L^2 METHODS AND VANISHING THEOREMS

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CONTENTS

INTRODUCTION

The aim of the present exposition is to give a brief introduction of L^2 methods in complex geometry. All the material presented is just a rework of what is done in [Dem+96] and [Hör90]. The paper is organised as follows.

The first section is just a very quick revision of the fundamentals of Kähler geometry and elliptic differential operators. It is assumed that the reader is already familiar with the results presented; in particular, results such as Gårding inequality, the finiteness theorem and (extended) commutation relations are not proved.

The central result of the second section is an L^2 existence theorem for the D''-equation. The proofs in this section are not straightforward, they are the core of the exposition and require the reader to have some prerequisites in functional analysis.

In the third part, applications of L^2 estimates are presented. After having introduced the reader to positivity concepts for vector bundles, we give a brief description of the "philosophy" behind the Levi problem and we present proofs for three celebrated vanishing theorems (Nakano, Cartan B, Kodaira-Serre).

1 PREPARATION

As the name "L^2 methods" suggests, our aim is to find a suitable Hilbert space to work on. Since we will mainly deal with applications in cohomology, it is natural to build this space as a subspace of the (p,q)-forms. In this section, we will explain in detail this process and recall other basic facts about Kähler geometry and operators on manifolds.

Some of the results presented in the paper still hold in the more general case of hermitian manifolds. In the Kähler setting the proofs are essentially unchanged, but the computations involved are simpler.

1.1 Kähler manifolds

We start with some basic remarks in linear algebra.

Let V be a real vector space of dimension $2n$ equipped with a linear complex structure J (i.e., $J^2 = -1$). We can take the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ and extend J in such a way that $v \otimes z \mapsto (Jv) \otimes z$, when $v \in V$ and z is a complex number. We also have conjugation in $V_{\mathbb{C}}$: simply set $\overline{v \otimes z} = v \otimes \bar{z}$.

We have a direct sum decomposition in eigenspaces of J. Hence, we can write $V_{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$, where $V^{(1,0)}$ stands for the i-eigenspace. Clearly, $V^{(0,1)} = \overline{V^{(1,0)}}$. Using J, we can build a complex n-dimensional vector space V_J, by setting $(a + Jb)v = (a + ib)v$, for $v \in V$ and $a, b \in \mathbb{R}$.

It is not difficult to see a complex isomorphism between V_J and $V^{(1,0)}$, given by the map $v \mapsto \frac{1}{2}(v - iJv)$. Analogously, $\overline{V_J}$ is identified with $V^{(0,1)}$, therefore we have a second decomposition $V_{\mathbb{C}} \cong V_J \oplus \overline{V_J}$.

The space of r-forms of $V_{\mathbb{C}}$ admits a decomposition in (p, q)-forms:

$$\Lambda^r(V_{\mathbb{C}}) = \bigoplus_{p+q=r} \Lambda^{p,q}(V_J) = \bigoplus_{p+q=r} \Lambda^p(V_J) \otimes \Lambda^q(\overline{V_J}).$$

We see that a (p, q)-form ω can be defined in two equivalent ways:

- ω is a real multilinear map $V_J \to \mathbb{C}$ which is complex linear alternating in the first p terms and conjugate linear alternating in the last q terms;
- ω is a complex multilinear alternating map $V_{\mathbb{C}} \to \mathbb{C}$ which vanish on homogeneous elements unless p are from $V^{(1,0)}$ and q are from $V^{(0,1)}$.

Let W, W' be complex vector spaces equipped with hermitian products B, B'. We can define a hermitian product on the tensor product $W \otimes W'$ by setting

$$\widehat{B}(v \otimes v', w \otimes w') = B(v, v') \cdot B'(w, w') \quad v, w \in W \text{ and } v', w' \in W'.$$

Moreover, B induces an isomorphism $W^* \cong \overline{W}$.

If a hermitian product H is defined on V_J, we can extend it to $\overline{V_J}$ simply setting $H(\bar{v}, \bar{w}) = \overline{H(v, w)}$. We use the map $\phi\colon TV_J \to \Lambda V_J$, and define

$$v_1 \wedge \cdots \wedge v_p \overset{\phi}{\mapsto} \sum_{\sigma \in S_p} (-1)^{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}$$
$$H(\nu, \omega) = H(\phi(\nu), \phi(\omega)) \quad \nu, \omega \in \Lambda^p V_J.$$

As a consequence of these definitions, H is also defined on $\Lambda^{p,q} V_J$.

If (X, h) is an n-dimensional hermitian manifold, its real tangent space to a point is to be thought as V. The holomorphic tangent space T^*X is then identified with V_J. In local holomorphic coordinates (z^{α}) the metric H can be written (using Einstein notation)

$$h = h_{\alpha\beta}\, dz^{\alpha} \otimes d\bar{z}^{\beta}$$

where $(h_{\alpha\beta})$ is a hermitian matrix.

The real part of h defines a riemannian metric $g = \frac{1}{2}(h + \bar{h})$ on the underlying real manifold. The complexification of g, defined on $TM_{\mathbb{C}}$, in local coordinates is written as

$$g = \frac{1}{2} h_{\alpha\beta} \, (dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha).$$

Analogously, minus the imaginary part of h defines a $(1,1)$-form $\Omega = \frac{i}{2}(h - \bar{h})$,

$$\Omega = \frac{i}{2} h_{\alpha\beta} \, dz^\alpha \wedge d\bar{z}^\beta.$$

When Ω is closed, we say that X is a Kähler manifold. Remember that the canonical volume form given by g is given in terms of Ω by

$$\frac{\Omega^n}{n!}$$

and induces a measure which we will denote dV.

1.2 Analysis on vector bundles

Let E be a complex vector bundle of rank r with hermitian structure h'. Then we can define a product on the space $\mathcal{M}(X, \Lambda^{p,q} T^* X \otimes E)$ of E-valued measurable (p,q)-forms by integration on X:

$$\langle v, \omega \rangle = \int_X \hat{h}(v, \omega) \, dV.$$

Finally, we can build the Hilbert space

$$L^2(X, \Lambda^{p,q} T^* X \otimes E) = \{\omega \in \mathcal{M}(X, \Lambda^{p,q} T^* X \otimes E) \mid \|\omega\|^2 = \langle \omega, \omega \rangle < \infty\}.$$

Let (F, h'') be a hermitian bundle of rank r' on X. We denote smooth sections by $C^\infty(X, F)$. Recall that a (linear) differential operator of degree δ is a $\mathbb{C}$-linear operator $P \colon C^\infty(X, E) \to C^\infty(X, F)$ of the form

$$Pu(z) = a_\alpha(z) D^\alpha u(z) \quad |\alpha| \leq \delta$$

where E, F are locally trivialised on some open chart $U \subseteq X$ with coordinates $z = (z_1, \dots, z_n)$, a_α are $(r' \times r)$-matrices with smooth coefficients on U, D^α stands for

$$\frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

and $u, D^\alpha u$ are viewed as column r-vectors.

If $t \in \mathbb{C}$ is a parameter, taken $f \in C^\infty(X, \mathbb{C}), u \in C^\infty(X, E)$, a simple calculation shows that

$$e^{-tf(z)} P(e^{tf(z)} u(z)) = t^\delta \sigma_P(z, df(z)) \cdot u(z) + \text{lower order terms}$$

is a polynomial of degree δ in t, where σ_P is a homogeneous polynomial map $T^* X \to \mathrm{Hom}(E, F)$ defined by

$$T_x^* X \ni \xi \mapsto \sigma_P(x, \xi) \in \mathrm{Hom}(E_x, F_x), \qquad \sigma_P(x, \xi) = \sum_{|\alpha| = \delta} a_\alpha \xi^\alpha.$$

The map σ_P is called the principal symbol of P, it is a smooth function in (x, ξ) and does not depend on coordinates or trivialisations.

The formal adjoint of P is the unique operator $P^*\colon C^\infty(X,F) \to C^\infty(X,E)$ satisfying

$$\langle Pu, v\rangle = \langle u, P^*v\rangle$$

for sections u, v such that $\operatorname{Supp} u \cap \operatorname{Supp} v \Subset X$. It is easy to see that

$$\sigma_{P^*}(x,\xi) = (-1)^\delta \sigma_P(x,\xi)^*.$$

By computing P in the sense of distributions, we get the maximal (hilbertian) extension $P_{\mathcal{H}}\colon L^2(X,E) \to L^2(X,F)$, whose domain is formed by sections $u \in L^2(X,E)$ such that $P_{\mathcal{H}}u$ is in $L^2(X,F)$.

Since $P_{\mathcal{H}}$ is closed and densely defined, we can consider its Hilbert adjoint $(P_{\mathcal{H}})^*$ (which is still closed and densely defined). The relationship between the formal and Hilbert adjoint is given by

$$\operatorname{Dom}(P_{\mathcal{H}})^* \subseteq \operatorname{Dom}(P^*)_{\mathcal{H}},$$

and they coincide where they are both defined.

The operator P is said to be elliptic if $\sigma_P(x,\xi)$ is injective for every $x \in X$ and $\xi \in T^*_xX \setminus \{0\}$. In the sequel, the notation H^s will be used to denote the s-th Sobolev space.

The following theorems lay the basis of elliptic PDE theory and Hodge theory.

Theorem 1 (Gårding)**.** *Suppose E, F have equal rank and let $\widetilde{P}$ denote the extension of P in the sense of distributions. Then, for every $u \in H^0(X,E)$ such that $\widetilde{P}u \in H^s(X,F)$, it holds $u \in H^{s+\delta}(X,E)$ and*

$$\|u\|_{s+\delta} \le C_s(\|\widetilde{P}u\|_s + \|u\|_0)$$

where C_s is a constant depending exclusively on s.

Theorem 2 (finiteness)**.** *In the same hypothesis of the preceding theorem, it holds:*

- $\operatorname{Ker} P$ *is of finite dimension;*
- $\operatorname{Ran} P = P(C^\infty(X,E))$ *is closed and finite-dimensional in* $C^\infty(X,F)$.

Moreover, the following decomposition is true:

$$C^\infty(X,F) = \operatorname{Ran} P \oplus \operatorname{Ker} P^*$$

where the sum is orthogonal in $H^0(X,F)$.

We briefly recall the basic properties of Chern connection. A connection is a $\mathbb{C}$-linear differential operator

$$D\colon C^\infty(X, \Lambda^r T^*X \otimes E) \to C^\infty(X, \Lambda^{r+1} T^*X \otimes E)$$

satisfying the Leibniz rule

$$D(\omega \wedge s) = d\omega \otimes u + (-1)^s \omega \wedge Du$$

for all $\omega \in C^\infty(X, \Lambda^s T^*X), u \in C^\infty(X, \Lambda^r T^*X \otimes E)$.

Clearly, we can split D in a unique way as a sum of the $(1,0)$ and $(0,1)$ part:

$$D'\colon C^\infty(X, \Lambda^{p,q} T^*X \otimes E) \to C^\infty(X, \Lambda^{p+1,q} T^*X \otimes E)$$
$$D''\colon C^\infty(X, \Lambda^{p,q} T^*X \otimes E) \to C^\infty(X, \Lambda^{p,q+1} T^*X \otimes E).$$

In a local trivialisation given by a smooth frame, one can write

$$\begin{aligned} Du &= du + \Gamma \wedge u \\ D'u &= d' + \Gamma' \wedge u \\ D''u &= d''u + \Gamma'' \wedge u \end{aligned}$$

where d acts componentwise (d is the exterior derivative and d', d'' its $(1,0)$ and $(0,1)$ parts, i.e., $d = d' + d''$), and $\Gamma = \Gamma' + \Gamma''$ is a matrix of local 1-forms.

We say that D is hermitian if $\Gamma' = -(\Gamma'')^*$ in any orthonormal frame. Thus, a hermitian connection is completely determined by its $(0,1)$ part.

When E is a holomorphic vector bundle, we can extend d'' by requiring that it acts componentwise on local trivialisations; setting $D'' = d''$ uniquely determines a connection, called the Chern connection. From now on D will always denote the Chern connection.

The Chern curvature is just D^2. On a local trivialisation, one checks

$$D^2 = (d\Gamma + \Gamma \wedge \Gamma)u,$$

hence there exists a matrix of global 2-forms Θ such that

$$D^2 u = \Theta \wedge u.$$

As a consequence of the definition of D, we have

$$\begin{aligned} D^2 &= D'D'' + D''D' \\ i\Theta &\in C^\infty(C, \Lambda^{1,1} T^*X \otimes \mathrm{Herm}(E,E)). \end{aligned}$$

For later use, we note that by the isomorphisms $T^*X \cong \overline{TX}, E^* \cong \overline{E}$, we can identify the curvature tensor to a hermitian form $\widetilde{\Theta}$ on $TX \otimes E$.

Finally, we define the Laplace-Beltrami differential operator to be

$$\Delta = DD^* + D^*D.$$

In order to check that it is an elliptic operator, we introduce vector field contraction:

$$(\theta \lrcorner \omega)(\eta_1, \ldots, \eta_{r-1}) = \omega(\theta, \eta_1, \ldots, \eta_{r-1})$$

where $\theta, \eta_1, \ldots, \eta_{r-1} \in TX$ are tangent vectors and $\omega \in \Lambda^r T^*X$ is an r-form. Note that if $\tilde{\theta} = h(\cdot, \theta) \in T^*X$, the operator $\theta \lrcorner \cdot$ is the adjoint of $\tilde{\theta} \wedge \cdot$. For all smooth functions f, Leibniz rule yields

$$\sigma_e^{-tf} D(e^{tf} s) = t\, df \wedge s + Ds.$$

Hence, by definition, we find

$$\sigma_D(x, \xi) \cdot s = \xi \wedge s, \quad \forall \xi \in T^*_x X, \forall s \in \Lambda^r T^*X \otimes E.$$

We obtain $\sigma_{D^*} = -(\sigma_D)^*$, therefore

$$\sigma_{D^*}(x, \xi) \cdot s = -\tilde{\xi} \lrcorner s.$$

The equality $\sigma_D \sigma_{D^*} + \sigma_{D^*} \sigma_D$ implies

$$\begin{aligned} \sigma_\Delta(x, \xi) \cdot s &= -\xi \wedge (\tilde{\xi} \lrcorner s) - \tilde{\xi} \lrcorner (\xi \wedge s) = -(\tilde{\xi} \lrcorner \xi) s \\ \sigma_\Delta(x, \xi) \cdot s &= -h(\xi, \xi) s = -|\xi|^2 s \end{aligned}$$

which concludes the proof of the ellipticity of Δ. One can also prove ellipticity for the variants

$$\Delta' = D'D'^* + D'^*D',$$
$$\Delta'' = D''D''^* + D''^*D''.$$

We conclude the section with a useful theorem on commutation identities, whose proof relies on the properties of $\lrcorner$ and on the possibility to choose a convenient coordinate system around a point.

Remember that an operator $C^\infty(\Lambda^r T^*X \otimes E) \to C^\infty(\Lambda^{r+s}T^*X \otimes E)$ is said to have degree s. The following notation for operators A, B of degree a, b will be used:

$$[A, B] = AB - (-1)^{ab}BA.$$

A simple computation shows that the Jacobi identity is valid for $[\cdot, \cdot]$.

Theorem 3 (commutation relations)**.** *Define the operator L by $Lu = \Omega \wedge u$, and set $\Lambda = L^*$. Then, it holds:*

$$[D''^*, L] = iD', \qquad [D'^*, L] = -iD'',$$
$$[\Lambda, D''] = -iD'^*, \qquad [\Lambda, D'] = iD''^*.$$

2 L^2 ESTIMATES

The goal of this section is to prove a central L^2 existence theorem, which is essentially due to Hörmander [Hör65]. The proofs presented are rather technical, but an effort has been made to clean up the exposition.

The reader is supposed to know the basics of functional analysis (e.g., the Hahn-Banach theorem), riemannian geometry (e.g., the Hopf-Rinow theorem) and some standard facts about plurisubharmonic functions. Moreover, we will freely use the knowledge provided by the previous chapter.

Some of the proofs below are completed by means of a regularisation argument. Basically, one needs to use convolution with regularising kernels, possibly after using a partitions of unity, so as to divide the support of some interesting function. This is the only case in which details are skipped, as we feel this process is quite standard, and does not add anything "new" to the ideas presented.

Theorem 4 (Bochner-Kodaira-Nakano)**.** *If X is a Kähler manifold, the complex Laplace operators Δ' and Δ'' acting on E-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [i\Theta, \Lambda].$$

Proof. The last commutation identity yields $D''^* = -i[\Lambda, D']$, hence

$$\Delta'' = [D'', D''^*] = -i[D'', [\Lambda, D']].$$

By means of the Jacobi identity, we get

$$[D'', [\Lambda, D']] = [\Lambda, [D', D'']] + [D', [D'', \Lambda]] = [\Lambda, \Theta] + i[D', D'^*].$$

To conclude, just remember that $[D', D''] = D^2 = \Theta$. □

Theorem 5 (basic a priori inequality)**.** *Assume that X is compact. The following inequality holds:*

$$\|D''u\|^2 + \|D''^*u\|^2 \geq \int_X \hat{h}([i\Theta, \Lambda]u, u)\, dV.$$

Proof. If $u \in C^\infty(X, \Lambda^{p,q}T^*X \otimes E)$ is an arbitrary (p,q)-form, an integration by parts yields
$$\langle \Delta' u, u\rangle = \|D'u\|^2 + \|D'^*u\|^2 \geq 0$$
and similarly for Δ'', hence by using the result above we obtain
$$\|D''u\|^2 + \|D''^*u\|^2 = \|D'u\|^2 + \|D'^*u\|^2 + \int_X \hat{h}([i\Theta, \Lambda]u, u)\, dV.$$
□

Lemma 6. *We denote by δ the geodesic distance of X. The distance δ is complete if and only if all balls $\overline{B(x_0, r)} = \{x \in X \mid \delta(x, x_0) \leq r\}$ are compact. Moreover, under these hypotheses, there exists a sequence of compact sets K_ν with $X = \bigcup K_\nu$ and $K_\nu \subseteq K^\circ_{\nu+1}$, and a sequence of cut-off functions ψ_ν such that $|d\psi_\nu| \leq 1$, $\psi_\nu = 1$ on K_ν and* $\operatorname{Supp}\psi_\nu \subseteq K_{\nu+1}$.

Proof. The first part of the statement derives from the Hopf-Rinow theorem. For the second part, set $K_\nu = \overline{B(x_0, 3^\nu)}$ and define ψ_ν by
$$\psi_\nu = \theta(3^{-\nu} d(x_0, x))$$
where $\theta\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with (for example)
$$-\frac{9}{10} \leq \theta' \leq 0$$
$$\theta(t) = 1 \quad t \leq 1 + \frac{1}{10}$$
$$\theta(t) = 0 \quad t \geq 3 - \frac{1}{10}$$
Since the distance function $x \mapsto \delta(x_0, x)$ is a 1-Lipschitz function, it is almost everywhere differentiable, with a differential of norm ≤ 1, hence we get
$$|d\psi_\nu| \leq 1 - \frac{1}{10}.$$
Now, in order to get ψ_ν smooth, we need to take convolution with regularising kernels. □

Lemma 7. *Assume that δ is complete. For every form with measurable coefficients $u \in \mathcal{M}(X, \Lambda^{p,q}T^*X \otimes E)$ such that*
$$u \in L^2, \quad D''u \in L^2, \quad D''^*u \in L^2$$
there exists a sequence of smooth forms u_ν with compact support, such that
$$u_\nu \to u, \quad D''u_\nu \to D''u, \quad D''^*u_\nu \to D''^*u$$
in L^2 sense.

Proof. Let us denote by ψ_ν a sequene of cut-off functions as provided by the preceding lemma. If $u \in L^2$ and $D''u \in L^2$, then $\psi_\nu u \in L^2$ and it holds that
$$D''(\psi_\nu u) = \psi_\nu D''u + d''\psi_\nu \wedge u \in L^2.$$
Moreover, since $\psi_\nu \to 1$ and $|d''\psi_\nu| \to 0$ pointwise with 1 as a uniform bound, by Lebesgue's bounded convergence theorem we find
$$\psi_\nu D''u \to D''u, \quad d''\psi_\nu \wedge u \to 0$$

in L^2 sense. By adjunction, provided that $D''^* u \in L^2$, we can conclude

$$D''^* \psi_\nu u \to D''^* u.$$

So far, we have been able to approximate u by the compactly supported elements $\psi_\nu u$. If we want to obtain smooth approximants u_ν, we have to go through a regularisation process, as explained in the beginning of the section. □

These corollaries follow immediately.

Corollary 8. *If the distance δ is complete, that the Hilbert adjoint $(D''_{\mathcal{H}})^*$ and the hilbertian extension of the formal adjoint $(D''^*)_{\mathcal{H}}$ coincide.*

Corollary 9. *The basic a priori inequality extends to arbitrary forms u such that*

$$u \in L^2, \quad D'' u \in L^2, \quad D''^* u \in L^2.$$

We now introduce the reader to the notion of positivity for vector bundles.

Definition 10. Let E be a hermitian vector bundle over X. One says that

- E is *strongly positive* if $A = [i\Theta, \Lambda]$ is positive definite;
- E is *Nakano positive* if $\widetilde{\Theta}$ is positive definite;
- E is *Griffiths positive* if $\widetilde{\Theta}(\xi \otimes v, \xi \otimes v) > 0$ for all decomposable tensors $\xi \otimes v \in TX \otimes E$.

The third definition has been included for completeness reasons, we will not need it in the rest of the paper.

The following theorem is the central result of our exposition.

Theorem 11 (L^2 existence theorem)**.** *Let X be a Kähler manifold and δ be complete. Let E be a strongly positive hermitian vector bundle over X. Then for every form $g \in L^2(X, \Lambda^{p,q} T^* X \otimes E)$ satisfying $D'' g = 0$ and*

$$\int_X \hat{h}(A^{-1} g, g)\, dV < \infty,$$

there exists $f \in L^2(X, \Lambda^{p,q-1} T^* X \otimes E)$ such that $D'' f = g$ and

$$\int_X |f|^2\, dV \le \int_X \hat{h}(A^{-1} g, g)\, dV.$$

Proof. We start by considering the Hilbert space decomposition

$$L^2(X, \Lambda^{p,q} T^* X \otimes E) = \operatorname{Ker} D'' \oplus (\operatorname{Ker} D'')^\perp.$$

Note that $\operatorname{Ker} D''$ is weakly closed, hence closed. If $v \in \mathcal{D}(X, \Lambda^{p,q} T^* X \otimes E)$ is a smooth form with compact support, we can write

$$v = v_1 + v_2$$

according to the above decomposition. Since $(\operatorname{Ker} D'')^\perp \subseteq \operatorname{Ker} D''^*$ by duality and $g, v_1 \in \operatorname{Ker} D''$ by hypothesis, we obtain $D''^* v_2 = 0$ and

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 = |\langle A A^{-1} g, v_1\rangle|^2 \le \langle A^{-1} g, g\rangle \langle A v_1, v_1\rangle$$

where the Cauchy-Schwarz inequality has been used.

The next step is applying the basic a priori inequality to $u = v_1$:

$$\langle Av_1, v_1\rangle \leq \|D''v_1\|^2 + \|D''^*v_1\|^2 = \|D''^*v_1\|^2 = \|D''^*v\|^2.$$

By making use of both inequalities, we find

$$|\langle g, v\rangle|^2 \leq \Big(\int_X \hat{h}(A^{-1}g, g)\, dV\Big)\|D''^*v\|^2$$

for every smooth (p, q)-form with compact support. Thus, we have defined a linear functional

$$D''^*(\mathcal{D}(X, \Lambda^{p,q}T^*X \otimes E) \to \mathbb{C}, \quad w = D''^*v \mapsto \langle v, g\rangle.$$

This functional is continuous in L^2 and its norm is bounded by a constant C, where

$$C = \Big(\int_X \hat{h}(A^{-1}g, g)\, dV\Big)^{\frac{1}{2}}.$$

By the Hahn-Banach theorem, there is an element $f \in L^2(X, \Lambda^{p,q}T^*X \otimes E)$ with $\|f\| \leq C$, such that

$$\langle v, g\rangle = \langle D''^*v, f\rangle$$

for every v. We conclude that $D''^*f = g$ in the sense of distribution, which is sufficient. Note that the inequality $\|f\| \leq C$ is precisely the last estimate in the statement of the theorem. □

Remark 12. It is always possible to take a solution f which also satisifies the property $f \in (\operatorname{Ker} D'')^\perp$. It is sufficient to replace f by its orthogonal projection on $(\operatorname{Ker} D'')^\perp$. Clearly, this solution is unique and has minimal L^2 norm in the set of forms satisfying the equation $D''f = g$. Since

$$(\operatorname{Ker} D'')^\perp = \overline{\operatorname{Ran} D''^*} \subseteq \operatorname{Ker} D''^*,$$

we conclude that the minimal L^2 solution satisfies the additional equation $D''^*f = 0$. As a consequence,

$$\Delta''f = D''^*D''f = D''^*g.$$

Also note that if g is of class C^∞, the ellipticity of Δ'' automatically implies the smoothness of f.

Our aim is to extend the main L^2 theorem to the case of non-complete metrics (i.e., the induced distance δ is not complete). We briefly recall an important definition.

Definition 13. Let X be a complex manifold. A continuous function $\psi\colon X \to \mathbb{R}$ is called an *exhaustion* if the sets $\{z \in X \mid \psi(z) < c\}$ are relatively compact for every constant c.

Equivalently, $\psi(z) \to \infty$ with respect to the filter of complements of compact sets. The manifold X is *weakly* (resp. *strongly*) *pseudoconvex* if there exists a smooth plurisubharmonic (resp. strongly plurisubharmonic) exhaustion function ψ on X.

Examples of pseudoconvex manifolds include:

- compact complex manifolds (take $\psi = 0$);
- closed analytic submanifold of $\mathbb{C}^N$ (take $\psi(z) = |z|^2$);
- open balls $B(z_0, r)$ (take $\psi(z) = 1/(r - |z - z_0|^2)$).

Proposition 14. *Every weakly pseudoconvex Kähler manifold X carries a complete metric.*

Proof. Let ψ be a plurisubharmonic exhaustion function. The idea is to introduce a slight modification of the fundamental form Ω:

$$\widehat{\Omega} = \Omega + id'd''(\chi \circ \psi) = \Omega + i(\chi' \circ \psi)d'd''\psi + i(\chi'' \circ \psi)d'\psi \wedge d''\psi$$

where $\chi \colon \mathbb{R} \to \mathbb{R}$ is a convex increasing function. The third term in the expression of $\widehat{\Omega}$ implies the norm of

$$\sqrt{\chi'' \circ \psi}\, d\psi$$

with respect to $\widehat{\Omega}$ is less or equal than 1. Hence, if ρ is a primitive of $\sqrt{\chi''}$, we have the inequality

$$|d(\rho \circ \psi)|_{\widehat{\Omega}} \leq 1.$$

By integrating along paths, we see that

$$|\rho(\psi(x)) - \rho(\psi(y))|_{\widehat{\Omega}} \leq \delta_{\widehat{\Omega}}(x, y)$$

for all $x, y \in X$. Therefore, the geodesic ball $\overline{B(z_0, r)} \subseteq \{z \in X \mid \delta_{\widehat{\Omega}}(z, z_0) \leq \psi(z_0) + r\}$ is relatively compact if $\rho \circ \psi$ is exhaustive, i.e., if

$$\lim_{t \to +\infty} \rho(t) = +\infty.$$

Thus, we obtain the sufficient condition

$$\int_{t_0}^{+\infty} = \sqrt{\chi''(t)} = +\infty,$$

which is realised (for example) by choosing $\chi(t) = t^2$. □

Theorem 15. *The L^2 existence theorem still holds if the metric is not complete.*

Proof. We developed the main ideas in the preceding proposition. If we apply the theorem for the complete metrics associated to

$$\widehat{\Omega} = \Omega + \varepsilon i d'd''(\psi^2) \qquad \varepsilon > 0,$$

we get solution forms f_ε, which are uniformly bounded in L^2 norm. Since the close unit ball of an Hilbert space is weakly compact, we can extract a subsequence

$$f_{\varepsilon_k} \to f \in L^2$$

converging weakly in L^2. By weak continuity of differentiations, by passing to the limit we obtain the desired equality

$$D'' f = g.$$

□

3 VANISHING THEOREMS

The most important applications of L^2 estimates are

- vanishing results for Dolbeaut cohomology groups;

- existence theorems for holomorphic functions;
- approximation theorems for holomorphic functions.

All three aspects are in fact intimately related, but a complete explanation of this statement goes beyond the scope of the present exposition. Existence results are at the heart of the solution of the celebrated Levi problem, which gives a characterisation of Stein manifolds in terms of strongly pseudoconvex manifolds. As we will (partly) see, Stein manifolds enjoy a very strong vanishing theorem about cohomology groups.

In the sequel we will prove three vanishing theorems.

Theorem 16 (Nakano vanishing theorem – weak form)**.** *Let E be a strongly positive holomorphic vector bundle over a weakly pseudoconvex manifold X of dimension n. Then*

$$H^{n,q}(X,E) = 0$$

for every $q \geq 1$.

Proof. The L^2 theorem shows that the equation $D''f = g$ can be solved provided that g is D''-closed and satisfies a suitable bound. Moreover, we know that f is smooth if g is smooth. We would like to solve the equation for a closed smooth form, whatever is its behaviour at infinity. To this end, let ψ be a smooth exhuastive plurisubharmonic function on X. If we multiply the metric of E by the weight factor

$$e^{-\chi\circ\psi},$$

where χ is as usual a convex increasing function, the resulting curvature tensor is written

$$i\Theta_\chi = i\Theta + id'd''(\chi\circ\psi) = i\Theta + i(\chi'\circ\psi)d'd''\psi + i(\chi''\circ\psi)d'\psi\wedge d''\psi.$$

Note that both terms $d'd''\psi$ and $d'\psi\wedge d''\psi$ yield nonnegative in the sense of Nakano. More precisely, the curvature operator A_χ on (n,q) forms satisfies

$$A_\chi \geq A, \qquad A_\chi^{-1} \leq A^{-1},$$

hence we get

$$\int_X \hat{h}(A_\chi{}^{-1}g,g)e^{-\chi\circ\psi}\,dV \leq \int_X \hat{h}(A^{-1}g,g)e^{-\chi\circ\psi}\,dV < +\infty$$

when χ grows quickly enough, e.g., if χ is such that

$$e^{-\chi(k)}\int_{\{k\leq\psi\leq k+1\}} \hat{h}(A^{-1}g,g)\,dV \leq 2^{-k}$$

for every integer $k \geq 0$. This allows to get a smooth and minima L^2 solution f, which implies $H^{n,q}(X,E) = 0$ for $q \geq 1$ as we wanted. □

By carrying out a deeper analysis of the local coordinate expression of A, one can prove a stronger version of the previous statement, where strong positiveness is substituted by Nakano positiveness. Indeed, if E is positive in the sense of *Nakano*, then A is positive definite.

Theorem 17 (Cartan theorem B – weak form)**.** *Let E be a holomorphic vector bundle over a weakly pseudoconvex manifold X. Then*

$$H^{p,q}(X,E) = 0$$

for every $q \geq 1$.

Proof. We choose an arbitrary metric h' on E. By the abobe formula for $i\Theta_\chi$, we see that the curvature of E can be made positive if the first derivative of χ grows quickly enough. As a consequence, we have the vanishing of the groups when $p = n$. To obtain the general conclusion for (p,q)-forms, we use the canonical duality pairing

$$\Lambda^k TX \otimes \Lambda^k T^*X \to \mathbb{C},$$

and the contraction pairing

$$\Lambda^n TX \otimes \Lambda^p T^*X \otimes E \to \Lambda^{n-p} TX.$$

These allow us to perform the following trick:

$$\Lambda^{p,q} T^*X \otimes E = \Lambda^{0,q} T^*X \otimes \Lambda^p T^*X \otimes \Lambda^n T^*X \otimes \Lambda^n TX \otimes E = \Lambda^{n,q} T^*X \otimes F$$

where F is defined as

$$\Lambda^{n-p} TX \otimes E.$$

Now just observe that the Dolbeaut complex $\Lambda^{p,\bullet} T^*X \otimes E$ is isomorphic to the Dolbeau complex $\Lambda^{n,\bullet} T^*X \otimes F$, hence

$$H^{p,q}(X,E) = H^{n,q}(X,F) = 0.$$

□

Our last vanishing result concerns line bundles. There are missing details, which can be filled by simple (but tedious-to-write) computations.

Theorem 18 (Kodaira-Serre vanishing theorem). *Let E be a hermitian holomorphic line bundle over a compact complex manifold X. Assume that $\widetilde{\Theta} > 0$. Then for every holomorphic vector bundle F, there exists an integer $k_0 = k_0(F)$ such that*

$$H^{p,q}(X, E^{\otimes k} \otimes F) = 0$$

for every $p \geq 0, q \geq 1, k \geq k_0$.

Sketch of the proof. Given hermitian holomorphic vector bundles L, N, one can express the Chern connection of the tensor product in terms of the connections of E and F. The resulting formula is the following:

$$D_{L\otimes N}(u \otimes v) = D_L u \otimes v + (-1)^{\deg u} u \otimes D_N v.$$

This easily implies

$$\Theta_{L\otimes N} = \Theta_L \otimes \mathrm{Id}_N + \mathrm{Id}_L \otimes \Theta_N.$$

In the case of our line bundle E, we find, after the identification $\mathrm{End}(E) = \mathbb{C}$,

$$\Theta_{E^{\otimes k}} = k\Theta_E,$$

hence

$$\Theta_{E^{\otimes k}\otimes F} = k\Theta_E \otimes \mathrm{Id}_F + \Theta_F.$$

This means that the associated hermitian form satisfies

$$\widetilde{\Theta}_{E^{\otimes k}\otimes F}(\eta,\eta) \geq k|\eta|^2 + \widetilde{\Theta}_F(\eta,\eta)$$

where $i\Theta_E$ is considered as the Kähler metric on X. Therefore, the bundle $E^{\otimes k} \otimes F$ is Nakano positive for $k \geq k_0$ large enough, and we conclude

$$H^{n,q}(X, E^{\otimes k} \otimes F) = 0$$

for $q \geq 1$ and $k \geq k_0$. The case $p \neq n$ is carried out by using the same trick of the previous proof. □

REFERENCES

[Dem+96] J.-P. Demailly et al. *Introduction à la théorie de Hodge*. Société Mathématiques de France, 1996 (cit. on p. 1).

[Dem12] J.-P. Demailly. *Complex analytic and differential geometry*. 2012. URL: www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[GH78] P. Griffits and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience Publication, 1978.

[Hör65] L. Hörmander. "L^2 estimates and existence theorems for the $\bar{\partial}$ operator". In: *Acta Math.* (1965) (cit. on p. 6).

[Hör90] L. Hörmander. *An introduction to Complex Analysis in several variables*. North-Holland Math, 1990 (cit. on p. 1).