsymbol{A}'(X),\boldsymbol{B}(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}) \\ &\quad - \alpha\big(\pi\big(\boldsymbol{A}'(X),\boldsymbol{B}'(Y)\big)\big)\sigma_+(s_{\boldsymbol{A}(X)}+s_{\boldsymbol{B}(Y)}) \end{aligned} \tag{227}$$

and, using once again the definition of α and of the product of ordered partitions, we find

$$\begin{aligned}\pi\big(\alpha\big(\boldsymbol{A}(X)\big),\alpha\big(\boldsymbol{B}(Y)\big)\big) &= \alpha\big(\pi\big(\boldsymbol{A}(X),\boldsymbol{B}'(Y)\big).B(Y)\big) \\ &\quad + \alpha\big(\pi\big(\boldsymbol{A}'(X),\boldsymbol{B}(Y)\big).A(X)\big) \\ &\quad + \alpha\big(\pi\big(\boldsymbol{A}'(X),\boldsymbol{B}'(Y)\big).\big(A(X)\cup B(Y)\big)\big) \\ &= \alpha\big(\pi\big(\boldsymbol{A}(X),\boldsymbol{B}(Y)\big)\big), \end{aligned} \tag{228}$$

so that the result follows by induction. □

For a standard partition $\boldsymbol{P}=(P_1,\dots,P_r)$ (with $\max(\boldsymbol{P})=n$ and $l(\boldsymbol{P})=r$), one can identify $m_{\boldsymbol{P}}$ with $\boldsymbol{P}(X)$ where X is "minimal" ($|X|=\max(\boldsymbol{P})$). For example, if

$$\boldsymbol{P}=\{1,3,3,3\}\{2,2,3\}\{3,3,3\}\{1,3,3\}, \tag{229}$$

then $\max(\boldsymbol{P})=3$, $X=\{x_1,x_2,x_3\}$, and

$$\mathbf{1}_{K_{\boldsymbol{P}}}=\sigma_+(x_1+3x_3)\sigma_+(x_1+2x_2+4x_3)\sigma_+(x_1+2x_2+7x_3)\sigma_+(2x_1+2x_2+9x_3). \tag{230}$$

Corollary 8.2. *The induced linear map from* **MQSym** *to* $\mathcal{F}$ (*also denoted by* α)

$$\alpha(m_{\boldsymbol{P}})=(-1)^{l(\boldsymbol{P})}\mathbf{1}_{K_{\boldsymbol{P}}} \tag{231}$$

is an algebra morphism (*character*).

9. The Rota–Baxter approach

9.1. Convolution and iterated integrals

The construction of R_f, L_f and D_f (see Section 3) relies upon iterated integrals involving convolution of functions. More precisely, let $\mathcal{A}$ be the vector space of bounded integrable functions whose restrictions to $\mathbb{R}^+$ and $\mathbb{R}^{-*}$ are continuous. Thanks to the regularization effect of the convolution

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy, \tag{232}$$

the convolution product of two functions of $\mathcal{A}$ is in $\mathcal{A}$ (and even continuous), so that $\mathcal{A}$ is a non-unital commutative algebra. The second step to define these iterated integrals is to mix the convolution product with the operations ($\varepsilon = \pm$)

$$\forall f \in \mathcal{A}, \quad (P_\varepsilon f)(x) = f(x)\sigma_\varepsilon(x). \tag{233}$$

On the one hand, using the operators P_ε, the functions defined in Section 3 for $n \geqslant 1$

$$\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f_1 & \cdots & f_n \end{bmatrix}(x) = \left(\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_{n-1} \\ f_1 & \cdots & f_{n-1} \end{bmatrix} * f_n\right)(x)\sigma_{\varepsilon_n}(x) \tag{234}$$

read

$$\begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f_1 & \cdots & f_n \end{bmatrix} = P_{\varepsilon_n}\big(f_n * P_{\varepsilon_{n-1}}\big(f_{n-1} * \cdots * P_{\varepsilon_1}(f_1)\ldots\big)\big). \tag{235}$$

On the other hand, if $\mathcal{A}_\varepsilon = P_\varepsilon(\mathcal{A})$, $\mathcal{A}_+$ and $\mathcal{A}_-$ are two subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ and using Eq. (221), we get

$$\forall f, g \in \mathcal{A}, \quad P_+(f * g) + P_+(f) * P_+(g) = P_+\big(f * P_+(g) + P_+(f) * g\big). \tag{236}$$

In other words (see [14] or [36]), $(\mathcal{A}, P_+)$ is a Rota–Baxter algebra of weight 1 and the properties of our iterated integrals can be derived from the properties of such Rota–Baxter algebras.

9.2. A short reminder on Rota–Baxter algebras

For a given commutative $\mathbb{K}$-algebra A, let us consider the tensor algebra $T(A) = \mathbb{K}\mathbf{1} \oplus (\bigoplus_{n \geqslant 1} A^{\otimes^n})$, with the quasi-shuffle product π: for $(\boldsymbol{a}, \boldsymbol{b}) \in T(A)^2$ and $(a, b) \in A^2$,

$$\pi(\boldsymbol{a} \otimes a, \boldsymbol{b} \otimes b) = \pi(\boldsymbol{a} \otimes a, \boldsymbol{b}) \otimes b + \pi(\boldsymbol{a}, \boldsymbol{b} \otimes b) \otimes a + \pi(\boldsymbol{a}, \boldsymbol{b}) \otimes (ab). \tag{237}$$

This algebra is indeed a Hopf algebra for the deconcatenation coproduct (see [28]), thus we can consider the group of characters from $T(A)$ to the unitarization $\mathbb{K}1 \oplus A$ of A.

Assume now that $A = A_+ \oplus A_-$ where A_+ and A_- are subalgebras of A, then A is a Rota–Baxter algebra: if R is the projection of A on A_+, parallel to A_- then we get the Rota–Baxter identity

$$\forall (x, y) \in A^2, \quad R(xy) + R(x)R(y) = R\big(xR(y) + R(x)y\big). \tag{238}$$

Rota–Baxter algebras have been deeply studied in the framework of renormalization in perturbative quantum field theory (see [14,36]) and

Proposition 9.1. *The map from $T(A)$ to $\mathbb{K}1 \oplus A$ defined by $C(\mathbf{1}) = 1$ and*

$$C(a_1 \otimes \cdots \otimes a_s) = (-1)^s R\big(R\big(\ldots R\big(R(a_1)a_2\big)\ldots\big)a_s\big) \tag{239}$$

is a character on $T(A)$.

This follows immediately from the Rota–Baxter relation (238) and the recursive definition of the quasi-shuffle product (237).

Now Theorem 3.2 can be easily deduced from the following corollary:

Corollary 9.2. *Let I and V be two maps from A to $\mathbb{K}$ such that $I(ab) = I(a)I(b)$ and $V(R(a)R(b)) = 0$, then:*

- *The map $I \circ C$ is a character from $T(A)$ to $\mathbb{K}$.*
- *The map $V \circ C$ is an infinitesimal character from $T(A)$ to $\mathbb{K}$.*

This corollary is closely related to our previous iterated integrals since, for any given $a \in A$, the coefficients

$$M_I(a) = M_{i_1,\ldots,i_r} = C\big(a^{i_1} \otimes \cdots \otimes a^{i_r}\big) \tag{240}$$

define a symmetrel mould (with values in A) on $\mathbb{K}\langle\mathbb{N}^*\rangle$ or, equivalently, an A-valued character on the quasi-shuffle Hopf algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$ (see Section 4.2).

Using this character, if there exist maps I and D on A with values in $\mathbb{K}$ such that $I(ab) = I(a)I(b)$ and $D(R(a)R(b)) = 0$, then the coefficients

$$M_a^{n_1,\ldots,n_s} = I\big(C\big(a^{n_1} \otimes \cdots \otimes a^{n_s}\big)\big) \tag{241}$$

define a character of the quasi-shuffle Hopf algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$ and the coefficients

$$D_a^{n_1,\ldots,n_s} = D\big(C\big(a^{n_1} \otimes \cdots \otimes a^{n_s}\big)\big) \tag{242}$$

define an infinitesimal character.

9.3. A proof of Theorem 3.2

These results provide a proof of Theorem 3.2 when applied to the Rota–Baxter algebra $\mathcal{A}$ defined in Section 9.1, with $I : \mathcal{A} \to \mathbb{R}$ the integral over $\mathbb{R}$ and $D : \mathcal{A} \to \mathbb{R}$ the evaluation at $x = 0$ ($D(f) = f(0)$).

For example, recall that, for a function f in A, the associated grouplike element R_f can be written

$$R_f = \sum \langle R_f, \Lambda\rangle_I \Lambda^I \tag{243}$$

where, for $I = (i_1, \ldots, i_r) \vDash n$,

$$\langle R_f, \Lambda\rangle_I = (-1)^{r+n} \int_{\mathbb{R}} \begin{bmatrix} + & \cdots & + \\ f^{*i_1} & \cdots & f^{*i_r} \end{bmatrix}(x)\, dx \tag{244}$$

but

$$\langle R_f, \Lambda\rangle_I = (-1)^n I\big(C\big(f^{i_1} \otimes \cdots \otimes f^{i_r}\big)\big) \tag{245}$$

is a character on the quasi-shuffle algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$, so R_f is grouplike. The same holds for the primitive element D_f.

10. The Catalan idempotents

Such iterated integrals are difficult to compute in general but for a specific family of functions, these integrals can be evaluated in closed form and yield a new family of primitive elements of **Sym**, originally introduced in [15] and [39] with a different interpretation. Up to a normalization, they provide new Lie idempotents whose combinatorial meaning is still under investigation.[3]

10.1. The Catalan triangle

Consider the generating series

$$\mathrm{ca}(a,b,t)=\frac{1-(a+b)t-\sqrt{1-2(a+b)t+(b-a)^2t^2}}{2abt}=\sum_{n\geqslant 1}\mathrm{ca}_n(a,b)t^n. \tag{246}$$

The coefficients $\mathrm{ca}_n(a,b)$ are homogeneous and symmetric polynomials in a, b of degree $n-1$:

$$\begin{aligned}
\mathrm{ca}_1(a,b)&=1,\\
\mathrm{ca}_2(a,b)&=a+b,\\
\mathrm{ca}_3(a,b)&=a^2+3ab+b^2,\\
\mathrm{ca}_4(a,b)&=a^3+6a^2b+6ab^2+b^3,\\
\mathrm{ca}_5(a,b)&=a^4+10a^3b+20a^2b^2+10ab^3+b^4
\end{aligned} \tag{247}$$

and we recognize the Catalan triangle of Narayana numbers $T(n,k)=\frac{1}{k}C_{n-1}^{k-1}C_n^{k-1}$ (see [21]):

$$\forall n\geqslant 1,\quad \mathrm{ca}_n(a,b)=\sum_{i=0}^{n-1}T(n,i+1)a^ib^{n-1-i}. \tag{248}$$

For any sequence of signs $\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_n)$ $(n\geqslant 1)$, consider its minimal decomposition into stacks of identical signs

$$\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_n)=(\eta_1)^{n_1}\dots(\eta_s)^{n_s}, \tag{249}$$

with $\eta_i\neq\eta_{i+1}$ and $n_1+\dots+n_s=n$. Then,

Theorem 10.1. *For $n\geqslant 1$, the element of $\mathbf{Sym}_{n+1}$ defined by*

$$D_{a,b}^{n+1}=\sum_{\boldsymbol{\varepsilon}=(\eta_1)^{n_1}\dots(\eta_s)^{n_s}}\Bigl(\prod_{\substack{\eta_i=+\\ i<s}}a\Bigr)\Bigl(\prod_{\substack{\eta_i=-\\ i<s}}b\Bigr)ca_{n_1}(a,b)\dots ca_{n_s}(a,b)R_{\boldsymbol{\varepsilon}\bullet} \tag{250}$$

is primitive.

[3] Since the first version of the present paper was released as a preprint, F. Chapoton [10] has found a combinatorial interpretation of the coefficients of these idempotents on the natural basis of the free preLie algebra on one generator.

For example, using the correspondence with the usual noncommutative ribbon Schur functions,

$$\begin{aligned} D^2_{a,b} &= R_2 - R_{11}, \\ D^3_{a,b} &= (a+b)R_3 - aR_{21} - bR_{12} + (a+b)R_{111}, \\ D^4_{a,b} &= \left(a^2+3ab+b^2\right)R_4 - a(a+b)R_{31} - abR_{22} - (a+b)bR_{13} \\ &\quad + a(a+b)R_{211} + abR_{121} + (a+b)bR_{112} - \left(a^2+3ab+b^2\right)R_{1111}. \end{aligned}$$

10.2. Proof of Theorem 10.1

We apply Theorem 3.2 to

$$f(x) = 2\big(a\sigma_+(x) + b\sigma_-(x)\big)e^{-|x|} \tag{251}$$

where a, b are two real numbers. Note that

$$\int_{\mathbb{R}} f(x)\,dx = 2(a+b). \tag{252}$$

To explicit the (Catalan) operators associated with this function, we essentially need to compute the function

$$f^{\varepsilon_1,\dots,\varepsilon_n} = \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ f & \cdots & f \end{bmatrix}. \tag{253}$$

Lemma 10.2. *We have, for $n \geqslant 1$,*

$$\begin{aligned} f^{\overbrace{+\cdots+}^{n}}(x) &= P_n(a,b,x)\sigma_+(x)e^{-|x|}, \\ f^{\overbrace{-\cdots-}^{n}}(x) &= P_n(b,a,x)\sigma_-(x)e^{-|x|}, \end{aligned} \tag{254}$$

where $P_n(a,b,x) \in \mathbb{R}[a,b,x]$ is of degree $n-1$ in x and homogeneous of degree n in a, b. Moreover,

$$P(a,b,x,t) = \sum_{n\geqslant 1} P_n(a,b,x)t^n = 2u(t)e^{2u(t)x}, \tag{255}$$

with

$$u(t) = \frac{1-(b-a)t - \sqrt{1-2(a+b)t+(b-a)^2t^2}}{2}. \tag{256}$$

Proof. Let $g_n(x) = f^{\overbrace{+\cdots+}^{n}}(x) = P_n(a,b,x)\sigma_+(x)e^{-|x|}$ and $P_n(x) = P_n(a,b,x)$. We have

$$g_{n+1}(x) = \sigma_+(x)\int_{\mathbb{R}} g_n(y)f(x-y)\,dy = \sigma_+(x)\left(\int_{\mathbb{R}^+} g_n(y)f^+(x-y)\,dy + \int_{\mathbb{R}^+} g_n(y)f^-(x-y)\,dy\right)$$

$$= 2\sigma_+(x)\left(\int_0^x P_n(y)e^{-y}ae^{-(x-y)}\,dy + \int_x^{+\infty} P_n(y)e^{-y}be^{+(x-y)}\,dy\right)$$

$$= 2\sigma_+(x)e^{-x}\left(\int_0^x aP_n(y)\,dy + e^{2x}\int_x^{+\infty} bP_n(y)e^{-2y}\,dy\right). \tag{257}$$

This equation defines recursively the polynomials P_n and, for the generating function ($P_1 = 2a$), we have

$$P(x,t) = 2t\left(a + \int_0^x aP(y,t)\,dy + e^{2x}\int_x^{+\infty} bP(y,t)e^{-2y}\,dy\right). \tag{258}$$

If we substitute

$$P(x,t) = 2u(t)e^{2u(t)x}, \tag{259}$$

we get

$$2u(t)e^{2u(t)x} = 2at + 2at\left(e^{2u(t)x} - 1\right) + 2bt\frac{2u(t)}{2-2u(t)}e^{2u(t)x}, \tag{260}$$

so that

$$u = at + bt\frac{u}{1-u} \tag{261}$$

which gives the expected generating function. □

Lemma 10.3. *Let*

$$\mathrm{ca}(a,b,t) = \frac{1-(a+b)t-\sqrt{1-2(a+b)t+(b-a)^2t^2}}{2abt} = \sum_{n\geqslant 1}\mathrm{ca}_n(a,b)t^n. \tag{262}$$

Then, for $n \geqslant 1$,

$$f^{\overbrace{+\cdots+}^{n}-}(x) = a\,\mathrm{ca}_n(a,b)f^-(x),$$

$$f^{\overbrace{-\cdots-}^{n}+}(x) = b\,\mathrm{ca}_n(a,b)f^+(x). \tag{263}$$

Proof. This is the same kind of computation with generating functions as in the previous lemma. Let $h_n = f^{\overbrace{+\cdots+}^{n}-}(x)$. Then

$$\begin{aligned}
H(t,x) &= \sum_{n\geqslant 1} h_n(x)t^n \\
&= \sigma_-(x)\int_{\mathbb{R}} P(a,b,y,t)e^{-|y|}\sigma_+(y)\big(f^+(x-y)+f^-(x-y)\big)\,dy \\
&= \sigma_-(x)\int_{\mathbb{R}^+} P(a,b,y,t)e^{-y}2be^{x-y}\,dy \\
&= 2b\sigma_-(x)e^x\int_{\mathbb{R}^+} 2u(t)e^{(2u(t)-2)y}\,dy \\
&= f^-(x)\frac{u(t)}{1-u(t)} \\
&= \frac{1}{bt}\big(u(t)-at\big)f^-(x) \\
&= a\,\mathrm{ca}(a,b,t)f^-(x). \qquad \square
\end{aligned} \tag{264}$$

Roughly speaking, when there is a change of sign, as in $f^{\overbrace{+\cdots+}^{n}-}(x)$, we recover, up to a scalar, the initial function. Theorem 10.1 follows easily from the previous lemma: for $n \geqslant 1$, the coefficients of $D_{a,b}^{n+1}$ in the basis $R_{\boldsymbol{\varepsilon}\bullet}$ are given by

$$\frac{1}{2}f^{\varepsilon_1,\ldots,\varepsilon_n,+}(0). \tag{265}$$

If $\varepsilon_1,\ldots,\varepsilon_n$ is decomposed into stacks of identical signs

$$\boldsymbol{\varepsilon} = \varepsilon_1,\ldots,\varepsilon_n = (\eta_1)^{n_1}\ldots(\eta_s)^{n_s} \tag{266}$$

with $\eta_i \neq \eta_{i+1}$ and $n_1+\cdots+n_s = n$ and $\eta_s = +$, then

$$f^{\varepsilon_1,\ldots,\varepsilon_n,+}(0) = \Bigg(\prod_{\substack{\eta_i=+\\ i<s}} a\Bigg)\Bigg(\prod_{\substack{\eta_i=-\\ i<s}} b\Bigg)\mathrm{ca}_{n_1}(a,b)\ldots\mathrm{ca}_{n_{s-1}}(a,b)P_{n_s+1}(a,b,0) \tag{267}$$

and $P_{n_s+1}(a,b,0) = 2ab\,\mathrm{ca}_{n_s}(a,b)$. If $\varepsilon_1,\ldots,\varepsilon_n$ is decomposed into stacks of identical signs,

$$\boldsymbol{\varepsilon} = \varepsilon_1,\ldots,\varepsilon_n = (\eta_1)^{n_1}\ldots(\eta_s)^{n_s}, \tag{268}$$

with $\eta_i \neq \eta_{i+1}$ and $n_1+\cdots+n_s = n$ and $\eta_s = -$, then

$$f^{\varepsilon_1,\ldots,\varepsilon_n,+}(0) = \Bigg(\prod_{\substack{\eta_i=+\\ i<s}} a\Bigg)\Bigg(\prod_{\substack{\eta_i=-\\ i<s}} b\Bigg)\mathrm{ca}_{n_1}(a,b)\ldots\mathrm{ca}_{n_{s-1}}(a,b)\,\mathrm{ca}_{n_s}(a,b)b.2a \tag{269}$$

which ends the proof of the theorem.

11. Alien calculus and noncommutative symmetric functions

We shall conclude this paper with a brief introduction to resummation theory, and explain how **Sym** appears in this context as a Hopf algebra of analytic continuation operators.

Alien calculus provides a deep understanding of resummation schemes that allow to interpret a formal power series as the asymptotic expansion of a function. This theory due to J. Ecalle, is developed in [16–18]. An introduction can be found in [7].

Let $\tilde{\varphi}(z) \in \mathbb{R}[[z^{-1}]]$ be a divergent series of "natural origin": for instance, the formal solution of a local analytic equation or system:

$$E(\tilde{\varphi}) = 0. \tag{270}$$

The simplest real resummation scheme for $\tilde{\varphi}(z)$ goes like this:

$$\begin{array}{ccc} \tilde{\varphi}(z) & - \quad - \quad \rightarrow & \varphi(z) \\ & \searrow \qquad \nearrow & \\ & \hat{\varphi}(\zeta). & \end{array} \tag{271}$$

We begin by subjecting $\tilde{\varphi}(z)$ to the formal Borel transform (to obtain $\hat{\varphi}(\zeta)$), which turns each monomial $z^{-\sigma}$ into $\zeta^{\sigma-1}\Gamma(\sigma)$ $(\sigma > 0)$. Under some growth condition on the coefficients of $\tilde{\varphi}(z)$, its Borel transform is a germ near $\zeta = 0$ and it converges only for small enough values of ζ. If this germ can be analytically continued along $\mathbb{R}^+$ then, under some growth condition, we can carry out a Laplace transform:

$$\hat{\varphi}(\zeta) \rightarrow \varphi(z) = \int_0^{+\infty} e^{-z\zeta} \hat{\varphi}(\zeta)\, d\zeta \tag{272}$$

which converges for $\Re(z) \gg 0$, is real for z real, and whose asymptotic expansion is $\tilde{\varphi}(z)$.

When it is possible, this procedure for turning a real formal object $\tilde{\varphi}(z)$ into a real geometric one $\varphi(z)$ is the simplest one and preserves the product of functions. Unfortunately, the analytic continuation of the germ $\hat{\varphi}(\zeta)$ often gives rise to analytic singularities on the real axis, which prevents from carrying out the Laplace transform. When this is the case, a careful analysis of the singularities is needed. It is provided by alien calculus.

In many instances, the Borel transform of a formal series $\tilde{\varphi}(z)$ lives in an algebra of functions whose product reflects the product of formal power series. We will focus here on the following algebra.

Definition 11.1. Let $\mathrm{Res}_{\mathbb{N}}$ be the vector space of functions $\hat{\varphi}(\zeta)$ such that:

- $\hat{\varphi}(\zeta)$ is defined and holomorphic at the root of $\mathbb{R}^+$, that is, on a domain

$$S = \{0 < |\zeta| < \varepsilon,\ |\arg \zeta| < \theta\}. \tag{273}$$

- $\hat{\varphi}(\zeta)$ is analytically continuable along any path that follows $\mathbb{R}^+$ and dodges each point of $\mathbb{N}^*$ to the left or to the right, but without ever going back.
- All the determinations of $\hat{\varphi}(\zeta)$ are locally integrable on $\mathbb{R}^+$.

This space is an algebra for the convolution product:

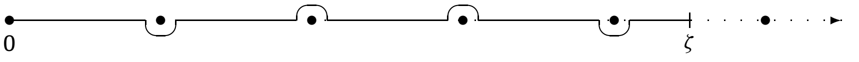
$$\hat{\varphi}_3(\zeta) = (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) = \int_0^{\zeta} \hat{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \zeta_1)\, d\zeta_1 \quad (0 < \zeta < 1) \tag{274}$$

where $\hat{\varphi}_1, \hat{\varphi}_2 \in \mathrm{Res}_{\mathbb{N}}$.

Note that this expression is purely local (at $\zeta = 0$) so that the germ $\hat{\varphi}_3(\zeta)$ must then be extended, by analytic continuation, to a global function. For details, see [15].

We can *label* the different determinations of a function of $\mathrm{Res}_{\mathbb{N}}$ as follows. Let $\hat{\varphi}(\zeta) \in \mathrm{Res}_{\mathbb{N}}$ and $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{E}$ be a sequence of n plus or minus signs. For ζ in $]n, n+1[$, we will denote by $\hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_n}(\zeta)$ the analytic continuation of $\hat{\varphi}$ from 0 to ζ along the path that follows $\mathbb{R}^+$ and dodges each singularity k (with $1 \leqslant k \leqslant n$) to the *right* (resp. to the *left*) if $\varepsilon_k = +$ (resp. $\varepsilon_k = -$).

For example, if $\zeta \in]4, 5[$, then $\hat{\varphi}^{+,-,-,+}(\zeta)$ is the analytic continuation of $\hat{\varphi}$ along the following path:



Of course, $\hat{\varphi}^{\emptyset}(\zeta)$ $(0 < \zeta < 1)$ is the unique determination of $\hat{\varphi}$ on $]0, 1[$ and, for any integer n, a function $\hat{\varphi}$ of $\mathrm{Res}_{\mathbb{N}}$ has 2^n possibly different determinations $\hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_n}(\zeta)$ *over* the interval $]n, n+1[$.

There exists an algebra of operators (alien operators) which allows to analyze the singularities of such functions.

For $\boldsymbol{\varepsilon} \in \mathcal{E}$, the endomorphism $D_{\boldsymbol{\varepsilon}\bullet}$ of $\mathrm{Res}_{\mathbb{N}}$ is defined as follows. For $\hat{\varphi} \in \mathrm{Res}_{\mathbb{N}}$ and $\zeta \in]0, 1[$,

$$\hat{\psi}(\zeta) = (D_{\boldsymbol{\varepsilon}\bullet}\hat{\varphi})(\zeta) = \hat{\varphi}^{\boldsymbol{\varepsilon}+}\big(\zeta + l(\boldsymbol{\varepsilon}\bullet)\big) - \hat{\varphi}^{\boldsymbol{\varepsilon}-}\big(\zeta + l(\boldsymbol{\varepsilon}\bullet)\big) \tag{275}$$

where $l(\boldsymbol{\varepsilon}\bullet) = n$ if $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{n-1})$. We also denote by $D_{\emptyset}$ the identity map on $\mathrm{Res}_{\mathbb{N}}$.

It follows from the definition that the composition of such operators is given by

$$\forall (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{E}^2, \quad D_{\boldsymbol{a}\bullet} D_{\boldsymbol{b}\bullet} = D_{\boldsymbol{b}+\boldsymbol{a}\bullet} - D_{\boldsymbol{b}-\boldsymbol{a}\bullet}, \tag{276}$$

which is reminiscent of (13), and there is a natural gradation ν on these operators defined by $\nu(D_{\emptyset}) = 0$ $(l(\emptyset) = 0)$ and $\nu(D_{\boldsymbol{\varepsilon}\bullet}) = l(\boldsymbol{\varepsilon}\bullet)$.

The fundamental theorem is

Theorem 11.2. *The graded algebra of alien operators*

$$\mathbf{Alien} = \bigoplus_{n \geqslant 0} \mathrm{Vect}_{\mathbb{Q}}\big\{D_{\boldsymbol{\varepsilon}\bullet};\ l(\boldsymbol{\varepsilon}\bullet) = n\big\} \tag{277}$$

is a Hopf algebra (with basis $D_{\boldsymbol{\varepsilon}\bullet}$) for the coproduct induced by the convolution:

$$\mathbf{Op}(\hat{\varphi} * \hat{\psi}) = \sum \mathbf{Op}_{(1)}(\hat{\varphi}) * \mathbf{Op}_{(2)}(\hat{\psi}) \quad \textit{for } \mathbf{Op} \in \mathbf{Alien} \textit{ and } (\hat{\varphi}, \hat{\psi}) \in (\mathrm{Res}_{\mathbb{N}})^2. \tag{278}$$

The proof of this nontrivial result can be found in [16] and is also clearly illustrated in [44]. It follows from a careful combinatorial and analytic study of the analytic continuation of functions of $\mathrm{Res}_{\mathbb{N}}$, which can be found in [16]. We shall only summarize some key points.

- It is not so simple to prove that the $D_{\boldsymbol{\varepsilon}\bullet}$ are free. Roughly speaking, this was proved by Ecalle, using the fact that for any linear combination of such operators, there exists a function in $\mathrm{Res}_{\mathbb{N}}$ which is not annihilated by the action of this linear combination.
- The construction of such functions involves the use of some specific elements of **Alien** such as

$$\Delta_n^+ = D_{\underbrace{+\cdots+}_{n-1}\bullet},$$

$$\Delta_n^- = -D_{\underbrace{-\cdots-}_{n-1}\bullet},$$
$$\Delta_n = \sum_{\boldsymbol{\varepsilon}\in\mathcal{E}_{n-1}} \lambda^{\boldsymbol{\varepsilon}} D_{\boldsymbol{\varepsilon}\bullet} \tag{279}$$

where $\lambda^{\boldsymbol{\varepsilon}} = \frac{p!q!}{(p+q+1)!}$ with p (resp. q) the number of plus (resp. minus) signs in $\boldsymbol{\varepsilon}$. It happens that each of the three families above is a family of generators.

– The existence of a coproduct δ is proved in [16]. The main idea is that $\hat{\varphi} * \hat{\psi}$ is defined by a path integral in the neighborhood of 0. To compute $\mathbf{Op}(\hat{\varphi} * \hat{\psi})$, the analytic continuations of $\hat{\varphi} * \hat{\psi}$ must be known. But, once again, these analytic continuations can be defined as path integrals on "self-symmetric shrinkable paths" (see [15]) and a careful decomposition of such paths (with respect to the involved analytic continuations of $\hat{\varphi}$ and $\hat{\psi}$) yields formula (278). Indeed, we get, for $n \geqslant 0$,

$$\delta(\Delta_n^+) = \sum_{k=0}^{n} \Delta_k^+ \otimes \Delta_{n-k}^+. \tag{280}$$

Given these properties of **Alien**, it is now clear that it is isomorphic to **Sym**, under the identification of S_n and Δ_n^+. Under this isomorphism, $D_{\varepsilon_1,\ldots,\varepsilon_k\bullet}$ is associated with $R_{\varepsilon_k,\ldots,\varepsilon_1\bullet}$ (note the reversion of the sequence, corresponding to the anti-involution in **Sym**). To summarize, if α is the isomorphism, then

$$\alpha(\Delta_n^+) = S_n,$$
$$\alpha(\Delta_n^-) = (-1)^n \Lambda_n,$$
$$\alpha(\Delta_n) = \frac{\Phi_n}{n}. \tag{281}$$

We have thus an explicit correspondence, and both worlds of resurgence and noncommutative symmetric functions can now interact. Especially any grouplike (resp. primitive) element of **Alien** (or **Sym**) provides a grouplike (resp. primitive) element of **Sym** (or **Alien**).

For example, the iterated integrals introduced in this paper were used in the framework of real resummation.

Let us go back to the above resummation scheme and assume that we are dealing with a formal power series $\tilde{\varphi}$ whose Borel transform $\hat{\varphi}$ is in $\mathrm{Res}_{\mathbb{N}}$. In order to perform the Laplace transform along $\mathbb{R}^+$, we need to *uniformize* the resurgent function. This can be done by averaging, above each interval $]n, n+1[$ the 2^n analytic continuations of $\hat{\varphi}$, but with many analytic and algebraic constraints, *e.g.*, this averaging must preserve the algebra structure but also provide a function whose Laplace transform on $\mathbb{R}^+$ converges (see [15,19]).

Since there are 2^n determinations of $\hat{\varphi}$ labeled by sequences $\varepsilon_1, \ldots, \varepsilon_n$ of signs, we can now understand the origin of the coefficients $m^{\boldsymbol{\varepsilon}}$ introduced in Section 3. Indeed, if such weights are given by the probabilities of some random walk, they fulfill all the algebraic and analytic properties required in real resummation theory.

12. Complements on the coproduct of Alien

Thanks to the product in **Alien**, it can be identified (as an algebra) to **Sym**, and it remains to understand why the coproduct of

$$\Delta_n^+ = D\underbrace{+\cdots+}_{n-1}\bullet \tag{282}$$

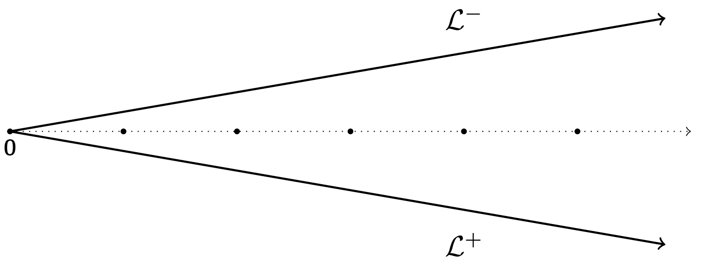
corresponds to the coproduct of S_n, that is

$$\delta(\Delta_n^+) = \sum_{k=0}^{n} \Delta_k^+ \otimes \Delta_{n-k}^+. \tag{283}$$

Going back to the Laplace transform, let us illustrate how this coproduct appears.

12.1. The Laplace transform

Assuming that all integrals are well-defined and convergent for z large enough, let us consider, for a given sign $\varepsilon = \pm$ the Laplace transform on a half-line going from 0 to infinity "on the same side as ε", precisely in the direction $\arg(\zeta) = -\varepsilon \cdot \alpha$ where $\alpha > 0$ is small enough:

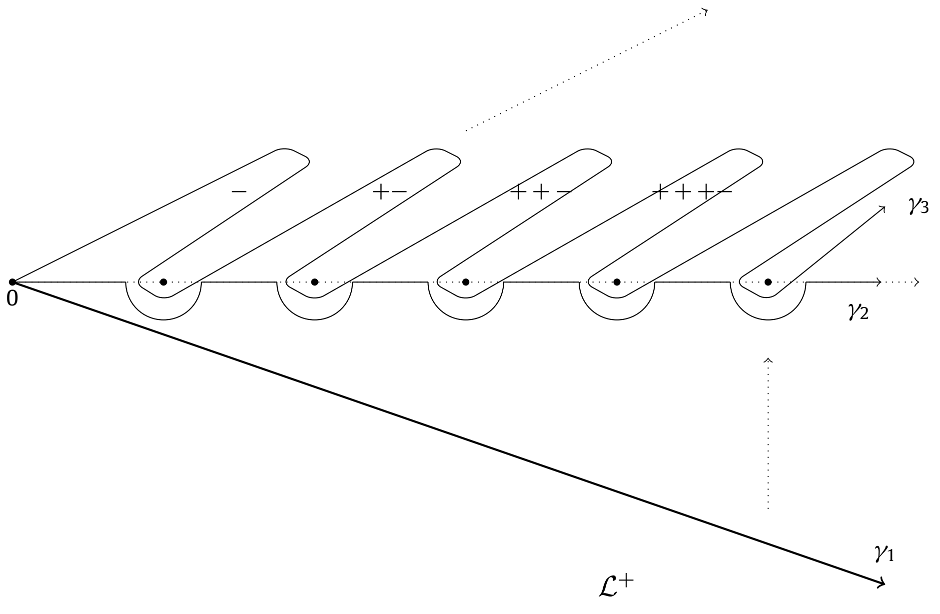


Thanks to the definition of the convolution in $\mathrm{Res}_{\mathbb{N}}$ (and to Fubini's theorem),

$$\begin{aligned}
\mathcal{L}^+(\hat{\varphi}_1 * \hat{\varphi}_2)(z) &= \int_0^{e^{-i\alpha}\infty=\infty^+} (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta)e^{-z\zeta}\, d\zeta \\
&= \int_0^{\infty^+}\int_0^{\zeta} \hat{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta-\zeta_1)\, d\zeta_1\, e^{-z\zeta}\, d\zeta \\
&= \int_0^{\infty^+}\int_0^{\zeta} \hat{\varphi}_1(\zeta_1)e^{-z\zeta_1}\hat{\varphi}_2(\zeta-\zeta_1)\, d\zeta_1\, e^{-z(\zeta-\zeta_1)}\, d\zeta \\
&= \left(\int_0^{\infty^+} \hat{\varphi}_1(\zeta_1)e^{-z\zeta_1}\, d\zeta_1\right)\left(\int_0^{\infty^+} \hat{\varphi}_2(\zeta_2)e^{-z\zeta_2}\, d\zeta_2\right) \\
&= \mathcal{L}^+(\hat{\varphi}_1)\cdot\mathcal{L}^+(\hat{\varphi}_2)
\end{aligned} \tag{284}$$

and the same holds for $\mathcal{L}^-$.

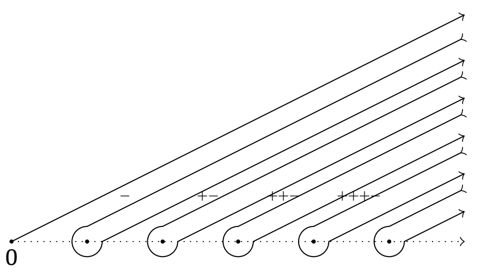
In order to compare these two Laplace transforms let us try to deform the path defining $\mathcal{L}^+$ so that it goes to infinity in the upper half plane. If we push the path without going through the singularities in $\mathbb{N}^*$, then, thanks to the Cauchy integral theorem, the function obtained after summation remains the same. For example, in the following picture:



we have

$$\mathcal{L}^+(\hat{\varphi})(z) = \int_{\gamma_1} \hat{\varphi}(\zeta)e^{-z\zeta}\,d\zeta = \int_{\gamma_2} \hat{\varphi}(\zeta)e^{-z\zeta}\,d\zeta = \int_{\gamma_3} \hat{\varphi}(\zeta)e^{-z\zeta}\,d\zeta. \tag{285}$$

We have written on the path γ_3 the determinations of $\hat{\varphi}$ that are involved when integrating along γ_3. If we stretch this path to infinity in the direction $e^{+i\alpha}\infty = \infty^-$, then we get the following picture:



The first path on the half-line from 0 corresponds to $\mathcal{L}^-(\hat{\varphi})$. For the second integral I_1 (that goes around 1), since the functions are integrable at the singularities, the circle around 1 can be shrinked and then, it is clear that one integrates from ∞^- to 1 the determination φ^- of φ then from 1 to ∞^- the determination φ^+ of φ:

$$I_1(\hat{\varphi}) = \int_1^{\infty^-} \big(\hat{\varphi}^+ - \hat{\varphi}^-\big)(\zeta)e^{-z\zeta}\,d\zeta. \tag{286}$$

Changing ζ into $\zeta - 1$, we get

$$I_1(\hat{\varphi}) = e^{-z} \int_0^{\infty^-} \left(\hat{\varphi}^+ - \hat{\varphi}^-\right)(\zeta + 1) e^{-z\zeta}\, d\zeta = e^{-z} \mathcal{L}^-\left(\Delta_1^+ \hat{\varphi}\right)(z). \tag{287}$$

In the same way, for the path that goes around 2, the corresponding integral is

$$\begin{aligned} I_2(\hat{\varphi}) &= \int_2^{\infty^-} \left(\hat{\varphi}^{++} - \hat{\varphi}^{+-}\right)(\zeta) e^{-z\zeta}\, d\zeta \\ &= e^{-2z} \int_0^{\infty^-} \left(\hat{\varphi}^{++} - \hat{\varphi}^{+-}\right)(\zeta + 2) e^{-z\zeta}\, d\zeta \\ &= e^{-2z} \mathcal{L}^-\left(\Delta_2^+ \hat{\varphi}\right)(z). \end{aligned} \tag{288}$$

Finally, we get

$$\mathcal{L}^+(\hat{\varphi}) = \mathcal{L}^-(\hat{\varphi}) + \sum_{k \geqslant 1} e^{-kz} \mathcal{L}^-\left(\Delta_k^+(\hat{\varphi})\right). \tag{289}$$

If we combine this with the action of the Laplace transforms $\mathcal{L}^+$ and $\mathcal{L}^-$, then

$$\begin{aligned} \mathcal{L}^+(\hat{\varphi}_1 * \hat{\varphi}_2) &= \mathcal{L}^-(\hat{\varphi}_1 * \hat{\varphi}_2) + \sum_{n \geqslant 1} e^{-nz} \mathcal{L}^-\left(\Delta_n^+(\hat{\varphi}_1 * \hat{\varphi}_2)\right) \\ &= \mathcal{L}^+(\hat{\varphi}_1).\mathcal{L}^+(\hat{\varphi}_2) \\ &= \left(\mathcal{L}^-(\hat{\varphi}_1) + \sum_{k \geqslant 1} e^{-kz} \mathcal{L}^-\left(\Delta_k^+(\hat{\varphi}_1)\right)\right) \times \left(\mathcal{L}^-(\hat{\varphi}_2) + \sum_{l \geqslant 1} e^{-lz} \mathcal{L}^-\left(\Delta_l^+(\hat{\varphi}_2)\right)\right) \\ &= \mathcal{L}^-(\hat{\varphi}_1 * \hat{\varphi}_2) + \sum_{n \geqslant 1} e^{-nz} \sum_{k+l=n} \mathcal{L}^-\left(\Delta_k^+(\hat{\varphi}_1) * \Delta_l^+ \hat{\varphi}_2\right) \end{aligned} \tag{290}$$

and the coefficient of e^{-nz} is precisely given by the proposed coproduct formula for Δ_n^+. The actual proof of the existence of this coproduct is also based on path deformation. We will illustrate it in the following subsection.

12.2. Path deformation and coproduct

In the definition of $\mathrm{Res}_{\mathbb{N}}$, the convolution was defined in the neighborhood of 0 by the path integral:

$$\hat{\varphi}_3(\zeta) = (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) = \int_0^{\zeta} \hat{\varphi}_1(\zeta_1) \hat{\varphi}_2(\zeta - \zeta_1)\, d\zeta_1 \quad (0 < \zeta < 1), \tag{291}$$

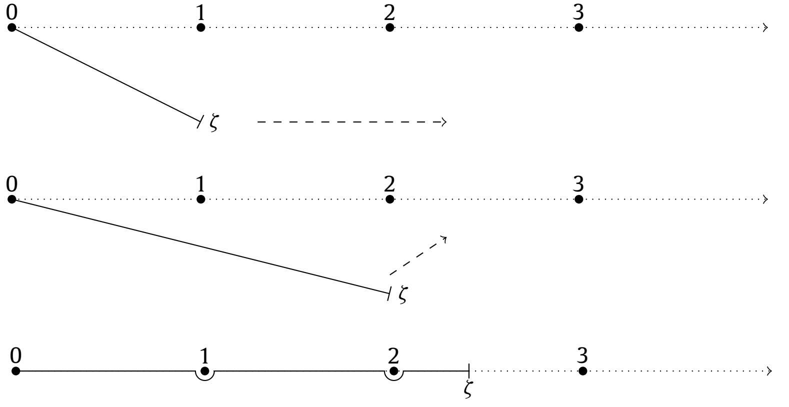
where $\hat{\varphi}_1, \hat{\varphi}_2 \in \mathrm{Res}_{\mathbb{N}}$.

In order to let Δ_n^+ act on the convolution product, the germ $\hat{\varphi}_3(\zeta)$ must be extended by analytic continuation, and, as this germ is defined as a path integral, the continuation of the germ is obtained by deformation of the path defining the convolution. But this deformation must be done carefully since one has to avoid the singularities of $\hat{\varphi}_1(\zeta_1)$ but also the singularities of $\hat{\varphi}_2(\zeta - \zeta_1)$, namely the set $\{\zeta - n,\ n \geqslant 1\}$. Moreover, in order to respect the commutativity of the convolution product, we have to take a self-symmetric path of analytic continuation from 0 to ζ, that is path such that, if ζ_1 is on the path, $\zeta - \zeta_1$ is also on the path.

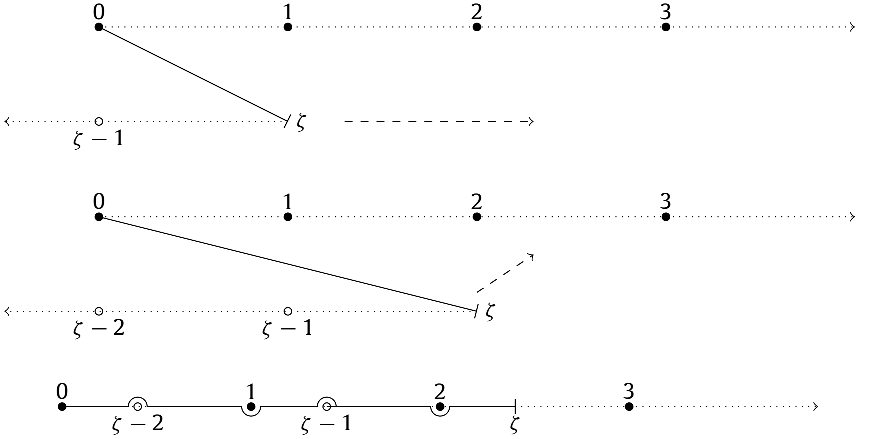
In order to do so, we can apply the following procedure (see [16]): Starting from ζ near 0, we deform the path to get the attempted analytic continuation, without going through the singularities in $\mathbb{N}^*$ *and* $\{\zeta - n,\ n \geqslant 1\}$. So we draw, these sets for a given ζ, and try to deform the path. For example, let us compute

$$\Delta_2(\hat{\varphi}_3)(\zeta) = \left(\hat{\varphi}_3^{++} - \hat{\varphi}_3^{+-}\right)(\zeta + 2). \tag{292}$$

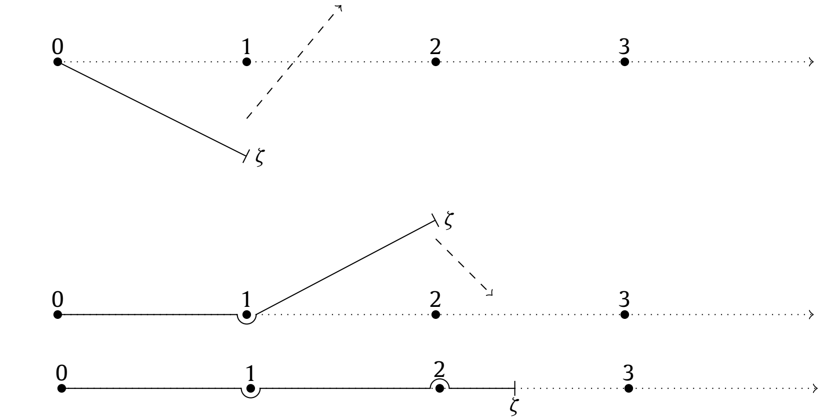
To do so, we need to know $\hat{\varphi}_3^{++}(\zeta)$ and $\hat{\varphi}_3^{+-}(\zeta)$ for $\zeta \in]2,3[$. Assuming $\hat{\varphi}_3$ in $\mathrm{Res}_{\mathbb{N}}$, $\hat{\varphi}_3^{++}$ is obtained by deformation of paths, starting from ζ near 0:



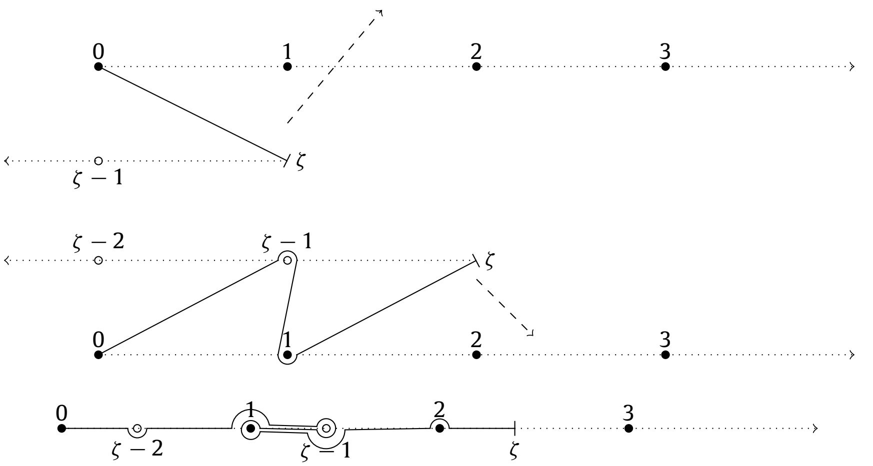
Since $\varphi_3(\zeta)$ is given by a convolution integral, we must deform the path of analytic continuation in a self-symmetric way and avoid the singularities over $\mathbb{N}^*$ and over their symmetrics $\zeta - \mathbb{N}^*$. If we draw these singularities, we get the following path of analytic continuation, which gives $\varphi_3(\zeta)$ as an integral:



For $\hat{\varphi}_3^{+-}(\zeta)$ ($\zeta \in]2,3[$), the natural way to get $\hat{\varphi}_3^{++}$ is obtained by deformation, starting from ζ near 0:



Once again, since $\varphi_3(\zeta)$ is given by a convolution integral, we must deform the path of analytic continuation in a self-symmetric way:



For these symmetric paths, we can shrink the different circles and, using the integral expression of $\hat{\varphi}_3$ when ζ_1 runs along the path, we can mark the determination of $\hat{\varphi}_1(\zeta_1)$. Since the path is symmetric, the determination of $\hat{\varphi}_2(\zeta - \zeta_1)$ is given by symmetry. For $\hat{\varphi}_3^{++}$ the information is summarized in the following table

ζ_1	0	$\to$	$\zeta-2$	$\to$	1	$\to$	$\zeta-1$	$\to$	2	$\to$	ζ
$\hat{\varphi}_1(\zeta_1)$		Ø		Ø		$+$		$+$		$++$	
$\hat{\varphi}_2(\zeta-\zeta_1)$		$++$		$+$		$+$		Ø		Ø	

and for $\hat{\varphi}_3^{+-}$:

ζ_1	0	$\to$	$\zeta-2$	$\to$	1	$\to$	$\zeta-1$	$\to$	1	$\to$	$\zeta-1$	$\to$	2	$\to$	ζ
$\hat{\varphi}_1$		Ø		Ø		$-$		$-$		$+$		$+$		$+-$	
$\hat{\varphi}_2$		$+-$		$+$		$+$		$-$		$-$		Ø		Ø	

If we compute carefully the convolution integral defining the difference

$$\big(\hat{\varphi}_3^{++} - \hat{\varphi}_3^{+-}\big)(\zeta) = \big(\Delta_2^+ \hat{\varphi}_3\big)(\zeta - 2) \quad \big(\zeta \in (2,3)\big), \tag{293}$$

some cancellations occur and we get

$$\begin{aligned}\big(\hat{\varphi}_3^{++} - \hat{\varphi}_3^{+-}\big)(\zeta) = &\int_0^{\zeta-2} \hat{\varphi}_1^{\emptyset}(\zeta_1)\big(\hat{\varphi}_2^{++} - \hat{\varphi}_2^{+-}\big)(\zeta-\zeta_1)\,d\zeta_1 \\ &+ \int_1^{\zeta-1} \big(\hat{\varphi}_1^{+} - \hat{\varphi}_1^{-}\big)(\zeta_1)\big(\hat{\varphi}_2^{+} - \hat{\varphi}_2^{-}\big)(\zeta-\zeta_1)\,d\zeta_1 \\ &+ \int_2^{\zeta} \big(\hat{\varphi}_1^{++} - \hat{\varphi}_1^{+-}\big)(\zeta_1)\hat{\varphi}_2^{\emptyset}(\zeta-\zeta_1)\,d\zeta_1. \end{aligned} \tag{294}$$

If $\zeta = \xi + 2$ ($\xi \in]0,1[$), then, by translation of the variable in each integral,

$$\begin{aligned}\Delta_2^+(\hat{\varphi}_3)(\xi) &= \Delta_2^+(\hat{\varphi}_1 * \hat{\varphi}_2)(\xi) \\ &= \big(\hat{\varphi}_3^{++} - \hat{\varphi}_3^{+-}\big)(\xi+2) \\ &= \int_0^{\xi} \hat{\varphi}_1^{\emptyset}(\zeta_1)\big(\hat{\varphi}_2^{++} - \hat{\varphi}_2^{+-}\big)(\xi-\zeta_1+2)\,d\zeta_1 \\ &\quad + \int_0^{\xi} \big(\hat{\varphi}_1^{+} - \hat{\varphi}_1^{-}\big)(\zeta_1+1)\big(\hat{\varphi}_2^{+} - \hat{\varphi}_2^{-}\big)(\xi-\zeta_1+1)\,d\zeta_1 \\ &\quad + \int_0^{\xi} \big(\hat{\varphi}_1^{++} - \hat{\varphi}_1^{+-}\big)(\zeta_1+2)\hat{\varphi}_2^{\emptyset}(\xi-\zeta_1)\,d\zeta_1. \end{aligned} \tag{295}$$

This is precisely the expected result since, near the origin,

$$\Delta_2^+(\hat{\varphi}_1 * \hat{\varphi}_2) = \hat{\varphi}_1 * \big(\Delta_2^+\hat{\varphi}_2\big) + \big(\Delta_1^+\hat{\varphi}_1\big) * \big(\Delta_1^+\hat{\varphi}_2\big) + \big(\Delta_2^+\hat{\varphi}_1\big) * \hat{\varphi}_2. \tag{296}$$

This way of computing the analytic continuations of a convolution product can be shown to yield in all cases the claimed coproduct of the operators Δ_n^+.

References

[1] M. Aguiar, N. Bergeron, F. Sottile, Combinatorial Hopf algebras and generalized Dehn–Sommerville relations, Compos. Math. 142 (2006) 1–30.
[2] A. Baker, B. Richter, Quasisymmetric functions from a topological point of view, Math. Scand. 103 (2008) 208–242.
[3] M. Beck, S. Robins, Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra, Springer-Verlag, New York, 2006.
[4] F. Bergeron, N. Bergeron, A.M. Garsia, Idempotents for the free Lie algebra and q-enumeration, in: D. Stanton (Ed.), Invariant Theory and Tableaux, in: IMA Vol. Math. Appl., vol. 19, Springer, 1988.

[5] N. Bergeron, M. Zabrocki, The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree, J. Algebra Appl. 8 (2009) 581–600.
[6] V. Buchstaber, N. Erokhovets, Polytopes, Hopf algebras and quasi-symmetric functions, arXiv:1011.1536v1 [math.CO].
[7] B. Candelpergher, J.-C. Nosmas, F. Pham, Approche de la résurgence, Actualités Math., Hermann, Paris, 1993.
[8] F. Chapoton, The anticyclic operad of moulds, Int. Math. Res. Not. 20 (2007), Art. ID rnm078, 36 pp.
[9] F. Chapoton, A rooted-trees q-series lifting a one-parameter family of Lie idempotents, Algebra Number Theory 3 (2009) 611–636.
[10] F. Chapoton, Flows on rooted trees and the Menous–Novelli–Thibon idempotents, Math. Scand., in press, arXiv:1203.1780.
[11] F. Chapoton, F. Hivert, J.-C. Novelli, J.-Y. Thibon, An operational calculus for the mould operad, Int. Math. Res. Not. IMRN 9 (2008), Art. ID rnn018, 22 pp.
[12] G. Duchamp, F. Hivert, J.-Y. Thibon, Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras, Internat. J. Algebra Comput. 12 (2002) 671–717.
[13] E.B. Dynkin, Calculation of the coefficients in the Campbell–Baker–Hausdorff formula, Dokl. Akad. Nauk. SSSR (N.S.) 57 (1947) 323–326 (in Russian).
[14] K. Ebrahimi-Fard, L. Guo, Rota–Baxter algebras in renormalization of perturbative quantum field theory, in: Universality and Renormalization, in: Fields Inst. Commun., vol. 50, Amer. Math. Soc., Providence, RI, 2007, pp. 47–105.
[15] J. Ecalle, Well-behaved convolution averages and their applications to real resummation, appendix of [38].
[16] J. Ecalle, Les fonctions résurgentes, vol. 1: Les algèbres de fonctions résurgentes, Publ. Math. Orsay, vol. 81-05, 1981.
[17] J. Ecalle, Les fonctions résurgentes, vol. 2, Publ. Math. Orsay, 1981.
[18] J. Ecalle, Les fonctions résurgentes, vol. 3, Publ. Math. Orsay, 1985.
[19] J. Ecalle, F. Menous, Well-behaved convolution averages and the non-accumulation theorem for limit-cycles, in: B.L.J. Braaksma, G.K. Immink, M. van der Put (Eds.), The Stokes Phenomenon and Hilbert's 16th Problem, World Scientific, 1996.
[20] W. Feller, An Introduction to Probability Theory and Its Applications, vol. II, second ed., John Wiley & Sons, New York, 1971.
[21] A. Frabetti, Simplicial properties of the set of planar binary trees, J. Algebraic Combin. 13 (2001) 41–65.
[22] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348.
[23] I. Gessel, Multipartite P-partitions and inner product of skew Schur functions, Contemp. Math. 34 (1984) 289–301.
[24] M. Hazewinkel, Formal Groups and Applications, Academic Press, New York, 1978.
[25] F. Hivert, Combinatoire des fonctions quasi-symétriques, Thèse de Doctorat, Marne-La-Vallée, 1999.
[26] F. Hivert, J.-C. Novelli, J.-Y. Thibon, The algebra of binary search trees, Theoret. Comput. Sci. 339 (1) (2005) 129–165.
[27] F. Hivert, J.-C. Novelli, J.-Y. Thibon, Trees, functional equations, and combinatorial Hopf algebras, European J. Combin. 29 (7) (2008) 1682–1695.
[28] M.E. Hoffman, Quasi-shuffle products, J. Algebraic Combin. 11 (2000) 49–68.
[29] A.A. Klyachko, Lie elements in the tensor algebra, Sib. Math. J. 15 (1974) 1296–1304.
[30] D. Krob, B. Leclerc, J.-Y. Thibon, Noncommutative symmetric functions II: Transformations of alphabets, Internat. J. Algebra Comput. 7 (1997) 181–264.
[31] D. Krob, J.-Y. Thibon, Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$, J. Algebraic Combin. 6 (4) (1997) 339–376.
[32] J.-L. Loday, On the operad of associative algebras with derivation, Georgian Math. J. 17 (2010) 347–372.
[33] J.-L. Loday, M.O. Ronco, Hopf algebra of the planar binary trees, Adv. Math. 139 (2) (1998) 293–309.
[34] J.-L. Loday, M.O. Ronco, Trialgebras and families of polytopes, Contemp. Math. 346 (2004).
[35] C. Malvenuto, C. Reutenauer, Duality between quasi-symmetric functions and Solomon descent algebra, J. Algebra 177 (1995) 967–982.
[36] D. Manchon, Renormalization in connected graded Hopf algebras: an introduction, in: Alan Carey, et al. (Eds.), Motives, Quantum Field Theory, and Pseudodifferential Operators, in: Clay Math. Proc., vol. 12, 2010, pp. 73–95.
[37] A. Mansuy, Algèbres de greffes, arXiv:1110.4800.
[38] F. Menous, Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle, PhD thesis, 1996.
[39] F. Menous, Les bonnes moyennes uniformisantes et une application à la resommation réelle, Ann. Fac. Sci. Toulouse Math. (6) 8 (4) (1999) 579–628.
[40] B. Mielnik, J. Plebański, Combinatorial approach to Baker–Campbell–Hausdorff exponents, Ann. Inst. H. Poincaré, Sect. A XII (1970) 215–254.
[41] J.-C. Novelli, F. Patras, J.-Y. Thibon, Natural endomorphisms of quasi-shuffle Hopf algebras, Bull. Soc. Math. France 141 (2013) 107–130.
[42] J.-C. Novelli, J.-Y. Thibon, Polynomial realizations of some trialgebras, in: Proc. FPSAC'06, San Diego, 2006.
[43] J.-C. Novelli, J.-Y. Thibon, Hopf algebras and dendriform structures arising from parking functions, Fund. Math. 193 (2007) 189–241.
[44] D. Sauzin, Mould expansions for the saddle-node and resurgence monomials, in: Renormalization and Galois Theories, in: IRMA Lect. Math. Theor. Phys., vol. 15, 2009, pp. 83–163.
[45] L. Solomon, On the Poincaré–Birkhoff–Witt theorem, J. Comb. Theory 4 (1968) 363–375.
[46] L. Solomon, A Mackey formula in the group ring of a Coxeter group, J. Algebra 41 (1976) 255–268.
[47] W. Specht, Die linearen Beziehungen zwischen höheren Kommutatoren, Math. Z. 51 (1948) 367–376.
[48] F. Wever, Über Invarianten in Lieschen Ringen, Math. Ann. 120 (1949) 563–580.
[49] R.M. Wilcox, Exponential operators and parameter differentiation in quantum physics, J. Math. Phys. 8 (1967) 962–982.