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par

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Le problème d'équivalence pour les variétés de Cauchy-Riemann en dimension 5.

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INTRODUCTION

This memoir is a contribution to the solution of the equivalence problem for CR-manifolds in dimension up to 5. The first occurence of this problem goes back to 1906, when Henri Poincaré formulated the equivalence problem for hypersurfaces of $\mathbb{C}^2$ as follows [24]:

> *Given two (local) real hypersurfaces $M, M' \subset \mathbb{C}^2$, does there exist a (local) biholomorphism of $\mathbb{C}^2$ which sends M on M' ?*

Poincaré gave a heuristical argument to show that the answer to this problem should be negative in general, but the first rigorous proof came in 1932, when Elie Cartan [4, 5] constructed a “hyperspherical connection” on real hypersurfaces of $\mathbb{C}^2$, using the powerful technique of moving frames which is nowadays referred to as Cartan's equivalence method. In modern terminology, given a manifold M and some geometric data specified on M, which usually appears as a G-structure on M (i.e. a reduction of the bundle of coframes of M), Cartan's equivalence method seeks to provide a principal bundle P on M together with a coframe ω of 1-forms on P which is adapted to the geometric structure of M in the following sense: an isomorphism between two such geometric structures M and M' lifts to a unique isomorphism between P and P' which sends ω on ω'. The equivalence problem between M and M' is thus reduced to an equivalence problem between $\{e\}$-structures, which is well understood [19, 25].

The concept of CR-manifold enables a reformulation of the biholomorphic equivalence problem between real submanifolds of complex spaces in a more intrinsic manner. We recall [3] that a CR-structure on a real manifold M is the data of a subbundle L of $\mathbb{C} \otimes TM$ of even rank $2n$ such that

- $L \cap \overline{L} = \{0\}$
- L is formally integrable, i.e. $[L,\, L] \subset L$.

The integer n is the CR-dimension of M and $k = \dim M - 2n$ is the codimension of M. A generic real submanifold $M \subset \mathbb{C}^n$ is canonically endowed with a CR-structure when one defines the CR-bundle L as $T^{1,0}M$.

Given two CR-structures (M, L) and (M', L'), a diffeomorphism $\varphi : M \longrightarrow M'$ is said to be a CR-isomorphism between M and M' if φ sends the CR-bundle of M onto the CR-bundle of M', i.e. if $\varphi(L) = L'$. The equivalence problem for CR-manifolds can be formulated as follows:

Given two (local) CR-manifolds M, M', does there exist a (local) CR-isomorphism between M and M' ?

An important tool to answer this question is the concept of Levi form [1, 3]. The Levi form LF_p of a CR-manifold M at a point $p \in M$ is the skew-symmetric hermitian form defined on L_p by

$$LF(X,Y) = i\,[\widetilde{X}, \overline{\widetilde{Y}}]_p \mod L_p \oplus \overline{L}_p,$$

where $X,\, Y \in L_p$ and $\widetilde{X}$ and $\widetilde{Y}$ are two local sections $M \longrightarrow L$ such that $\widetilde{X}_p = X$ and $\widetilde{Y}_p = Y$. It is a CR-invariant of M in the sense that if $\varphi : M \longrightarrow M'$ is a CR-isomorphism between M and M', then $LF = \varphi^* LF'$. It is well known [15] that if LF does vanish identically on M, which means that the bundle $L \oplus \overline{L}$ is involutive, then M is CR-isomorphic to a product $M \cong \mathbb{C}^n \times \mathbb{R}^k$. We therefore exclude this degenerate case, referred to as Levi-flat, in the subsequent parts of this memoir.

Another CR-invariant plays a central role in our analysis. Let $(E^i)_{i \geq 1}$ be the sequence of subbundles of $\mathbb{C} \otimes TM$ defined by:

$$E^1 := L \oplus \overline{L}, \qquad\qquad E^{i+1} := E^i \oplus [L, E^i] \oplus [\overline{L}, E^i],$$

and let

$$r_i := \text{rank}_{\mathbb{C}}\, E^i.$$

For example we always have $r_1 = 2n$, where n is the CR-dimension of M, while $r_2 = r_1$ if and only if M is Levi-flat. The sequence $r := (r_i)_{i \geq 1}$ constitutes a CR-invrariant of M. As it is increasing and bounded by the dimension of M, it is stationary for i sufficiently large. For this reason, we will adopt the convention to write only the first distinct values of r, writing for example $(2,3)$ instead of $(2,3,3\dots)$.

Let us now restrict the analysis of the equivalence problem to CR-manifolds of dimension not greater than 5. The CR-dimension n and the codimension k shall satisfy $2n + k \leq 5$, which, setting apart the trivial cases of totally real and complex manifolds, only leaves 4 possible values for (n,k):

$$(1,1), \qquad (1,2), \qquad (1,3), \qquad (2,1),$$

which we refer to as the type of M.

A further investigation of the possible values for the sequence r and the rank of the Levi-form LF leads to the identification of 6 different classes of CR-manifolds of dimension ≤ 5 (see [15]).

General class I is constituted by non Levi-flat CR-manifolds of type $(1,1)$. In this case the Levi form is of constant rank 1, and $r = (2,3)$. The equivalence problem for this kind of CR-manifolds has been solved by Elie Cartan in the famous papers [4, 5] mentioned at the beginning of this introduction. This problem has been tackled again recently by Merker and

Sabzevari [12], in a way which explicits the CR-invariants of M in terms of its graphing function.

General class II is constituted by non Levi-flat CR-manifolds of type $(1,2)$ such that $r = (2,3,4)$, which are also referred to as Engel manifolds. The equivalence problem for this class has been solved by Beloshapka, Ezhov and Schmalz [2] in 2007. An alternative proof of the results contained in [2] has been provided by [22], which constitutes chapter 2 of this memoir.

We mention that the other possible value for r in the case of non-Levi flat CR-manifolds of type $(1,2)$ is $(2,3)$. However this case is set apart, and should be considered as degenerate, as it is known [15] that M is then biholomorphic to a product $M \cong N \times \mathbb{R}$, where N is a CR-manifold belonging to general class I.

For non Levi-flat CR-manifolds of type $(1,3)$, the possible outcomes for r are $(2,3)$, $(2,3,5)$ and $(2,3,4,5)$. Case $r = (2,3)$ is degenerate, as it corresponds to products $M \cong N \times \mathbb{R}^2$, with N belonging to general class I [15]. Case $r = (2,3,5)$ leads to the class of CR-manifolds which we refer to as class III_1. Cartan's equivalence method for this class has been conducted recently by Merker and Sabzevari [13], which has led to a complete set of invariants for these CR-manifolds. The third case $r = (2,3,4,5)$ defines what we refer to as general class III_2. To our knowledge, this class of 5-dimensional CR-manifolds has not been studied before, and chapter 3 of this memoir is devoted to solve the equivalence problem for this class by the construction of a Cartan connection [23].

In the case of non-Levi flat CR-manifolds of type $(2,1)$, the sequence r can only take the value $(4,5)$, and the distinction between general classes of CR-manifolds of this type depends on the rank of the Levi form. Levi nondegenerate CR-manifolds are said to belong to general class IV_1, while those whose Levi form is of constant rank 1 are said to belong to general class IV_2. One also assumes that the CR-manifolds which constitute this last class are 2 nondegenerate, i.e. that their Freeman form is nonzero (see [15], pp 70–94), as they would otherwise be biholomorphic to a product $N \times \mathbb{C}$, with N belonging to general class I.

The equivalence problem for Levi-nondegenerate CR-manifolds of codimension 1 has been solved in 1974 by Chern and Moser [6] through the use of Cartan's original approach. The case of class IV_1 is thus covered by the results contained in [6]. General class IV_2 however has concentrated a lot of research efforts recently. Ebenfelt gave a solution to this problem in 2001 [7], but it appeared that this approach should only be considered as a partial one [8]. Isaev-Zaitsev [10], Medori-Spiro [14] and Pocchiola [20] have independently provided solutions to the equivalence problem for this class of CR-manifolds in 2013. Reference [20] is included in chapter 1 of the present memoir.

Each of these 6 classes entails a distinguished manifold, the model, whose local CR-automorphisms group is of maximal dimension. It plays a special role, as CR-manifolds belonging to the same class can be viewed as its deformations, generally by the way of Cartan connection. The determination of the Lie algebra of infinitesimal automorphisms of the models can be conducted through Cartan's equivalence method, and it often provides a guide for the more complicated case of general CR-manifolds of the same class, as the same structure of normalizations of group parameters occurs in both cases. For this reason, we started the resolution of the equivalence problem for general classes II, III_2 and IV_2 by the determination of the Lie algebra of infinitesimal CR-automorphisms of the models for each of these 3 classes, which are respectively given by:

(1) Beloshapka's cubic in $\mathbb{C}^3$:

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z}), \end{aligned}$$

(2) the submanifold $\mathsf{N} \subset \mathbb{C}^4$:

$$\mathsf{N}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z}), \\ w_3 &= \overline{w_3} + 2i\,z\overline{z}(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2), \end{aligned}$$

(3) the tube over the future light cone, $\mathsf{LC} \subset \mathbb{C}^3$:

$$\mathsf{LC}: \qquad (\mathsf{Re}\,z_1)^2 - (\mathsf{Re}\,z_2)^2 - (\mathsf{Re}\,z_3)^2 = 0, \qquad \mathsf{Re}\,z_1 > 0.$$

The determination of their infinitesimal automorphisms is done in reference [21], which constitutes chapter 4 of this memoir.

To conclude, the present memoir entails the following parts:

- Chapter 1 contains two versions of the solution to the equivalence problem for 2-nondegenerate 5-dimensional CR-manifolds of constant Levi rank 1, a short one which summarizes the results and sketches the proofs, and a longer one, which provides the necessary details.
- Chapter 2 is constituted by reference [22], whose aim is to provide a solution to the equivalence problem for Engel CR-manifolds.
- Chapter 3 contains reference [23], which provides a solution to the equivalence problem for CR-manifolds belonging to general class III_2.

- Chapter 4 contains reference [21], which aims to determine the Lie algebra of infinitesimal CR-automorphisms of the model manifolds for general classes II, III_2 and IV_2 through Cartan's equivalence method.

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ABSOLUTE PARALLELISM FOR 2-NONDEGENERATE REAL HYPERSURFACES $M^5 \subset \mathbb{C}^3$ OF CONSTANT LEVI RANK 1

SAMUEL POCCHIOLA

ABSTRACT

We study the local equivalence problem for five dimensional real hypersurfaces M^5 of $\mathbb{C}^3$ which are 2-nondegenerate and of constant Levi rank 1 under biholomorphisms. We find two invariants, J and W, which are expressed explicitly in terms of the graphing function F of M, the annulation of which gives a necessary and sufficient condition for M to be locally biholomorphic to a model hypersurface, the tube over the light cone. If one of the two invariants J or W does not vanish on M, we show that the equivalence problem under biholomophisms reduces to an equivalence problem between $\{e\}$-structures, that is we construct an absolute parallelism on M.

1. Introduction

A smooth 5-dimensional real hypersurface $M \subset \mathbb{C}^3$ is locally represented as the graph of a smooth function F over the 5-dimensional real hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$:

$$u = F(z_1, z_2, \overline{z_1}, \overline{z_2}, v).$$

Such a hypersurface M is said to be of CR-dimension 2 if at each point p of M, the vector space

$$T_p^{1,0}M := \mathbb{C} \otimes T_pM \cap T_p^{1,0}\mathbb{C}$$

is of complex dimension 2.

We recall that the Levi form LF of M at a point p is the skew-symmetric hermitian form defined on $T_p^{1,0}M$ by

$$LF(X, Y) = i\,[\widetilde{X}, \overline{\widetilde{Y}}]_p \quad \mathrm{mod}\ T_p^{1,0}M \oplus T_p^{0,1}M,$$

where $\widetilde{X}$ and $\widetilde{Y}$ are two local sections $M \longrightarrow T^{1,0}M$ such that $\widetilde{X}_p = X$ and $\widetilde{Y}_p = Y$.

The aim of this paper is to study the equivalence problem under biholomorphisms of the hypersurfaces $M \subset \mathbb{C}^3$ which are of CR-dimension 2, and whose Levi form is degenerate and of constant rank 1. For well-known natural reasons, we will also assume that the hypersurfaces we consider are 2-nondegenerate, i.e. that their Freeman forms are non-zero.

We refer to [2] for a historical perspective on equivalence problems for hypersurfaces of complex spaces, where the emphasis is put on the importance and the lack of practical computations in the subject. For example, even in the Levi-nondegenerate case of hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$, which was tackled by the celebrated paper by S.S. Chern and J. Moser in 1974 (see [1]), a problem still open currently is to determine the Cartan-Chern-Moser invariants explicitely in terms of a fondamental datum, namely a (local) graphing function for the hypersurfaces. As a result, the problem to determine whether a given hypersurface is locally biholomorphic to a sphere is still open. It has been solved in 2000 in the case of an ellipsoid of $\mathbb{C}^n$ by S. M. Webster in [10], where he states:

> Despite their importance, until now [the invariants of pseudoconvex domains] have been fully computed, to our knowledge, only in the case of the unit ball $D = B^n$, where they all vanish!

The main result of this paper is an attempt to answer to Sidney Webster's dissatisfaction in the case of 2-nondegenerate, Levi rank 1 hypersurfaces of $\mathbb{C}^3$. It solves e.g. the problem to determine whether such a hypersurface is locally biholomorphic to the tube over the light cone:

$$\mathsf{LC}: \qquad (\mathsf{Re}\, z_1)^2 - (\mathsf{Re}\, z_2)^2 - (\mathsf{Re}\, z_3)^2 = 0, \qquad \mathsf{Re}\, z_1 > 0,$$

which is the most symmetric hypersurface of this class. It can be summarized as follows (the explicit expressions of the invariants J and W in terms of the graphing function F of M are given in section 4):

Theorem 1. *Two fundamental invariants, J and W, occur in the biholomorphic equivalence problem for 2-nondegenerate hypersurfaces $M \subset \mathbb{C}^3$ having Levi form of constant rank 1. M is locally biholomorphic to the tube over the future light cone,*

$$\mathsf{LC}: \qquad (\mathsf{Re}\, z_1)^2 - (\mathsf{Re}\, z_2)^2 - (\mathsf{Re}\, z_3)^2 = 0, \qquad \mathsf{Re}\, z_1 > 0,$$

having $\mathbf{10}$*-dimensional Lie algebra of infinitesimal CR-automorphisms* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{LC})$*, if and only if:*

$$J \equiv W \equiv 0.$$

If either $J \not\equiv 0$, or $W \not\equiv 0$, an absolute parallelism is constructed on M. In particular, the Lie algebra of infinitesimal CR-automorphisms of M satisfies:

$$\dim \mathsf{aut}_{\mathsf{CR}}(M) \leq 5.$$

The class of 2-nondegenerate, Levi rank 1 hypersurfaces $M \subset \mathbb{C}^3$ which are homogeneous (i.e have a transitive group of CR automorphisms) have been classified in 2007 by Fels-Kaup in [5]. Theorem 1 confirms the drop from 10 to 5 of the group dimension observed in this case by Felds-Kaup, and extends it to the case of CR manifolds which are not homogeneous. Our approach is to employ Cartan's equivalence method, whose strength is to provide explicit formulae for the invariants and to treat in a unified way *all* CR-manifolds, regardless of their symmetry group.

We note that the class of hypersurfaces we consider has been studied recently by [3], where an absolute parallelism is constructed on a 10-dimensional bundle, and [4], where a Cartan-connexion is provided through a purely Lie algebraic approach. To our knowledge, the Cartan's method we employ here is the only one which exhibits the bifurcation:

$$(J \equiv W \equiv 0) \qquad \text{or} \qquad (J \not\equiv 0 \ \text{ or } \ W \not\equiv 0)\,,$$

which characterizes explicitly the local equivalence to the model, and which provides the estimate

$$\mathsf{dim}\,\mathsf{aut}_{\mathsf{CR}}(M) \leq \mathsf{5}$$

when M is not locally biholomorphic to the light cone.

2. Initial G-structure

Let $M \subset \mathbb{C}^3$ be a smooth hypersurface locally represented as a graph over the 5-dimensional real hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$:

$$u = F(z_1, z_2, \overline{z_1}, \overline{z_2}, v),$$

where F is a local smooth function depending on 5 arguments. We assume that M is a CR-submanifold of CR dimension 2 which is 2-non degenerate and whose Levi form is of constant rank 1. The two vector fields $\mathscr{L}_1$ and $\mathscr{L}_2$ defined by:

$$\mathscr{L}_j = \frac{\partial}{\partial z_j} + A^j \frac{\partial}{\partial v}, \qquad A^j := -i\,\frac{F_{z_j}}{1 + i\,F_v}, \qquad\qquad j = 1, 2,$$

constitute a basis of $T^{1,0}_p M$ at each point p of M and thus provide an identification of $T^{1,0}_p M$ with $\mathbb{C}^2$ at each point. Moreover, the real 1-form σ defined by:

$$\sigma := dv - A^1\,dz_1 - A^2\,dz_2 - \overline{A^1}\,d\overline{z_1} - \overline{A^2}\,d\overline{z_2},$$

satisfies

$$\{\sigma = 0\} = T^{1,0}M \oplus T^{0,1}M,$$

and thus provides an identication of the projection

$$\mathbb{C} \otimes T_pM \longrightarrow \mathbb{C} \otimes T_pM \;/\; \big(T^{1,0}_p M \oplus T^{0,1}_p M\big)$$

with the map $\sigma_p\colon \quad \mathbb{C}\otimes T_pM \longrightarrow \mathbb{C}$. With these two identifications, the Levi form LF can be viewed at each point p as a skew hermitian form on $\mathbb{C}^2$ represented by the matrix:

$$LF = \begin{pmatrix} \sigma_p\left(i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]\right) & \sigma_p\left(i\,[\mathscr{L}_2, \overline{\mathscr{L}_1}]\right) \\ \sigma_p\left(i\,[\mathscr{L}_1, \overline{\mathscr{L}_2}]\right) & \sigma_p\left(i\,[\mathscr{L}_2, \overline{\mathscr{L}_2}]\right) \end{pmatrix}.$$

The fact that LF is supposed to be of constant rank 1 ensures the existence of a certain function k such that the vector field

$$\mathscr{K} := k\,\mathscr{L}_1 + \mathscr{L}_2$$

lies in the kernel of LF. Here are the expressions of $\mathscr{K}$ and k in terms of the graphing function F:

$$\mathscr{K} = k\,\partial_{z_1} + \partial_{z_2} - \frac{i}{1+i\,F_v}\,(k\,F_{z_1} + F_{z_2})\,\partial_v,$$

$$k = -\frac{F_{z_2,\overline{z_1}} + F_{z_2,\overline{z_1}}\,F_v^2 - i\,F_{\overline{z_1}}\,F_{z_2,v} - F_{\overline{z_1}}\,F_v\,F_{v,z_2} + i\,F_{z_2}\,F_{\overline{z_1}}\,F_{v,v} - F_{z_2}\,F_v\,F_{v,\overline{z_1}}}{F_{z_1,\overline{z_1}} + F_{z_1,\overline{z_1}}\,F_v^2 - i\,F_{\overline{z_1}}\,F_{z_1,v} - F_{\overline{z_1}}\,F_v\,F_{z_1,v} + i\,F_{z_1}\,F_{\overline{z_1},v} + F_{z_1}\,F_{\overline{z_1}}\,F_{v,v} - F_{z_1}\,F_v\,F_{v,\overline{z_1}}},$$

and it is emphasized that the expressions that appear in the subsequent formulae are expressed in terms of Lie derivatives of the function k by the vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$, hence in terms of F.

From the above construction, the four vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ constitute a basis of $T_p^{1,0}M \oplus T_p^{0,1}M$ at each point p of M. It turns out that the vector field $\mathscr{T}$ defined by:

$$\mathscr{T} := i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]$$

is linearly independant from $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$. With the five vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ and $\mathscr{T}$, we have thus exhibited a local section from M into $\mathbb{C}\otimes F(M)$, the complexification of the bundle $F(M)$ of frames of M, which is geometrically adapted to M in the following sense:

(1) the line bundle generated by $\mathscr{K}$ is the kernel of the Levi form of M,
(2) $\mathscr{L}_1$ and $\mathscr{K}$ constitute a basis of $T^{1,0}M$,
(3) $\mathscr{T}$ is defined by the formula $\mathscr{T} := i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]$.

We now introduce the coframe ω_0 of 1-forms:

$$\omega_0 := \left(\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0\right)$$

which is the dual coframe of the frame:

$$\left(\mathscr{L}_1, \mathscr{K}, \overline{\mathscr{L}_1}, \overline{\mathscr{K}}, \mathscr{T}\right).$$

The expression of the exterior derivatives of ρ_0, κ_0, ζ_0, $\overline{\kappa}_0$, $\overline{\zeta}_0$, which constitute the so-called structure equations of the coframe ω_0, involves another important function on M, that we denote by P in the sequel. We give here the expression of P in terms of the graphing function F because, as with the function k, all the subsequent formulae will involve terms expressed as derivatives of P by the fundamental vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$, $\mathscr{T}$, namely:

$$P = \frac{l_{z_1} + A^1\, l_v - l\, A^1_v}{l},$$

where:

$$l := i\left(\overline{A^1_{z_1}} - A^1_{\overline{z_1}} + A^1\overline{A^1_v} - \overline{A^1}A^1_v\right).$$

In terms of P and k, the structure equations enjoyed by ω_0 are the following:

(1)

$$\begin{aligned}
d\rho_0 &= P\ \rho_0 \wedge \kappa_0 - \mathscr{L}_1(k)\ \rho_0 \wedge \zeta_0 + \overline{P}\ \rho_0 \wedge \overline{\kappa_0} - \overline{\mathscr{L}_1}(\overline{k})\ \rho_0 \wedge \overline{\zeta_0} + i\ \kappa_0 \wedge \overline{\kappa_0}\ , \\
d\kappa_0 &= -\mathscr{T}(k)\ \ \rho_0 \wedge \zeta_0 - \mathscr{L}_1(k)\ \ \kappa_0 \wedge \zeta_0 + \overline{\mathscr{L}_1}(k)\ \ \zeta_0 \wedge \overline{\kappa}_0\ , \\
d\zeta_0 &= 0, \\
d\overline{\kappa}_0 &= -\mathscr{T}(\overline{k})\ \ \rho_0 \wedge \overline{\zeta_0} - \mathscr{L}_1(\overline{k})\ \ \kappa_0 \wedge \overline{\zeta}_0 - \overline{\mathscr{L}_1}(\overline{k})\ \ \overline{\kappa_0} \wedge \overline{\zeta}_0\ , \\
d\overline{\zeta_0} &= 0,
\end{aligned}$$

which are equivalent to the Lie bracket relations:

$$\begin{aligned}
&[\mathscr{T}, \mathscr{L}_1] = -P\, \mathscr{T}, && [\mathscr{T}, \mathscr{K}] = \mathscr{L}_1(k)\, \mathscr{T} + \mathscr{T}(k)\, \mathscr{L}_1, \\
&[\mathscr{T}, \overline{\mathscr{L}_1}] = -\overline{P}\, \mathscr{T}, && [\mathscr{T}, \overline{\mathscr{K}}] = \overline{\mathscr{L}_1}(\overline{k})\, \mathscr{T} + \mathscr{T}(\overline{k})\, \overline{\mathscr{L}_1}, \\
&[\mathscr{L}_1, \overline{\mathscr{L}_1}] = -i\, \mathscr{T}, && [\mathscr{L}_1, \mathscr{K}] = \mathscr{L}_1(k)\, \mathscr{L}_1, \\
&[\mathscr{L}_1, \overline{\mathscr{K}}] = \mathscr{L}_1(\overline{k})\, \overline{\mathscr{L}_1}, && [\overline{\mathscr{L}_1}, \mathscr{K}] = \overline{\mathscr{L}_1}(k)\, \mathscr{L}_1, \\
&[\overline{\mathscr{L}_1}, \overline{\mathscr{K}}] = \overline{\mathscr{L}_1}(\overline{k})\, \overline{\mathscr{L}_1}, && [\mathscr{K}, \overline{\mathscr{K}}] = 0.
\end{aligned}$$

We note that the Jacobi identity implies the following two additional relations:

$$\mathscr{K}(P) = -P\, \mathscr{L}_1(k) - \mathscr{L}_1\left(\mathscr{L}_1(k)\right),$$

and

$$\mathscr{K}(\overline{P}) = -P\, \overline{\mathscr{L}_1}(k) - \overline{\mathscr{L}_1}\left(\mathscr{L}_1(k)\right) - i\, \mathscr{T}(k)$$

The Freeman form of M at a point p might be identified with the $\mathbb{C}$-skew bilinear form:

$$\mathsf{FF}(p): \qquad (x, y) \longrightarrow x\overline{y} \cdot \kappa_0\left([\mathscr{K}, \overline{\mathscr{L}_1}]_p\right),$$

and it does vanish identically on M if and only if M is biholomorphic, locally in the neighbourhood of every point, to a product:

$$M = N \times \mathbb{C}^2,$$

where $N \subset \mathbb{C}^2$ is a smooth hypersurface of $\mathbb{C}^2$ (see, for example, arxiv.org/abs/1311.5669/, pp. 70–94).

From the above Lie brackets formulae, the fact that M is 2-nondegenerate at every point, i.e. that its Freeman-form is non-zero, is thus expressed by the biholomorphically invariant assumption that:

$$\overline{\mathscr{L}_1}(k) \ \textit{vanishes nowhere on } M;$$

notice here that $\overline{\mathscr{L}_1}(k)$ appears as the coefficient of $\zeta_0 \wedge \overline{\kappa_0}$ in $d\kappa_0$.

The equivalence problem under biholomophisms of hypersurfaces $M \subset \mathbb{C}^3$ which are 2-nondegenerate and of constant Levi rank 1 is now reinterpreted as an equivalence problem between G-structures. We recall that if $G \subset GL(n, \mathbb{R})$ is a Lie group, a G-structure on a manifold M of dimension n is a subbundle of the bundle $F(M)$ of frames of M, which is a principal G-bundle. We make the following observation: if ϕ is a local biholomorphism of $\mathbb{C}^3$ such that $\phi(M) = M$, then the restriction ϕ_M of ϕ to M is a local smooth diffeomorphism of M which satisfies the additional two conditions:

(1) ϕ_M stabilizes the bundle $T^{1,0}(M)$;
(2) ϕ_M stabilizes the kernel of the Levi form of M.

As a result, there are three functions f, c and e on M such that :

$$\phi_{M*}(\mathscr{K}) = \mathsf{f}\,\mathscr{K}\,,$$

and

$$\phi_{M*}(\mathscr{L}_1) = \mathsf{c}\,\mathscr{L}_1 + \mathsf{e}\,\mathscr{K}\,.$$

Of course, as ϕ_M is a real diffeomorphism, we shall also have :

$$\phi_{M*}(\overline{\mathscr{K}}) = \overline{\phi_{M*}(\mathscr{K})} = \overline{\mathsf{f}}\,\overline{\mathscr{K}}\,,$$

and

$$\phi_{M*}(\overline{\mathscr{L}_1}) = \overline{\phi_{M*}(\mathscr{L}_1)} = \overline{\mathsf{c}}\,\overline{\mathscr{L}_1} + \overline{\mathsf{e}}\,\overline{\mathscr{K}}\,.$$

Moreover, as we have:

$$\phi_{M*}(\mathscr{T}) = i\,[\phi_{M*}(\mathscr{L}_1), \phi_{M*}(\overline{\mathscr{L}_1})] \qquad \equiv \mathsf{c}\,\overline{\mathsf{c}}\,\mathscr{T} \qquad\qquad \mod T^{1,0}M,$$

there exist two functions b and d on M such that:

$$\phi_{M*}(\mathscr{T}) = \mathsf{c}\overline{\mathsf{c}}\,\mathscr{T} + \mathsf{b}\,\mathscr{L}_1 + \mathsf{d}\,\mathscr{K} + \overline{\mathsf{b}}\,\overline{\mathscr{L}_1} + \overline{\mathsf{d}}\,\overline{\mathscr{K}}\,.$$

Let G_1 be the 10-dimensional real matrix Lie group whose elements are of the form:

$$g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix},$$

where c and f are non-zero complex numbers, while b, d, e are arbitrary complex numbers. The equivalence problem for M is suitably encoded by the G_1-structure P^1 on M consisting of the coframes of 1-forms ω which satisfy the relation:

$$\omega = g \cdot \omega_0,$$

for some $g \in G_1$.

The rest of the present paper is devoted to solve the equivalence problem for P^1 using Cartan's theory, for which we use [8] and [9] as standard references.

3. Reductions of P^1.

The coframe ω_0 gives a natural (local) trivialisation $P^1 \xrightarrow{tr} M \times G_1$ from which we may consider any differential form on M (resp. G_1) as a differential form on P^1 through the pullback by the first (resp. the second) component of tr. With this identification, the structure equations of P^1 are naturally obtained by the formula:

$$d\omega = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0.$$

The term $g \cdot d\omega_0$ contains the so-called torsion coefficients of P^1. A 1-form $\tilde{\alpha}$ on P^1 is called a modified Maurer-Cartan form if its restriction to any fiber of P^1 is a Maurer-Cartan form of G_1, or equivalently, if it is of the form:

$$\tilde{\alpha} := \alpha - x_\rho\, \rho - x_\kappa\, \kappa - x_\zeta\, \zeta - x_{\overline{\kappa}}\, \overline{\kappa} - x_{\overline{\zeta}}\, \overline{\zeta},$$

where x_ρ, x_κ, x_ζ, $x_{\overline{\kappa}}$, $x_{\overline{\zeta}}$ are arbitrary complex-valued functions on M and where α is a Maurer-Cartan form of G_1. From the relations (3), we derive the following structure equations of P^1:

$$\begin{aligned}
d\rho &= \tilde{\alpha}^1 \wedge \rho + \overline{\tilde{\alpha}^1} \wedge \rho + i\, \kappa \wedge \overline{\kappa},\\
d\kappa &= \tilde{\alpha}^1 \wedge \kappa + \tilde{\alpha}^2 \wedge \rho + T\, \zeta \wedge \overline{\kappa},\\
d\zeta &= \tilde{\alpha}^3 \wedge \rho + \tilde{\alpha}^4 \wedge \kappa + \tilde{\alpha}^5 \wedge \zeta,\\
d\overline{\kappa} &= \overline{d\kappa},\\
d\overline{\zeta} &= \overline{d\zeta},
\end{aligned}$$

for some modified Maurer-Cartan forms $\tilde{\alpha}^1$, $\tilde{\alpha}^2$, $\tilde{\alpha}^3$, $\tilde{\alpha}^4$ and $\tilde{\alpha}^5$, where the essential torsion coefficient T is given by:

$$T = \frac{\mathsf{c}}{\overline{\mathsf{c}}\mathsf{f}}\, \overline{\mathscr{L}_1}(k).$$

From standard results on Cartan theory (see [8, 9]), a diffeomorphism of M is an isomorphism of the G_1-structure P^1 if and only if it is an isomorphism of the reduced bundle $P^2 \subset P^1$ consisting of those coframes ω on M such that

$$T = 1.$$

This is equivalent to the normalization:

$$\mathsf{f} = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\, \overline{\mathscr{L}_1}(k),$$

from which one can consider P^2 as a G_2-structure on M, where G_2 is the 8-dimensional matrix Lie group whose elements g take the form:

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

The next step is now to reduce the bundle P^2. To this aim, one determines its structure equations, which take the form:

$$\begin{aligned} d\rho &= \tilde{\beta}^1 \wedge \rho + \overline{\tilde{\beta}^1} \wedge \rho + i\, \kappa \wedge \overline{\kappa}, \\ d\kappa &= \tilde{\beta}^1 \wedge \kappa + \tilde{\beta}^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\ d\zeta &= \tilde{\beta}^3 \wedge \rho + \tilde{\beta}^4 \wedge \kappa + \tilde{\beta}^1 \wedge \zeta - \overline{\tilde{\beta}^1} \wedge \zeta + U\, \zeta \wedge \overline{\kappa}, \end{aligned}$$

for some modified Maurer-Cartan forms $\tilde{\beta}^1$, $\tilde{\beta}^2$, $\tilde{\beta}^3$ and $\tilde{\beta}^4$. Setting the essential torsion U to 0 yields the normalization:

$$\mathsf{b} = -i\, \overline{\mathsf{c}}\mathsf{e} + i\, \frac{\mathsf{c}}{3} \left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P} \right).$$

Introducing the subbundle $P^3 \subset P^2$ of those coframes on M such that b is defined by the above formula, we are reduced to the study of a G_3 equivalence problem where G_3 is the 6-dimensional matrix Lie group whose elements are of the form:

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\, \mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\, \overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

As for the first two steps, we determine the set of the new structure equations enjoyed by P^3:

$$
\begin{aligned}
d\rho &= \tilde\gamma^1\wedge\rho+\overline{\tilde\gamma^1}\wedge\rho+i\,\kappa\wedge\overline{\kappa},\\
d\kappa &= \tilde\gamma^1\wedge\kappa+\tilde\gamma^2\wedge\rho+\zeta\wedge\overline{\kappa}\\
d\zeta &= \tilde\gamma^3\wedge\rho+i\,\tilde\gamma^2\wedge\kappa+\tilde\gamma^1\wedge\zeta-\overline{\tilde\gamma^1}\wedge\zeta+V^1\,\kappa\wedge\zeta+V^2\,\kappa\wedge\overline{\kappa}\,,
\end{aligned}
$$

for some modified Maurer-Cartan forms $\tilde\gamma^1$, $\tilde\gamma^2$, $\tilde\gamma^3$ and two functions V^1 and V^2. The normalization of the group parameter d comes from the normalization $V^2=0$, which yields:

$$
\begin{aligned}
\mathsf{d} = -i\,\frac{1}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}}+i\,\frac{2}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2}+i\,\frac{1}{18}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)}\\
-i\,\frac{1}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{P}^2+i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathscr{L}_1}\left(\overline{P}\right)-i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}.
\end{aligned}
$$

Considering those 1-forms on M such that $V^2=0$, we introduce a subbundle P^4 which is a G_4-structure on M, where G_4 is the 4-dimensional Lie group whose elements are of the form:

$$
g=\begin{pmatrix}
\mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0\\
-i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0\\
-\frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0\\
i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0\\
\frac{i}{2}\,\frac{\overline{\mathsf{e}}^2\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}}
\end{pmatrix}.
$$

4. Main theorem

The fourth loop of reductions leads to a more advanced analysis than the three previous ones. The normalizations of the group parameters that are suggested at this stage depend on the vanishing or the non-vanishing of two functions, J and W, which appear to be two fundamental invariants of the problem. The new set of structure equations is indeed of the form:

$$
\begin{aligned}
d\rho &= \tilde\delta^1\wedge\rho+\overline{\tilde\delta^1}\wedge\rho+i\,\kappa\wedge\overline{\kappa},\\
d\kappa &= \tilde\delta^1\wedge\kappa+\tilde\delta^2\wedge\rho+\zeta\wedge\overline{\kappa}\\
d\zeta &= i\,\tilde\delta^2\wedge\kappa+\tilde\delta^1\wedge\zeta-\overline{\tilde\delta^1}\wedge\zeta+\frac{W}{\mathsf{c}}\,\rho\wedge\zeta+i\,\frac{\overline{J}}{\overline{\mathsf{c}}^3}\,\rho\wedge\overline{\kappa}\,,
\end{aligned}
$$

for some modified Maurer-Cartan forms $\tilde{\delta}^1$, $\tilde{\delta}^2$, where the functions J and W are defined on M by:

$$\begin{aligned} J = & \frac{5}{18}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)^2}{\mathscr{L}_1(\overline{k})^2}\,P + \frac{1}{3}\,P\,\mathscr{L}_1\left(P\right) - \frac{1}{9}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})}\,P^2 \\ & + \frac{20}{27}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)^3}{\mathscr{L}_1(\overline{k})^3} - \frac{5}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\,\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\right)}{\mathscr{L}_1(\overline{k})^2} \\ & + \frac{1}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\,\mathscr{L}_1(P)}{\mathscr{L}_1(\overline{k})} - \frac{1}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\right)}{\mathscr{L}_1(\overline{k})}\,P \\ & - \frac{2}{27}\,P^3 - \frac{1}{6}\,\mathscr{L}_1\left(\mathscr{L}_1\left(P\right)\right) + \frac{1}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\right)\right)}{\mathscr{L}_1(\overline{k})}, \end{aligned}$$

and

$$\begin{aligned} W := & \frac{2}{3}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})} \\ & + \frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} - \frac{1}{3}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)}. \end{aligned}$$

We thus observe a branching phenomenon at that point: if J and W are both identically vanishing on M, then no further reductions of the group parameters are allowed and the equivalence problem must be handled by a suitable prolongation. However, if J is non-vanishing we can normalize the parameter c by

$$\mathsf{c} = J^{\frac{1}{3}},$$

whereas if W is non vanishing we can perform the normalization

$$\mathsf{c} = W.$$

We notice here that we are not treating the cases where one of the two invariants J or W might vanish somewhere on M without beeing identically vanishing on M, that is we are making a genericity assumption M, which is a standard process when using Cartan's theory. This motivates the following definition:

Definition 1. *A 5-dimensional CR-submanifold of $\mathbb{C}^3$ of CR-dimension 2 which is 2-non degenerate, and whose Levi form is of constant rank 1 is said to be generic if the functions J and W are either 0 or non-vanishing on M.*

We are now in position to state the main theorem of the present paper:

Theorem 2. *Let $M \subset \mathbb{C}^3$ be a $\mathscr{C}^\infty$-smooth 5-dimensional hypersurface of CR-dimension 2, which is 2-non degenerate, whose Levi form is of constant rank 1 and which is generic in the sense of definition 1. Then*

(1) *if $W \not\equiv 0$ or if $J \not\equiv 0$ on M, then the local equivalence problem for M reduces to the equivalence problem for a five dimensional $\{e\}$-structure.*
(2) *if $W \equiv 0$ and $J \equiv 0$ identically on M, then M is locally biholomorphic to the tube over the light cone.*

Granted that the functions k and P are expressed in terms of partial derivatives of order ≤ 3 of the graphing function F, and that the two main invariants J and W are explicit in terms of k and P, we stress that the local biholomorphic equivalence to the light cone is explicitely characterised in terms of F.

It is well-known (see, for example, [6]) that the group of automorphisms $\mathscr{U}$ of an $\{e\}$-structure on a $\mathscr{C}^\infty$ manifold N is a Lie transformation group such that $\dim \mathscr{U} \leq \dim N$.

Corollary 1. *Let $M \subset \mathbb{C}^3$ be a $\mathscr{C}^\infty$ CR-manifold satisfying the hypotheses of theorem 2. If M is not locally equivalent to the tube over the light cone at a point $p \in M$, then the dimension of the Lie algebra of germs of CR-automorphisms of M at p is bounded by 5.*

The next 3 subsections are devoted to complete the proof of theorem 2, by distinguishing the 3 cases $J \not\equiv 0$, $W \not\equiv 0$ and $J \equiv W \equiv 0$. The following lemma is of crucial importance for the first two cases:

Lemma 1. *The invariants J and W satisfy the following two differential equations:*

$$\mathscr{K}(J) + 3\,\mathscr{L}_1(k)\,J = 0,$$

and

$$\overline{\mathscr{K}}(W) + 2\,\overline{\mathscr{L}_1}(k)\,\overline{W} = 0.$$

Proof. These equations are obtained by a direct computation of $\mathscr{K}(J)$ and $\overline{\mathscr{K}}(W)$, using the fact that $\overline{\mathscr{K}}(k) = 0$ and the commutation relations between the vector fields $\mathscr{L}_1$, $\overline{\mathscr{L}_1}$, $\mathscr{K}$, $\overline{\mathscr{K}}$ and $\mathscr{T}$. □

4.1. **Case** $J \not\equiv 0$**.** From the normalization $c^3 = J$, the expression of $d\rho$ becomes

$$d\rho = S^\rho_{\rho\kappa}\,\rho\wedge\kappa + S^\rho_{\rho\zeta}\,\rho\wedge\zeta + S^\rho_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + S^\rho_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\;\kappa\wedge\overline{\kappa}$$

for some essential torsion coefficients $S^{\rho}_{\rho\kappa}$, $S^{\rho}_{\rho\zeta}$, $S^{\rho}_{\rho\overline{\kappa}}$ and $S^{\rho}_{\rho\overline{\zeta}}$. On the other hand, the expression of $d\zeta$ is

$$\begin{aligned} d\zeta = i\,\delta_2 \wedge \kappa \\ &+ S^{\zeta}_{\rho\kappa}\,\rho\wedge\kappa + S^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta + \rho\wedge\overline{\zeta} \\ &+ S^{\zeta}_{\kappa\zeta}\,\kappa\wedge\zeta + S^{\zeta}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + S^{\zeta}_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + S^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + S^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

where the $S^{\bullet}_{\bullet\bullet}$ are new torsion coeficients. From the above equations, we get that $-S^{\rho}_{\rho\overline{\kappa}} + S^{\zeta}_{\zeta\overline{\kappa}}$ is an essential torsion coefficient, which can be normalized to zero. The careful computation of this coefficient, using lemma 1, gives the normalization of e:

$$\mathsf{e} = \frac{1}{3}\,\frac{J^{1/3}}{\overline{J}^{1/3}}\left(-\frac{\overline{\mathscr{L}_1}(\overline{J})}{\overline{J}} + 2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P}\right).$$

4.2. **Case $W \not\equiv 0$.** We now assume that the fonction W does not vanish on M, and we show how the group parameter e can be normalized. We choose the normalization $c := W$. The second structure equation takes the form:

$$\begin{aligned} d\kappa = -i\,d\epsilon \wedge \rho \\ &+ X^{\kappa}_{\rho\kappa}\;\rho\wedge\kappa + X^{\kappa}_{\rho\zeta}\;\rho\wedge\zeta + X^{\kappa}_{\rho\overline{\kappa}}\;\rho\wedge\overline{\kappa} + X^{\kappa}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} \\ &+ X^{\kappa}_{\kappa\zeta}\;\kappa\wedge\zeta + X^{\kappa}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + X^{\kappa}_{\kappa\overline{\zeta}}\;\kappa\wedge\overline{\zeta} + \zeta\wedge\overline{\kappa}, \end{aligned}$$

where $\epsilon = \frac{\mathsf{e}}{W}$, and for a new set of torsion coefficients $X^{\bullet}_{\bullet\bullet}$. The computation of the coefficient $X^{\kappa}_{\kappa\overline{\kappa}}$ gives, using lemma 1:

$$X^{\kappa}_{\kappa\overline{\kappa}} = -2\,\overline{\epsilon} - \frac{\overline{\mathscr{L}_1}\,(W)}{W\overline{W}} - \frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{W\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\overline{P}}{\overline{W}}.$$

Setting this coefficient to zero, we get a normalization of ϵ, and hence of e.

4.3. **Case $J \equiv W \equiv 0$.** We suppose that $W \equiv J \equiv 0$ identically on M. If we return to the structure equations that we have obtained for P^4 at the end of section 3, we have:

$$\begin{aligned} d\rho &= \tilde{\delta}^1 \wedge \rho + \overline{\tilde{\delta}^1} \wedge \rho + i\,\kappa\wedge\overline{\kappa}, \\ d\kappa &= \tilde{\delta}^1 \wedge \kappa + \tilde{\delta}^2 \wedge \rho + \zeta\wedge\overline{\kappa} \\ d\zeta &= i\,\tilde{\delta}^2 \wedge \kappa + \tilde{\delta}^1 \wedge \zeta - \overline{\tilde{\delta}^1} \wedge \zeta, \end{aligned} \tag{2}$$

for a certain choice of modified Maurer-Cartan forms $\tilde{\delta}^1$ and $\tilde{\delta}^2$ on P^4. We remark that this set of equations are invariant if we replace $\tilde{\delta}^1$ and $\tilde{\delta}^2$ by the 1-forms π^1 and π^2 defined by:

$$\begin{cases} \pi^1 := \hat{\delta}^1 + \mathsf{t}\,\rho \\ \pi^2 := \hat{\delta}^2 + \mathsf{t}\,\kappa, \end{cases}$$

where t is a real parameter. As no further reductions of the group parameters are allowed by the above structure equations, we perform a prolongation of the problem by considering the G_{prol}-structure on P^4 consisting of the coframes ω on P^4 of the form:

$$\omega_{\mathsf{t}} := \left(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \pi^1, \pi^2, \overline{\pi^1}, \overline{\pi^2}\right),$$

where G_{prol} is the 1-dimensional Lie group whoses elements g_{t} act on the coframes ω by:

$$g_{\mathsf{t}} \cdot \omega_{\mathsf{s}} = \omega_{\mathsf{t+s}}.$$

As P^4 is 9-dimensional, this introduce a 10-dimensional subbundle of the bundle of frames of P^4, that we denote by Π in the sequel. Our next aim is to determine the expressions of $d\pi^1$ and $d\pi^2$. Both of these expressions can be deduced by taking the exterior derivative of the equations (2). For example, taking the exterior derivative of both sides of the equation giving $d\rho$, we get after simplications:

$$0 = \left(d\pi^1 - i\,\kappa \wedge \overline{\pi^2} + \overline{d\pi^1} + i\,\overline{\kappa} \wedge \pi^2\right) \wedge \rho.$$

The same operations for $d\kappa$ and $d\zeta$ yield

$$0 = \left(d\pi^1 - \zeta \wedge \overline{\zeta}\right) \wedge \kappa + \left(d\pi^2 - \pi^2 \wedge \overline{\pi^1} - \zeta \wedge \overline{\pi^2}\right) \wedge \rho,$$
$$0 = \left(d\pi^1 - d\overline{\pi^1} - i\,\overline{\kappa} \wedge \pi^2\right) \wedge \zeta + i\,\left(d\pi^2 - \pi^2 \wedge \overline{\pi^1}\right) \wedge \kappa.$$

From these equations, we deduce the existence of a modified Maurer-Cartan form Λ on Π such that:

$$d\pi^1 = i\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta} + \Lambda \wedge \rho,$$
$$d\pi^2 = \pi^2 \wedge \overline{\pi^1} + \zeta \wedge \overline{\pi^2} + \Lambda \wedge \kappa.$$

By adding Λ to the set of 1-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, π^1, π^2, $\overline{\pi^1}$, $\overline{\pi^2}$, we get a 10-dimensional $\{e\}$-structure on Π which constitutes the second (and last) 1-dimensional prolongation to the equivalence problem. It remains to determine the exterior derivative of Λ, which is done by taking the exterior derivative of $d\pi^1$ and $d\pi^2$, which yields:

$$0 = \left(d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi^1} - i\,\pi^2 \wedge \overline{\pi^2}\right) \wedge \rho$$
$$0 = \left(d\Lambda - i\,\pi^2 \wedge \overline{\pi^2} - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi^1}\right) \wedge \kappa = 0.$$

From these last two equations, we deduce that:

$$d\Lambda = i\,\pi^2 \wedge \overline{\pi^2} + \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi^1}.$$

Summing up the results that we have obtained so far, the ten 1-differential forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, π^1, π^2, $\overline{\pi^1}$, $\overline{\pi^2}$, Λ, satisfy the structure equations:

$$\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa},\\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa},\\
d\zeta &= i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta,\\
d\pi^1 &= i\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta} + \Lambda \wedge \rho,\\
d\pi^2 &= \pi^2 \wedge \overline{\pi^1} + \zeta \wedge \overline{\pi^2} + \Lambda \wedge \kappa,\\
d\Lambda &= i\,\pi^2 \wedge \overline{\pi^2} + \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi^1}.
\end{aligned}$$

The torsion coefficients of these structure equations are all constant, and they do not depend on the graphing function F of M. This proves that all the hypersurfaces M which satisfy

$$J = W = 0$$

are locally biholomorphic. A direct computation shows that the tube over the future light cone is precisely such that $J = W = 0$. This completes the proof of theorem 2.

5. Extensions of theorem 2

We now give a slight extension of theorem 2. If M is a 5-dimensional abstract CR-manifold of CR dimension 2 then there exist a subbundle L of $\mathbb{C} \otimes TM$ of dimension 2 such that

(1) $L \cap \overline{L} = \{0\}$
(2) L is formally integrable.

It is then well-known that there exist local coordinates (x_1, x_2, x_3, x_4, v) on M and two local sections $\mathscr{L}_1$ and $\mathscr{L}_2$ of L, such that:

$$\mathscr{L}_1 = \frac{\partial}{\partial z_1} + A^1 \frac{\partial}{\partial v}, \qquad \mathscr{L}_2 = \frac{\partial}{\partial z_2} + A^2 \frac{\partial}{\partial v},$$

where A^1 and A^2 are two locally defined functions on M, and where the vector fields $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ are defined by the usual formulas:

$$\frac{\partial}{\partial z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right) \qquad \frac{\partial}{\partial z_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - i\,\frac{\partial}{\partial x_4}\right).$$

As a result, we can define the functions k and P together with the four vector fields $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ and $\mathscr{T}$ in terms of the fundamental functions A^1 and A^2 as in the embedded case, and all the subsequent structure equations at each step of Cartan's method are unchanged. Theorem 2 remains thus valid in the more general setting of abstract CR-manifolds.

Finally, the G-structures that we introduce at each step are in fact globally defined on M (as subbundles of $\mathbb{C} \otimes TM$). As a result, the first part of theorem 2 has the following global counterpart:

Theorem 3. *Let M be an abstract CR-manifold satisfying the hypotheses of theorem 2. Then J and W are globally defined on M. If J does not vanish on M or if W does not vanish on M, then there exist an absolute parallelism on M.*

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EXPLICIT ABSOLUTE PARALLELISM FOR 2-NONDEGENERATE REAL HYPERSURFACES $M^5 \subset \mathbb{C}^3$ OF CONSTANT LEVI RANK 1

SAMUEL POCCHIOLA

ABSTRACT

We study the local equivalence problem for five dimensional real hypersurfaces M^5 of $\mathbb{C}^3$ which are 2-nondegenerate and of constant Levi rank 1 under biholomorphisms. We find two invariants, J and W, which are expressed explicitly in terms of the graphing function F of M, the annulation of which give a necessary and sufficient condition for M to be locally biholomorphic to a model hypersurface, the tube over the light cone. If one of the two invariants J or W does not vanish on M, we show that the equivalence problem under biholomophisms reduces to an equivalence problem between $\{e\}$-structures, that is we construct an absolute parallelism on M.

1. Introduction

A smooth 5-dimensional real hypersurface $M \subset \mathbb{C}^3$ is locally represented as the graph of a smooth function F over the 5-dimensional real hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$:

$$u = F(z_1, z_2, \overline{z_1}, \overline{z_2}, v).$$

Such a hypersurface M is said to be of CR-dimension 2 if at each point p of M, the vector space

$$T_p^{1,0}M := \mathbb{C} \otimes T_pM \cap T_p^{1,0}\mathbb{C}$$

is of complex dimension 2 (for background, see [21, 4, 2]).

We recall that the Levi form LF of M at a point p is the skew-symmetric hermitian form defined on $T_p^{1,0}M$ by

$$LF(X, Y) = i\,[\widetilde{X}, \overline{\widetilde{Y}}]_p \mod T_p^{1,0}M \oplus T_p^{0,1}M,$$

where $\widetilde{X}$ and $\widetilde{Y}$ are two local sections $M \longrightarrow T^{1,0}M$ such that $\widetilde{X}_p = X$ and $\widetilde{Y}_p = Y$.

The aim of this paper is to study the equivalence problem under biholomorphisms of the hypersurfaces $M \subset \mathbb{C}^3$ which are of CR-dimension 2, and whose Levi form is degenerate and of constant rank 1. For well-known natural reasons, we will also assume that the hypersurfaces we consider are 2-nondegenerate, i.e. that their Freeman forms are non-zero (see for example [21], p. 91). Two other approaches on this problem have been recently provided by Isaev-Zaitsev and Medori-Spiro ([18, 10]). We refer to [9] for an historical perspective on equivalence problems for hypersurfaces of complex spaces.

We start by exhibiting two vector fields $\mathscr{L}_1$ and $\mathscr{L}_2$ which constitute a basis of $T_p^{1,0}M$ at each point p of M. This provides an identification of $T_p^{1,0}M$ with $\mathbb{C}^2$ at each point. We also exhibit a real 1-form σ on TM whose prolongation to $\mathbb{C} \otimes TM$ satisfies:

$$\{\sigma = 0\} = T^{1,0}M \oplus T^{0,1}M,$$

which provides an identication of the projection

$$\mathbb{C} \otimes T_pM \longrightarrow \mathbb{C} \otimes T_pM \; / \left(T_p^{1,0}M \oplus T_p^{0,1}M\right)$$

with the map $\sigma_p\colon \quad \mathbb{C} \otimes T_pM \longrightarrow \mathbb{C}$. With these two identifications, the Levi form LF can be viewed at each point p as a skew hermitian form on $\mathbb{C}^2$ represented by the matrix:

$$LF = \begin{pmatrix} \sigma_p\left(i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]\right) & \sigma_p\left(i\,[\mathscr{L}_2, \overline{\mathscr{L}_1}]\right) \\ \sigma_p\left(i\,[\mathscr{L}_1, \overline{\mathscr{L}_2}]\right) & \sigma_p\left(i\,[\mathscr{L}_2, \overline{\mathscr{L}_2}]\right) \end{pmatrix}.$$

The fact that LF is supposed to be of constant rank 1 ensures the existence of a certain function k such that the vector field

$$\mathscr{K} := k\,\mathscr{L}_1 + \mathscr{L}_2$$

lies in the kernel of LF. Our explicit computation of LF provides us with an explicit expression of k in terms of the graphing function F for M. In fact, here are the expressions of $\mathscr{L}_1$ and $\mathscr{K}$:

$$\begin{aligned} \mathscr{L}_1 &= \partial_{z_1} - i\,\frac{F_{z_1}}{1 + i\,F_v}\,\partial_v, \\ \mathscr{K} &= k\,\partial_{z_1} + \partial_{z_2} - \frac{i}{1 + i\,F_v}\,(k\,F_{z_1} + F_{z_2})\,\partial_v, \end{aligned}$$

and also of k:

$$k = -\frac{F_{z_2,\overline{z_1}} + F_{z_2,\overline{z_1}}\,F_v^2 - i\,F_{\overline{z_1}}\,F_{z_2,v} - F_{\overline{z_1}}\,F_v\,F_{v,z_2} + i\,F_{z_2}\,F_{\overline{z_1}}\,F_{v,v} - F_{z_2}\,F_v\,F_{v,\overline{z_1}}}{F_{z_1,\overline{z_1}} + F_{z_1,\overline{z_1}}\,F_v^2 - i\,F_{\overline{z_1}}\,F_{z_1,v} - F_{\overline{z_1}}\,F_v\,F_{z_1,v} + i\,F_{z_1}\,F_{\overline{z_1},v} + F_{z_1}\,F_{\overline{z_1}}\,F_{v,v} - F_{z_1}\,F_v\,F_{v,\overline{z_1}}},$$

and we want to emphasize that all our subsequent computations will be expressed in terms of Lie derivatives of the function k by the vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$, hence in terms of F.

From our construction, the four vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ constitute a basis of $T_p^{1,0}M \oplus T_p^{0,1}M$ at each point p of M. It turns out that the vector field $\mathscr{T}$ defined by:

$$\mathscr{T} := i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]$$

is linearly independant from $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$. With the five vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ and $\mathscr{T}$, we have thus exhibited a local section from M into $\mathbb{C} \otimes F(M)$, the complexification of the bundle $F(M)$ of frames of M, which is geometrically adapted to M in the following sense:

(1) the line bundle generated by $\mathscr{K}$ is the kernel of the Levi form of M,
(2) $\mathscr{L}_1$ and $\mathscr{K}$ constitute a basis of $T^{1,0}M$,
(3) $\mathscr{T}$ is defined by the formula $\mathscr{T} := i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]$.

Then we define the coframe of 1-forms:

$$\left(\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0\right)$$

which is the dual coframe of the frame:

$$\left(\mathscr{L}_1, \mathscr{K}, \overline{\mathscr{L}_1}, \overline{\mathscr{K}}, \mathscr{T}\right).$$

The computation of the exterior derivatives of ρ_0, κ_0, ζ_0, $\overline{\kappa}_0$, $\overline{\zeta}_0$, which constitute the so-called structure equations of the coframe, involves another important function on M, that we denote by P in the sequel. We give here the expression of P in terms of the graphing function F because, as with the function k, all our subsequent computations will involve terms expressed as derivatives of P by the fundamental vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$, $\mathscr{T}$, namely:

$$P = \frac{l_{z_1} + A^1\,l_v - l\,A^1_v}{l},$$

where:

$$A_1 = 2\,\frac{F_{z_1}}{1 + i\,F_v},$$

and where:

$$l := i\left(\overline{A^1_{z_1}} - A^1_{\overline{z_1}} + A^1\overline{A^1_v} - \overline{A^1}A^1_v\right).$$

Then in terms of P and k, the structure equations enjoyed by ρ_0, κ_0, ζ_0, $\overline{\kappa}_0$, $\overline{\zeta}_0$, are the following:

$$\begin{aligned}
d\rho_0 &= P\ \rho_0 \wedge \kappa_0 - \mathscr{L}_1(k)\ \rho_0 \wedge \zeta_0 + \overline{P}\ \rho_0 \wedge \overline{\kappa_0} - \overline{\mathscr{L}_1}(\overline{k})\ \rho_0 \wedge \overline{\zeta_0} + i\ \kappa_0 \wedge \overline{\kappa_0}\ , \\
d\kappa_0 &= -\mathscr{T}(k)\ \ \rho_0 \wedge \zeta_0 - \mathscr{L}_1(k)\ \ \kappa_0 \wedge \zeta_0 + \overline{\mathscr{L}_1}(k)\ \ \zeta_0 \wedge \overline{\kappa}_0\ , \\
d\zeta_0 &= 0, \\
d\overline{\kappa}_0 &= -\mathscr{T}(\overline{k})\ \ \rho_0 \wedge \overline{\zeta_0} - \mathscr{L}_1(\overline{k})\ \ \kappa_0 \wedge \overline{\zeta}_0 - \overline{\mathscr{L}_1}(\overline{k})\ \ \overline{\kappa_0} \wedge \overline{\zeta}_0\ , \\
d\overline{\zeta_0} &= 0.
\end{aligned}$$

The fact that M is 2-nondegenerate is expressed by the (biholomorphically invariant, see [21]) assumption that:

$$\overline{\mathscr{L}_1}(k) \ \textit{vanishes nowhere on } M;$$

notice here that $\overline{\mathscr{L}_1}(k)$ appears as the coefficient of $\zeta_0 \wedge \overline{\kappa_0}$ in $d\kappa_0$.

The end of section 2 is devoted to reinterpret the equivalence problem under biholomophisms of such hypersurfaces as an equivalence problem between G-structures. We recall that if $G \subset GL(n, \mathbb{R})$ is a Lie group, a G-structure on a manifold M of dimension n is a subbundle of the bundle $F(M)$ of frames of M, which is a principal G-bundle. The fact that we can express the equivalence problem in terms of equivalences between G-structures comes from the following observation: if ϕ is a local biholomorphism of $\mathbb{C}^3$ such that $\phi(M) = M$, then the restriction ϕ_M of ϕ to M is a local smooth diffeomorphism of M which satisfies the additional two conditions:

(1) ϕ_M stabilizes the bundle $T^{1,0}(M)$;
(2) ϕ_M stabilizes the kernel of the Levi form of M.

As a result, there are three functions f, c and e on M such that :

$$\phi_{M*}(\mathscr{K}) = \mathsf{f}\,\mathscr{K},$$

and

$$\phi_{M*}(\mathscr{L}_1) = \mathsf{c}\,\mathscr{L}_1 + \mathsf{e}\,\mathscr{K}.$$

Of course, as ϕ_M is a real diffeomorphism, we shall also have :

$$\phi_{M*}(\overline{\mathscr{K}}) = \overline{\phi_{M*}(\mathscr{K})} = \overline{\mathsf{f}}\,\overline{\mathscr{K}},$$

and

$$\phi_{M*}(\overline{\mathscr{L}_1}) = \overline{\phi_{M*}(\mathscr{L}_1)} = \overline{\mathsf{c}}\,\overline{\mathscr{L}_1} + \overline{\mathsf{e}}\,\overline{\mathscr{K}}.$$

It is then easy to show that the matrix Lie group which encodes suitably the problem is the 10 dimensional Lie group G_1 given by the matrices of the form:

$$g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix},$$

where c and f are non-zero complex numbers, while b, d, e are arbitrary complex numbers.

The rest of our article is devoted to the implementation of Cartan's equivalence method to reduce this G_1-equivalence problem to an absolute parallelism. We use [24] and [26] as standard references on Cartan's equivalence method. We develope the parametric version of Cartan's equivalence method, that is we perform all the computations and give explicit expressions of the functions involved in the normalizations of the group parameters, because we need to control carefully the expressions of these functions: some of them might indeed vanish identically on M, which is of crucial importance when deciding whether a potential normalization might be allowed or not. Our computations involves only terms which are derivatives of the functions k and P by the fundamental vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$, $\mathscr{T}$, and they become ramified by the fact that some relations exists between these derivatives: those that follow simply from the Jacobi identities, and those that follow from the fact that the Levi form of M is of rank 1 everywhere. We give a sum up of the relations that we use at the end of subsection 2.2. These relations imply important simplifications in the formulae we obtain for the torsion coefficients, and shall not be missed if one keeps in mind that we usually want to control whether these coefficients do vanish or not on M, which is a delicate task, even with the help of a computer algebra system.

We find in section 3 that the first normalization of the group parameters is:

$$\mathsf{f} = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathscr{L}_1}(k).$$

This enables us to reduce G_1 to a new matrix Lie group G_2, which is 8-dimensional and whose elements g take the form:

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} \end{pmatrix}.$$

We then perform a second loop in Cartan's equivalence method in section 4, which yields the normalization:

$$\mathsf{b} = -i\,\overline{\mathsf{c}}\mathsf{e} + i\,\frac{\mathsf{c}}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right),$$

and which therefore leads to a G_3-equivalence problem, where G_3 is the 6-dimensional matrix Lie group whose elements are of the form:

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

The third loop is done in section 5 and it gives us a normalization of the parameter d as:

$$\mathsf{d} = -i\,\frac{1}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} + i\,\frac{2}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + i\,\frac{1}{18}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)}$$
$$- i\,\frac{1}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{P}^2 + i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) - i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}.$$

This therefore reduces G_3 to the 4-dimensional group G_4, whose elements are of the form:

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ -\frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \frac{i}{2}\,\frac{\overline{\mathsf{e}}^2\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

The fourth loop of Cartan's method, which is done in section 6, leads to a more advanced analysis than the three previous ones. The normalizations of the group parameters that are suggested at this stage depend on the vanishing or the non-vanishing of two functions, J and W, which appear to be two fundamental invariants of the problem. The expressions of J and W are given below:

$$J = \frac{5}{18}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)^2}{\mathscr{L}_1(\overline{k})^2}\,P + \frac{1}{3}\,P\,\mathscr{L}_1\left(P\right) - \frac{1}{9}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})}\,P^2$$
$$+ \frac{20}{27}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)^3}{\mathscr{L}_1(\overline{k})^3} - \frac{5}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\,\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\right)}{\mathscr{L}_1(\overline{k})^2}$$
$$+ \frac{1}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\,\mathscr{L}_1(P)}{\mathscr{L}_1(\overline{k})} - \frac{1}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\right)}{\mathscr{L}_1(\overline{k})}\,P$$
$$- \frac{2}{27}\,P^3 - \frac{1}{6}\,\mathscr{L}_1\left(\mathscr{L}_1\left(P\right)\right) + \frac{1}{6}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)\right)\right)}{\mathscr{L}_1(\overline{k})},$$

and

$$W := \frac{2}{3}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})}$$
$$+ \frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} - \frac{1}{3}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)}.$$

We thus observe a branching phenomenon at that point: if J and W are both identically vanishing on M, then no further reductions of the group parameters are allowed by Cartan's method. However, if J is non-vanishing we can normalize the parameter c by

$$\mathsf{c} = J^{\frac{1}{3}},$$

whereas if W is non vanishing we can perform the normalization

$$\mathsf{c} = W.$$

We notice here that we are not treating the cases where one of the two invariants J or W might vanish somewhere on M without beeing identically vanishing on M, that is we are making a genericity assumption M, which is a standard process when using Cartan's technique. To be fully precise, we also suppose in section 8 that the function $\overline{\mathscr{K}}(W)$ is generic on M, that is it is either identically 0 or non-vanishing on M, in order to establish the results of this section. This motivates the following definition:

Definition 1. *A five dimensional CR-submanifold of $\mathbb{C}^3$ of CR-dimension 2 which is 2-non degenerate, and whose Levi form is of constant rank 1 is said to be generic if the functions J, W and $\overline{\mathscr{K}}(W)$ are either 0 or non-vanishing on M.*

Section 7 is devoted to show that in the case $J \neq 0$, one can normalize the last group parameter e, thus reducing the equivalence problem to the study of a five dimensional $\{e\}$-structure. Section 8 deals with the same issue in the case $W \neq 0$. To this end, we show that $W \neq 0$ implies $\overline{\mathscr{K}}(W) \neq 0$ under the genericity assumption (this is the purpose of Lemma 1). In both cases $J \neq 0$ and $W \neq 0$, the final $\{e\}$-structure that we obtain on M contains terms which are derivatives of the graphing function F up to order 8. Thus the results of these sections only require that M is $\mathscr{C}^8$-smooth.

Finally, in section 9, we show that when both J and W vanish identically on M, we can reduce the equivalence problem to a 10-dimensional $\{e\}$-structure after performing two suitable prolongations. The structure equations that we obtain are the same as those enjoyed by the tube over the future the light cone:

$$(\mathsf{Re}\, z_1)^2 - (\mathsf{Re}\, z_2)^2 - (\mathsf{Re}\, z_3)^2 = 0, \qquad\qquad \mathsf{Re}\, z_1 > 0,$$

which is locally biholomorphic (see [11, 13]) to the graphed hypersurface:

$$u = \frac{z_1\overline{z_1} + \frac{1}{2}z_1^2\overline{z_2} + \frac{1}{2}\overline{z_1^2}z_2}{1 - z_2\overline{z_2}}.$$

This proves the fact that when J and W are both vanishing, M is locally biholomorphic to the tube over the light cone. We summarize these results in the following theorem:

Theorem 1. *Let $M \subset \mathbb{C}^3$ be a $\mathscr{C}^8$-smooth 5-dimensional hypersurface of CR-dimension 2, which is 2-non degenerate, whose Levi form is of constant rank 1 and which is generic in the sense of definition 1. Then*

(1) *if $W \neq 0$ or if $J \neq 0$ on M, then the local equivalence problem for M reduces to the equivalence problem for a five dimensional $\{e\}$-structure.*
(2) *if $W = 0$ and $J = 0$ identically on M, then M is locally biholomorphic to the tube over the light cone.*

Granted that the functions k and P are expressed in terms of partial derivatives of order ≤ 3 of the graphing function F, and that the two main invariants J and W are explicit in terms of k and P, we stress that the local biholomorphic equivalence to the light cone is explicitely characterised in terms of F.

It is well-known (see, for example, [17]) that the group of automorphisms $\mathscr{U}$ of an $\{e\}$-structure on a $\mathscr{C}^\infty$ manifold N is a Lie transformation group such that $\dim \mathscr{U} \leq \dim N$. As a result of theorem 1, we thus have:

Corollary 1. *Let $M \subset \mathbb{C}^3$ be a $\mathscr{C}^\infty$ CR-manifold satisfying the hypotheses of theorem 1. If M is not locally equivalent to the tube over the light cone at a point $p \in M$, then the dimension of the Lie algebra of germs of CR-automorphisms of M at p is bounded by 5.*

We now give a slight extension of theorem 1. If M is a 5-dimensional abstract CR-manifold of CR dimension 2 then there exist a subbundle L of $\mathbb{C} \otimes TM$ of dimension 2 such that

(1) $L \cap \overline{L} = \{0\}$
(2) L is formally integrable.

It is then well-known that there exist local coordinates (x_1, x_2, x_3, x_4, v) on M and two local sections $\mathscr{L}_1$ and $\mathscr{L}_2$ of L, such that:

$$\mathscr{L}_1 = \frac{\partial}{\partial z_1} + A^1 \frac{\partial}{\partial v},$$

and

$$\mathscr{L}_2 = \frac{\partial}{\partial z_2} + A^2 \frac{\partial}{\partial v},$$

where A^1 and A^2 are two locally defined functions on M, and where the vector fields $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ are defined by the usual formulae:

$$\frac{\partial}{\partial z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right),$$

and

$$\frac{\partial}{\partial z_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - i\,\frac{\partial}{\partial x_4}\right).$$

As a result, we can define the functions k and P together with the four vector fields $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ and $\mathscr{T}$ in terms of the fundamental functions A^1 and A^2 as in the embedded case, and all the subsequent structure equations at each step of Cartan's method are unchanged. Theorem 1 remains thus valid in the more general setting of abstract CR-manifolds.

Finally, the G-structures that we introduce at each step are in fact globally defined on M (as subbundles of $\mathbb{C} \otimes TM$). As a result, the first part of theorem 1 has the following global counterpart:

Theorem 2. *Let M be an abstract CR-manifold satisfying the hypotheses of theorem 1. Then J and W are globally defined on M. If J does not vanish on M or if W does not vanish on M, then there exist an absolute parallelism on M.*

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2. Geometric and analytic set up

2.1. **Shape of the initial coframe.** Let $M \subset \mathbb{C}^3$ be a local real analytic hypersurface passing through the origin of $\mathbb{C}^3$. We recall that M can be represented as a graph over the 5-dimensional real hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$:

$$u = F(z_1, z_2, \overline{z_1}, \overline{z_2}, v),$$

where F is a local real analytic function depending on 5 arguments. We make the assumption that M is a CR-submanifold of CR dimension 2, that is the bundle $T^{1,0}M$ is of complex dimension 2. Let us look for a frame of $T^{1,0}M$ constituted of two vectors field of the form:

$$\mathscr{L}_1 = \frac{\partial}{\partial z_1} + A_1 \frac{\partial}{\partial w},$$
$$\mathscr{L}_2 = \frac{\partial}{\partial z_2} + A_2 \frac{\partial}{\partial w},$$

with two unknown functions A_1 and A_2. As M is the zero set of the function $G := u - F$, the condition that $\mathscr{L}_1$ and $\mathscr{L}_2$ belong to $T^{1,0}M$ take the form:

$$d\,G\,(\mathscr{L}_1) = 0 \qquad \text{and} \qquad d\,G\,(\mathscr{L}_2) = 0.$$

As we have:

$$dG = du - F_{z_1}\, dz_1 - F_{z_2}\, dz_2 - F_{\overline{z_1}}\, d\overline{z_1} - F_{\overline{z_2}}\, d\overline{z_2} - F_v\, dv$$

and

$$\partial_w = \frac{1}{2}\left(\partial_u - i\,\partial_v\right),$$

these two conditions read as:

$$F_{z_j} - \frac{1}{2}\, A_j - \frac{i}{2}\, F_v\, A_j = 0, \qquad\qquad j = 1, 2,$$

which lead to:

$$A_j = 2\, \frac{F_{z_j}}{1 + i\, F_v} \qquad\qquad j = 1, 2.$$

If π denotes the canonical projection $\mathbb{C}^3 \longrightarrow \mathbb{C}^2 \times \mathbb{R}$ which sends the variables (z_1, z_2, w) on (z_1, z_2, v), the fact that M is a graph over the hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$ makes the restriction of π to M a local diffeomorphism $M \longrightarrow \mathbb{C}^2 \times \mathbb{R}$, that is a local chart on M. All the subsequent computations will be made in coordinates (z_1, z_2, v), which means that they will be made through this local chart provided by π. The (extrinsic) vector fields $\mathscr{L}_j$ are mapped by π onto the (intrinsic) vector fields $\pi_*\,(\mathscr{L}_j)$. As $\pi_*\,(\partial_w) = -\frac{i}{2}\,\partial_v$, we have:

$$\pi_*\,(\mathscr{L}_j) = \partial_{z_j} + A^j\,\partial_v \qquad\qquad j = 1, 2$$

where

$$A^j := -i\, \frac{F_{z_j}}{1 + i\, F_v} \qquad\qquad j = 1, 2.$$

In order to simplify the notations, we will still denote $p_*\,(\mathscr{L}_j)$ by $\mathscr{L}_j$ in the sequel. If σ is a 1-form on M whose kernel at each point p is $T_p^{1,0}M \oplus T_p^{0,1}M$, we identify the projection

$$\mathbb{C} \otimes T_pM \longrightarrow \mathbb{C} \otimes T_pM \;/\; T_p^{1,0}M \oplus T_p^{0,1}M$$

with the map σ_p: $\mathbb{C} \otimes T_pM \longrightarrow \mathbb{C}$. An example of such a 1-form σ is given by:

$$\sigma := dv - A^1\, dz_1 - A^2\, dz_2 - \overline{A^1}\, d\overline{z_1} - \overline{A^2}\, d\overline{z_2}.$$

As an identification of $T_p^{1,0}M$ with $\mathbb{C}^2$ is also provided by the basis of vector fields $\mathscr{L}_1$ and $\mathscr{L}_2$, the Levi form of M can be viewed as the skew-symmetric hermitian form on $\mathbb{C}^2$ given by the matrix:

$$LF := \begin{pmatrix} \sigma_p\left(i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]\right) & \sigma_p\left(i\,[\mathscr{L}_2, \overline{\mathscr{L}_1}]\right) \\ \sigma_p\left(i\,[\mathscr{L}_1, \overline{\mathscr{L}_2}]\right) & \sigma_p\left(i\,[\mathscr{L}_2, \overline{\mathscr{L}_2}]\right) \end{pmatrix}.$$

The computation of the Lie bracket $[\mathscr{L}_1, \overline{\mathscr{L}_1}]$ gives:

$$\begin{aligned} [\mathscr{L}_1, \overline{\mathscr{L}_1}] &= [\partial_{z_1} + A^1\,\partial_v, \partial_{\overline{z_1}} + \overline{A^1}\,\partial_v] \\ &= \left(\overline{A^1}_{z_1} - A^1_{\overline{z_1}} + A^1\overline{A^1_v} - \overline{A^1}A^1_v\right)\partial_v. \end{aligned}$$

Similar computations of $[\mathscr{L}_1, \overline{\mathscr{L}_2}]$, $[\mathscr{L}_2, \overline{\mathscr{L}_1}]$ and $[\mathscr{L}_2, \overline{\mathscr{L}_2}]$ give that

$$[\mathscr{L}_1, \overline{\mathscr{L}_2}] = [\mathscr{L}_2, \overline{\mathscr{L}_1}] = [\mathscr{L}_2, \overline{\mathscr{L}_2}] = 0 \qquad \mod \partial_v.$$

In the sequel, we make the assumption that M is Levi degenerate of rank 1. There is therefore a function k defined on M such that $\binom{k}{1}$ gives a basis of the kernel of LF. As a result of the definition of k and LF, the four Lie brackets $[\mathscr{L}_1, \overline{\mathscr{L}_1}]$, $[\mathscr{L}_1, \overline{\mathscr{L}_2}]$, $[\mathscr{L}_2, \overline{\mathscr{L}_1}]$ and $[\mathscr{L}_2, \overline{\mathscr{L}_2}]$ enjoy the following two relations:

$$\begin{cases} k\,[\mathscr{L}_1, \overline{\mathscr{L}_1}] + [\mathscr{L}_2, \overline{\mathscr{L}_1}] = 0 \\ k\,[\mathscr{L}_1, \overline{\mathscr{L}_2}] + [\mathscr{L}_2, \overline{\mathscr{L}_2}] = 0. \end{cases} \tag{1}$$

Moreover, the vector field $\mathscr{K} := \mathscr{L}_2 + k\,\mathscr{L}_1$ gives a basis of the kernel of the Levi form on M and the four vectors fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$ and $\overline{\mathscr{K}}$ give a basis of $T^{1,0}M \oplus T^{0,1}M$. Let us introduce the fifth vector field

$$\mathscr{T} := i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}].$$

As $\mathscr{T}$ lies in the line bundle generated by ∂_v, the five vector fields $\mathscr{T}$, $\mathscr{L}_1$, $\overline{\mathscr{L}_1}$, $\mathscr{K}$ and $\overline{\mathscr{K}}$ give a basis of $\mathbb{C} \otimes_{\mathbb{R}} TM$.

2.2. **Lie bracket structure.** Let us explore the Lie bracket relations satisfied by this basis of $\mathbb{C} \otimes_{\mathbb{R}} TM$. We start with the computation of $[\mathscr{L}_1, \mathscr{L}_2]$.

$$\begin{aligned} [\mathscr{L}_1, \mathscr{L}_2] &= [\partial_{z_1} + A^1\partial_v, \partial_{z_2} + A^2\partial_v] \\ &\equiv 0 \qquad \mathrm{mod}\ \partial_v, \end{aligned}$$

which means that $[\mathscr{L}_1, \mathscr{L}_2]$ belongs to the line bundle generated by ∂_v. On the other hand, as $\mathscr{L}_1$ and $\mathscr{L}_2$ both belong to $T^{1,0}M$, and as it is a well known fact that $T^{1,0}M$ is involutive, $[\mathscr{L}_1, \mathscr{L}_2]$ belongs to $T^{1,0}M$, whose intersection with $\mathbb{C} \cdot \partial_v$ is reduced to zero. We thus have:

$$[\mathscr{L}_1, \mathscr{L}_2] = 0.$$

As a result, we can compute $[\mathscr{K}, \mathscr{L}_1]$. Indeed we have:

$$[\mathscr{K}, \mathscr{L}_1] = [k\,\mathscr{L}_1 + \mathscr{L}_2, \mathscr{L}_1] = -\mathscr{L}_1(k)\,\mathscr{L}_1.$$

We now turn our attention on the computation of the bracket $[\mathscr{K}, \overline{\mathscr{L}_1}]$. Using the relation (1), we get:

$$\begin{aligned} [\mathscr{K}, \overline{\mathscr{L}_1}] &= [k\,\mathscr{L}_1 + \mathscr{L}_2, \overline{\mathscr{L}_1}] \\ &= k\,[\mathscr{L}_1, \overline{\mathscr{L}_1}] + [\mathscr{L}_2, \overline{\mathscr{L}_1}] - \overline{\mathscr{L}_1}(k)\,\mathscr{L}_1 \\ &= -\overline{\mathscr{L}_1}(k)\,\mathscr{L}_1. \end{aligned}$$

To compute further brackets, we need to determine the value of $\mathscr{K}(\overline{k})$.

Taking the Lie bracket between $\mathscr{K}$ and the complex conjugate of the first equation of (1) gives:

$$\mathscr{K}(\overline{k})\,[\mathscr{L}_1, \overline{\mathscr{L}_1}] + \overline{k}\,[\mathscr{K}, [\mathscr{L}_1, \overline{\mathscr{L}_1}]] + [\mathscr{K}, [\mathscr{L}_1, \overline{\mathscr{L}_2}]] = 0.$$

As $\mathscr{K}(\overline{k})\,[\mathscr{L}_1,\overline{\mathscr{L}_1}]$ belongs to $\mathbb{C}\cdot\partial_v$, the vector field

$$S := \overline{k}\,[\mathscr{K},[\mathscr{L}_1,\overline{\mathscr{L}_1}]] + [\mathscr{K},[\mathscr{L}_1,\overline{\mathscr{L}_2}]]$$

is equal to its projection on $\mathbb{C}\cdot\partial_v$. It is thus sufficient to perform its computation mod $T^{1,0}M$. The Jacobi identity gives:

$$S = \overline{k}\,[[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{L}_1}] + \overline{k}\,[\mathscr{L}_1,[\mathscr{K},\overline{\mathscr{L}_1}]] + [[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{L}_2}] + [\mathscr{L}_1,[\mathscr{K},\overline{\mathscr{L}_2}]].$$

As $[\mathscr{K},\overline{\mathscr{L}_1}] = -\overline{\mathscr{L}_1}(k)\,\mathscr{L}_1$, we have $[\mathscr{L}_1,[\mathscr{K},\overline{\mathscr{L}_1}]] \equiv 0 \mod T^{1,0}M$. Similarly we have $[\mathscr{K},\overline{\mathscr{L}_2}] = -\overline{\mathscr{L}_2}(k)\,\mathscr{L}_1$, from which we deduce that $[\mathscr{L}_1,[\mathscr{K},\overline{\mathscr{L}_2}]] \equiv 0 \mod T^{1,0}M$. We thus have:

$$\begin{aligned} S &\equiv \overline{k}\,[[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{L}_1}] + [[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{L}_2}] && \mod T^{1,0}M \\ &\equiv [[\mathscr{K},\mathscr{L}_1],\overline{k}\,\overline{\mathscr{L}_1}] + [[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{L}_2}] && \mod T^{1,0}M \\ &\equiv [[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{K}}] && \mod T^{1,0}M. \end{aligned}$$

The involutivity of the bundle $T^{1,0}M$ implies that $[\mathscr{K},\mathscr{L}_1]$ belongs to $T^{1,0}M$. As $\mathscr{K}$ has been choosen to belong to the kernel of the Levi form of M, $[[\mathscr{K},\mathscr{L}_1],\overline{\mathscr{K}}]$ belongs to $T^{1,0}M$. We thus have $S \equiv 0 \mod T^{1,0}M$, from which we deduce:

(2) $$\mathscr{K}(\overline{k}) = 0.$$

We are now ready to compute $[\mathscr{K},\overline{\mathscr{K}}]$:

$$\begin{aligned} [\mathscr{K},\overline{\mathscr{K}}] &= [k\,\mathscr{L}_1 + \mathscr{L}_2, \overline{k}\,\overline{\mathscr{L}_1} + \overline{\mathscr{L}_2}] \\ &= k\,\overline{k}\,[\mathscr{L}_1,\overline{\mathscr{L}_1}] + k\,[\mathscr{L}_1\overline{\mathscr{L}_2}] + \overline{k}\,[\mathscr{L}_2,\mathscr{L}_1] + \overline{k}\,[\mathscr{L}_2,\overline{\mathscr{L}_1}] + [\mathscr{L}_2,\overline{\mathscr{L}_2}] \\ &\quad + k\,\mathscr{L}_1(\overline{k})\,\overline{\mathscr{L}_1} + \mathscr{L}_2(\overline{k})\,\overline{\mathscr{L}_1} - \overline{\mathscr{L}_2}(k)\,\mathscr{L}_1 - \overline{k}\,\overline{\mathscr{L}_1}(k)\,\mathscr{L}_1 \\ &= k\left(\overline{k}\,[\mathscr{L}_1,\overline{\mathscr{L}_1}] + [\mathscr{L}_1\overline{\mathscr{L}_2}]\right) + \left(\overline{k}\,[\mathscr{L}_2,\overline{\mathscr{L}_1}] + [\mathscr{L}_2,\overline{\mathscr{L}_2}]\right) + \mathscr{K}(\overline{k})\,\overline{\mathscr{L}_1} - \overline{\mathscr{K}}(k)\,\mathscr{L}_1 \\ &= 0 \qquad \text{by } (1) \text{ and } (2). \end{aligned}$$

We now compute $[\mathscr{L}_1,\mathscr{T}]$. We recall that from the definition of $\mathscr{T}$ we have $\mathscr{T} = l\,\partial_v$, where the function l is defined by

$$l := i\left(\overline{A^1_{z_1}} - A^1_{\overline{z_1}} + A^1\overline{A^1_v} - \overline{A^1}A^1_v\right).$$

We thus have:

$$\begin{aligned} [\mathscr{L}_1,\mathscr{T}] &= [\partial_{z_1} + A^1\,\partial_v, l\,\partial_v] \\ &= \left(l_{z_1} + A^1\,l_v - l\,A^1_v\right)\partial_v \\ &= P\,\mathscr{T}. \end{aligned}$$

where P is the function defined on M by

$$P = \frac{l_{z_1} + A^1\,l_v - l\,A^1_v}{l}.$$

The last bracket that we need to compute is $[\mathscr{K}, \mathscr{T}]$. Using the Jacobi identity, we get:

$$\begin{aligned}
[\mathscr{K}, \mathscr{T}] &= i[\mathscr{K}, [\mathscr{L}_1, \overline{\mathscr{L}_1}]] \\
&= i\,[[\mathscr{K}, \mathscr{L}_1], \overline{\mathscr{L}_1}] + i\,[\mathscr{L}_1, [\mathscr{K}, \overline{\mathscr{L}_1}]] \\
&= i\,[\,-\mathscr{L}_1(k)\,\mathscr{L}_1, \overline{\mathscr{L}_1}] + i\,[\mathscr{L}_1, -\overline{\mathscr{L}_1}(k)\mathscr{L}_1] \\
&= -\mathscr{L}_1(k)\,\mathscr{T} + i\,\overline{\mathscr{L}_1}\,(\mathscr{L}_1(k))\,\mathscr{L}_1 - i\,\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1 \\
&= -\mathscr{L}_1(k)\,\mathscr{T} - i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}](k)\,\mathscr{L}_1 \\
&= -\mathscr{L}_1(k)\,\mathscr{T} - \mathscr{T}(k)\,\mathscr{L}_1.
\end{aligned}$$

The Jacobi identity actually implies other relations between the functions P, k and their derivatives with respect to the five vector fields $\mathscr{T}$, $\mathscr{L}_1$, $\overline{\mathscr{L}_1}$, $\mathscr{K}$ and $\overline{\mathscr{K}}$. The following computation of $[\mathscr{K}, [\mathscr{T}, \mathscr{L}_1]]$ aims to determine an expression of $\mathscr{K}(P)$.

$$\begin{aligned}
[\mathscr{K}, [\mathscr{T}, \mathscr{L}_1]] &= -[\mathscr{K}, P\,\mathscr{T}] \\
&= -\mathscr{K}(P)\,\mathscr{T} - P\,[\,-\mathscr{L}_1(k)\,\mathscr{T} - \mathscr{T}(k)\,\mathscr{L}_1] \\
&= -\mathscr{K}(P)\,\mathscr{T} + P\,\mathscr{L}_1(k)\,\mathscr{T} + P\,\mathscr{T}(k)\,\mathscr{L}_1.
\end{aligned}$$

On the other hand, the Jacobi identity gives:

$$\begin{aligned}
[\mathscr{K}, [\mathscr{T}, \mathscr{L}_1]] &= [[\mathscr{K}, \mathscr{T}], \mathscr{L}_1] + [\mathscr{T}, [\mathscr{K}, \mathscr{L}_1]] \\
&= [\,-\mathscr{L}_1(k)\,\mathscr{T} - \mathscr{T}(k)\,\mathscr{L}_1, \mathscr{L}_1] + [\mathscr{T}, -\mathscr{L}_1(k)\,\mathscr{L}_1] \\
&= \mathscr{L}_1\left(\mathscr{T}(k)\right)\,\mathscr{L}_1 - \mathscr{L}_1(k)\,[\mathscr{T}, \mathscr{L}_1] + \mathscr{L}_1\left(\mathscr{L}_1(k)\right)\,\mathscr{T} \\
&\quad - \mathscr{L}_1(k)\,[\mathscr{T}, \mathscr{L}_1] - \mathscr{T}\left(\mathscr{L}_1(k)\right)\,\mathscr{L}_1 \\
&= [\mathscr{L}_1, \mathscr{T}](k)\,\mathscr{L}_1 + 2\,\mathscr{L}_1(k)\,[\mathscr{L}_1, \mathscr{T}] + \mathscr{L}_1\left(\mathscr{L}_1(k)\right)\,\mathscr{T} \\
&= P\,\mathscr{T}(k)\,\mathscr{L}_1 + 2\,\mathscr{L}_1(k)\,P\,\mathscr{T} + \mathscr{L}_1\left(\mathscr{L}_1(k)\right)\,\mathscr{T} \\
&= P\,\mathscr{T}(k)\,\mathscr{L}_1 + \left(2\,\mathscr{L}_1(k)\,P + \mathscr{L}_1\left(\mathscr{L}_1(k)\right)\right)\mathscr{T}.
\end{aligned}$$

By identification of both results, we have:

$$-\mathscr{K}(P) + P\,\mathscr{L}_1(k) = 2\,\mathscr{L}_1(k)\,P + \mathscr{L}_1\left(\mathscr{L}_1(k)\right),$$

that is:

$$\mathscr{K}(P) = -P\,\mathscr{L}_1(k) - \mathscr{L}_1\left(\mathscr{L}_1(k)\right).$$

We compute $\mathscr{K}(\overline{P})$ in a similar way. We start with a direct computation of $[\mathscr{K}, [\mathscr{T}, \overline{\mathscr{L}_1}]]$:

$$\begin{aligned}
[\mathscr{K}, [\mathscr{T}, \overline{\mathscr{L}_1}]] &= -[\mathscr{K}, \overline{P}\,\mathscr{T}] \\
&= -\mathscr{K}(\overline{P})\,\mathscr{T} - \overline{P}\,[\,-\mathscr{L}_1(k)\,\mathscr{T} - \mathscr{T}(k)\,\mathscr{L}_1] \\
&= -\mathscr{K}(\overline{P})\,\mathscr{T} + \overline{P}\,\mathscr{L}_1(k)\,\mathscr{T} + \overline{P}\,\mathscr{T}(k)\,\mathscr{L}_1.
\end{aligned}$$

The computation using the Jacobi identity gives:

$$\begin{aligned}
[\mathscr{K}, [\mathscr{T}, \overline{\mathscr{L}_1}]] &= [[\mathscr{K}, \mathscr{T}], \overline{\mathscr{L}_1}] + [\mathscr{T}, [\mathscr{K}, \overline{\mathscr{L}_1}]] \\
&= [-\mathscr{L}_1(k)\,\mathscr{T} - \mathscr{T}(k)\,\mathscr{L}_1, \overline{\mathscr{L}_1}] + [\mathscr{T}, -\overline{\mathscr{L}_1}\,\mathscr{L}_1] \\
&= \overline{\mathscr{L}_1}\left(\mathscr{L}_1(k)\right)\,\mathscr{T} - \mathscr{L}_1(k)\,[\mathscr{T}, \overline{\mathscr{L}_1}] + \overline{\mathscr{L}_1}\left(\mathscr{T}(k)\right)\,\mathscr{L}_1 \\
&\quad - \mathscr{T}(k)[\mathscr{L}_1, \overline{\mathscr{L}_1}] - \mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)\,\mathscr{L}_1 - \overline{\mathscr{L}_1}(k)[\mathscr{T}, \mathscr{L}_1] \\
&= \overline{\mathscr{L}_1}\left(\mathscr{L}_1(k)\right)\,\mathscr{T} + \overline{P}\,\mathscr{L}_1(k)\,\mathscr{T} + [\overline{\mathscr{L}_1}, \mathscr{T}](k)\,\mathscr{L}_1 + i\,\mathscr{T}(k)\,\mathscr{T} + P\,\overline{\mathscr{L}_1}(k)\,\mathscr{T} \\
&= \left(\overline{\mathscr{L}_1}\left(\mathscr{L}_1(k)\right) + \overline{P}\,\mathscr{L}_1(k) + P\,\overline{\mathscr{L}_1}(k) + i\,\mathscr{T}(k)\right)\mathscr{T} + \overline{P}\,\mathscr{T}(k)\,\mathscr{L}_1.
\end{aligned}$$

Identification of both results gives:

$$\mathscr{K}(\overline{P}) = -P\,\overline{\mathscr{L}_1}(k) - \overline{\mathscr{L}_1}\left(\mathscr{L}_1(k)\right) - i\,\mathscr{T}(k).$$

Let us summarize the results that we have obtained so far. The five vector fields $\mathscr{T}$, $\mathscr{L}_1$, $\overline{\mathscr{L}_1}$, $\mathscr{K}$ and $\overline{\mathscr{K}}$ enjoy the following Lie bracket structure:

$$\begin{aligned}
[\mathscr{T}, \mathscr{L}_1] &= -P\,\mathscr{T}, \\
[\mathscr{T}, \overline{\mathscr{L}_1}] &= -\overline{P}\,\mathscr{T}, \\
[\mathscr{T}, \mathscr{K}] &= \mathscr{L}_1(k)\,\mathscr{T} + \mathscr{T}(k)\,\mathscr{L}_1, \\
[\mathscr{T}, \overline{\mathscr{K}}] &= \overline{\mathscr{L}_1}(\overline{k})\,\mathscr{T} + \mathscr{T}(\overline{k})\,\overline{\mathscr{L}_1}, \\
[\mathscr{L}_1, \overline{\mathscr{L}_1}] &= -i\,\mathscr{T}, \\
[\mathscr{L}_1, \mathscr{K}] &= \mathscr{L}_1(k)\,\mathscr{L}_1, \\
[\mathscr{L}_1, \overline{\mathscr{K}}] &= \mathscr{L}_1(\overline{k})\,\overline{\mathscr{L}_1}, \\
[\overline{\mathscr{L}_1}, \mathscr{K}] &= \overline{\mathscr{L}_1}(k)\,\mathscr{L}_1, \\
[\overline{\mathscr{L}_1}, \overline{\mathscr{K}}] &= \overline{\mathscr{L}_1}(\overline{k})\,\overline{\mathscr{L}_1}, \\
[\mathscr{K}, \overline{\mathscr{K}}] &= 0,
\end{aligned} \tag{3}$$

where P is a function defined on M. The Jacobi identity implies the following two additional relations:

$$\mathscr{K}(P) = -P\,\mathscr{L}_1(k) - \mathscr{L}_1\left(\mathscr{L}_1(k)\right),$$

and

$$\mathscr{K}(\overline{P}) = -P\,\overline{\mathscr{L}_1}(k) - \overline{\mathscr{L}_1}\left(\mathscr{L}_1(k)\right) - i\,\mathscr{T}(k).$$

2.3. **Structure equations of the initial coframe.** From the formula

$$d\omega(X, Y) = X\left(\omega(Y)\right) - Y\left(\omega(X)\right) - \omega\left([X, Y]\right),$$

where X and Y are two arbitrary vector fields and ω is a 1-form, we deduce from equation (3) the structure equations enjoyed by the base coframe

$(\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0)$, that is:
(4)
$$\begin{aligned}
d\rho_0 &= P\ \rho_0 \wedge \kappa_0 - \mathscr{L}_1(k)\ \rho_0 \wedge \zeta_0 + \overline{P}\ \rho_0 \wedge \overline{\kappa_0} - \overline{\mathscr{L}_1}(\overline{k})\ \rho_0 \wedge \overline{\zeta_0} + i\ \kappa_0 \wedge \overline{\kappa_0}\ , \\
d\kappa_0 &= -\mathscr{T}(k)\ \ \rho_0 \wedge \zeta_0 - \mathscr{L}_1(k)\ \ \kappa_0 \wedge \zeta_0 + \overline{\mathscr{L}_1}(k)\ \ \zeta_0 \wedge \overline{\kappa}_0\ , \\
d\zeta_0 &= 0, \\
d\overline{\kappa}_0 &= -\mathscr{T}(\overline{k})\ \ \rho_0 \wedge \overline{\zeta_0} - \mathscr{L}_1(\overline{k})\ \ \kappa_0 \wedge \overline{\zeta_0} - \overline{\mathscr{L}_1}(\overline{k})\ \ \overline{\kappa_0} \wedge \overline{\zeta_0}\ , \\
d\overline{\zeta_0} &= 0.
\end{aligned}$$

2.4. **Equivalence under biholomorphisms.** Let ϕ be a local biholomorphism of $\mathbb{C}^3$ such that $\phi(0) = 0$ which preserves M, i.e. such that $\phi(M) = M$. Then the restriction ϕ_M of ϕ to M is a local real analytic diffeomorphism of M which satisfies the following two additional conditions:

(1) ϕ_M stabilizes the bundle $T^{1,0}M$.
(2) ϕ_M stabilizes the kernel of the Levi form of M.

As a result, there are three functions f, c and e on M such that:
$$\phi_{M*}(\mathscr{K}) = \mathsf{f}\,\mathscr{K}\,,$$
and
$$\phi_{M*}(\mathscr{L}_1) = \mathsf{c}\,\mathscr{L}_1 + \mathsf{e}\,\mathscr{K}\,.$$
Of course, as ϕ_M is a real diffeomorphism, we shall also have:
$$\phi_{M*}(\overline{\mathscr{K}}) = \overline{\phi_{M*}(\mathscr{K})} = \overline{\mathsf{f}}\,\overline{\mathscr{K}}\,,$$
and
$$\phi_{M*}(\overline{\mathscr{L}_1}) = \overline{\phi_{M*}(\mathscr{L}_1)} = \overline{\mathsf{c}}\,\overline{\mathscr{L}_1} + \overline{\mathsf{e}}\,\overline{\mathscr{K}}\,.$$
On the other hand there is a priori no special condition that shall be satisfied by $\phi_{M*}(\mathscr{T})$, except the fact that it shall be a real vector field, because $\mathscr{T}$ is real. There are thus a real function a and two complex valued functions b and d such that:
$$\phi_{M*}(\mathscr{T}) = \mathsf{a}\,\mathscr{T} + \mathsf{b}\,\mathscr{L}_1 + \mathsf{d}\,\mathscr{K} + \overline{\mathsf{b}}\,\overline{\mathscr{L}_1} + \overline{\mathsf{d}}\,\overline{\mathscr{K}}\,.$$
We sum up these relations with the following matrix notation:
$$\phi_{M*}\begin{pmatrix} \mathscr{T} \\ \mathscr{L}_1 \\ \mathscr{K} \\ \overline{\mathscr{L}_1} \\ \overline{\mathscr{K}} \end{pmatrix} = \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{d}} \\ 0 & \mathsf{c} & \mathsf{e} & 0 & 0 \\ 0 & 0 & \mathsf{f} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathsf{c}} & \overline{\mathsf{e}} \\ 0 & 0 & 0 & 0 & \overline{\mathsf{f}} \end{pmatrix} \cdot \begin{pmatrix} \mathscr{T} \\ \mathscr{L}_1 \\ \mathscr{K} \\ \overline{\mathscr{L}_1} \\ \overline{\mathscr{K}} \end{pmatrix}.$$
As ϕ_{M*} is invertible, the functions a, c and f shall not vanish on M. The relation between the coframe $(\rho_0, \kappa_0, \zeta_0, \overline{\kappa_0}, \overline{\zeta_0})$ and the coframe

$\phi_M^* \left(\rho_0, \kappa_0, \zeta_0, \overline{\kappa_0}, \overline{\zeta_0}\right)$ is thus given by a plain transposition of the previous equation, that is:

$$\phi_M^* \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa_0} \\ \overline{\zeta_0} \end{pmatrix} = \begin{pmatrix} \mathsf{a} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix} \cdot \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa_0} \\ \overline{\zeta_0} \end{pmatrix}.$$

In fact the function a shall satisfy another condition. As $\mathscr{T} = i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]$, we have

$$\begin{aligned} \phi_{M*}(\mathscr{T}) &= i\,[\phi_{M*}(\mathscr{L}_1), \phi_{M*}(\overline{\mathscr{L}_1})] \\ &= i\,[\mathsf{c}\,\mathscr{L}_1 + \mathsf{e}\,\mathscr{K}, \overline{\mathsf{c}}\,\overline{\mathscr{L}_1} + \overline{\mathsf{e}}\,\overline{\mathscr{K}}] \\ &\equiv \mathsf{c}\,\overline{\mathsf{c}}\,\mathscr{T} \qquad\qquad \mod T^{1,0}M, \end{aligned}$$

On the other hand we have from the definition of a that $\phi_{M*}(\mathscr{T}) \equiv \mathsf{a}\,\mathscr{T} \mod T^{1,0}M$, which implies:

$$\mathsf{a} = \mathsf{c}\,\overline{\mathsf{c}}.$$

2.5. **Initial G-structure.** Let G_1 be the 10 dimensional real matrix Lie group whose elements are of the form:

$$g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix},$$

where c and f are non-zero complex numbers whereas b, d and e are arbitrary complex numbers.

Following [24], let us introduce 5 new one-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$ in accordance with the shape of the ambiguity matrix related to local biholomorphic equivalences of such kinds of hypersurfaces:

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \end{pmatrix} := g \cdot \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa_0} \\ \overline{\zeta_0} \end{pmatrix},$$

that is to say, in expanded form:

$$\begin{aligned}
\rho &:= \mathsf{c}\overline{\mathsf{c}}\,\rho_0,\\
\kappa &:= \mathsf{b}\,\rho_0 + \mathsf{c}\,\kappa_0,\\
\zeta &:= \mathsf{d}\,\rho_0 + \mathsf{e}\,\kappa_0 + \mathsf{f}\,\zeta_0,\\
\overline{\kappa} &:= \overline{\mathsf{b}}\,\rho_0 + \overline{\mathsf{c}}\,\overline{\kappa_0},\\
\overline{\zeta}v &:= \overline{\mathsf{d}}\,\rho_0 + \overline{\mathsf{e}}\,\overline{\kappa_0} + \overline{\mathsf{f}}\,\overline{\zeta_0}.
\end{aligned}$$

By inverting the matrix:

$$\begin{pmatrix}\rho_0\\ \kappa_0\\ \zeta_0\\ \overline{\kappa_0}\\ \overline{\zeta_0}\end{pmatrix} = \begin{pmatrix}
\frac{1}{\mathsf{c}\overline{\mathsf{c}}} & 0 & 0 & 0 & 0\\
\frac{-\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} & \frac{1}{\mathsf{c}} & 0 & 0 & 0\\
\frac{\mathsf{be}-\mathsf{cd}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}} & -\frac{\mathsf{e}}{\mathsf{cf}} & \frac{1}{\mathsf{f}} & 0 & 0\\
\frac{-\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2} & 0 & 0 & \frac{1}{\overline{\mathsf{c}}} & 0\\
\frac{\overline{\mathsf{b}}\overline{\mathsf{e}}-\overline{\mathsf{c}}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}^2\overline{\mathsf{f}}} & 0 & 0 & -\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}\overline{\mathsf{f}}} & \frac{1}{\overline{\mathsf{f}}}
\end{pmatrix}\begin{pmatrix}\rho\\ \kappa\\ \zeta\\ \overline{\kappa}\\ \overline{\zeta}\end{pmatrix},$$

we find how the $\{\}_0$-indexed forms express in terms of the lifted complete forms:

$$\begin{aligned}
\rho_0 &= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\rho,\\
\kappa_0 &= -\frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\rho + \frac{1}{\mathsf{c}}\,\kappa,\\
\zeta_0 &= \frac{\mathsf{be}-\mathsf{cd}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}\,\rho - \frac{\mathsf{e}}{\mathsf{cf}}\,\kappa + \frac{1}{\mathsf{f}}\,\zeta,\\
\overline{\kappa_0} &= -\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\rho + \frac{1}{\overline{\mathsf{c}}}\,\overline{\kappa},\\
\overline{\zeta_0} &= \frac{\overline{\mathsf{b}}\overline{\mathsf{e}}-\overline{\mathsf{c}}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}^2\overline{\mathsf{f}}}\,\rho - \frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\overline{\kappa} + \frac{1}{\overline{\mathsf{f}}}\,\overline{\zeta}.
\end{aligned} \tag{5}$$

3. Absorption of torsion and normalization: first loop

3.1. **Lifted structure equations.** We apply the Cartan's method as explained in [24]. The first step is to compute the structure equations for the lifted coframe. With the matrix notations

$$\omega_0 := \begin{pmatrix}\rho_0\\ \kappa_0\\ \zeta_0\\ \overline{\kappa}_0\\ \overline{\zeta}_0\end{pmatrix}, \qquad \omega := \begin{pmatrix}\rho\\ \kappa\\ \zeta\\ \overline{\kappa}\\ \overline{\zeta}\end{pmatrix},$$

we have

$$\omega = g\cdot\omega_0.$$

As a result, the structure equations for the lifted coframe are related to those of the base coframe by the relation:

$$d\omega = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0. \tag{6}$$

The term $dg \cdot g^{-1} \wedge \omega$ depends only on the structure equations of G_1 and is expressed through its Maurer-Cartan forms. The term $g \cdot d\omega_0$ contains the so-called torsion coefficients of the G_1-structure. It is computed easily in terms of the forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, by applying the linear change (5) in the expression of $d\omega_0$, which is given by the set of equations (4), and a matrix multiplication by g.

We start with the expression of the Maurer-Cartan forms of G_1. They are given by the linear independant entries of the matrix $dg \cdot g^{-1}$. An easy computation gives:

$$dg \cdot g^{-1} = \begin{pmatrix} \alpha^1 + \overline{\alpha^1} & 0 & 0 & 0 & 0 \\ \alpha^2 & \alpha^1 & 0 & 0 & 0 \\ \alpha^3 & \alpha^4 & \alpha^5 & 0 & 0 \\ \overline{\alpha^2} & 0 & 0 & \overline{\alpha^1} & 0 \\ \overline{\alpha^3} & 0 & 0 & \overline{\alpha^4} & \overline{\alpha^5} \end{pmatrix},$$

where

$$\begin{aligned} \alpha^1 &:= \frac{d\mathsf{c}}{\mathsf{c}}, \\ \alpha^2 &:= \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\,d\mathsf{c}}{\mathsf{c}^2}\overline{\mathsf{c}}, \\ \alpha^3 &:= \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\,d\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{(-\mathsf{dc} + \mathsf{eb})\,d\mathsf{f}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}, \\ \alpha^4 &:= \frac{d\mathsf{e}}{\mathsf{c}} - \frac{\mathsf{e}\,d\mathsf{f}}{\mathsf{cf}}, \\ \alpha^5 &:= \frac{d\mathsf{f}}{\mathsf{f}}. \end{aligned}$$

The next step is to express the structure equations of the lifted coframe from equation (6) as explained above. Rather lenghty but straigtforward computations give:

$$\begin{aligned} d\rho = {} & \alpha^1 \wedge \rho + \overline{\alpha^1} \wedge \rho \\ & + T^{\rho}_{\rho\kappa}\, \rho \wedge \kappa + T^{\rho}_{\rho\zeta}\, \rho \wedge \zeta + T^{\rho}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + T^{\rho}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\ d\kappa = {} & \alpha^1 \wedge \kappa + \alpha^2 \wedge \rho \\ & + T^{\kappa}_{\rho\kappa}\, \rho \wedge \kappa + T^{\kappa}_{\rho\zeta}\, \rho \wedge \zeta + T^{\kappa}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\ & + T^{\kappa}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + T^{\kappa}_{\kappa\zeta}\, \kappa \wedge \zeta + T^{\kappa}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + T^{\kappa}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa}, \end{aligned}$$

$$\begin{aligned} d\zeta = \alpha^3 \wedge \rho + \alpha^4 \wedge \kappa + \alpha^5 \wedge \zeta \\ + T^{\zeta}_{\rho\kappa}\, \rho \wedge \kappa + T^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta + T^{\zeta}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\ + T^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + T^{\zeta}_{\kappa\zeta}\, \kappa \wedge \zeta + T^{\zeta}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + T^{\zeta}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa}, \end{aligned}$$

where the expressions of the torsion coefficients $T^{\bullet}_{\bullet\bullet}$ are given by the following equations:

$$T^{\rho}_{\rho\kappa} = i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}}{\mathsf{c}\mathsf{f}}\,\mathscr{L}_1(k) + \frac{P}{\mathsf{c}},$$

$$T^{\rho}_{\rho\zeta} = -\frac{\mathscr{L}_1(k)}{\mathsf{f}},$$

$$T^{\rho}_{\rho\overline{\kappa}} = -i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\overline{\mathscr{L}_1}(\overline{k}) + \frac{\overline{P}}{\overline{\mathsf{c}}},$$

$$T^{\rho}_{\rho\overline{\zeta}} = -\frac{\overline{\mathscr{L}_1}(\overline{k})}{\overline{\mathsf{f}}},$$

$$T^{\kappa}_{\rho\kappa} = -\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{T}(k) - \frac{\overline{\mathsf{b}}\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) - \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{L}_1(k) + i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{L}_1(k) + \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\,P,$$

$$T^{\kappa}_{\rho\zeta} = \frac{\overline{\mathsf{b}}}{\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) - \frac{1}{\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{T}(k),$$

$$T^{\kappa}_{\rho\overline{\kappa}} = -\frac{\mathsf{d}}{\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) - i\,\frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}^2\overline{\mathsf{f}}}\,\overline{\mathscr{L}_1}(\overline{k}) + \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\overline{P},$$

$$T^{\kappa}_{\rho\overline{\zeta}} = -\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\overline{\mathscr{L}_1}(\overline{k})$$

$$T^{\kappa}_{\kappa\zeta} = -\frac{\mathscr{L}_1(k)}{\mathsf{f}},$$

$$T^{\kappa}_{\kappa\overline{\kappa}} = -\frac{\mathsf{e}}{\mathsf{c}\mathsf{f}}\,\overline{\mathscr{L}_1}(k) + i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$T^{\kappa}_{\zeta\overline{\kappa}} = \frac{\mathsf{c}}{\overline{\mathsf{c}}\mathsf{f}}\,\overline{\mathscr{L}_1}(k),$$

$$T^{\zeta}_{\rho\kappa} = -\frac{\mathsf{e}^2}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{T}(k) - \frac{\mathsf{b}\mathsf{e}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) + i\,\frac{\overline{\mathsf{b}}\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,P,$$

$$T^{\zeta}_{\rho\zeta} = -\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{T}(k) + \frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{L}_1(k) - \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\mathscr{L}_1(k),$$

$$T^{\zeta}_{\rho\overline{\kappa}} = -\frac{\mathsf{de}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) + \frac{\mathsf{be}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2\mathsf{f}}\,\overline{\mathscr{L}_1}(k) - i\,\frac{\mathsf{bd}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{d}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}^2\overline{\mathsf{f}}}\,\overline{\mathscr{L}_1}(\overline{k}) + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\overline{P},$$

$$T^{\zeta}_{\rho\overline{\zeta}} = -\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\overline{\mathscr{L}_1}(\overline{k}),$$

$$T^{\zeta}_{\kappa\zeta} = -\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\mathscr{L}_1(k),$$

$$T^{\zeta}_{\kappa\overline{\kappa}} = -\frac{\mathsf{e}^2}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\overline{\mathscr{L}_1}(k) + i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$T^{\zeta}_{\zeta\overline{\kappa}} = \frac{\mathsf{e}}{\overline{\mathsf{c}}\mathsf{f}}\,\overline{\mathscr{L}_1}(k).$$

3.2. **Normalization of the group parameter f.** We now proceed with the absorption step of Cartan's method. We introduce the modified Maurer-Cartan forms $\tilde{\alpha}^i$, which are a related to the 1-forms α^i by the relations:

$$\tilde{\alpha}^i := \alpha^i - x^i_{\rho}\,\rho - x^i_{\kappa}\,\kappa - x^i_{\zeta}\,\zeta \,-\, x^i_{\overline{\kappa}}\,\overline{\kappa} \,-\, x^i_{\overline{\zeta}}\,\overline{\zeta},$$

where x^1, x^2, x^3, x^4 and x^5 are arbitrary complex-valued functions. The previously written structure equations take the new form:

$$\begin{aligned}
d\rho = {} & \tilde{\alpha}^1 \wedge \rho + \overline{\tilde{\alpha}^1} \wedge \rho \\
& + \left(T^{\rho}_{\rho\kappa} - x^1_{\kappa} - x^1_{\overline{\kappa}}\right)\rho\wedge\kappa \,+\, \left(T^{\rho}_{\rho\zeta} - x^1_{\kappa} - \overline{x^1_{\overline{\zeta}}}\right)\rho\wedge\zeta \\
& + \left(T^{\rho}_{\rho\overline{\kappa}} - x^1_{\overline{\kappa}} - \overline{x^1_{\kappa}}\right)\rho\wedge\overline{\kappa} \,+\, \left(T^{\rho}_{\rho\overline{\zeta}} - x^1_{\zeta} - x^1_{\overline{\zeta}}\right)\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa}, \\
d\kappa = {} & \tilde{\alpha}^1 \wedge \kappa + \tilde{\alpha}^2 \wedge \rho \\
& + \left(T^{\kappa}_{\rho\kappa} - x^2_{\kappa} + x^1_{\rho}\right)\rho\wedge\kappa \,+\, \left(T^{\kappa}_{\rho\zeta} - x^2_{\kappa}\right)\,\rho\wedge\zeta \\
& + \left(T^{\kappa}_{\rho\overline{\kappa}} - x^2_{\overline{\kappa}}\right)\rho\wedge\overline{\kappa} + \left(T^{\kappa}_{\rho\overline{\zeta}} - x^2_{\overline{\zeta}}\right)\rho\wedge\overline{\zeta} \,+\, \left(T^{\kappa}_{\kappa\zeta} + x^1_{\zeta}\right)\,\kappa\wedge\zeta \\
& + \left(T^{\kappa}_{\kappa\overline{\kappa}} - x^1_{\overline{\kappa}}\right)\kappa\wedge\overline{\kappa} \,+\, T^{\kappa}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} \,+\, \left(T^{1}_{\kappa\overline{\zeta}} - x^{1}_{\kappa\overline{\zeta}}\right)\,\kappa\wedge\zeta, \\
d\zeta = {} & \tilde{\alpha}^3 \wedge \rho + \tilde{\alpha}^4 \wedge \kappa + \tilde{\alpha}^5 \wedge \zeta \\
& + \left(T^{\zeta}_{\rho\kappa} - x^3_{\kappa} + x^4_{\rho}\right)\rho\wedge\kappa + \left(T^{\zeta}_{\rho\zeta} - x^3_{\zeta} + x^5_{\rho}\right)\rho\wedge\zeta + \left(T^{\zeta}_{\rho\kappa} - x^3_{\overline{\kappa}}\right)\,\rho\wedge\overline{\kappa} \\
& + \left(T^{\zeta}_{\rho\overline{\zeta}} - x^3_{\overline{\zeta}}\right)\rho\wedge\overline{\zeta} \,+\, \left(T^{\zeta}_{\kappa\overline{\kappa}} - x^4_{\overline{\kappa}}\right)\kappa\wedge\overline{\kappa} + \left(T^{\zeta}_{\zeta\overline{\kappa}} - x^5_{\overline{\kappa}}\right)\zeta\wedge\overline{\kappa} \\
& + \left(x^5_{\kappa} - x^4_{\zeta}\right)\kappa\wedge\zeta - x^4_{\overline{\kappa}}\,\kappa\wedge\overline{\kappa} \,+\, \left(x^5_{\overline{\kappa}} - x^4_{\overline{\zeta}}\right)\overline{\kappa}\wedge\zeta - x^5_{\overline{\zeta}}\,\zeta\wedge\overline{\zeta}.
\end{aligned}$$

We then choose x^1, x^2, x^3, x^4 and x^5 in a way that eliminate as many torsion coefficients as possible. We easily see that the only coefficient which can not be absorbed is the one in front of $\zeta \wedge \overline{\kappa}$ in $d\kappa$, because it does not depend on the x^i's. We choose the normalization

$$T^{\kappa}_{\zeta\overline{\kappa}} = 1,$$

which yields to:

$$\mathsf{f} = \frac{\mathsf{c}}{\overline{\mathsf{c}}} \overline{\mathscr{L}_1}(k).$$

We notice that the absorbed structure equations take the form:

$$\begin{aligned} d\rho &= \tilde{\alpha}^1 \wedge \rho + \overline{\tilde{\alpha}^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}, \\ d\kappa &= \tilde{\alpha}^1 \wedge \kappa + \tilde{\alpha}^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\ d\zeta &= \tilde{\alpha}^3 \wedge \rho + \tilde{\alpha}^4 \wedge \kappa + \tilde{\alpha}^5 \wedge \zeta. \end{aligned}$$

As a preliminary step towards the second loop of the algorithm, we return to the expression of the lifted coframe. The normalization of f gives the new relation:

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \end{pmatrix} = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}}\overline{\mathscr{L}_1}(k) & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}}\mathscr{L}_1(k) \end{pmatrix} \cdot \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa}_0 \\ \overline{\zeta}_0 \end{pmatrix}. \tag{7}$$

Let us interpret this in the framework of G-structures. We introduce the new one-form

$$\hat{\zeta}_0 = \overline{\mathscr{L}_1}(k) \cdot \zeta_0, \tag{8}$$

such that the previous equation rewrites:

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \end{pmatrix} = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix} \cdot \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \hat{\zeta}_0 \\ \overline{\kappa}_0 \\ \overline{\hat{\zeta}}_0 \end{pmatrix}. \tag{9}$$

We thus have reduced the G_1 equivalence problem to a G_2 equivalence problem, where G_2 is the 8 dimensional real matrix Lie group whose elements are of the form

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

The last task that we need to perform before the second loop of the algorithm is to compute the new structures equations enjoyed by the base coframe $(\rho_0, \kappa_0, \hat{\zeta}_0, \kappa_0, \overline{\hat{\zeta}_0})$. We easily get:

$$
\begin{aligned}
d\rho_0 &= P\,\rho_0\wedge\kappa_0 - \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)}\,\rho_0\wedge\hat{\zeta}_0 + \overline{P}\,\rho_0\wedge\overline{\kappa_0} - \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\,\rho_0\wedge\overline{\hat{\zeta}_0} + i\,\kappa_0\wedge\overline{\kappa_0},\\
d\kappa_0 &= -\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)}\,\rho_0\wedge\hat{\zeta}_0 - \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)}\,\kappa_0\wedge\hat{\zeta}_0 + \hat{\zeta}_0\wedge\overline{\kappa_0},\\
d\hat{\zeta}_0 &= \frac{\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\rho_0\wedge\hat{\zeta}_0 + \frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\kappa_0\wedge\hat{\zeta}_0\\
&\quad - \frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\hat{\zeta}_0\wedge\overline{\kappa_0} + \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\,\hat{\zeta}_0\wedge\overline{\hat{\zeta}_0}\,.
\end{aligned}
$$

4. Absorption of torsion and normalization: second loop

4.1. Lifted structure equations. The Maurer forms of the G_2 are given by the independant entries of the matrix $dg\cdot g^{-1}$. A straightforward computation gives

$$
dg\cdot g^{-1} = \begin{pmatrix}
\beta^1+\overline{\beta^1} & 0 & 0 & 0 & 0\\
\beta^2 & \beta^1 & 0 & 0 & 0\\
\beta^3 & \beta^4 & \beta^1-\overline{\beta^1} & 0 & 0\\
\overline{\beta^2} & 0 & 0 & \overline{\beta^1} & 0\\
\overline{\beta^3} & 0 & 0 & \overline{\beta^4} & -\beta^1+\overline{\beta^1}
\end{pmatrix},
$$

where the forms β^1, β^2, β^3 and β^4 are defined by

$$
\begin{aligned}
\beta^1 &:= \frac{d\mathsf{c}}{\mathsf{c}},\\
\beta^2 &:= \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}},\\
\beta^3 &:= \frac{(-\mathsf{dc}+\mathsf{eb})\,d\mathsf{c}}{\mathsf{c}^3\overline{\mathsf{c}}} - \frac{(-\mathsf{dc}+\mathsf{eb})\,d\overline{\mathsf{c}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}},\\
\beta^4 &:= -\frac{\mathsf{e}d\mathsf{c}}{\mathsf{c}^2} + \frac{\mathsf{e}d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\mathsf{c}} + \frac{d\mathsf{e}}{\mathsf{c}}.
\end{aligned}
$$

Using formula (6), we get the structure equations for the lifted coframe $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta})$ from those of the base coframe $(\rho_0, \kappa_0, \hat{\zeta}_0, \overline{\kappa_0}, \overline{\hat{\zeta}_0})$ by a matrix multiplication and a linear change of coordinates, as in the first loop:

$$\begin{aligned}
d\rho &= \beta^1 \wedge \rho + \overline{\beta^1} \wedge \rho \\
&\quad + U^{\rho}_{\rho\kappa}\, \rho \wedge \kappa + U^{\rho}_{\rho\zeta}\, \rho \wedge \zeta + U^{\rho}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + U^{\rho}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \beta^1 \wedge \kappa + \beta^2 \wedge \rho \\
&\quad + U^{\kappa}_{\rho\kappa}\, \rho \wedge \kappa + U^{\kappa}_{\rho\zeta}\, \rho \wedge \zeta + U^{\kappa}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\
&\quad + U^{\kappa}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + U^{\kappa}_{\kappa\zeta}\, \kappa \wedge \zeta + U^{\kappa}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \beta^3 \wedge \rho + \beta^4 \wedge \kappa + \beta^1 \wedge \zeta - \overline{\beta^1} \wedge \zeta \\
&\quad + U^{\zeta}_{\rho\kappa}\, \rho \wedge \kappa + U^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta + U^{\zeta}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + U^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} \\
&\quad + U^{\zeta}_{\kappa\zeta}\, \kappa \wedge \zeta + U^{\zeta}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + U^{\zeta}_{\kappa\overline{\zeta}}\, \kappa \wedge \overline{\zeta} + U^{\zeta}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa} + U^{\zeta}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}.
\end{aligned}$$

The torsion coefficients $U^{\bullet}_{\bullet\bullet}$ are given by:

$$U^{\rho}_{\rho\kappa} = i\, \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} + \frac{P}{\mathsf{c}},$$

$$U^{\rho}_{\rho\zeta} = -\frac{\overline{\mathsf{c}}}{\mathsf{c}}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\rho}_{\rho\overline{\kappa}} = -i \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2}\, \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{\overline{P}}{\overline{\mathsf{c}}},$$

$$U^{\rho}_{\rho\overline{\zeta}} = -\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},$$

$$U^{\kappa}_{\rho\kappa} = -\frac{\mathsf{e}}{\mathsf{c}^2}\, \frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{\mathsf{d}}{\mathsf{c}^2}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} + i\, \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^3}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} + \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\, P,$$

$$U^{\kappa}_{\rho\zeta} = \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{1}{\mathsf{c}}\, \frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\kappa}_{\rho\overline{\kappa}} = -\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - i\, \frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^3}\, \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2}\, \overline{P},$$

$$U^{\kappa}_{\rho\overline{\zeta}} = -\frac{\mathsf{b}}{\overline{\mathsf{c}}^2}\, \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},$$

$$U^{\kappa}_{\kappa\zeta} = -\frac{\overline{\mathsf{c}}}{\mathsf{c}}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\kappa}_{\kappa\overline{\kappa}} = -\frac{\mathsf{e}}{\mathsf{c}} + i\, \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$U^{\zeta}_{\rho\kappa} = \frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{\mathsf{e}\overline{\mathsf{e}}\overline{\mathsf{b}}}{\overline{\mathsf{c}}^3\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}(k)}{\mathscr{L}_1(\overline{k})} + \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)}$$
$$- \frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{T}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} + \frac{\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}^2\overline{\mathsf{b}}}{\overline{\mathsf{c}}\mathsf{c}^3} + i\,\frac{\mathsf{d}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,P,$$

$$U^{\zeta}_{\rho\zeta} = \frac{\overline{\mathsf{d}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} - \frac{\overline{\mathsf{e}}\overline{\mathsf{b}}}{\overline{\mathsf{c}}^3}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}\,(k)}$$
$$+ \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\mathscr{T}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} + \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^3}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{d}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\zeta}_{\rho\overline{\kappa}} = 2\,\frac{\overline{\mathsf{e}}\mathsf{d}}{\overline{\mathsf{c}}^3}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} - \frac{\mathsf{e}\overline{\mathsf{e}}\mathsf{b}}{\overline{\mathsf{c}}^3\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)}$$
$$- \frac{\mathsf{e}\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{\mathsf{e}^2\mathsf{b}}{\overline{\mathsf{c}}\mathsf{c}^3} - i\,\frac{\mathsf{d}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\overline{P},$$

$$U^{\zeta}_{\rho\overline{\zeta}} = -2\,\frac{\mathsf{d}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},$$

$$U^{\zeta}_{\kappa\zeta} = \frac{1}{\mathsf{c}}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1\,(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\zeta}_{\kappa\overline{\kappa}} = \frac{\mathsf{e}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{\mathsf{e}}{\overline{\mathsf{c}}\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}^2}{\mathsf{c}^2} + i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$U^{\zeta}_{\kappa\overline{\zeta}} = -\frac{\mathsf{e}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},$$

$$U^{\zeta}_{\zeta\overline{\kappa}} = -\frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} - \frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\big(\overline{\mathscr{L}_1}(k)\big)}{\overline{\mathscr{L}_1}(k)} + \frac{\mathsf{e}}{\mathsf{c}},$$

$$U^{\zeta}_{\zeta\overline{\zeta}} = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}.$$

4.2. **Normalization of the group parameter** b**.** We can now perform the absorption step. As for the first loop, we introduce the modified Maurer-Cartan forms $\tilde{\beta}^i$ which differ from the β^i by a linear combination of the 1-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, i.e. that is:

$$\tilde{\beta}^i = \beta^i - y^i_\rho\,\rho - y^i_\kappa\,\kappa - y^i_\zeta\,\zeta - y^i_{\overline{\kappa}}\,\overline{\kappa} - y^i_{\overline{\zeta}}\,\overline{\zeta}.$$

The structure equations rewrite:

$$
\begin{aligned}
d\rho = {} & \tilde{\beta}^1 \wedge \rho + \overline{\tilde{\beta}^1} \wedge \rho \\
& + \left(U^{\rho}_{\rho\kappa} - y^1_{\kappa} - \overline{y}^1_{\overline{\kappa}}\right) \rho \wedge \kappa + \left(U^{\rho}_{\rho\zeta} - y^1_{\zeta} - \overline{y}^1_{\overline{\zeta}}\right) \rho \wedge \zeta \\
& + \left(U^{\rho}_{\rho\overline{\kappa}} - y^1_{\overline{\kappa}} - \overline{y}^1_{\kappa}\right) \rho \wedge \overline{\kappa} + \left(U^{\rho}_{\rho\zeta} - y^1_{\overline{\zeta}} - \overline{y}^1_{\zeta}\right) \rho \wedge \zeta + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa = {} & \tilde{\beta}^1 \wedge \kappa + \tilde{\beta}^2 \wedge \rho \\
& + \left(U^{\kappa}_{\rho\kappa} + y^1_{\rho} - y^2_{\kappa}\right) \rho \wedge \kappa + \left(U^{\kappa}_{\rho\zeta} - y^2_{\zeta}\right) \rho \wedge \zeta + \left(U^{\kappa}_{\rho\overline{\kappa}} - y^2_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} \\
& + \left(U^{\kappa}_{\rho\overline{\zeta}} - y^2_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} + \left(U^{\kappa}_{\kappa\zeta} - y^1_{\zeta}\right) \kappa \wedge \zeta \\
& + \left(U^{\kappa}_{\kappa\overline{\kappa}} - y^1_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} - y^1_{\overline{\zeta}}\, \kappa \wedge \overline{\zeta} + \zeta \wedge \overline{\kappa}, \\
d\zeta = {} & \tilde{\beta}^3 \wedge \rho + \tilde{\beta}^4 \wedge \kappa + \tilde{\beta}^1 \wedge \zeta - \overline{\tilde{\beta}^1} \wedge \zeta \\
& + \left(U^{\zeta}_{\rho\kappa} - y^3_{\kappa} + y^4_{\rho}\right) \rho \wedge \kappa + \left(U^{\zeta}_{\rho\zeta} - y^3_{\zeta} + y^1_{\rho} - \overline{y}^1_{\rho}\right) \rho \wedge \zeta \\
& + \left(U^{\zeta}_{\rho\overline{\kappa}} - y^3_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(U^{\zeta}_{\kappa\zeta} - y^4_{\zeta} + y^1_{\kappa} - \overline{y}^1_{\kappa}\right) \kappa \wedge \zeta + \left(U^{\zeta}_{\kappa\overline{\kappa}} - y^4_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} \\
& + \left(U^{\zeta}_{\kappa\overline{\zeta}} - y^4_{\overline{\zeta}}\right) \kappa \wedge \overline{\zeta} + \left(U^{\zeta}_{\zeta\overline{\kappa}} - y^1_{\overline{\kappa}} + \overline{y}^1_{\kappa}\right) \zeta \wedge \overline{\kappa} + \left(U^{\zeta}_{\zeta\overline{\zeta}} - y^1_{\overline{\zeta}} + \overline{y}^1_{\zeta}\right) \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

We get the following absorption equations:

$$
\begin{array}{rrr}
y^1_{\kappa} + \overline{y}^1_{\overline{\kappa}} = U^{\rho}_{\rho\kappa}, & y^1_{\zeta} + \overline{y}^1_{\overline{\zeta}} = U^{\rho}_{\rho\zeta}, & y^1_{\overline{\kappa}} + \overline{y}^1_{\kappa} = U^{\rho}_{\rho\overline{\kappa}}, \\
y^1_{\overline{\zeta}} + \overline{y}^1_{\zeta} = U^{\rho}_{\rho\zeta}, & -y^1_{\rho} + y^2_{\kappa} = U^{\kappa}_{\rho\kappa}, & y^2_{\zeta} = U^{\kappa}_{\rho\zeta}, \\
y^2_{\overline{\kappa}} = U^{\kappa}_{\rho\overline{\kappa}}, & y^2_{\overline{\zeta}} = U^{\kappa}_{\rho\overline{\zeta}}, & y^1_{\zeta} = U^{\kappa}_{\kappa\zeta}, \\
y^1_{\overline{\kappa}} = U^{\kappa}_{\kappa\overline{\kappa}}, & y^1_{\overline{\zeta}} = 0, & y^3_{\kappa} - y^4_{\rho} = U^{\zeta}_{\rho\kappa}, \\
y^3_{\zeta} - y^1_{\rho} + \overline{y}^1_{\rho} = U^{\zeta}_{\rho\zeta}, & y^3_{\overline{\kappa}} = U^{\zeta}_{\rho\overline{\kappa}}, & y^4_{\zeta} - y^1_{\kappa} + \overline{y}^1_{\kappa} = U^{\zeta}_{\kappa\zeta}, \\
y^4_{\overline{\kappa}} = U^{\zeta}_{\kappa\overline{\kappa}}, & y^4_{\overline{\zeta}} = U^{\zeta}_{\kappa\overline{\zeta}}, & y^1_{\overline{\kappa}} - \overline{y}^1_{\kappa} = U^{\zeta}_{\zeta\overline{\kappa}}, \\
y^1_{\overline{\zeta}} - \overline{y}^1_{\zeta} = U^{\zeta}_{\zeta\overline{\zeta}}. & &
\end{array}
$$

Eliminating the $y^{\bullet}_{\bullet}$ among these equations leads to the following relations between the torsion coefficients:

$$
\begin{aligned}
U^{\rho}_{\rho\overline{\kappa}} &= \overline{U^{\rho}_{\rho\kappa}}, \\
U^{\rho}_{\rho\overline{\zeta}} &= \overline{U^{\rho}_{\rho\zeta}}, \\
U^{\rho}_{\rho\zeta} &= U^{\kappa}_{\kappa\zeta}, \\
U^{\zeta}_{\zeta\overline{\zeta}} &= -U^{\rho}_{\rho\overline{\zeta}}, \\
2\, U^{\kappa}_{\kappa\overline{\kappa}} &= U^{\zeta}_{\zeta\overline{\kappa}} + U^{\rho}_{\rho\overline{\kappa}}.
\end{aligned}
$$

We verify easily that the first four equations do not depend on the group coefficients and are already satisfied. However, the last one does depend on the group coefficients. It gives us the normalization of b as it rewrites:

$$\mathsf{b} = -i\,\bar{\mathsf{c}}\mathsf{e} + i\,\frac{\mathsf{c}}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right).$$

We now look at the new relation between the coframe $(\rho_0, \kappa_0, \hat{\zeta}_0, \overline{\kappa_0}, \overline{\hat{\zeta}_0})$ and the lifted coframe $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta})$, when one takes into account the normalization (4.2). Indded we have:

$$\begin{aligned}
\rho &= \mathsf{c}\bar{\mathsf{c}}\,\rho_0\\
\kappa &= -i\,\mathsf{e}\bar{\mathsf{c}}\,\rho_0 + \mathsf{c}\left(\kappa_0 + \frac{i}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right)\rho_0\right)\\
\zeta &= \mathsf{d}\,\rho_0 + \mathsf{e}\,\kappa_0 + \frac{\mathsf{c}}{\bar{\mathsf{c}}}\,\hat{\zeta}_0.
\end{aligned}$$

As in the first loop of the method, we modify the base coframe to get an interpretation of these equations as a G-structure. Let us introduce:

$$\hat{\kappa}_0 := \kappa_0 + \frac{i}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right)\rho_0.$$

The first two equations become

$$\rho = \mathsf{c}\bar{\mathsf{c}}\,\rho_0 \quad \text{and} \quad \kappa = -i\,\mathsf{e}\bar{\mathsf{c}}\,\rho_0 + \mathsf{c}\,\hat{\kappa}_0,$$

while the third one rewrites:

$$\zeta = \left[\mathsf{d} - i\,\frac{\mathsf{e}}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right)\right]\rho_0 + \mathsf{e}\,\hat{\kappa}_0 + \frac{\mathsf{c}}{\bar{\mathsf{c}}}\,\hat{\zeta}_0.$$

Let us introduce the new group parameter $\mathsf{d}' := \mathsf{d} - i\,\frac{\mathsf{e}}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right)$. We note that d' describes $\mathbb{C}$ when d describes $\mathbb{C}$. We have thus reduced the problem to an equivalence of G_3-structure, described by the coframe $(\rho, \hat{\kappa}, \hat{\zeta}, \hat{\kappa}, \overline{\hat{\zeta}})$ and the relations:

$$\begin{pmatrix}\rho\\ \kappa\\ \zeta\\ \overline{\kappa}\\ \overline{\zeta}\end{pmatrix} = \begin{pmatrix}\mathsf{c}\bar{\mathsf{c}} & 0 & 0 & 0 & 0\\ -i\,\mathsf{e}\bar{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0\\ \mathsf{d}' & \mathsf{e} & \frac{\mathsf{c}}{\bar{\mathsf{c}}} & 0 & 0\\ i\,\bar{\mathsf{e}}\mathsf{c} & 0 & 0 & \bar{\mathsf{c}} & 0\\ \overline{\mathsf{d}}' & 0 & 0 & \bar{\mathsf{e}} & \frac{\bar{\mathsf{c}}}{\mathsf{c}}\end{pmatrix} \cdot \begin{pmatrix}\rho_0\\ \hat{\kappa}_0\\ \hat{\zeta}_0\\ \overline{\hat{\kappa}_0}\\ \overline{\hat{\zeta}_0}\end{pmatrix}.$$

To simplify the notations, we simply drop the $'$ and write d instead of d' in the sequel. G_3 is the matrix Lie group whose elements are of the form

$$\mathsf{g} = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

It is a six dimensional real Lie group. We compute its Maurer Cartan forms with the usual formula

$$d\mathsf{g}\cdot\mathsf{g}^{-1} = \begin{pmatrix} \gamma^1+\overline{\gamma}^1 & 0 & 0 & 0 & 0 \\ \gamma^2 & \gamma^1 & 0 & 0 & 0 \\ \gamma^3 & i\,\gamma^2 & \gamma^1-\overline{\gamma}^1 & 0 & 0 \\ \overline{\gamma}^2 & 0 & 0 & \overline{\gamma}^1 & 0 \\ -\gamma^3 & 0 & 0 & -i\,\overline{\gamma}^2 & -\gamma^1+\overline{\gamma}^1 \end{pmatrix}$$

where

$$\gamma^1 := \frac{d\mathsf{c}}{\mathsf{c}},$$

$$\gamma^2 := i\,\mathsf{e}\frac{d\mathsf{c}}{\mathsf{c}^2} - i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{d\mathsf{e}}{\mathsf{c}}$$

and

$$\gamma^3 := \left(\frac{\mathsf{dc} + i\,\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}^2\overline{\mathsf{c}}}\right)\left(\frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}} - \frac{d\mathsf{c}}{\mathsf{c}}\right) + \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{\mathsf{e}d\mathsf{e}}{\mathsf{c}^2}.$$

As a preliminary step before the third loop of absorption and normalization, we compute the structure equations for the coframe $(\rho_0, \hat{\kappa}_0, \hat{\zeta}_0, \overline{\hat{\kappa}_0}, \overline{\hat{\zeta}_0})$. From the formula :

$$\begin{aligned} d\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right) = &\left(-\mathscr{T}\left(\overline{P}\right) - \frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{\mathscr{T}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}\right)\rho_0 \\ &+ \left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} - \mathscr{L}_1\left(\overline{P}\right)\right)\kappa_0 \\ &+ \left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} - \mathscr{K}\left(\overline{P}\right) + \frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}\right)\zeta_0 \\ &+ \left(-\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + \frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{\mathscr{L}_1}\left(\overline{P}\right)\right)\overline{\kappa_0} \\ &+ \left(-\frac{\overline{\mathscr{L}_1}(\overline{k})\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{\mathscr{L}_1}(\overline{k})\overline{P}\right)\overline{\zeta_0}, \end{aligned}$$

we get:

$$
\begin{aligned}
d\rho_0 = {} & \left(\frac{1}{3}\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})} + \frac{2}{3}P\right)\rho_0 \wedge \hat{\kappa}_0 - \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)}\rho_0 \wedge \hat{\zeta}_0 \\
& + \left(\frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\overline{P}\right)\rho_0 \wedge \overline{\hat{\kappa}_0} - \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\rho_0 \wedge \overline{\hat{\zeta}_0} + i\,\hat{\kappa}_0 \wedge \overline{\hat{\kappa}_0},
\end{aligned}
$$

$$
\begin{aligned}
d\hat{\kappa}_0 = {} & \Bigg(\frac{i}{9}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\overline{\mathscr{L}_1}(k)\mathscr{L}_1(\overline{k})} + i\,\frac{2}{9}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}P \\
& - \frac{i}{9}\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})}\overline{P} - i\,\frac{2}{9}P\overline{P} + \frac{i}{3}\mathscr{L}_1\left(\overline{P}\right) - \frac{i}{3}\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} \\
& + \frac{i}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2}\Bigg)\,\rho_0 \wedge \hat{\kappa}_0 \\
& + \Bigg(-\frac{i}{3}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{i}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} \\
& - \frac{i}{3}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} - \frac{i}{3}\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})} - \frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)}\Bigg)\,\rho_0 \wedge \hat{\zeta}_0 \\
& + \Bigg(i\,\frac{4}{9}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + \frac{i}{9}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)} - i\,\frac{2}{9}\,\overline{P}^2 \\
& + i\,\frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}\Bigg)\,\rho_0 \wedge \overline{\hat{\kappa}_0} \\
& - \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)}\hat{\kappa}_0 \wedge \hat{\zeta}_0 + \left(\frac{1}{3}\,\overline{P} - \frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\right)\hat{\kappa}_0 \wedge \overline{\hat{\kappa}_0} + \hat{\zeta}_0 \wedge \overline{\hat{\kappa}_0},
\end{aligned}
$$

and

$$
\begin{aligned}
d\hat{\zeta}_0 = {} & \Bigg(\frac{i}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\overline{\mathscr{L}_1}(k)\mathscr{L}_1(\overline{k})} - \frac{i}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}P \\
& - \frac{i}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{i}{3}\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\overline{P} + \frac{\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\Bigg)\,\rho_0 \wedge \hat{\kappa}_0 \\
& + \frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\hat{\kappa}_0 \wedge \hat{\zeta}_0 - \frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\;\hat{\zeta}_0 \wedge \overline{\hat{\kappa}_0} + \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\,\hat{\zeta}_0 \wedge \overline{\hat{\zeta}_0}.
\end{aligned}
$$

5. Absorption of torsion and normalization: third loop

5.1. **Lifted structure equations.** We are now ready to perform the third loop of Cartan's method. We begin with the structure equations for the lifted coframe. We have:

$$\begin{aligned}
d\rho &= \gamma^1 \wedge \rho + \overline{\gamma^1} \wedge \rho \\
&\quad + V^{\rho}_{\rho\kappa}\, \rho \wedge \kappa + V^{\rho}_{\rho\zeta}\, \rho \wedge \zeta + V^{\rho}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + V^{\rho}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \gamma^1 \wedge \kappa + \gamma^2 \wedge \rho \\
&\quad + V^{\kappa}_{\rho\kappa}\, \rho \wedge \kappa + V^{\kappa}_{\rho\zeta}\, \rho \wedge \zeta + V^{\kappa}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\
&\quad + V^{\kappa}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + V^{\kappa}_{\kappa\zeta}\, \kappa \wedge \zeta + V^{\kappa}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \gamma^3 \wedge \rho + i\, \gamma^2 \wedge \kappa + \gamma^1 \wedge \zeta - \overline{\gamma^1} \wedge \zeta \\
&\quad + V^{\zeta}_{\rho\kappa}\, \rho \wedge \kappa + V^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta + V^{\zeta}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + V^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} \\
&\quad + V^{\zeta}_{\kappa\zeta}\, \kappa \wedge \zeta + V^{\zeta}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + V^{\zeta}_{\kappa\overline{\zeta}}\, \kappa \wedge \overline{\zeta} + V^{\zeta}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa} + V^{\zeta}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta},
\end{aligned}$$

where

$$V^{\rho}_{\rho\kappa} = -\frac{\overline{\mathsf{e}}}{\mathsf{c}} + \frac{1}{3\mathsf{c}}\, \frac{\mathscr{L}_1\left(\mathscr{L}_1\left(\overline{k}\right)\right)}{\mathscr{L}_1\left(\overline{k}\right)} + \frac{2}{3}\, \frac{P}{\mathsf{c}} + \frac{\mathsf{e}\,\overline{\mathsf{c}}}{\mathsf{c}^2}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},$$

$$V^{\rho}_{\rho\zeta} = -\frac{\overline{\mathsf{c}}}{\mathsf{c}}\, \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}\,(k)},$$

$$V^{\rho}_{\rho\overline{\kappa}} = -\frac{\mathsf{e}}{\mathsf{c}} + \frac{1}{3\overline{\mathsf{c}}}\, \frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\, \frac{\overline{P}}{\overline{\mathsf{c}}} + \frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2}\, \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},$$

$$V^{\rho}_{\rho\overline{\zeta}} = -\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},$$

$$
\begin{aligned}
V^{\kappa}_{\rho\kappa} = {} & \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} - \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} \\
& - \frac{i}{3}\,\frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{9}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)P}{\overline{\mathscr{L}_1}(k)} - \frac{2i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,P \\
& + \frac{i}{3}\,\frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2}\,\overline{P} + \frac{1}{3}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,\mathscr{L}_1(\overline{P}) - \frac{2}{9}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,P\overline{P} - i\,\frac{\bar{\mathsf{c}}\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} \\
& + \frac{1}{9}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)}{\overline{\mathscr{L}_1}(k)\,\mathscr{L}_1(\bar{k})} - \frac{1}{9}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)\overline{P}}{\mathscr{L}_1(\bar{k})} \\
& + \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{d}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} \\
& + \frac{1}{3}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} - \frac{1}{3}\,\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}
\end{aligned}
$$

$$
\begin{aligned}
V^{\kappa}_{\rho\zeta} = {} & \frac{i}{3\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\left(\overline{\mathscr{L}_1}(k)\right)^3} - \frac{i}{3\mathsf{c}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\
& - \frac{i}{3\mathsf{c}}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\left(\overline{\mathscr{L}_1}(k)\right)^2} - \frac{1}{3\mathsf{c}}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} + i\,\frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}} - \frac{i}{3\mathsf{c}}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)}{\mathscr{L}_1(\bar{k})},
\end{aligned}
$$

$$
\begin{aligned}
V^{\kappa}_{\rho\bar{\kappa}} = {} & -\frac{2i}{3}\,\frac{\mathsf{e}}{\bar{\mathsf{c}}\,\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{4i}{9}\,\frac{\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)^2}{\bar{\mathsf{c}}^2\left(\overline{\mathscr{L}_1}(k)\right)^2} \\
& + \frac{i}{9\bar{\mathsf{c}}^2}\,\frac{\overline{P}\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{i}{3}\,\frac{\overline{P}\mathsf{e}}{\bar{\mathsf{c}}\mathsf{c}} - \frac{2i}{9}\,\frac{\overline{P}^2}{\bar{\mathsf{c}}^2} + \frac{i}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{P}\right)}{\bar{\mathsf{c}}^2} \\
& - \frac{i}{3\bar{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{d}}{\mathsf{c}\bar{\mathsf{c}}} - i\,\frac{\mathsf{e}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\bar{k}\right)}{\mathscr{L}_1(\bar{k})},
\end{aligned}
$$

$$
V^{\kappa}_{\rho\bar{\zeta}} = i\,\frac{\mathsf{e}}{\bar{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}(\bar{k})}{\mathscr{L}_1(\bar{k})},
$$

$$
V^{\kappa}_{\kappa\zeta} = -\frac{\bar{\mathsf{c}}}{\mathsf{c}}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},
$$

$$
V^{\kappa}_{\kappa\bar{\kappa}} = -\frac{1}{3\bar{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\overline{P}}{\bar{\mathsf{c}}},
$$

$$
\begin{aligned}
V^{\zeta}_{\rho\kappa} = {} & \frac{2i}{3}\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{i}{3}\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\,\overline{\mathsf{c}}^2}\,\overline{P} - \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\overline{P}\mathscr{L}_1\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} \\
& + \frac{i}{3}\,\frac{\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\
& - \frac{\mathsf{e}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{2i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} \\
& - \frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\overline{\mathscr{L}_1}(\overline{P}) + \frac{5i}{9}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\frac{P\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\
& - \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} \\
& + \frac{i}{3}\,\frac{\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})} - \frac{\mathsf{d}\,\overline{\mathsf{e}}}{\mathsf{c}\,\overline{\mathsf{c}}^2} + \frac{2}{3}\,\frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,P - \frac{i}{9}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\overline{P}\mathscr{L}_1\left(\mathscr{L}_1\left(\overline{k}\right)\right)}{\mathscr{L}_1\left(\overline{k}\right)} \\
& - \frac{i}{3}\,\frac{\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\left(\overline{\mathscr{L}_1}\,(k)\right)^3} + \frac{i}{3}\,\frac{\mathsf{e}^2}{\mathsf{c}^3}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\left(\overline{\mathscr{L}_1}(k)\right)^2} \\
& - \frac{2i}{9}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\overline{P}P - \frac{2i}{9}\,\frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\overline{\mathscr{L}_1}(k)\mathscr{L}_1(\overline{k})} \\
& + i\,\frac{\mathsf{e}\overline{\mathsf{e}}^2}{\overline{\mathsf{c}}^3}\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1(\overline{k})} - i\,\frac{\mathsf{e}^2\overline{\mathsf{e}}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{1}{3}\,\frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})},
\end{aligned}
$$

$$
\begin{aligned}
V^{\zeta}_{\rho\zeta} = {} & -\frac{1}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} - \frac{\overline{\mathsf{d}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{i}{3}\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{P}\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\
& + i\,\frac{\mathsf{e}\,\overline{\mathsf{e}}}{\mathsf{c}\,\overline{\mathsf{c}}} - i\,\frac{\overline{\mathsf{c}}\mathsf{e}^2}{\mathsf{c}^3}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} - \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} \\
& - \frac{i}{3}\,\frac{1}{\mathsf{c}\,\overline{\mathsf{c}}}\,\frac{P\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - i\,\frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\frac{\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\
& - \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1\left(\overline{k}\right)} - \frac{i}{3}\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\,(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} \\
& + \frac{i}{3}\,\frac{1}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\overline{\mathscr{L}_1}(k)\mathscr{L}_1(\overline{k})} + \frac{i}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} \\
& + i\,\frac{2}{3}\,\frac{\mathsf{e}}{\mathsf{c}^2}\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - i\,\frac{\mathsf{c}\overline{\mathsf{e}}^2}{\overline{\mathsf{c}}^3}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1\left(\overline{k}\right)} - \frac{\mathsf{d}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},
\end{aligned}
$$

$$
\begin{aligned}
V^{\zeta}_{\rho\overline{\kappa}} = {} & 2\,\frac{\mathsf{d}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1(\overline{k})} + i\,\frac{\overline{\mathsf{e}}\mathsf{e}^2}{\overline{\mathsf{c}}^2\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{4}{3}\,\frac{\mathsf{d}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\
& + \frac{2i}{3}\,\frac{\mathsf{e}^2}{\overline{\mathsf{c}}\,\mathsf{c}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{4i}{9}\,\frac{\mathsf{e}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\frac{\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)^2}{\overline{\mathscr{L}_1}(k)^2} \\
& + \frac{i}{9}\,\frac{\mathsf{e}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\frac{\overline{P}\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} + \frac{i}{3}\,\frac{\mathsf{e}^2}{\overline{\mathsf{c}}\mathsf{c}^2}\,\overline{P} - \frac{2i}{9}\,\frac{\mathsf{e}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\overline{P}^2 + \frac{i}{3}\,\frac{\mathsf{e}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) \\
& - \frac{i}{3}\,\frac{\mathsf{e}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} - 2\,\frac{\mathsf{e}\mathsf{d}}{\overline{\mathsf{c}}\mathsf{c}^2} - i\,\frac{\mathsf{e}^3}{\mathsf{c}^3} + \frac{2}{3}\,\frac{\mathsf{d}}{\overline{\mathsf{c}}^2\mathsf{c}}\,\overline{P},
\end{aligned}
$$

$$
V^{\zeta}_{\rho\overline{\zeta}} = -2\,\frac{\mathsf{d}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1\left(\overline{k}\right)} - i\,\frac{\mathsf{e}^2}{\mathsf{c}\,\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},
$$

$$
V^{\zeta}_{\kappa\zeta} = \frac{1}{\mathsf{c}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathsf{e}\,\overline{\mathsf{c}}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1\,(k)}{\overline{\mathscr{L}_1}(k)},
$$

$$
V^{\zeta}_{\kappa\overline{\kappa}} = \frac{\mathsf{e}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} + \frac{2}{3}\,\frac{\mathsf{e}\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{P}}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\mathsf{e}}{\mathsf{c}\,\overline{\mathsf{c}}}\,\overline{P} - \frac{\mathsf{e}^2}{\mathsf{c}^2} + i\,\frac{\mathsf{d}}{\mathsf{c}\,\overline{\mathsf{c}}},
$$

$$
V^{\zeta}_{\kappa\overline{\zeta}} = -\frac{\mathsf{e}}{\overline{\mathsf{c}}}\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})},
$$

$$
V^{\zeta}_{\zeta\overline{\kappa}} = -\frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} - \frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{\mathsf{e}}{\mathsf{c}},
$$

$$
V^{\zeta}_{\zeta\overline{\zeta}} = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}.
$$

5.2. **Normalization of the group parameter** d**.** As for the previous steps, we now start the absorption step. We introduce:

$$
\tilde{\gamma}^i := \gamma^i - z^i_\rho\,\rho - z^i_\kappa\,\kappa - z^i_\zeta\,\zeta - z^i_{\overline{\kappa}}\,\overline{\kappa} - z^i_{\overline{\zeta}}\,\overline{\zeta}.
$$

The structure equations are modified accordingly:

$$\begin{aligned}
d\rho &= \tilde{\gamma}^1 \wedge \rho + \overline{\tilde{\gamma}^1} \wedge \rho \\
&\quad + \left(V^\rho_{\rho\kappa} - z^1_\kappa - \overline{z^1_{\overline{\kappa}}}\right) \rho \wedge \kappa + \left(V^\rho_{\rho\zeta} - z^1_\zeta - \overline{z^1_{\overline{\zeta}}}\right) \rho \wedge \zeta \\
&\quad + \left(V^\rho_{\rho\overline{\kappa}} - z^1_{\overline{\kappa}} - \overline{z^1_\kappa}\right) \rho \wedge \overline{\kappa} + \left(V^\rho_{\rho\overline{\zeta}} - \overline{z^1_\zeta} - z^1_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \tilde{\gamma}^1 \wedge \kappa + \tilde{\gamma}^2 \wedge \rho \\
&\quad + \left(V^\kappa_{\rho\kappa} - z^2_\kappa + z^1_\rho\right) \rho \wedge \kappa + \left(V^\kappa_{\rho\zeta} - z^2_\zeta\right) \rho \wedge \zeta \\
&\quad + \left(V^\kappa_{\rho\overline{\kappa}} - z^2_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(V^\kappa_{\rho\overline{\zeta}} - z^2_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} + \left(V^\kappa_{\kappa\zeta} - z^1_\zeta\right) \kappa \wedge \zeta \\
&\quad + \left(V^\kappa_{\kappa\overline{\kappa}} - z^1_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa} - z^1_{\overline{\zeta}}\, \kappa \wedge \overline{\zeta}\,,
\end{aligned}$$

and

$$\begin{aligned}
d\zeta &= \tilde{\gamma}^3 \wedge \rho + i\, \tilde{\gamma}^2 \wedge \kappa + \tilde{\gamma}^1 \wedge \zeta - \overline{\tilde{\gamma}^1} \wedge \zeta \\
&\quad + \left(V^\zeta_{\rho\kappa} - z^3_\kappa + i\, z^2_\rho\right) \rho \wedge \kappa + \left(V^\zeta_{\rho\zeta} + z^1_\rho - z^3_\zeta - \overline{z^1_\rho}\right) \rho \wedge \zeta \\
&\quad + \left(V^\zeta_{\rho\overline{\kappa}} - z^3_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(V^\zeta_{\rho\overline{\zeta}} - z^3_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} + \left(V^\zeta_{\kappa\zeta} - i\, z^2_\zeta + z^1_\kappa - \overline{z^1_{\overline{\kappa}}}\right) \kappa \wedge \zeta \\
&\quad + \left(V^\zeta_{\kappa\overline{\kappa}} - i\, z^2_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} + \left(V^\zeta_{\kappa\overline{\zeta}} - i\, z^2_{\overline{\zeta}}\right) \kappa \wedge \overline{\zeta} \\
&\quad + \left(V^\zeta_{\zeta\overline{\kappa}} - z^1_{\overline{\kappa}} + \overline{z^1_\kappa}\right) \zeta \wedge \overline{\zeta} + \left(V^\zeta_{\zeta\overline{\zeta}} - z^1_{\overline{\zeta}} + \overline{z^1_{\overline{\zeta}}}\right) \zeta \wedge \overline{\zeta}\,.
\end{aligned}$$

We thus want to solve the system of linear equations:

$$\begin{aligned}
z^1_\kappa + \overline{z^1_{\overline{\kappa}}} &= V^\rho_{\rho\kappa}, & z^1_{\overline{\kappa}} + \overline{z^1_\kappa} &= V^\rho_{\rho\overline{\kappa}}, & z^1_\zeta + \overline{z^1_{\overline{\zeta}}} &= V^\rho_{\rho\zeta}, \\
\overline{z^1_\zeta} + z^1_{\overline{\zeta}} &= V^\rho_{\rho\overline{\zeta}}, & z^2_\kappa - z^1_\rho &= V^\kappa_{\rho\zeta}, & z^2_{\overline{\kappa}} &= V^\kappa_{\rho\overline{\kappa}}, \\
z^2_\zeta &= V^\kappa_{\rho\zeta}, & z^2_{\overline{\zeta}} &= V^\kappa_{\rho\overline{\zeta}}, & z^1_\zeta &= V^\kappa_{\kappa\zeta}, \\
z^1_{\overline{\zeta}} &= 0, & z^1_{\overline{\kappa}} &= V^\kappa_{\kappa\overline{\kappa}}, & z^3_\kappa - i\, z^2_\rho &= V^\zeta_{\rho\kappa}, \\
-z^1_\rho + \overline{z^1_\rho} + z^3_\zeta &= V^\zeta_{\rho\zeta}, & z^1_\kappa - \overline{z^1_{\overline{\kappa}}} - i\, z^2_\zeta &= -V^\zeta_{\kappa\zeta}, & i\, z^2_{\overline{\kappa}} &= V^\zeta_{\kappa\overline{\kappa}}, \\
z^3_{\overline{\kappa}} &= V^\zeta_{\rho\overline{\kappa}}, & z^3_{\overline{\zeta}} &= V^\zeta_{\rho\overline{\zeta}}, & i\, z^2_{\overline{\zeta}} &= V^\zeta_{\kappa\overline{\zeta}}, \\
z^1_{\overline{\kappa}} - \overline{z^1_\kappa} &= V^\zeta_{\zeta\overline{\kappa}}, & z^1_{\overline{\zeta}} - \overline{z^1_\zeta} &= V^\zeta_{\zeta\overline{\zeta}}.
\end{aligned}$$

This is easily done as:

$$\begin{cases} z^1_\kappa = \overline{V^\rho_{\rho\bar\kappa}} - \overline{V^\kappa_{\kappa\bar\kappa}}, \\ z^1_{\bar\kappa} = V^\kappa_{\kappa\bar\kappa}, \\ z^1_\zeta = V^\rho_{\rho\zeta}, \\ z^1_{\bar\zeta} = 0, \\ z^2_{\bar\kappa} = V^\kappa_{\rho\bar\kappa}, \\ z^2_{\bar\zeta} = V^\kappa_{\rho\bar\zeta}, \\ z^2_\zeta = V^\kappa_{\rho\zeta}, \\ z^3_{\bar\kappa} = V^\zeta_{\rho\bar\kappa}, \\ z^3_{\bar\zeta} = V^\zeta_{\rho\bar\zeta}, \\ z^3_\zeta = V^\zeta_{\rho\zeta} + z^1_\rho - z^1_\rho, \\ z^3_\kappa = V^\zeta_{\rho\kappa} + i\, z^2_\rho, \\ z^2_\kappa = V^\kappa_{\rho\zeta} + z^1_\rho, \end{cases}$$

where z^1_ρ and z^2_ρ may be choosen freely. Eliminating the $z^\bullet_\bullet$ we get the following additional conditions on the $V^\bullet_{\bullet\bullet}$:

$$\text{(10)} \qquad \begin{cases} V^\rho_{\rho\bar\kappa} = \overline{V^\rho_{\rho\kappa}}, \\ V^\rho_{\rho\bar\zeta} = \overline{V^\rho_{\rho\zeta}}, \\ V^\rho_{\rho\zeta} = V^\kappa_{\kappa\zeta}, \\ i\, V^\kappa_{\rho\bar\zeta} = V^\zeta_{\kappa\bar\zeta}, \\ V^\rho_{\rho\zeta} = -\overline{V^\zeta_{\zeta\bar\zeta}}, \\ 2\, V^\kappa_{\kappa\bar\kappa} = V^\rho_{\rho\bar\kappa} + V^\zeta_{\zeta\bar\kappa}. \end{cases}$$

and

$$\text{(11)} \qquad \begin{cases} i\, V^\kappa_{\rho\bar\kappa} = V^\zeta_{\kappa\bar\kappa}, \\ V^{\bar\zeta}_{\kappa\bar\zeta} + V^\zeta_{\kappa\zeta} = i\, V^\kappa_{\rho\zeta}. \end{cases}$$

We easily verify that the equations (10) are indeed satisfied. However the remaining two equations are not and they provide two essential torsion coefficients, namely $i\, V^\kappa_{\rho\bar\kappa} - V^\zeta_{\kappa\bar\kappa}$ and $V^{\bar\zeta}_{\kappa\bar\zeta} + V^\zeta_{\kappa\zeta} - i\, V^\kappa_{\rho\zeta}$, which will give us at least one new normalization of the group coefficients. Indeed we have

$$i\,V^{\kappa}_{\rho\overline{\kappa}} - V^{\zeta}_{\kappa\overline{\kappa}} = -\frac{4}{9}\,\frac{1}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} - \frac{1}{9}\,\frac{1}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)} + \frac{2}{9}\,\frac{\overline{P}^2}{\overline{\mathsf{c}}^2}$$
$$- \frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{P}\right)}{\overline{\mathsf{c}}^2} + \frac{1}{3}\,\frac{1}{\overline{\mathsf{c}}^2}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} - 2i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}^2}{\mathsf{c}^2}.$$

Setting this expression to 0, we get the normalization of the parameter d:

$$\mathsf{d} = -i\,\frac{1}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} + i\,\frac{2}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + i\,\frac{1}{18}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)}$$
$$- i\,\frac{1}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{P}^2 + i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) - i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}.$$

The other equation gives the essential torsion coefficient:

$$\frac{1}{\mathsf{c}}\left(\frac{2}{3}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})} + \frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3}\right.$$
$$\left. - \frac{1}{3}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)}\right).$$

In the sequel we define the functions H and W on M^5 by:

$$H := \frac{2}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + \frac{1}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)}$$
$$- \frac{1}{9}\,\overline{P}^2 + \frac{1}{6}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) - \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}$$

and

$$W := \frac{2}{3}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})}$$
$$+ \frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} - \frac{1}{3}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)}. \tag{12}$$

We do not use the normalization $\mathsf{c} = W$ at the moment, because this is allowed only if W does not vanish. We will deal with this discussion further during the fourth loop of the algorithm. With these notations, we have

$$\mathsf{d} = -\frac{i}{2}\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,H.$$

As a result, the relations between the base coframe $(\rho_0, \hat{\kappa}_0, \hat{\zeta}_0, \overline{\hat{\kappa}_0}, \overline{\hat{\zeta}_0})$ and the lifted coframe $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta})$ take the form:

$$\begin{cases} \rho = \mathsf{c}\overline{\mathsf{c}}\,\rho_0 \\ \kappa = -i\,\mathsf{e}\overline{\mathsf{c}}\,\rho_0 + \overline{\mathsf{c}}\,\hat{\kappa}_0 \\ \zeta = -i\,\frac{1}{2}\,\frac{\mathsf{e}^2}{\overline{\mathsf{c}}\mathsf{c}}\,\rho_0 + \mathsf{e}\,\hat{\kappa}_0 + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\left(\hat{\zeta}_0 + i\,H\,\rho_0\right) \end{cases}$$

Here again we explicitly exhibit the new G-structure by letting

$$\check{\zeta}_0 := \hat{\zeta}_0 + i\,H\,\rho_0.$$

With these notations, we have:

$$\begin{cases} \rho = \mathsf{c}\overline{\mathsf{c}}\,\rho_0 \\ \kappa = -i\,\mathsf{e}\overline{\mathsf{c}}\,\rho_0 + \overline{\mathsf{c}}\,\hat{\kappa}_0 \\ \zeta = -i\,\frac{1}{2}\,\frac{\mathsf{e}^2}{\overline{\mathsf{c}}\mathsf{c}}\,\rho_0 + \mathsf{e}\,\hat{\kappa}_0 + \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\check{\zeta}_0. \end{cases}$$

We have reduced the previous G_3-structure to a G_4-structure, where G_4 is the four dimensional matrix Lie group whose elements are of the form:

$$\begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ -\frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \frac{i}{2}\,\frac{\overline{\mathsf{e}}^2\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}$$

The basis for the Maurer-Cartan forms of G_4 is provided by the four forms

$$\delta^1 := \frac{d\mathsf{c}}{\mathsf{c}} \quad , \quad \delta^2 := i\,\mathsf{e}\frac{d\mathsf{c}}{\mathsf{c}^2} - i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{d\mathsf{e}}{\mathsf{c}} \quad , \quad \overline{\delta^1} \quad , \quad \overline{\delta^2}.$$

6. Absorption of torsion and normalisation: fourth loop

At this stage we could compute the structure equations enjoyed by the base coframe $(\rho_0, \hat{\kappa}_0, \check{\zeta}_0, \overline{\hat{\kappa}_0}, \overline{\check{\zeta}_0})$, but as this involves rather lenghty computations, we procceed slightly differently from here. We just substitute the parameter d by its normalization in the set of structure equations at the third loop. We have to take into account the fact that $d\mathsf{d}$ is modified accordingly. Indeed we have:

$$d\mathsf{d} = -i\mathsf{e}\frac{\overline{\mathsf{c}}}{\mathsf{c}} - \frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}}\left(\frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}} - \frac{d\mathsf{c}}{\mathsf{c}}\right) + i\,H\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\left(\frac{d\mathsf{c}}{\mathsf{c}} - \frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}}\right) + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,dH$$

The forms γ^1 and γ^2 are not modified as they do not involve terms in $d\mathsf{d}$, but this is not the case for γ^3 which is transformed as:

$$\begin{aligned}\gamma^3 &= \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{\mathsf{e}}{\mathsf{c}^2} - \frac{\mathsf{d}\,d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - i\,\mathsf{e}^2\frac{d\mathsf{c}}{\mathsf{c}^3} + \frac{\mathsf{d}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}^2} + i\,\frac{\mathsf{e}^2\,d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\mathsf{c}^2}\\ &= i\,\frac{dH}{\overline{\mathsf{c}}^2}\end{aligned}$$

The expressions of $d\rho$ and $d\kappa$ are thus unchanged from the expressions given by the structure equations at the third step, except the fact that we shall replace d by $-\frac{i}{2}\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,H$ in the expression of each torsion coefficient $V_{\bullet\bullet}^{\bullet}$ and the fact that the forms γ^1 and γ^2 shall be replaced by the forms δ^1 and δ^2, that is:

$$\begin{aligned}d\rho &= \delta^1\wedge\rho + \overline{\delta^1}\wedge\rho\\ &\quad + V^{\rho}_{\rho\kappa}\,\rho\wedge\kappa + V^{\rho}_{\rho\zeta}\,\rho\wedge\zeta + V^{\rho}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + V^{\rho}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa},\end{aligned}$$

and

$$\begin{aligned}d\kappa &= \delta^1\wedge\kappa + \delta^2\wedge\rho\\ &\quad + V^{\kappa}_{\rho\kappa}\,\rho\wedge\kappa + V^{\kappa}_{\rho\zeta}\,\rho\wedge\zeta + V^{\kappa}_{\rho\overline{\kappa}}\;\rho\wedge\overline{\kappa}\\ &\quad + V^{\kappa}_{\rho\overline{\zeta}}\;\rho\wedge\overline{\zeta} + V^{\kappa}_{\kappa\zeta}\,\kappa\wedge\zeta + V^{\kappa}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + \zeta\wedge\overline{\kappa}.\end{aligned}$$

The computation of $d\zeta$ involves the expression of the form γ^3 and is therefore modified as

$$\begin{aligned}d\zeta &= i\,\frac{dH}{\overline{\mathsf{c}}^2}\wedge\rho + i\,\delta_2\wedge\kappa + \delta_1\wedge\zeta - \overline{\delta_1}\wedge\zeta\\ &\quad + V^{\zeta}_{\rho\kappa}\,\rho\wedge\kappa + V^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta + V^{\zeta}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + V^{\zeta}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta}\\ &\quad + V^{\zeta}_{\kappa\zeta}\,\kappa\wedge\zeta + V^{\zeta}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + V^{\zeta}_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + V^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + V^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}.\end{aligned}$$

The term $\frac{dH}{\overline{\mathsf{c}}^2}\wedge\rho$ involves torsion terms in $\rho\wedge\kappa$, $\rho\wedge\zeta$, $\rho\wedge\overline{\kappa}$ and $\rho\wedge\overline{\zeta}$, which only affect the expressions of the coefficients $V^{\zeta}_{\rho\kappa}$, $V^{\zeta}_{\rho\zeta}$, $V^{\zeta}_{\rho\overline{\kappa}}$ and $V^{\zeta}_{\rho\zeta}$. If we write $W^{\zeta}_{\rho\kappa}$, $W^{\zeta}_{\rho\zeta}$, $W^{\zeta}_{\rho\overline{\kappa}}$ and $W^{\zeta}_{\rho\zeta}$ for these modified torsion coefficients, we get

$$\begin{aligned}d\zeta &= i\,\delta_2\wedge\kappa + \delta_1\wedge\zeta - \overline{\delta_1}\wedge\zeta\\ &\quad + W^{\zeta}_{\rho\kappa}\,\rho\wedge\kappa + W^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta + W^{\zeta}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + W^{\zeta}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta}\\ &\quad + V^{\zeta}_{\kappa\zeta}\;\kappa\wedge\zeta + V^{\zeta}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + V^{\zeta}_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + V^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + V^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}.\end{aligned}$$

Before computing the actual value of the coefficients $W_{\bullet\bullet}^{\bullet}$, we proceed with the absorption phase. We make the two substitutions

$$\begin{aligned}\delta^1 &:= \tilde{\delta}^1 + w^1_{\rho}\,\rho + w^1_{\kappa}\,\kappa + w^1_{\zeta}\,\zeta + w^1_{\overline{\kappa}}\,\overline{\kappa} + w^1_{\overline{\zeta}}\,\overline{\zeta},\\ \delta^2 &:= \tilde{\delta}^2 + w^2_{\rho}\,\rho + w^2_{\kappa}\,\kappa + w^2_{\zeta}\,\zeta + w^2_{\overline{\kappa}}\,\overline{\kappa} + w^2_{\overline{\zeta}}\,\overline{\zeta}\end{aligned}$$

in the previous equations. We get:

$$\begin{aligned} d\rho = {} & \tilde{\delta}^1 \wedge \rho + \overline{\tilde{\delta}^1} \wedge \rho \\ & + \left(V^{\rho}_{\rho\kappa} - w^1_{\kappa} - \overline{w^1_{\overline{\kappa}}}\right) \rho \wedge \kappa + \left(V^{\rho}_{\rho\zeta} - w^1_{\zeta} - \overline{w^1_{\overline{\zeta}}}\right) \rho \wedge \zeta \\ & + \left(V_{\rho\overline{\kappa}} - w^1_{\overline{\kappa}} - \overline{w^1_{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(V^{\rho}_{\rho\overline{\zeta}} - \overline{w^1_{\zeta}} - w^1_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} \ , \end{aligned}$$

$$\begin{aligned} d\kappa = {} & \tilde{\delta}^1 \wedge \kappa + \tilde{\delta}^2 \wedge \rho \\ & + \left(V^{\kappa}_{\rho\kappa} - w^2_{\kappa} + w^1_{\rho}\right) \rho \wedge \kappa + \left(V^{\kappa}_{\rho\zeta} - w^2_{\zeta}\right) \rho \wedge \zeta \\ & + \left(V^{\kappa}_{\rho\overline{\kappa}} - w^2_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(V_{\rho\overline{\zeta}} - w^2_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} + \left(V^{\kappa}_{\kappa\zeta} - w^1_{\zeta}\right) \kappa \wedge \zeta \\ & + \left(V^{\kappa}_{\kappa\overline{\kappa}} - w^1_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa} - w^1_{\overline{\zeta}}\, \kappa \wedge \overline{\zeta} \ , \end{aligned}$$

and

$$\begin{aligned} d\zeta = {} & i\,\tilde{\delta}_2 \wedge \kappa + \tilde{\delta}_1 \wedge \zeta - \overline{\tilde{\delta}_1} \wedge \zeta \\ & + \left(W^{\zeta}_{\rho\kappa} + i\,w^2_{\rho}\right) \rho \wedge \kappa + \left(W^{\zeta}_{\rho\zeta} + w^1_{\rho} - \overline{w^1_{\rho}}\right) \rho \wedge \zeta \\ & + W^{\zeta}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + W^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + \left(V^{\zeta}_{\kappa\overline{\kappa}} - i\,w^2_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} \\ & + \left(V^{\zeta}_{\kappa\overline{\zeta}} - i\,w^2_{\overline{\zeta}}\right) \kappa \wedge \overline{\zeta} + \left(V^{\zeta}_{\zeta\overline{\kappa}} - w^1_{\overline{\kappa}} + \overline{w^1_{\kappa}}\right) \zeta \wedge \overline{\zeta} \ . \end{aligned}$$

From the last equation, we immediately see that $W^{\zeta}_{\rho\overline{\kappa}}$ and $W^{\zeta}_{\rho\overline{\zeta}}$ are two new essential torsion coefficients. We find the remaining ones by solving the set of equations:

$$\begin{aligned} & w^1_{\kappa} + \overline{w^1_{\overline{\kappa}}} = V^{\rho}_{\rho\kappa}, && w^1_{\overline{\kappa}} + \overline{w^1_{\kappa}} = V^{\rho}_{\rho\overline{\kappa}}, && w^1_{\zeta} + \overline{w^1_{\overline{\zeta}}} = V^{\rho}_{\rho\zeta}, \\ & \overline{w^1_{\zeta}} + w^1_{\overline{\zeta}} = V^{\rho}_{\rho\overline{\zeta}}, && w^2_{\kappa} - w^1_{\rho} = V^{\kappa}_{\rho\kappa}, && w^2_{\overline{\kappa}} = V^{\kappa}_{\rho\overline{\kappa}}, \\ & w^2_{\zeta} = V^{\kappa}_{\rho\zeta}, && w^2_{\overline{\zeta}} = V^{\kappa}_{\rho\overline{\zeta}}, && w^1_{\zeta} = V^{\kappa}_{\kappa\zeta}, \\ & w^1_{\overline{\zeta}} = 0, && w^1_{\overline{\kappa}} = V^{\kappa}_{\kappa\overline{\kappa}}, && -i\,w^2_{\rho} = V^{\zeta}_{\rho\kappa}, \\ & -w^1_{\rho} + \overline{w^1_{\rho}} = V^{\zeta}_{\rho\zeta}, && w^1_{\kappa} - \overline{w^1_{\overline{\kappa}}} - i\,w^2_{\zeta} = -V^{\zeta}_{\kappa\zeta}, && i\,w^2_{\overline{\kappa}} = V^{\zeta}_{\kappa\overline{\kappa}}, \\ & w^1_{\kappa} - \overline{w^1_{\kappa}} = V^{\zeta}_{\zeta\overline{\kappa}}, && i\,w^2_{\overline{\zeta}} = V^{\zeta}_{\kappa\overline{\zeta}}, && w^1_{\overline{\zeta}} - \overline{w^1_{\zeta}} = V^{\zeta}_{\zeta\overline{\zeta}}, \end{aligned}$$

which lead easily as before to:

$$
(13) \qquad \begin{cases}
w^1_\kappa = \overline{V^\rho_{\rho\overline{\kappa}}}, \\
w^1_{\overline{\kappa}} = V^\kappa_{\kappa\overline{\kappa}}, \\
w^1_\zeta = V^\rho_{\rho\zeta}, \\
w^1_{\overline{\zeta}} = 0, \\
w^2_{\overline{\kappa}} = V^\kappa_{\rho\overline{\kappa}}, \\
w^2_{\overline{\zeta}} = V^\kappa_{\rho\overline{\zeta}}, \\
w^2_\zeta = V^\kappa_{\rho\zeta}, \\
w^2_\kappa = V^\kappa_{\rho\kappa} + w^1_\rho, \\
w^2_\rho = W^\zeta_{\rho\kappa} \\
-w^1_\rho + \overline{w^1_\rho} = W^\zeta_{\rho\zeta}.
\end{cases}
$$

Eliminating the $w^{\bullet}_{\bullet}$ from (13), we get one additionnal condition on the $W^{\bullet}_{\bullet\bullet}$ which has not yet been checked, namely that $W^\zeta_{\rho\zeta}$ shall be purely imaginary. We now need to compute the two essential torsion coefficients $W^\zeta_{\rho\overline{\kappa}}$ and $W^\zeta_{\rho\overline{\zeta}}$. As they both involves the term $dH \wedge \rho$, we start with the computation of this term. Standard differentiation with respect to base coframe $(\rho_0, \kappa_0, \zeta_0, \overline{\kappa_0}, \overline{\zeta_0})$ yields:

$$
dH = \mathscr{T}(H)\,\rho_0 + \mathscr{L}_1(H)\,\kappa_0 + \mathscr{K}(H)\,\zeta_0 + \overline{\mathscr{L}_1}(H)\,\overline{\kappa_0} + \overline{\mathscr{K}}(H)\,\overline{\zeta_0}.
$$

Taking the wedge product with ρ and using the fact that

$$
\kappa_0 \wedge \rho = \hat{\kappa}_0 \wedge \rho
$$

and

$$
\zeta_0 \wedge \rho = \frac{\check{\zeta}_0}{\overline{\mathscr{L}_1}(k)} \wedge \rho,
$$

which is easily seen from the definitions of $\hat{\kappa}_0$ and $\check{\zeta}_0$, we get:

$$
dH \wedge \rho = \left(\mathscr{L}_1(H)\,\hat{\kappa}_0 + \frac{\mathscr{K}(H)}{\overline{\mathscr{L}_1}(k)}\,\check{\zeta}_0 + \overline{\mathscr{L}_1}(H)\,\overline{\hat{\kappa}_0} + \frac{\overline{\mathscr{K}}(H)}{\mathscr{L}_1(\overline{k})}\,\overline{\check{\zeta}_0} \right) \wedge \rho.
$$

We now use the expressions of the 1-forms $\hat{\kappa}_0$ and $\check{\zeta}_0$ in terms of ρ, κ and ζ, which are deduced by the use of (5), that is:

$$
\begin{cases}
\hat{\kappa}_0 = i\,\dfrac{\mathsf{e}}{\mathsf{c}^2}\,\rho + \dfrac{1}{\mathsf{c}}\,\kappa \\
\check{\zeta}_0 = -i\,\dfrac{1}{2}\dfrac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}^3}\,\rho - \dfrac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2}\,\kappa + \dfrac{\overline{\mathsf{c}}}{\mathsf{c}}\,\zeta.
\end{cases}
$$

As a result, we get:

$$dH\wedge\rho=\left(\frac{\mathsf{e}\bar{\mathsf{c}}}{\mathsf{c}^2}\frac{\mathscr{K}(H)}{\overline{\mathscr{L}_1}(k)}-\frac{\mathscr{L}_1(H)}{\mathsf{c}}\right)\rho\wedge\kappa-\frac{\bar{\mathsf{c}}}{\mathsf{c}}\frac{\mathscr{K}(H)}{\overline{\mathscr{L}_1}(k)}\,\rho\wedge\zeta$$
$$+\left(\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2}\frac{\mathscr{K}(H)}{\overline{\mathscr{L}_1}(k)}-\frac{\overline{\mathscr{L}_1}(H)}{\bar{\mathsf{c}}}\right)\rho\wedge\overline{\kappa}-\frac{\mathsf{c}}{\bar{\mathsf{c}}}\frac{\overline{\mathscr{K}}(H)}{\overline{\mathscr{L}_1}(k)}\,\rho\wedge\overline{\zeta}\,.$$

Inserting this equation in the expression of $d\zeta$, we find that:

$$d\zeta=i\,\delta_2\wedge\kappa+\delta_1\wedge\zeta-\overline{\delta_1}\wedge\zeta$$
$$+\left(V^{\zeta}_{\rho\kappa}+\frac{i}{\mathsf{c}^2\bar{\mathsf{c}}}\frac{\mathscr{K}(H)}{\overline{\mathscr{L}_1}(k)}-\frac{i}{\mathsf{c}\bar{\mathsf{c}}^2}\mathscr{L}_1(H)\right)\rho\wedge\kappa$$
$$+\left(V^{\zeta}_{\rho\zeta}-\frac{i}{\mathsf{c}\bar{\mathsf{c}}}\frac{\mathscr{K}(H)}{\overline{\mathscr{L}_1}(k)}\right)\rho\wedge\zeta$$
$$+\left(V^{\zeta}_{\rho\overline{\kappa}}+i\,\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^4}\frac{\overline{\mathscr{K}}(H)}{\mathscr{L}_1(\overline{k})}-\frac{i}{\bar{\mathsf{c}}^3}\overline{\mathscr{L}_1}(H)\right)\rho\wedge\overline{\kappa}$$
$$+\left(V^{\zeta}_{\rho\overline{\zeta}}-i\,\frac{\mathsf{c}}{\bar{\mathsf{c}}^3}\frac{\overline{\mathscr{K}}(H)}{\mathscr{L}_1(\overline{k})}\right)\rho\wedge\overline{\zeta}+V^{\zeta}_{\kappa\zeta}\;\kappa\wedge\zeta$$
$$+V^{\zeta}_{\kappa\overline{\kappa}}\;\kappa\wedge\overline{\kappa}+V^{\zeta}_{\kappa\overline{\zeta}}\;\kappa\wedge\overline{\zeta}+V^{\zeta}_{\zeta\overline{\kappa}}\;\zeta\wedge\overline{\kappa}+V^{\zeta}_{\zeta\overline{\zeta}}\;\zeta\wedge\overline{\zeta}\,.$$

We thus have

(14) $$W^{\zeta}_{\rho\overline{\zeta}}=V^{\zeta}_{\rho\overline{\zeta}}-i\,\frac{\mathsf{c}}{\bar{\mathsf{c}}^3}\frac{\overline{\mathscr{K}}(H)}{\mathscr{L}_1(\overline{k})}$$

and

(15) $$W^{\zeta}_{\rho\overline{\kappa}}=V^{\zeta}_{\rho\overline{\kappa}}+i\,\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^4}\frac{\overline{\mathscr{K}}(H)}{\mathscr{L}_1(\overline{k})}-\frac{i}{\bar{\mathsf{c}}^3}\overline{\mathscr{L}_1}(H).$$

We first compute $W^{\zeta}_{\rho\overline{\zeta}}$. Performing the substitution $\mathsf{d}=-\frac{i}{2}\frac{e^2\bar{\mathsf{c}}}{cc}+i\,\frac{\mathsf{c}}{\bar{\mathsf{c}}}\,H$ in $V^{\zeta}_{\rho\overline{\zeta}}$ gives

(16) $$V^{\zeta}_{\rho\overline{\zeta}}=-2i\,\frac{\mathsf{c}}{\bar{\mathsf{c}}^3}\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\,H.$$

On the other hand, straightforward computations using the commutation relations given by the set of equations (3) lead to:

$$\overline{\mathscr{K}}(H)=-\frac{4}{9}\,\overline{\mathscr{L}_1}(\overline{k})\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2}-\frac{1}{9}\,\overline{\mathscr{L}_1}(\overline{k})\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P}+\frac{2}{9}\,\overline{\mathscr{L}_1}(\overline{k})^2\,\overline{P}^2$$
$$+\frac{1}{3}\,\overline{\mathscr{L}_1}(\overline{k})\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}-\frac{1}{3}\,\overline{\mathscr{L}_1}(\overline{k})\,\overline{\mathscr{L}_1}\left(\overline{P}\right),$$

that is:

$$\overline{\mathscr{K}}(H) = -2\,\overline{\mathscr{L}_1}(\overline{k})\,H.$$

Combining this with (14) and (16) leads to

$$W^{\zeta}_{\rho\bar{\zeta}} = 0,$$

which therefore do not provide any new normalization of the group parameters. We now turn our attention on $W^{\zeta}_{\rho\overline{\kappa}}$. As before, the substitution $\mathsf{d} = -\frac{i}{2}\,\frac{e^2\bar{\mathsf{c}}}{cc} + i\,\frac{\mathsf{c}}{\bar{\mathsf{c}}}\,H$ gives

$$V^{\zeta}_{\rho\overline{\kappa}} = 2i\,\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^4}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\,H + \frac{i}{\bar{\mathsf{c}}^3}\left(\frac{4}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P}\right)H$$
$$+i\,\frac{\mathsf{e}}{\bar{\mathsf{c}}^2\mathsf{c}}\left(-\frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{2}{9}\,\overline{P}^2 + \frac{1}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P} + \frac{4}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + \frac{1}{3}\,\overline{\mathscr{L}_1}(P) - 2\,H\right),$$

that is, taking into account the expression of H,

$$V^{\zeta}_{\rho\overline{\kappa}} = 2i\,\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^4}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}\,H + \frac{i}{\bar{\mathsf{c}}^3}\left(\frac{4}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P}\right)H.$$

Combining this equation with (15), we thus get the value of $W^{\zeta}_{\rho\overline{\kappa}}$:

$$\begin{aligned} W^{\zeta}_{\rho\overline{\kappa}} &= i\,\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^4}\,\frac{1}{\mathscr{L}_1(\overline{k})}\left(2\,\overline{\mathscr{L}_1}(\overline{k})\,H + \overline{\mathscr{K}}(H)\right) + \frac{i}{\bar{\mathsf{c}}^3}\left[\frac{2}{3}\left(2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P}\right)H - \overline{\mathscr{L}_1}(H)\right] \\ &= \frac{i}{\bar{\mathsf{c}}^3}\left[\frac{2}{3}\left(2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P}\right)H - \overline{\mathscr{L}_1}(H)\right], \end{aligned}$$

as the last equality follows from the relation (16). This provide us with a new essential torsion coefficient, leading to a new invariant of the problem. Indeed we define the function J by:

$$\overline{J} := \left[\frac{2}{3}\left(2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P}\right)H - \overline{\mathscr{L}_1}(H)\right].$$

If J does not vanish, one can perform the normalization $\bar{\mathsf{c}}^3 := \overline{J}$. We now give the expression of the invariant J in terms of the functions k, $\overline{P}$ and their coframe derivatives. Straightforward computations lead to

$$\begin{aligned}\overline{\mathscr{L}_1}(H) = &-\frac{4}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^3}{\overline{\mathscr{L}_1}(k)^3} + \frac{11}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} \\ &-\frac{1}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2}\,\overline{P} + \frac{1}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P} \\ &+\frac{1}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\,\overline{\mathscr{L}_1}\left(\overline{P}\right)}{\overline{\mathscr{L}_1}(k)} - \frac{2}{9}\,\overline{P}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) \\ &-\frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{6}\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{P}\right)\right),\end{aligned}$$

which in turn gives the expression of $\overline{J}$:

$$\begin{aligned}\overline{J} = &\frac{5}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2}\,\overline{P} + \frac{1}{3}\,\overline{P}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) - \frac{1}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P}^2 \\ &+\frac{20}{27}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^3}{\overline{\mathscr{L}_1}(k)^3} - \frac{5}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} \\ &+\frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\,\overline{\mathscr{L}_1}(\overline{P})}{\overline{\mathscr{L}_1}(k)} - \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P} \\ &-\frac{2}{27}\,\overline{P}^3 - \frac{1}{6}\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{P}\right)\right) + \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)\right)}{\overline{\mathscr{L}_1}(k)}.\end{aligned}$$

7. CASE $J \neq 0$

We now turn our attention on the case $J \neq 0$. We show here how the last group parameter e can be normalized, reducing thus the G-equivalence problem to the study of an e-structure. From the normalization $c^3 = J$, we get

$$\frac{d\mathsf{c}}{\mathsf{c}} = \frac{1}{3}\,\frac{dJ}{J}.$$

The expression of $d\rho$ is thus modified as:

$$d\rho = \frac{1}{3}\left(\frac{dJ}{J} + \frac{d\overline{J}}{\overline{J}}\right)\wedge\rho + V^{\rho}_{\rho\kappa}\,\rho\wedge\kappa + V^{\rho}_{\rho\zeta}\,\rho\wedge\zeta + V^{\rho}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + V^{\rho}_{\rho\zeta}\,\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa},$$

which rewrites

$$d\rho = S^{\rho}_{\rho\kappa}\,\rho\wedge\kappa + S^{\rho}_{\rho\zeta}\,\rho\wedge\zeta + S^{\rho}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + S^{\rho}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\;\kappa\wedge\overline{\kappa}\,.$$

From this expression, we see that $S^{\rho}_{\rho\kappa}$, $S^{\rho}_{\rho\zeta}$, $S^{\rho}_{\rho\overline{\kappa}}$ and $S^{\rho}_{\rho\overline{\zeta}}$ are essential torsion coefficients. We now turn our attention on the computation of $S^{\rho}_{\rho\overline{\kappa}}$.

The expression of $dJ \wedge \rho$ is obtained in a similar way as that of $dH \wedge \rho$, namely:

$$dJ \wedge \rho = \left(\frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2} \frac{\mathscr{K}(J)}{\overline{\mathscr{L}_1}(k)} - \frac{\mathscr{L}_1(J)}{\mathsf{c}} \right) \rho \wedge \kappa - \frac{\overline{\mathsf{c}}}{\mathsf{c}} \frac{\mathscr{K}(J)}{\overline{\mathscr{L}_1}(k)} \rho \wedge \zeta \\ + \left(\frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2} \frac{\overline{\mathscr{K}}(J)}{\mathscr{L}_1(\overline{k})} - \frac{\overline{\mathscr{L}_1}(J)}{\overline{\mathsf{c}}} \right) \rho \wedge \overline{\kappa} - \frac{\mathsf{c}}{\overline{\mathsf{c}}} \frac{\overline{\mathscr{K}}(J)}{\mathscr{L}_1(\overline{k})} \rho \wedge \overline{\zeta} \, .$$

Replacing c by $J^{1/3}$, we thus get that

$$\left(\frac{dJ}{J} + \frac{d\overline{J}}{\overline{J}} \right) \wedge \rho = \\ \left[\frac{\mathsf{e}}{\overline{\mathscr{L}_1}(k)} \frac{\overline{J}^{1/3}}{J^{2/3}} \left(\frac{\mathscr{K}(J)}{J} + \frac{\mathscr{K}\left(\overline{J}\right)}{\overline{J}} \right) - \frac{\mathscr{L}_1(J)}{J^{4/3}} - \frac{\mathscr{L}_1\left(\overline{J}\right)}{J^{1/3}\,\overline{J}} \right] \rho \wedge \kappa \\ - \frac{1}{\overline{\mathscr{L}_1}(k)} \frac{\overline{J}^{1/3}}{J^{1/3}} \left(\frac{\mathscr{K}(J)}{J} + \frac{\mathscr{K}\left(\overline{J}\right)}{\overline{J}} \right) \rho \wedge \zeta \\ + \left[\frac{\overline{\mathsf{e}}}{\mathscr{L}_1(\overline{k})} \frac{J^{1/3}}{\overline{J}^{2/3}} \left(\frac{\overline{\mathscr{K}}(J)}{J} + \frac{\overline{\mathscr{K}}\left(\overline{J}\right)}{\overline{J}} \right) - \frac{\overline{\mathscr{L}_1}(J)}{J\overline{J}^{1/3}} - \frac{\overline{\mathscr{L}_1}\left(\overline{J}\right)}{\overline{J}^{4/3}} \right] \rho \wedge \overline{\kappa} \\ - \frac{1}{\mathscr{L}_1(\overline{k})} \frac{J^{1/3}}{\overline{J}^{1/3}} \left(\frac{\overline{\mathscr{K}}(J)}{J} + \frac{\overline{\mathscr{K}}\left(\overline{J}\right)}{\overline{J}} \right) \rho \wedge \overline{\zeta}$$

On the other hand, after replacing c by its normalization in $V^{\rho}_{\rho\overline{\kappa}}$, we get:

$$V^{\rho}_{\rho\overline{\kappa}} = -\frac{\mathsf{e}}{J^{1/3}} + \frac{1}{3} \frac{1}{\overline{J}^{1/3}} \frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3} \frac{\overline{P}}{\overline{J}^{1/3}} + \overline{\mathsf{e}} \frac{J^{1/3}}{\overline{J}^{2/3}} \frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})}.$$

We thus obtain the following essential torsion coefficient, which depends on e and $\overline{\mathsf{e}}$:

$$S^{\rho}_{\rho\overline{\kappa}} = -\frac{\mathsf{e}}{J^{1/3}} + \frac{\overline{\mathsf{e}}}{\mathscr{L}_1(\overline{k})} \frac{J^{1/3}}{\overline{J}^{2/3}} \left(\overline{\mathscr{L}_1}(\overline{k}) + \frac{1}{3} \frac{\overline{\mathscr{K}}(\overline{J})}{\overline{J}} + \frac{1}{3} \frac{\overline{\mathscr{K}}(J)}{J} \right) \\ + \frac{1}{3} \frac{1}{\overline{J}^{1/3}} \left(2\,\overline{P} + \frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{\overline{\mathscr{L}_1}(J)}{J} - \frac{\overline{\mathscr{L}_1}(\overline{J})}{\overline{J}} \right).$$

The actual computation of the other essential torsion coefficients $S^{\rho}_{\rho\kappa}$, $S^{\rho}_{\rho\zeta}$ and $S^{\rho}_{\rho\overline{\zeta}}$ do not lead to any useful equation depending in e. On the other hand, the study of the third structure equation provides us with another meaningful essential torsion coefficient. Indeed we have:

$$\begin{aligned} d\zeta &= i\,\delta_2\wedge\kappa + \frac{1}{3}\left(\frac{dJ}{J} - \frac{d\overline{J}}{\overline{J}}\right)\wedge\zeta \\ &\quad + W^{\zeta}_{\rho\kappa}\,\rho\wedge\kappa + W^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta + W^{\zeta}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + W^{\zeta}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} \\ &\quad + V^{\zeta}_{\kappa\zeta}\,\kappa\wedge\zeta + V^{\zeta}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + V^{\zeta}_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + V^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + V^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

which, taking into account the facts that $W^{\zeta}_{\rho\overline{\kappa}} = 0$ and that $W^{\zeta}_{\rho\overline{\zeta}}$ as been normalized to 1, can be rewritten as

$$\begin{aligned} d\zeta &= i\,\delta_2\wedge\kappa \\ &\quad + S^{\zeta}_{\rho\kappa}\,\rho\wedge\kappa + S^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta + \rho\wedge\overline{\zeta} \\ &\quad + S^{\zeta}_{\kappa\zeta}\,\kappa\wedge\zeta + S^{\zeta}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + S^{\zeta}_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + S^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + S^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

where the $S^{\bullet}_{\bullet\bullet}$ are new torsion coeficients. We easily deduce from this equation that

$$S^{\zeta}_{\zeta\overline{\kappa}} = V^{\zeta}_{\zeta\overline{\kappa}} + \frac{1}{3}\left[\frac{\overline{\mathsf{e}}}{\mathscr{L}_1(\overline{k})}\,\frac{J^{1/3}}{\overline{J}^{2/3}}\left(\frac{\overline{\mathscr{K}}(J)}{J} - \frac{\overline{\mathscr{K}}\left(\overline{J}\right)}{\overline{J}}\right) - \frac{\overline{\mathscr{L}_1}(J)}{J\overline{J}^{1/3}} + \frac{\overline{\mathscr{L}_1}\left(\overline{J}\right)}{\overline{J}^{4/3}}\right]$$

is an essential torsion coefficient. From the expression of $V^{\zeta}_{\zeta\overline{\kappa}}$ obtained by performing the substitution $c := J^{\frac{1}{3}}$, we have

$$\begin{aligned} S^{\zeta}_{\zeta\overline{\kappa}} &= \frac{\mathsf{e}}{J^{1/3}} - \overline{\mathsf{e}}\,\frac{J^{1/3}}{\overline{J}^{2/3}}\,\frac{\overline{\mathscr{L}_1}(\overline{k})}{\mathscr{L}_1(\overline{k})} - \frac{1}{\overline{J}^{1/3}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} \\ &\quad + \frac{1}{3}\left[\frac{\overline{\mathsf{e}}}{\mathscr{L}_1(\overline{k})}\,\frac{J^{1/3}}{\overline{J}^{2/3}}\left(\frac{\overline{\mathscr{K}}(J)}{J} - \frac{\overline{\mathscr{K}}\left(\overline{J}\right)}{\overline{J}}\right) - \frac{\overline{\mathscr{L}_1}(J)}{J\overline{J}^{1/3}} + \frac{\overline{\mathscr{L}_1}\left(\overline{J}\right)}{\overline{J}^{4/3}}\right]. \end{aligned}$$

We now substract the two essential torsion coefficients that we have get so far:

$$\begin{aligned} -S^{\rho}_{\rho\overline{\kappa}} + S^{\zeta}_{\zeta\overline{\kappa}} &= 2\,\frac{\mathsf{e}}{J^{1/3}} - 2\,\overline{\mathsf{e}}\,\frac{J^{1/3}}{\overline{J}^{2/3}}\,\frac{1}{\mathscr{L}_1(\overline{k})}\left(\overline{\mathscr{L}_1}(\overline{k}) + \frac{1}{3}\,\frac{\overline{\mathscr{K}}(\overline{J})}{\overline{J}}\right) \\ &\quad + \frac{2}{3}\,\frac{1}{\overline{J}^{1/3}}\left(\frac{\overline{\mathscr{L}_1}(\overline{J})}{\overline{J}} - 2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right). \end{aligned}$$

From the full expression of $\mathscr{K}(J)$ in terms of the coframe derivatives, obtained by using extensively the commutations relations (3), we find the relation:

$$\frac{1}{3}\,\overline{\mathscr{K}}\left(\overline{J}\right) + \overline{\mathscr{L}_1}(\overline{k})\cdot\overline{J} = 0,$$

from which we deduce that the following expression:

$$\frac{\mathsf{e}}{J^{1/3}} + \frac{1}{3}\,\frac{1}{\overline{J}^{1/3}} \left(\frac{\overline{\mathscr{L}_1}(\overline{J})}{\overline{J}} - 2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P} \right)$$

is an essential torsion coefficient. Setting this coefficient to zero, gives the normalization of e:

$$\mathsf{e} = \frac{1}{3}\,\frac{J^{1/3}}{\overline{J}^{1/3}} \left(-\frac{\overline{\mathscr{L}_1}(\overline{J})}{\overline{J}} + 2\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \overline{P} \right).$$

8. Case $W \neq 0$

We now assume that the fonction W does not vanish on M, and we show how the group parameter e can be normalized. We choose the normalization $c := W$. We recall that prior to this last normalization, the structure equations read:

$$\begin{aligned} d\rho = {} & \delta^1 \wedge \rho + \overline{\delta^1} \wedge \rho \\ & + V^{\rho}_{\rho\kappa}\ \rho \wedge \kappa + V^{\rho}_{\rho\zeta}\ \rho \wedge \zeta + V^{\rho}_{\rho\overline{\kappa}}\ \rho \wedge \overline{\kappa} + V^{\rho}_{\rho\overline{\zeta}}\ \rho \wedge \overline{\zeta} + i\ \kappa \wedge \overline{\kappa}\,, \end{aligned}$$

$$\begin{aligned} d\kappa = {} & \delta^1 \wedge \kappa + \delta^2 \wedge \rho \\ & + V^{\kappa}_{\rho\kappa}\ \rho \wedge \kappa + V^{\kappa}_{\rho\zeta}\ \rho \wedge \zeta + V^{\kappa}_{\rho\overline{\kappa}}\ \rho \wedge \overline{\kappa} \\ & + V^{\kappa}_{\rho\overline{\zeta}}\ \rho \wedge \overline{\zeta} + V^{\kappa}_{\kappa\zeta}\ \kappa \wedge \zeta + V^{\kappa}_{\kappa\overline{\kappa}}\ \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa} \end{aligned}$$

and

$$\begin{aligned} d\zeta = {} & i\ \delta_2 \wedge \kappa + \delta_1 \wedge \zeta - \overline{\delta_1} \wedge \zeta \\ & + W^{\zeta}_{\rho\kappa}\ \rho \wedge \kappa + W^{\zeta}_{\rho\zeta}\ \rho \wedge \zeta + W^{\zeta}_{\rho\overline{\kappa}}\ \rho \wedge \overline{\kappa} + V^{\zeta}_{\kappa\zeta}\ \kappa \wedge \zeta \\ & + V^{\zeta}_{\kappa\overline{\kappa}}\ \kappa \wedge \overline{\kappa} + V^{\zeta}_{\kappa\overline{\zeta}}\ \kappa \wedge \overline{\zeta} + V^{\zeta}_{\zeta\overline{\kappa}}\ \zeta \wedge \overline{\kappa} + V^{\zeta}_{\zeta\overline{\zeta}}\ \zeta \wedge \overline{\zeta}\,, \end{aligned}$$

where

$$\delta^1 = \frac{d\mathsf{c}}{\mathsf{c}} \quad , \quad \delta^2 = i\,\mathsf{e}\frac{d\mathsf{c}}{\mathsf{c}^2} - i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{d\mathsf{e}}{\mathsf{c}},$$

and

$$W^{\zeta}_{\rho\overline{\kappa}} = i\,\frac{\overline{J}}{\overline{\mathsf{c}}^3}.$$

As we have

$$\delta^2 = -i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,d\left(\frac{\mathsf{e}}{\mathsf{c}}\right),$$

it is convenient to introduce the new parameter ϵ defined by

$$\epsilon := \frac{\mathsf{e}}{\mathsf{c}}.$$

With the normalization $c := W$, we get:

$$\delta^1 = \frac{dW}{W},$$

$$\delta^2 = -i\, d\epsilon - i\epsilon \frac{d\overline{W}}{W}$$

and

$$W^{\zeta}_{\rho\overline{\kappa}} = i\,\frac{\overline{J}}{W^3}.$$

As a result, the new structure equations take the form:

$$d\rho = X^{\rho}_{\rho\kappa}\,\rho\wedge\kappa + X^{\rho}_{\rho\zeta}\,\rho\wedge\zeta + X^{\rho}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + X^{\rho}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\ \kappa\wedge\overline{\kappa}\ ,$$

$$\begin{aligned} d\kappa = {} & -i\, d\epsilon\wedge\rho \\ & + X^{\kappa}_{\rho\kappa}\ \rho\wedge\kappa + X^{\kappa}_{\rho\zeta}\ \rho\wedge\zeta + X^{\kappa}_{\rho\overline{\kappa}}\ \rho\wedge\overline{\kappa} + X^{\kappa}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} \\ & + X^{\kappa}_{\kappa\zeta}\,\kappa\wedge\zeta + X^{\kappa}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + X^{\kappa}_{\kappa\overline{\zeta}}\ \kappa\wedge\overline{\zeta} + \zeta\wedge\overline{\kappa}, \end{aligned}$$

$$\begin{aligned} d\zeta = {} & d\epsilon\wedge\kappa \\ & + X^{\zeta}_{\rho\kappa}\ \rho\wedge\kappa + X^{\zeta}_{\rho\zeta}\ \rho\wedge\zeta + X^{\zeta}_{\rho\overline{\kappa}}\ \rho\wedge\overline{\kappa} + X^{\zeta}_{\kappa\zeta}\ \kappa\wedge\zeta \\ & + X^{\zeta}_{\kappa\overline{\kappa}}\ \kappa\wedge\overline{\kappa} + X^{\zeta}_{\kappa\overline{\zeta}}\ \kappa\wedge\overline{\zeta} + X^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + X^{\zeta}_{\zeta\overline{\zeta}}\ \zeta\wedge\overline{\zeta}\ , \end{aligned}$$

for a new set of torsion coefficients $X^{\bullet}_{\bullet\bullet}$. The absorption process is straightforward and leads to the following essential torsion coefficients:

$$\begin{array}{llll} X^{\rho}_{\rho\kappa}, & X^{\rho}_{\rho\zeta}, & X^{\rho}_{\rho\overline{\kappa}}, & X^{\rho}_{\rho\overline{\zeta}}, \\ X^{\kappa}_{\kappa\zeta}, & X^{\kappa}_{\kappa\overline{\kappa}}, & X^{\kappa}_{\kappa\overline{\zeta}}, & X^{\zeta}_{\rho\zeta}, \\ X^{\zeta}_{\rho\overline{\kappa}}, & X^{\zeta}_{\zeta\overline{\kappa}}, & X^{\zeta}_{\zeta\overline{\zeta}}, & i\,X^{\zeta}_{\kappa\zeta} + X^{\kappa}_{\rho\zeta}, \\ i\,X^{\zeta}_{\kappa\overline{\kappa}} + X^{\kappa}_{\rho\overline{\kappa}}, & & i\,X^{\zeta}_{\kappa\overline{\zeta}} + X^{\kappa}_{\rho\overline{\zeta}}. & \end{array}$$

The careful computation of the coefficient $X^{\kappa}_{\kappa\overline{\kappa}}$ gives:

$$X^{\kappa}_{\kappa\overline{\kappa}} = \overline{\epsilon}\,\frac{\overline{\mathscr{K}}\,(W)}{\overline{W}\mathscr{L}_1(\overline{k})} - \frac{\overline{\mathscr{L}_1}\,(W)}{W\overline{W}} - \frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{W\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\overline{P}}{\overline{W}}.$$

The expression of $\overline{\mathscr{K}}\,(W)$ can be simplified by using the commutations relations (3), as in the case of $\mathscr{K}\,(J)$. We find the relation:

$$\overline{\mathscr{K}}(W) + 2\,\overline{\mathscr{L}_1}(k)\,\overline{W} = 0,$$

from which we deduce that $X^{\kappa}_{\kappa\overline{\kappa}}$ rewrites:

$$X^{\kappa}_{\kappa\overline{\kappa}} = -2\,\overline{\epsilon} - \frac{\overline{\mathscr{L}_1}\,(W)}{W\overline{W}} - \frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{W\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\overline{P}}{\overline{W}}.$$

Setting this coefficient to zero, we get a normalization of ϵ, and hence of e, provided that $\overline{\mathscr{K}}\,(W)$ does not vanish on M, which is given by the following lemma:

Lemma 1. $\overline{\mathscr{K}}\,(W)$ *does not vanish identically on* M.

Proof. The computation of $\overline{\mathscr{K}}(W)$, using the commutation relations (3) leads to the following formula:

$$\overline{\mathscr{K}}(W) + 2\,\mathscr{L}_1(\overline{k})\,\overline{W} + 2i\,\mathscr{T}(\overline{k}) = 0.$$

If $\overline{\mathscr{K}}(W) = 0$ then $W = i\frac{\mathscr{T}(k)}{\mathscr{L}_1(k)}$ which implies $\overline{\mathscr{K}}(W) = -i\,\mathscr{T}(k)$ (using (3) once again), that is

$$\overline{\mathscr{K}}(W) = \overline{W}\mathscr{L}_1(\overline{k}),$$

which gives a contradiction with the fact that $W \neq 0$. □

9. Case $J = 0$ and $W = 0$

We show that in this case, M is biholomorphically equivalent to the light cone. We start by showing that the coefficient $W_{\rho\zeta}^{\zeta}$ is purely imaginary, which implies that no further group reductions are allowed at this stage.

The full computation of this coefficient leads to:

$$
\begin{aligned}
i\,\mathsf{c}\bar{\mathsf{c}}\,W^{\zeta}_{\rho\zeta} = {} & -\frac{1}{6}\,\overline{\mathscr{L}_1}(P) - \frac{1}{6}\,\mathscr{L}_1(\overline{P}) - \frac{2}{3}\,\frac{\bar{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{2}\,\frac{\bar{\mathsf{c}}^2\,\mathsf{e}^2}{\mathsf{c}^2}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} \\
& + i\,\frac{\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\bar{\mathsf{c}}\mathsf{e}}{\mathsf{c}}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{1}{3}\,\frac{\bar{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)}{\mathscr{L}_1(\bar{k})} \\
& + \frac{1}{2}\,\frac{\mathsf{c}^2\,\bar{\mathsf{e}}^2}{\bar{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}(\bar{k})}{\mathscr{L}_1(\bar{k})} - \frac{1}{3}\,\frac{\bar{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} \\
& + \frac{1}{18}\,\frac{\overline{\mathscr{L}_1}(\bar{k})\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)}{\mathscr{L}_1(\bar{k})^2}\,P - \frac{1}{3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)}{\overline{\mathscr{L}_1}(k)\mathscr{L}_1(\bar{k})} \\
& - \frac{1}{18}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3}\,\overline{P} - \mathsf{e}\,\bar{\mathsf{e}} \\
& + \frac{2}{9}\,P\,\overline{P} + \frac{4}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^3} \\
& - \frac{1}{6}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} - \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)} \\
& - \frac{1}{9}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)}\,\overline{P}^2 - \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}(\bar{k})\mathscr{L}_1\left(\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)\right)}{\mathscr{L}_1(\bar{k})^2} + \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}(\bar{k})\mathscr{L}_1(P)}{\mathscr{L}_1(\bar{k})} \\
& + \frac{1}{6}\,\frac{\mathscr{L}_1(k)\overline{\mathscr{L}_1}(\overline{P})}{\overline{\mathscr{L}_1}(k)} - \frac{4}{9}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^4} - \frac{1}{9}\,\frac{\overline{\mathscr{L}_1}(\bar{k})P^2}{\mathscr{L}_1(\bar{k})} \\
& + \frac{1}{18}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)^2} + \frac{2}{9}\,\frac{\overline{\mathscr{L}_1}(\bar{k})\mathscr{L}_1\left(\mathscr{L}_1(\bar{k})\right)^2}{\mathscr{L}_1(\bar{k})^3} \\
& - \frac{1}{6}\,\frac{\mathscr{L}_1(k)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{5}{18}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} \\
& + \frac{2}{9}\,\frac{\mathscr{L}_1(k)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^3} + \frac{1}{6}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^3} \\
& - \frac{1}{9}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P} + \frac{1}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,P \\
& + \frac{1}{18}\,\frac{\mathscr{L}_1(k)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2}\,\overline{P} + \frac{\mathsf{c}\bar{\mathsf{e}}}{\bar{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{i}{3}\,\frac{\mathsf{e}\bar{\mathsf{c}}}{\mathsf{c}}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} \\
& - \frac{i}{3}\,\frac{\overline{\mathscr{L}_1}\left(\mathscr{T}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{i}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2}\,\mathscr{T}(k) + \frac{4i}{9}\,\frac{\overline{P}}{\overline{\mathscr{L}_1}(k)}\,\mathscr{T}(k).
\end{aligned}
$$

As we shall check that this expression is real, we just drop the terms which come together with their conjugate counterpart, i.e., we perform a computation mod $\mathbb{R}$. We thus get:

$$\begin{aligned} i\,\mathsf{c}\overline{\mathsf{c}}\,W^{\zeta}_{\rho\zeta} \equiv & -\frac{2}{3}\,\frac{\overline{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + i\,\frac{\mathscr{T}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\,\frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{c}}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} \\ & + \frac{1}{3}\,\frac{\overline{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}}\,\frac{\mathscr{L}_1\left(\mathscr{L}_1(\overline{k})\right)}{\mathscr{L}_1(\overline{k})} - \frac{1}{3}\,\frac{\overline{\mathsf{c}}\,\mathsf{e}}{\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} \\ & - \frac{1}{18}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3}\,\overline{P} \\ & + \frac{4}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^3} \\ & - \frac{1}{6}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} - \frac{4}{9}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^4} \\ & + \frac{1}{18}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)^2} + \frac{5}{18}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2} \\ & + \frac{1}{6}\,\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^3} - \frac{1}{9}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P} \\ & + \frac{1}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,P + \frac{\mathsf{c}\overline{\mathsf{e}}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{i}{3}\,\frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}_1}(k)} \\ & - \frac{i}{3}\,\frac{\overline{\mathscr{L}_1}\left(\mathscr{T}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \frac{i}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^2}\,\mathscr{T}(k) + \frac{4i}{9}\,\frac{\overline{P}}{\overline{\mathscr{L}_1}(k)}\,\mathscr{T}(k). \end{aligned}$$

We now give an expression of $i\,\mathsf{c}\overline{\mathsf{c}}\,W^{\zeta}_{\rho\zeta}$ in terms of the function W and its derivative by $\overline{\mathscr{L}_1}$. Using the expression of W given by (12) and dropping once again the terms which come with their conjugate counterpart, we get the formula:

$$i\,\mathsf{c}\overline{\mathsf{c}}\,W^{\zeta}_{\rho\zeta} \equiv \frac{1}{6}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right)W + \frac{1}{2}\,\overline{\mathscr{L}_1}(W) - \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}}\,W,$$

from which we get that $W^{\zeta}_{\rho\zeta}$ is purely imaginary under that assumption that W does vanish identically on M.

The normalization step of Cartan's algorithm stops here and we shall now perform a prolongation of the problem. We introduce the modified Maurer Cartan forms of the group G_4, namely:

$$\begin{cases} \hat{\delta}^1 := \delta^1 - w^1_{\rho}\,\rho - w^1_{\kappa}\,\kappa - w^1_{\zeta}\,\zeta - w^1_{\overline{\kappa}}\,\overline{\kappa} - w^1_{\overline{\zeta}}\,\overline{\zeta} \\ \hat{\delta}^2 := \delta^2 - w^2_{\rho}\,\rho - w^2_{\kappa}\,\kappa - w^2_{\zeta}\,\zeta - w^2_{\overline{\kappa}}\,\overline{\kappa} - w^2_{\overline{\zeta}}\,\overline{\zeta} \end{cases}$$

where w^i_ρ, w^i_κ, w^i_ζ, $w^i_{\overline{\kappa}}$, $w^i_{\overline{\zeta}}$, $i = 1, 2$, are the solutions of the system of equations (13) corresponding to $w^1_\rho + \overline{w^1_\rho} = 0$, that is:

$$\begin{cases} \hat{\delta}^1 := \delta^1 + \dfrac{1}{2} V^\zeta_{\rho\zeta}\,\rho - \overline{V^\rho_{\rho\kappa}}\,\kappa - V^\rho_{\rho\zeta}\,\zeta - V^\kappa_{\kappa\overline{\kappa}}\,\overline{\kappa} \\ \hat{\delta}^2 := \delta^2 - V^\zeta_{\rho\kappa}\,\rho - \left(V^\kappa_{\rho\kappa} - \dfrac{1}{2} V^\zeta_{\rho\zeta}\right)\kappa - V^\kappa_{\rho\zeta}\,\zeta - V^\kappa_{\rho\overline{\kappa}}\,\overline{\kappa} - V^\kappa_{\rho\overline{\zeta}}\,\overline{\zeta}. \end{cases}$$

We also introduce the modified Maurer Cartan forms which correspond to solutions of the system (13) when $\mathsf{Re}(w^1_\rho)$ is not necessarily set to zero, namely:

$$\begin{cases} \pi^1 := \hat{\delta}^1 - \mathsf{Re}(w^1_\rho)\,\rho \\ \pi^2 := \hat{\delta}^2 - \mathsf{Re}(w^1_\rho)\,\kappa. \end{cases}$$

Let P^9 be the nine dimensionnal G_4-structure constituted by the set of all coframes of the form $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta})$ on M. The initial coframe $(\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0)$ gives a natural trivialisation $P^9 \xrightarrow{p} M \times G_4$ which allows us to consider any differential form on M or G^4 as a differential form on P^9. If ω is a differential form on M for example, we just consider $p^*(pr_1^*(\omega))$, where pr_1 is the projection on the first component $M \times G_4 \xrightarrow{pr_1} M$. We still denote this form by ω in the sequel. Following [24], we introduce the two coframes $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \delta^1, \delta^2, \overline{\delta^1}, \overline{\delta^2})$ and $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \pi^1, \pi^2, \overline{\pi^1}, \overline{\pi^2})$ on P^9. Setting $t := -\mathsf{Re}(w^1_\rho)$, they relate to each other by the relation:

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \\ \pi^1 \\ \pi^2 \\ \overline{\pi^1} \\ \overline{\pi^2} \end{pmatrix} = g_t \cdot \begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \\ \delta^1 \\ \delta^2 \\ \overline{\delta^1} \\ \overline{\delta^2} \end{pmatrix}$$

where g_t is defined by

$$g_t := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The set $\{g_t, t \in \mathbb{R}\}$ defines a one dimensional Lie group G_{prol}, whose Maurer Cartan form is given by dt. We now start the absorption-normalization procedure in Cartan's method on P^9.

From the definition of π^1 and π^2 as the solutions of the absorption equations (13), the five first structure equations read as

$$\begin{aligned} d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}, \\ d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\ d\zeta &= i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta, \\ d\overline{\kappa} &= \overline{\pi^1} \wedge \overline{\kappa} + \overline{\pi^2} \wedge \rho - \kappa \wedge \overline{\zeta}, \\ d\overline{\zeta} &= -i\,\overline{\pi^2} \wedge \overline{\kappa} + \overline{\pi^1} \wedge \overline{\zeta} - \pi^1 \wedge \overline{\zeta}. \end{aligned} \tag{17}$$

The computations that follow aim to determine the expressions of $d\pi^1$ and $d\pi^2$. Both of these expressions can be deduced from the the set of equations (17). For example, taking the exterior derivative of both sides of the equation giving $d\rho$, we get:

$$0 = d\pi^1 \wedge \rho - \pi^1 \wedge d\rho + d\overline{\pi^1} \wedge \rho - \overline{\pi^1} \wedge d\rho + i\,d\kappa \wedge \overline{\kappa} - i\,\kappa \wedge d\overline{\kappa}.$$

Replacing each two-form $d\rho$, $d\kappa$ and $d\overline{\kappa}$ by its expression given by (17) yields:

$$\begin{aligned} 0 = d\pi^1 \wedge \rho + d\overline{\pi^1} \wedge \rho - \pi^1 \wedge \left(\pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}\right) \\ - \overline{\pi^1} \wedge \left(\pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}\right) + i\,\left(\pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}\right) \wedge \overline{\kappa} \\ - i\,\kappa \wedge \left(\overline{\pi^1} \wedge \overline{\kappa} + \overline{\pi^2} \wedge \rho - \kappa \wedge \overline{\zeta}\right), \end{aligned}$$

which can be simplified as:

$$0 = \left(d\pi^1 - i\,\kappa \wedge \overline{\pi^2} + d\overline{\pi^1} + i\,\overline{\kappa} \wedge \pi^2\right) \wedge \rho.$$

Performing the same computation from the equation giving $d\kappa$, we get:

$$0 = d\pi^1 \wedge \kappa - \pi^1 \wedge d\kappa + d\pi^2 \wedge \rho - \pi^2 \wedge d\rho + d\zeta \wedge \overline{\kappa} - \zeta \wedge d\overline{\kappa},$$

that is:

$$\begin{aligned} 0 &= d\pi^1 \wedge \kappa - \pi^1 \wedge \left(\pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}\right) \\ &\quad + d\pi^2 \wedge \rho - \pi^2 \wedge \left(\pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}\right) \\ &\quad + \left(i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta\right) \wedge \overline{\kappa} - \zeta \wedge \left(\overline{\pi^1} \wedge \overline{\kappa} + \overline{\pi^2} \wedge \rho - \kappa \wedge \overline{\zeta}\right), \end{aligned}$$

which yields:

$$0 = \left(d\pi^1 - \zeta \wedge \overline{\zeta}\right) \wedge \kappa + \left(d\pi^2 - \pi^2 \wedge \overline{\pi^1} - \zeta \wedge \overline{\pi^2}\right) \wedge \rho.$$

On the other hand, the same computation with the equation giving $d\zeta$ leads to

$$\begin{aligned} 0 &= i\,d\pi^2 \wedge \kappa - i\,\pi^2 \wedge \left(\pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}\right) + d\pi^1 \wedge \zeta \\ &\quad - d\overline{\pi^1} \wedge \zeta + \left(\overline{\pi^1} - \pi^1\right) \wedge \left(i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta\right), \end{aligned}$$

that is:

$$0 = \left(d\pi^1 - d\overline{\pi^1} - i\,\overline{\kappa} \wedge \pi^2\right) \wedge \zeta + i\,\left(d\pi^2 - \pi^2 \wedge \overline{\pi^1}\right) \wedge \kappa.$$

Let us introduce the two-forms Ω_1 and Ω_2 defined by

$$\Omega_1 := d\pi^1 - i\,\kappa \wedge \overline{\pi^2} - \zeta \wedge \overline{\zeta},$$

and

$$\Omega_2 := d\pi^2 - \pi^2 \wedge \overline{\pi^1} - \zeta \wedge \overline{\pi^2}.$$

With these notations, the three equations that we have obtained so far rewrite:

$$\text{(18)} \qquad \begin{cases} 0 = \left(\Omega_1 + \overline{\Omega_1}\right) \wedge \rho, \\ 0 = \Omega_1 \wedge \kappa + \Omega_2 \wedge \rho, \\ 0 = \left(\Omega_1 - \overline{\Omega_1}\right) \wedge \zeta + i\,\Omega_2 \wedge \kappa. \end{cases}$$

Taking the exterior product with κ in the second equation gives:

$$0 = \Omega_2 \wedge \rho \wedge \kappa,$$

from which we can deduce the two relations:

$$\begin{aligned} 0 &= \left(\Omega_1 + \overline{\Omega_1}\right) \wedge \rho \wedge \zeta, \\ 0 &= \left(\Omega_1 - \overline{\Omega_1}\right) \wedge \rho \wedge \zeta, \end{aligned}$$

which yields:

$$\Omega_1 \wedge \rho \wedge \zeta = 0.$$

This implies the existence of two 1-forms α and β such that:

$$\Omega_1 = \alpha \wedge \rho + \beta \wedge \zeta.$$

Similarly, there exist two 1-form γ and δ such that:

$$\Omega_2 = \gamma \wedge \rho + \delta \wedge \kappa.$$

Inserting these two expressions in the second equation of (18), we obtain the existence of a real 1-form Λ such that:

$$\begin{aligned}\Omega_1 &= \Lambda \wedge \rho,\\ \Omega_2 &= \Lambda \wedge \kappa.\end{aligned}$$

If we come back to the expression of $d\pi^1$ and $d\pi^2$, we get the two following additional structure equations:

$$\begin{aligned}d\pi^1 &= i\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta} + \Lambda \wedge \rho,\\ d\pi^2 &= \pi^2 \wedge \overline{\pi^1} + \zeta \wedge \overline{\pi^2} + \Lambda \wedge \kappa.\end{aligned}$$

From the definition of π^1 and π^2, Λ shall involve a term in dt. By adding Λ to the set of 1-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, π^1, π^2, $\overline{\pi^1}$, $\overline{\pi^2}$, we thus get a 10-dimensional $\{e\}$-structure on $G_{prol} \times P^9$, which constitutes the second (and last) 1-dimensional prolongation to the equivalence problem. It remains to compute the exterior derivative of Λ, which is done in what follows.

Taking the exterior derivative of the equation giving $d\pi^1$, we get:

$$0 = i\,d\kappa \wedge \overline{\pi^2} - i\,\kappa \wedge \overline{d\pi^2} + d\zeta \wedge \overline{\zeta} - \zeta \wedge d\overline{\zeta} + d\Lambda \wedge \rho - \Lambda \wedge d\rho,$$

that is

$$\begin{aligned}0 = i\left(\pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}\right) \wedge \overline{\pi^2} - i\,\kappa \wedge \left(\overline{\pi^2} \wedge \pi^1 + \overline{\zeta} \wedge \pi^2 + \Lambda \wedge \overline{\kappa}\right)\\ + \left(i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta\right) \wedge \overline{\zeta} - \zeta \wedge \left(-i\,\overline{\pi^2} \wedge \overline{\kappa} + \overline{\pi^1} \wedge \overline{\zeta} - \pi^1 \wedge \overline{\zeta}\right)\\ + d\Lambda \wedge \rho - \Lambda \wedge \left(\pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}\right),\end{aligned}$$

which yields:

$$0 = \left(d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi^1} - i\,\pi^2 \wedge \overline{\pi^2}\right) \wedge \rho.$$

On the other hand, a similar computation starting from the expression of $d\pi^2$ gives:

$$0 = d\pi^2 \wedge \overline{\pi^1} - \pi^2 \wedge \overline{d\pi^1} + d\zeta \wedge \overline{\pi^2} - \zeta \wedge d\overline{\pi^2} + d\Lambda \wedge \kappa - \Lambda \wedge d\kappa,$$

that is

$$\left(\pi^2 \wedge \overline{\pi^1} + \zeta \wedge \overline{\pi^2} + \Lambda \wedge \kappa\right) \wedge \overline{\pi^1} - \pi^2 \wedge \left(-\overline{\kappa} \wedge \pi^2 + \overline{\zeta} \wedge \zeta + \Lambda \wedge \rho\right)$$
$$+ \left(i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta\right) \wedge \overline{\pi^2} - \zeta \wedge \left(\overline{\pi^2} \wedge \pi^1 + \overline{\zeta} \wedge \pi^2 + \Lambda \wedge \overline{\kappa}\right)$$
$$+ d\Lambda \wedge \kappa - \Lambda \wedge \left(\pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}\right),$$

or

$$\left(d\Lambda - i\,\pi^2 \wedge \overline{\pi^2} - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi^1}\right) \wedge \kappa = 0.$$

From these last two equations, we deduce that:

$$d\Lambda = i\,\pi^2 \wedge \overline{\pi^2} + \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi^1}.$$

Summing up the results that we have obtained so far, the ten 1-differential forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, π^1, π^2, $\overline{\pi^1}$, $\overline{\pi^2}$, Λ, satisfy the structure equations:

$$\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa},\\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa},\\
d\zeta &= i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta,\\
d\overline{\kappa} &= \overline{\pi^1} \wedge \overline{\kappa} + \overline{\pi^2} \wedge \rho - \kappa \wedge \overline{\zeta},\\
d\overline{\zeta} &= -i\,\overline{\pi^2} \wedge \overline{\kappa} + \overline{\pi^1} \wedge \overline{\zeta} - \pi^1 \wedge \overline{\zeta},\\
d\pi^1 &= i\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta} + \Lambda \wedge \rho,\\
d\pi^2 &= \pi^2 \wedge \overline{\pi^1} + \zeta \wedge \overline{\pi^2} + \Lambda \wedge \kappa,\\
d\Lambda &= i\,\pi^2 \wedge \overline{\pi^2} + \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi^1}.
\end{aligned}$$

The torsion coefficients of these structure equations are all constant, and they do not depend on the graphing function F of M. This proves that all the hypersurfaces M which satisfy

$$J = W = 0$$

are locally biholomorphic. A direct computation shows that the hypersurface defined by

$$u = \frac{z_1\overline{z_1} + \frac{1}{2}z_1^2\overline{z_2} + \frac{1}{2}\overline{z_1^2}z_2}{1 - z_2\overline{z_2}}$$

is precisely such that $J = W = 0$. This completes the proof of theorem 1.

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CANONICAL CARTAN CONNECTION FOR 4-DIMENSIONAL CR-MANIFOLDS BELONGING TO GENERAL CLASS II

SAMUEL POCCHIOLA

ABSTRACT

We study the equivalence problem for 4-dimensional CR-manifolds of CR-dimension 1 and codimension 2 which have been referred to as belonging to general class II in [9], and which are also known as Engel CR-manifolds. We construct a canonical Cartan connection on such CR-manifolds through Cartan equivalence's method, thus providing an alternative approach to the results contained in [1]. In particular, we give the explicit expression of 4 biholomorphic invariants, the annulation of which is a necessary and sufficient condition for an Engel manifold to be locally biholomorphic to Beloshapka's cubic in $\mathbb{C}^3$.

1. INTRODUCTION

As highlighted by Henri Poincaré [14] in 1907, the (local) biholomorphic equivalence problem between two submanifolds M and M' of $\mathbb{C}^N$ is to determine whether or not there exists a (local) biholomorphism ϕ of $\mathbb{C}^N$ such that $\phi(M) = M'$. Elie Cartan [2, 3] solved this problem for hypersurfaces $M^3 \subset \mathbb{C}^2$ in 1932, as he constructed a "hyperspherical connection" on such hypersurfaces by using the powerful technique which is now referred to as Cartan's equivalence method.

Given a manifold M and some geometric data specified on M, which usually appears as a G-structure on M (i.e. a reduction of the bundle of coframes of M), Cartan's equivalence method seeks to provide a principal bundle P on M together with a coframe ω of 1-forms on P which is adapted to the geometric structure of M in the following sense: an isomorphism between two such geometric structures M and M' lifts to a unique isomorphism between P and P' which sends ω on ω'. The equivalence problem between M and M' is thus reduced to an equivalence problem between $\{e\}$-structures, which is well understood [10, 15].

We recall that a CR-manifold M is a real manifold endowed with a subbundle L of $\mathbb{C} \otimes TM$ of even rank $2n$ such that

(1) $L \cap \overline{L} = \{0\}$

(2) L is formally integrable, i.e. $[L,\, L] \subset L$.

The integer n is the CR-dimension of M and $k = \dim M - 2n$ is the codimension of M. In a recent attempt [9] to solve the equivalence problem for CR-manifolds up to dimension 5, it has been shown that one can restrict the study to six different general classes of CR-manifolds of dimension ≤ 5, which have been referred to as general classes I, II, III_1, III_2, IV_1 and IV_2. The aim of this paper is to provide a solution to the equivalence problem for CR-manifolds which belong to general class II, that is the CR-manifolds of dimension 4 and of CR-dimension 1 whose CR-bundle L satisfy the additional non-degeneracy condition:

$$\mathbb{C} \otimes TM = L + \overline{L} + [L,\, \overline{L}] + [L, [L,\, \overline{L}]],$$

meaning that $\mathbb{C} \otimes TM$ is spanned by L, $\overline{L}$ and their Lie brackets up to order 3.

This problem has already been solved by Beloshapka, Ezhov and Schmalz in [1], where the CR-manifolds we study are called Engel manifolds. The present paper provides thus an alternative solution to the results contained in [1]. The main result is the following:

Theorem 1. *Let M be a CR-manifold belonging to general class II. There exists a 5-dimensional subbundle P of the bundle of coframes $\mathbb{C} \otimes F(M)$ of M and a coframe $\omega := (\Lambda, \sigma, \rho, \zeta, \overline{\zeta})$ on P such that any CR-diffeomorphism h of M lifts to a bundle isomorphism h^* of P which satisfy $h^*(\omega) = \omega$. Moreover the structure equations of ω on P are of the form:*

$$\begin{aligned}
d\sigma &= 3\,\Lambda \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta},\\
d\rho &= 2\,\Lambda \wedge \rho + i\,\zeta \wedge \overline{\zeta}\\
d\zeta &= \Lambda \wedge \zeta + \mathfrak{I}_1\,\sigma \wedge \rho + \mathfrak{I}_2\,\sigma \wedge \zeta + \mathfrak{I}_3\,\sigma \wedge \overline{\zeta} + \mathfrak{I}_4\,\rho \wedge \zeta + \mathfrak{I}_5\,\rho \wedge \overline{\zeta},\\
d\overline{\zeta} &= \Lambda \wedge \overline{\zeta} + \overline{\mathfrak{I}_1}\,\sigma \wedge \rho + \overline{\mathfrak{I}_3}\,\sigma \wedge \zeta + \overline{\mathfrak{I}_2}\,\sigma \wedge \overline{\zeta} + \overline{\mathfrak{I}_5}\,\rho \wedge \zeta + \overline{\mathfrak{I}_4}\,\rho \wedge \overline{\zeta},\\
d\Lambda &= \frac{i}{2}\mathfrak{I}_1\,\sigma \wedge \overline{\zeta} - \frac{i}{2}\,\overline{\mathfrak{I}_1}\,\sigma \wedge \zeta - \frac{1}{3}\left(\mathfrak{I}_2 + \overline{\mathfrak{I}_3}\right)\rho \wedge \zeta - \frac{1}{3}\left(\overline{\mathfrak{I}_2} + \mathfrak{I}_3\right)\rho \wedge \zeta\\
&\quad + \mathfrak{I}_0\,\sigma \wedge \zeta\,,
\end{aligned}$$

where $\mathfrak{I}_0$, $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{I}_3$, $\mathfrak{I}_4$, $\mathfrak{I}_5$, are functions on P.

An example of CR-manifold belonging to general class II is provided by Beloshapka's cubic $\mathsf{B} \subset \mathbb{C}^3$, which is defined by the equations:

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z},\\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z + \overline{z})\,. \end{aligned}$$

Cartan's equivalence method has been applied to Beloshapka's cubic in [12] where it has been shown that the coframe $(\Lambda, \sigma, \rho, \zeta, \overline{\zeta})$ of theorem 1 satisfy the simplified structure equations:

$$\begin{aligned} d\sigma &= 3\,\Lambda\wedge\sigma + \rho\wedge\zeta + \rho\wedge\overline{\zeta}\,, \\ d\rho &= 2\,\Lambda\wedge\rho + i\;\zeta\wedge\overline{\zeta}\,, \\ d\zeta &= \Lambda\wedge\zeta\,, \\ d\overline{\zeta} &= \Lambda\wedge\overline{\zeta}\,, \\ d\Lambda &= 0, \end{aligned}$$

corresponding to the case where the biholomorphic invariants $\mathfrak{I}_i$ vanish identically. From this result together with theorem 1, we deduce the existence of a Cartan connection on CR-manifolds belonging to general class II in section 4.

We start in section 2 with the construction of a canonical G-structure P^1 on M, (e.g. a subbundle of the bundle of coframes of M), which encodes the equivalence problem for M under CR-automorphisms in the following sense: a diffeomorphism

$$h : M \longrightarrow M$$

is a CR-automorphism of M if and only if

$$h^* : P^1 \longrightarrow P^1$$

is a G-structure isomorphism of P^1. We refer to [9, 6, 7] for details on the results summarized in this section and to [15] for an introduction to G-structures. Section 3 is devoted to reduce successively P^1 to three subbundles:

$$P^4 \subset P^3 \subset P^2 \subset P^1,$$

which are still adapted to the biholomorphic equivalence problem for M. We use Cartan equivalence method, for which we refer to [10]. Eventually a Cartan connection is constructed on P^4 in section 4.

2. Initial G-structure

Let M be a 4-dimensional CR-manifold belonging to general class II and $\mathscr{L}$ be a local generator of the CR-bundle L of M. As M belongs to general class II, the two vector fields $\mathscr{T}$, $\mathscr{S}$, defined by:

$$\begin{aligned} \mathscr{T} &:= i\,[\mathscr{L},\overline{\mathscr{L}}], \\ \mathscr{S} &:= [\mathscr{L},\mathscr{T}], \end{aligned}$$

are such that:

$$4 = \text{rank}_{\mathbb{C}}\left(\mathscr{L},\overline{\mathscr{L}},\mathscr{T},\mathscr{S}\right),$$

namely

$$\left(\mathscr{L},\overline{\mathscr{L}},\mathscr{T},\mathscr{S}\right) \text{ is a frame on } M.$$

As a result there exist two functions A and B such that:

$$\overline{\mathscr{S}} = A \cdot \mathscr{T} + B \cdot \mathscr{S}.$$

From the fact that $\overline{\overline{\mathscr{S}}} = \mathscr{S}$, the functions A and B satisfy the relations:

$$(1) \qquad \begin{aligned} B\overline{B} &= 1, \\ \overline{A} + \overline{B}A &= 0. \end{aligned}$$

There also exist two functions P, Q such that:

$$[\mathscr{L}, \mathscr{S}] = P \cdot \mathscr{T} + Q \cdot \mathscr{S}.$$

The conjugate of P and Q, $\overline{P}$ and $\overline{Q}$, are given by the relations:
(2)

$$\overline{Q} = \mathscr{L}(B) + B\,Q + 2A + \frac{\overline{\mathscr{L}}(B)}{B},$$

$$\overline{P} = B\,\mathscr{L}(A) - A\,\mathscr{L}(B) - B\,A\,Q - A^2 - A\,\frac{\overline{\mathscr{L}}(B)}{B} + \overline{\mathscr{L}}(A) + B^2\,P.$$

The four functions A, B, P, Q appear to be fundamental as all other Lie brackets between the vector fields $\mathscr{L}$, $\overline{\mathscr{L}}$, $\mathscr{T}$ and $\mathscr{S}$ are expressed in terms of these five functions and their $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivatives ([7]).

In the case of an embedded CR-manifold $M \subset \mathbb{C}^3$, we can give an explicit formula for the fundamental vector field $\mathscr{L}$, and hence for the functions A, B, P, Q, in terms of a graphing function of M. We refer to [8] for details on this question. Let us just mention that the submanifold $M \subset \mathbb{C}^3$ is represented in local coordinates:

$$(z, w_1, w_2) := (x + i\,y,\ u_1 + i\,v_1,\ u_2 + i\,v_2)$$

as a graph:

$$\begin{aligned} v_1 &= \phi_1(x, y, u_1, u_2) \\ v_2 &= \phi_2(x, y, u_1, u_2). \end{aligned}$$

There exists then a unique local generator $\mathscr{L}$ of $T^{1,0}M$ of the form:

$$\mathscr{L} = \frac{\partial}{\partial z} + A^1\,\frac{\partial}{\partial u_1} + A^2\,\frac{\partial}{\partial u_2}$$

having conjugate:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \overline{A^1}\,\frac{\partial}{\partial u_1} + \overline{A^2}\,\frac{\partial}{\partial u_2}$$

which is a generator of $T^{0,1}M$, where the functions A^1 and A^2 are given by the determinants:

$$A^1 := \frac{\begin{vmatrix} -\phi_{1,z} & \phi_{1,u_2} \\ -\phi_{2,z} & i+\phi_{2,u_2} \end{vmatrix}}{\begin{vmatrix} i+\phi_{1,u_1} & \phi_{1,u_2} \\ \phi_{2,u_1} & i+\phi_{2,u_2} \end{vmatrix}}, \qquad A^2 := \frac{\begin{vmatrix} i+\phi_{1,u_1} & -\phi_{1,z} \\ \phi_{2,u_1} & -\phi_{2,z} \end{vmatrix}}{\begin{vmatrix} i+\phi_{1,u_1} & \phi_{1,u_2} \\ \phi_{2,u_1} & i+\phi_{2,u_2} \end{vmatrix}}.$$

Returning to the general case of abstract CR-manifolds, let us introduce the coframe

$$\omega_0 := \left(\sigma_0, \rho_0, \zeta_0, \overline{\zeta}_0\right),$$

as the dual coframe of $\left(\mathscr{S}, \mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}\right)$. We have [7]:

Lemma 1. *The structure equations enjoyed by ω_0 are of the form:*

$$\begin{aligned}
d\sigma_0 &= H\,\sigma_0\wedge\rho_0 + F\,\sigma_0\wedge\overline{\zeta}_0 + Q\,\sigma_0\wedge\zeta_0 + B\,\rho_0\wedge\overline{\zeta}_0 + \rho_0\wedge\zeta_0,\\
d\rho_0 &= G\,\sigma_0\wedge\rho_0 + E\,\sigma_0\wedge\overline{\zeta}_0 + P\,\sigma_0\wedge\zeta_0 + A\,\rho_0\wedge\overline{\zeta}_0 + i\,\zeta_0\wedge\overline{\zeta}_0,\\
d\zeta_0 &= 0,\\
d\overline{\zeta}_0 &= 0,
\end{aligned}$$

where the four functions:

$$E,\ F,\ G,\ H,$$

can be expressed in terms of the four fundamental functions:

$$A,\ B,\ P,\ Q,$$

and their $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivatives as:

$$\begin{aligned}
E &:= \mathscr{L}(A) + B\,P,\\
F &:= \mathscr{L}(B) + B\,Q + A,\\
G &:= i\,\mathscr{L}(\mathscr{L}(A)) + i\,P\,\mathscr{L}(B) - i\,\mathscr{L}(P) - i\,Q\,\mathscr{L}(A) + i\,P\,\mathscr{L}(B) + i\,B\mathscr{L}(P),\\
H &:= i\,\mathscr{L}(\mathscr{L}(B)) + i\,Q\,\mathscr{L}(B) + i\,B\,\mathscr{L}(Q) + 2i\,\mathscr{L}(A) - i\,\mathscr{L}(Q).
\end{aligned}$$

Let $h:\ M \longrightarrow M$ be a CR-automorphism of M. As we have

$$h_*\,(L) = L,$$

there exists a non-vanishing complex-valued function a on M such that:

$$h_*\,(\mathscr{L}) = \mathsf{a}\,\mathscr{L}.$$

From the definition of $\mathscr{T}$, $\mathscr{S}$, and the invariance

$$h_*\,([X,Y]) = [h_*(X), h_*(Y)]$$

for any vector fields X, Y on M, we easily get the existence of four functions

$$\mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e} : M \longrightarrow \mathbb{C},$$

such that:

$$h_* \begin{pmatrix} \mathscr{L} \\ \overline{\mathscr{L}} \\ \mathscr{T} \\ \mathscr{S} \end{pmatrix} = \begin{pmatrix} \mathsf{a} & 0 & 0 & 0 \\ 0 & \overline{\mathsf{a}} & 0 & 0 \\ \mathsf{b} & \overline{\mathsf{b}} & \mathsf{a}\overline{\mathsf{a}} & 0 \\ \mathsf{e} & \mathsf{d} & \mathsf{c} & \mathsf{a}^2\overline{\mathsf{a}} \end{pmatrix} \cdot \begin{pmatrix} \mathscr{L} \\ \overline{\mathscr{L}} \\ \mathscr{T} \\ \mathscr{S} \end{pmatrix}.$$

This is summarized in the following lemma [6]:

Lemma 2. *Let $h : M \longrightarrow M$ a CR-automorphism of M and let G_1 be the subgroup of* $\mathsf{GL}_4(\mathbb{C})$

$$G_1 := \left\{ \begin{pmatrix} \mathsf{a}^2\overline{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \mathsf{d} & \mathsf{b} & \mathsf{a} & 0 \\ \mathsf{e} & \overline{\mathsf{b}} & 0 & \overline{\mathsf{a}} \end{pmatrix}, \, \mathsf{a} \in \mathbb{C} \setminus \{0\}, \, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e} \in \mathbb{C} \right\}.$$

Then the pullback ω of ω_0 by h, $\omega := h^* \omega_0$, satisfies:

$$\omega = g \cdot \omega_0,$$

where g is smooth (locally defined) function $M \xrightarrow{g} G_1$.

This motivates the introduction of the subbundle P^1 of the bundle of coframes on M constituted by the coframes ω of the form

$$\omega := g \cdot \omega_0, \qquad g \in G_1.$$

The next section is devoted to reduce successively P^1 to three subbundles:

$$P^4 \subset P^3 \subset P^2 \subset P^1,$$

which are adapted to the biholomorphic equivalence problem for M.

3. Reductions of P^1

The coframe ω_0 gives a natural (local) trivialisation $P^1 \xrightarrow{tr} M \times G_1$ from which we may consider any differential form on M (resp. G_1) as a differential form on P^1 through the pullback by the first (resp. the second) component of tr. With this identification, the structure equations of P^1 are naturally obtained by the formula:

$$d\omega = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0. \tag{3}$$

The term $g \cdot d\omega_0$ contains the so-called torsion coefficients of P^1. A 1-form $\widetilde{\alpha}$ on P^1 is called a modified Maurer-Cartan form if its restriction to any fiber of P^1 is a Maurer-Cartan form of G_1, or equivalently, if it is of the form:

$$\widetilde{\alpha} := \alpha - x_\sigma \, \sigma - x_\rho \, \rho - x_\zeta \, \zeta - x_{\overline{\zeta}} \, \overline{\zeta},$$

where x_σ, x_ρ, x_ζ, $x_{\overline{\zeta}}$, are arbitrary complex-valued functions on M and where α is a Maurer-Cartan form of G_1.

A basis for the Maurer-Cartan forms of G_1 is given by the following 1-forms:

$$\alpha^1 := \frac{d\mathsf{a}}{\mathsf{a}},$$
$$\alpha^2 := -\frac{\mathsf{b}d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}},$$
$$\alpha^3 := -\frac{\mathsf{c}d\mathsf{a}}{\overline{\mathsf{a}}\,\mathsf{a}^3} - \frac{\mathsf{c}d\overline{\mathsf{a}}}{\overline{\mathsf{a}}^2\mathsf{a}^2} + \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}},$$
$$\alpha^4 = -\frac{(\mathsf{d}\mathsf{a}\overline{\mathsf{a}} - \mathsf{b}\mathsf{c})\, d\mathsf{a}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{c}d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}},$$
$$\alpha^5 = -\frac{\left(\mathsf{e}\mathsf{a}\overline{\mathsf{a}} - \overline{\mathsf{b}}\,\mathsf{c}\right) d\overline{\mathsf{a}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\mathsf{c}d\overline{\mathsf{b}}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},$$

together with their conjugate.

We derive the structure equations of P^1 from the relations (3), from which we extract the expression of $d\sigma$:

$$d\sigma = 2\,\alpha^1 \wedge \sigma + \overline{\alpha^1} \wedge \sigma$$
$$+ T^{\sigma}_{\sigma\rho}\,\sigma \wedge \rho - T^{\sigma}_{\sigma\zeta}\,\sigma \wedge \zeta - T^{\sigma}_{\sigma\overline{\zeta}}\,\sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \frac{\mathsf{a}}{\overline{\mathsf{a}}} B\ \rho \wedge \overline{\zeta}\,,$$

or equivalently:

$$d\sigma = 2\,\widetilde{\alpha}^1 \wedge \sigma + \overline{\widetilde{\alpha}^1} \wedge \sigma + \rho \wedge \zeta + \frac{\mathsf{a}}{\overline{\mathsf{a}}} B\ \rho \wedge \overline{\zeta}\,,$$

for a modified Maurer-Cartan form $\widetilde{\alpha}^1$. The coefficient

$$\frac{\mathsf{a}}{\overline{\mathsf{a}}} B,$$

which can not be absorbed for any choice of the modified Maurer-Cartan form $\widetilde{\alpha}^1$, is referred to as an essential torsion coefficient. From standard results on Cartan theory (see [10, 15]), a diffeomorphism of M is an isomorphism of the G_1-structure P^1 if and only if it is an isomorphism of the reduced bundle $P^2 \subset P^1$ consisting of those coframes ω on M such that

$$\frac{\mathsf{a}}{\overline{\mathsf{a}}} B = 1.$$

This is equivalent to the normalization:

$$\overline{\mathsf{a}} = \mathsf{a}B.$$

A coframe $\omega \in P^2$ is related to the coframe ω_0 by the relations:

$$\begin{cases} \sigma = \mathsf{a}^3\, B\, \sigma_0 \\ \rho = \mathsf{c}\,\sigma_0 + \mathsf{a}^2\, B\, \rho_0 \\ \zeta = \mathsf{d}\,\sigma_0 + \mathsf{b}\,\rho_0 + \mathsf{a}\,\zeta_0 \\ \overline{\zeta} = \mathsf{e}\,\sigma_0 + \overline{\mathsf{b}}\,\rho_0 + \mathsf{a}\,B\,\overline{\zeta}_0, \end{cases}$$

which are equivalent to:

$$\begin{cases} \sigma = \mathsf{a}'^{\,3}\,\sigma_1 \\ \rho = \mathsf{c}'\,\sigma_1 + \mathsf{a}'^{\,2}\,\rho_1 \\ \zeta = \mathsf{d}'\,\sigma_1 + \mathsf{b}\,\rho_1 + \mathsf{a}'\,\zeta_1 \\ \overline{\zeta} = \mathsf{e}'\,\sigma_1 + \overline{\mathsf{b}}\,\rho_1 + \mathsf{a}'\,\overline{\zeta}_1, \end{cases}$$

where:

$$\sigma_1 := \frac{\sigma_0}{B^{\frac{1}{2}}}, \qquad \rho_1 := \rho_0, \qquad \zeta_1 := \frac{\zeta_0}{B^{\frac{1}{2}}},$$

and

$$x' := x \cdot B^{\frac{1}{2}}, \qquad \text{for} \quad x = \mathsf{a},\, \mathsf{c},\, \mathsf{d},\, \mathsf{e}.$$

We notice that a' is a real parameter, and that σ_1 is a real 1-form. Let ω_1 be the coframe $\omega_1 := \big(\sigma_1, \rho_1, \zeta_1, \overline{\zeta}_1\big)$, and G_2 be the subgroup of G_1:

$$G_2 := \left\{ \begin{pmatrix} \mathsf{a}^3 & 0 & 0 & 0 \\ \mathsf{c} & \mathsf{a}^2 & 0 & 0 \\ \mathsf{d} & \mathsf{b} & \mathsf{a} & 0 \\ \mathsf{e} & \overline{\mathsf{b}} & 0 & \mathsf{a} \end{pmatrix},\ \mathsf{a} \in \mathbb{R}\setminus\{0\},\ \mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e} \in \mathbb{C} \right\}.$$

A coframe ω on M belongs to P^2 if and only if there is a local function $g : M \xrightarrow{g} G_2$ such that $\omega = g \cdot \omega_1$, namely P^2 is a G_2 structure on M.

The Maurer-Cartan forms of G_2 are given by:

$$\begin{aligned}
\beta^1 &:= \frac{d\mathsf{a}}{\mathsf{a}}, \\
\beta^2 &:= -\frac{\mathsf{b}d\mathsf{a}}{\mathsf{a}^3} + \frac{d\mathsf{b}}{\mathsf{a}^2}, \\
\beta^3 &:= -2\,\frac{\mathsf{c}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{c}}{\mathsf{a}^3}, \\
\beta^4 &= -\frac{\left(\mathsf{d}\mathsf{a}^2 - \mathsf{b}\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\mathsf{b}}{\mathsf{a}^5} + \frac{d\mathsf{d}}{\mathsf{a}^3}, \\
\beta^5 &= -\frac{\left(\mathsf{e}\mathsf{a}^2 - \overline{\mathsf{b}}\,\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\overline{\mathsf{b}}}{\mathsf{a}^5} + \frac{d\mathsf{e}}{\mathsf{a}^3},
\end{aligned}$$

together with $\overline{\beta^2}$, $\overline{\beta^3}$, $\overline{\beta^4}$, $\overline{\beta^5}$. Using formula (3), we get the structure equations of P^2:

$$\begin{aligned}
d\sigma = {} & 3\,\beta^1 \wedge \sigma \\
& + U^{\sigma}_{\sigma\rho}\,\sigma\wedge\rho + U^{\sigma}_{\sigma\zeta}\,\sigma\wedge\zeta + U^{\sigma}_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta}
\end{aligned}$$

$$\begin{aligned} d\rho = 2\beta^1 \wedge \rho + \beta^3 \wedge \sigma \\ + U^{\rho}_{\sigma\rho}\, \sigma \wedge \rho + U^{\rho}_{\sigma\zeta}\, \sigma \wedge \zeta + U^{\rho}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} \\ + U^{\rho}_{\rho\zeta}\, \rho \wedge \zeta + U^{\rho}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \zeta \wedge \overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\zeta = \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma \\ + U^{\zeta}_{\sigma\rho}\, \sigma \wedge \rho + U^{\zeta}_{\sigma\zeta}\, \sigma \wedge \zeta + U^{\zeta}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + U^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta \\ + U^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + U^{\zeta}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}. \end{aligned}$$

Introducing the modified Maurer-Cartan forms:

$$\widetilde{\beta}^i = \beta^i - y_\sigma\, \sigma - y^i_\rho\, \rho - y^i_\zeta\, \zeta - y^i_{\overline{\zeta}}\, \overline{\zeta},$$

the structure equations rewrite:

$$\begin{aligned} d\sigma = 3\, \widetilde{\beta}^1 \wedge \sigma \\ + \left(U^{\sigma}_{\sigma\rho} - 3\, y^1_\rho\right)\, \sigma \wedge \rho + \left(U^{\sigma}_{\sigma\zeta} - 3\, y^1_\zeta\right)\, \sigma \wedge \zeta \\ + \left(U^{\sigma}_{\sigma\overline{\zeta}} - 3\, y^1_{\overline{\zeta}}\right)\, \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta} \end{aligned}$$

$$\begin{aligned} d\rho = 2\widetilde{\beta}^1 \wedge \rho + \widetilde{\beta}^3 \wedge \sigma \\ + \left(U^{\rho}_{\sigma\rho} + 2\, y^1_\sigma - y^3_\rho\right)\, \sigma \wedge \rho + \left(U^{\rho}_{\sigma\zeta} - y^3_\zeta\right)\, \sigma \wedge \zeta \\ + \left(U^{\rho}_{\sigma\overline{\zeta}} - y^3_{\overline{\zeta}}\right)\, \sigma \wedge \overline{\zeta} + \left(U^{\rho}_{\rho\zeta} - 2\, y^1_\zeta\right)\, \rho \wedge \zeta \\ + \left(U^{\rho}_{\rho\overline{\zeta}} - 2\, y^1_{\overline{\zeta}}\right)\, \rho \wedge \overline{\zeta} + i\; \zeta \wedge \overline{\zeta}\, , \end{aligned}$$

$$\begin{aligned} d\zeta = \widetilde{\beta^1} \wedge \zeta + \widetilde{\beta^2} \wedge \rho + \widetilde{\beta^4} \wedge \sigma \\ + \left(U^{\zeta}_{\sigma\rho} + y^2_\sigma - y^4_\rho\right)\, \sigma \wedge \rho + \left(U^{\zeta}_{\sigma\zeta} + y^1_\sigma - y^4_\zeta\right)\, \sigma \wedge \zeta \\ + \left(U^{\zeta}_{\sigma\overline{\zeta}} - y^4_{\overline{\zeta}}\right)\, \sigma \wedge \overline{\zeta} + \left(U^{\zeta}_{\rho\zeta} + y^1_\rho - y^2_\zeta\right)\, \rho \wedge \zeta \\ + \left(U^{\zeta}_{\rho\overline{\zeta}} - y^2_{\overline{\zeta}}\right)\, \rho \wedge \overline{\zeta} + \left(U^{\zeta}_{\zeta\overline{\zeta}} - y^1_{\overline{\zeta}}\right)\, \zeta \wedge \overline{\zeta}\, , \end{aligned}$$

which leads to the following absorbtion equations:

$$\begin{aligned} 3\,y^1_\rho &= U^\sigma_{\sigma\rho}, & 3\,y^1_\zeta &= U^\sigma_{\sigma\zeta}, & 3\,y^1_{\overline{\zeta}} &= U^\sigma_{\sigma\overline{\zeta}}, \\ -2\,y^1_\sigma + y^3_\rho &= U^\rho_{\sigma\rho}, & y^3_\zeta &= U^\rho_{\sigma\zeta}, & y^3_{\overline{\zeta}} &= U^\rho_{\sigma\overline{\zeta}}, \\ 2\,y^1_\zeta &= U^\rho_{\rho\zeta}, & 2\,y^1_{\overline{\zeta}} &= U^\rho_{\rho\overline{\zeta}}, & -y^2_\sigma + y^4_\rho &= U^\zeta_{\sigma\rho}, \\ -y^1_\sigma + y^4_\zeta &= U^\zeta_{\sigma\zeta}, & y^4_{\overline{\zeta}} &= U^\zeta_{\sigma\overline{\zeta}}, & -y^1_\rho + y^2_\zeta &= U^\zeta_{\rho\zeta}, \\ y^2_{\overline{\zeta}} &= U^\zeta_{\rho\overline{\zeta}}, & y^1_{\overline{\zeta}} &= U^\zeta_{\zeta\overline{\zeta}}. \end{aligned}$$

Eliminating $y^1_{\overline{\zeta}}$ among these equations leads to:

$$U^\zeta_{\zeta\overline{\zeta}} = \frac{1}{2}\,U^\rho_{\rho\overline{\zeta}} = \frac{1}{3}\,U^\sigma_{\sigma\overline{\zeta}},$$

from which we deduce the following normalizations:

$$\mathsf{c} = \mathsf{a}^2\,\mathbf{C}_0,$$

and

$$\mathsf{b} = \mathsf{a}\,\mathbf{B}_0,$$

where:

$$\mathbf{C}_0 := \left(\frac{1}{2}\,\frac{\mathscr{L}(B)}{B^{\frac{1}{2}}} + \frac{1}{2}\,Q B^{\frac{1}{2}}\right),$$

and

$$\mathbf{B}_0 := \left(\frac{i}{3}\,\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}} - \frac{i}{3}\,\frac{A}{B^{\frac{1}{2}}} - \frac{i}{6}\,B^{\frac{1}{2}}Q - \frac{i}{6}\,\frac{\mathscr{L}(B)}{B^{\frac{1}{2}}}\right).$$

We introduce the coframe $\omega_2 := \big(\sigma_2, \rho_2, \zeta_2, \overline{\zeta}_2\big)$ on M, defined by:

$$\begin{cases} \sigma_2 := \sigma_1, \\ \rho_2 := \rho_1 + \mathbf{C}_0\,\sigma_1, \\ \zeta_2 := \zeta_1 + \mathbf{B}_0\,\rho_1, \end{cases}$$

and the 3-dimensional subgroup $G_3 \subset G_2$:

$$G_3 := \left\{ \begin{pmatrix} \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & \mathsf{a}^2 & 0 & 0 \\ \mathsf{d} & 0 & \mathsf{a} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \mathsf{a} \end{pmatrix},\ \mathsf{a} \in \mathbb{R}\setminus\{0\},\ \mathsf{d} \in \mathbb{C} \right\}.$$

The normalizations:

$$\mathsf{b} := \mathsf{a}\,\mathbf{B}_0, \qquad\qquad \mathsf{c} := \mathsf{a}^2\,\mathbf{C}_0,$$

amount to consider the subbundle $P^3 \subset P^2$ consisting of those coframes ω of the form

$$\omega := g\cdot\omega_2, \qquad \text{where } g \text{ is a function } \ g : M \xrightarrow{\ g\ } G_3.$$

A basis of the Maurer Cartan forms of G_3 is given by:

$$\gamma^1 := \frac{d\mathsf{a}}{\mathsf{a}}, \qquad \gamma^2 := -\frac{\mathsf{d}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{d}}{\mathsf{a}^3}, \qquad \overline{\gamma}_2.$$

The structure equations of P^3 are:

$$\begin{aligned} d\sigma &= 3\,\gamma^1 \wedge \sigma \\ &\qquad + V^{\sigma}_{\sigma\rho}\,\sigma\wedge\rho + V^{\sigma}_{\sigma\zeta}\,\sigma\wedge\zeta + V^{\sigma}_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\rho &= 2\gamma^1\wedge\rho + V^{\rho}_{\sigma\rho}\,\sigma\wedge\rho + V^{\rho}_{\sigma\zeta}\,\sigma\wedge\zeta + V^{\rho}_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} \\ &\qquad + V^{\rho}_{\rho\zeta}\,\rho\wedge\zeta + V^{\rho}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\zeta &= \gamma^1\wedge\zeta + \gamma^2\wedge\sigma \\ &\qquad + V^{\zeta}_{\sigma\rho}\,\sigma\wedge\rho + V^{\zeta}_{\sigma\zeta}\,\sigma\wedge\zeta + V^{\zeta}_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + V^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta \\ &\qquad + V^{\zeta}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + V^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}. \end{aligned}$$

$$\begin{aligned} d\overline{\zeta} &= \gamma^1\wedge\zeta + \gamma^3\wedge\sigma \\ &\qquad + V^{\overline{\zeta}}_{\sigma\rho}\,\sigma\wedge\rho + V^{\overline{\zeta}}_{\sigma\zeta}\,\sigma\wedge\zeta + V^{\overline{\zeta}}_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + V^{\overline{\zeta}}_{\rho\zeta}\,\rho\wedge\zeta \\ &\qquad + V^{\overline{\zeta}}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + V^{\overline{\zeta}}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}. \end{aligned}$$

It is straightforward to notice that $V^{\rho}_{\sigma\zeta}$ and $V^{\rho}_{\sigma\overline{\zeta}}$ are two essential torsion coefficients. The first one leads to the normalization:

$$\overline{\mathsf{d}} = \mathsf{a}\,\overline{\mathbf{D}}_0,$$

with

$$\overline{\mathbf{D}}_0 := \frac{i}{2}\,\frac{\mathscr{L}(B)^2}{B} + \frac{i}{3}\,Q\mathscr{L}(B) - \frac{i}{2}\,\mathscr{L}\left(\mathscr{L}(B)\right) - \frac{i}{2}\,B\mathscr{L}(Q) + \frac{i}{2}\,A\frac{\mathscr{L}(B)}{B} + \frac{i}{6}\,AQ + iBP,$$

while the second essential torsion coefficient gives the normalization:

$$\mathsf{d} = \mathsf{a}\,\mathbf{D}_0,$$

with:

$$\begin{aligned} \mathbf{D}_0 := &-\frac{2i}{3}\,\mathscr{L}(B)\,Q - \frac{i}{6}\,\frac{\mathscr{L}(B)\,A}{B} - \frac{i}{6}\,A\,Q + \frac{i}{6}\,\frac{\overline{\mathscr{L}}(B)\,Q}{B} - \frac{i}{3}\,\frac{\mathscr{L}(B)^2}{B} - \frac{i}{3}\,B\,Q^2 \\ &- i\,\mathscr{L}(A) - \frac{i}{3}\,\frac{\overline{\mathscr{L}}(B)\,\mathscr{L}(B)}{B^2} + \frac{i}{2}\,\frac{\overline{\mathscr{L}}\left(\mathscr{L}(B)\right)}{B} + \frac{i}{2}\,\overline{\mathscr{L}}(Q) - i\,B\,P. \end{aligned}$$

The coherency of the above formulae can be checked using the relations (1) and (2).

Let G_4 be the 1-dimensional Lie subgroup of G_3 whose elements g are of the form:

$$g := \begin{pmatrix} \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & \mathsf{a}^2 & 0 & 0 \\ 0 & 0 & \mathsf{a} & 0 \\ 0 & 0 & 0 & \mathsf{a} \end{pmatrix}, \qquad \mathsf{a} \in \mathbb{R} \setminus \{0\},$$

and let $\omega_3 := \big(\sigma_3, \rho_3, \zeta_3, \overline{\zeta}_3\big)$ be the coframe defined on M by:

$$\sigma_3 := \sigma_2, \qquad \rho_3 := \rho_2, \qquad \zeta_3 := \zeta_2 + \mathbf{D}_0\, \sigma_2.$$

The normalization of d is equivalent to the reduction of P^3 to a subbundle P^4 consisting of those coframes ω on M such that:

$$\omega := g \cdot \omega_3, \qquad \text{where } g \text{ is a function } \; g : M \xrightarrow{g} G_4.$$

The Maurer-Cartan forms of G_4 are spanned by:

$$\alpha := \frac{d\mathsf{a}}{\mathsf{a}}.$$

Proceeding as in the previous steps, we compute the structure equations of P^4:

$$\begin{aligned} d\sigma = 3\, \frac{d\mathsf{a}}{\mathsf{a}} \wedge \sigma & \\ & + W^{\sigma}_{\sigma\rho}\, \sigma \wedge \rho + W^{\sigma}_{\sigma\zeta}\, \sigma \wedge \zeta + W^{\sigma}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \end{aligned}$$

$$d\rho \;=\; 2\, \frac{d\mathsf{a}}{\mathsf{a}} \wedge \rho + W^{\rho}_{\sigma\rho}\, \sigma \wedge \rho + W^{\rho}_{\rho\zeta}\, \rho \wedge \zeta + W^{\rho}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \zeta \wedge \overline{\zeta},$$

$$\begin{aligned} d\zeta = \frac{d\mathsf{a}}{\mathsf{a}} \wedge \zeta & \\ & + W^{\zeta}_{\sigma\rho}\, \sigma \wedge \rho + W^{\zeta}_{\sigma\zeta}\, \sigma \wedge \zeta + W^{\zeta}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + W^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta \\ & + W^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + W^{\zeta}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\overline{\zeta} = \frac{d\mathsf{a}}{\mathsf{a}} \wedge \overline{\zeta} & \\ & + W^{\overline{\zeta}}_{\sigma\rho}\, \sigma \wedge \rho + W^{\overline{\zeta}}_{\sigma\zeta}\, \sigma \wedge \zeta + W^{\overline{\zeta}}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + W^{\overline{\zeta}}_{\rho\zeta}\, \rho \wedge \zeta \\ & + W^{\overline{\zeta}}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + W^{\overline{\zeta}}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}. \end{aligned}$$

Introducing the modified Maurer-Cartan form Λ:

$$\Lambda := \frac{d\mathsf{a}}{\mathsf{a}} + \frac{W^{\rho}_{\sigma\rho}}{2}\, \rho - \frac{W^{\sigma}_{\sigma\rho}}{3}\, \sigma - \frac{W^{\sigma}_{\sigma\rho}}{3}\, \zeta - \frac{W^{\sigma}_{\sigma\overline{\zeta}}}{3}\, \overline{\zeta},$$

these equations rewrite in the absorbed form as:
(4)

$$
\begin{aligned}
d\sigma &= 3\,\Lambda\wedge\sigma + \rho\wedge\zeta + \rho\wedge\overline{\zeta},\\
d\rho &= 2\,\Lambda\wedge\rho + i\,\zeta\wedge\overline{\zeta},\\
d\zeta &= \Lambda\wedge\zeta + \frac{I_1}{\mathsf{a}^4}\,\sigma\wedge\rho + \frac{I_2}{\mathsf{a}^3}\,\sigma\wedge\zeta + \frac{I_3}{\mathsf{a}^3}\,\sigma\wedge\overline{\zeta} + \frac{I_4}{\mathsf{a}^2}\,\rho\wedge\zeta + \frac{I_5}{\mathsf{a}^2}\,\rho\wedge\overline{\zeta},\\
d\overline{\zeta} &= \Lambda\wedge\overline{\zeta} + \frac{\overline{I_1}}{\mathsf{a}^4}\,\sigma\wedge\rho + \frac{\overline{I_3}}{\mathsf{a}^3}\,\sigma\wedge\zeta + \frac{\overline{I_2}}{\mathsf{a}^3}\,\sigma\wedge\overline{\zeta} + \frac{\overline{I_5}}{\mathsf{a}^2}\,\rho\wedge\zeta + \frac{\overline{I_4}}{\mathsf{a}^2}\,\rho\wedge\overline{\zeta},
\end{aligned}
$$

where the invariants $I_i,\ i = 2\dots5$, are given by:

$$
\begin{aligned}
\overline{I_2} = \frac{i}{8}\,\frac{Q\mathscr{L}(B)^2}{B^{\frac{1}{2}}} - \frac{i}{8}\,B^{\frac{1}{2}}\mathscr{L}(B)Q^2 - \frac{3i}{4}\,\frac{\mathscr{L}(\mathscr{L}(B))\mathscr{L}(B)}{B^{\frac{1}{2}}} + \frac{i}{4}\,B^{\frac{1}{2}}\mathscr{L}(B)\mathscr{L}(Q)\\
- \frac{i}{2}\,B^{\frac{1}{2}}P\mathscr{L}(B) - \frac{i}{4}\,B^{\frac{1}{2}}Q\mathscr{L}(\mathscr{L}(B)) - \frac{i}{4}B^{\frac{1}{2}}Q\mathscr{L}(\mathscr{L}(B))\\
- \frac{3i}{4}\,B^{\frac{3}{2}}Q\mathscr{L}(Q) + \frac{i}{2}\,B^{\frac{3}{2}}PQ + \frac{3i}{8}\,\frac{\mathscr{L}(B)^3}{B^{\frac{3}{2}}} + \frac{i}{8}\,B^{\frac{3}{2}}Q^3\\
+ \frac{i}{2}\,B^{\frac{3}{2}}\mathscr{L}(\mathscr{L}(Q)) + \frac{i}{2}\,B^{\frac{1}{2}}\,\mathscr{L}\left(\mathscr{L}\left(\mathscr{L}(B)\right)\right) - iB^{\frac{3}{2}}\mathscr{L}(P)
\end{aligned}
$$

$$
\begin{aligned}
I_3 = -\mathbf{D}_0\mathbf{C}_0 + \frac{\mathscr{L}(B)}{B^{\frac{1}{2}}}\,\mathbf{D}_0 + B^{\frac{1}{2}}Q\mathbf{D}_0 + \frac{A}{B^{\frac{1}{2}}}\,\mathbf{D}_0 - \frac{\overline{\mathscr{L}}\,(\mathbf{D}_0)}{B^{\frac{1}{2}}} - i\,\mathbf{B}_0\mathbf{D}_0 + i\,\mathbf{B}_0^2\mathbf{C}_0\\
- \frac{A}{B^{\frac{1}{2}}}\,\mathbf{B}_0\mathbf{C}_0 + \mathbf{B}_0\,\mathscr{L}(A) + BP\,\mathbf{B}_0 + \frac{\overline{\mathscr{L}}\,(\mathbf{B}_0)}{B^{\frac{1}{2}}}\,\mathbf{C}_0 + \frac{1}{2}\,\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}}\,\mathbf{B}_0\mathbf{C}_0
\end{aligned}
$$

$$
\overline{I_4} = \frac{3}{4}\,i\,\frac{\mathscr{L}(B)^2}{B} + \frac{1}{6}\,i\,\mathscr{L}(B)\,Q + \frac{11}{36}\,i\,B\,Q^2 - i\,\mathscr{L}(\mathscr{L}(B)) - \frac{2}{3}\,i\,B\,\mathscr{L}(Q) + i\,BP,
$$

$$
\begin{aligned}
\overline{I_5} = \frac{i}{3}\,\mathscr{L}(A) + \frac{i}{3}\,\overline{\mathscr{L}}(Q) - i\,\frac{\overline{\mathscr{L}}(\mathscr{L}(B))}{B} + \frac{5}{12}\,i\frac{\mathscr{L}(B)^2}{B} - \frac{i}{3}\,B\mathscr{L}(Q)\\
+ \frac{11}{36}\,i\,B\,Q^2 + i\,B\,P + \frac{2}{3}\,i\,\frac{\mathscr{L}(\overline{\mathscr{L}}(B))}{B} - \frac{i}{3}\,\mathscr{L}(\mathscr{L}(B))\\
+ \frac{i}{3}\,\frac{A\,\mathscr{L}(B)}{B} + \frac{7}{18}\,i\,\mathscr{L}(B)\,Q - \frac{i}{9}\,\frac{\overline{\mathscr{L}}(B)\,Q}{B} + \frac{i}{9}\,A\,Q,
\end{aligned}
$$

and I_1 is given by:

$$
I_1 = \frac{2i}{3}\,\left(I_3\right)_\zeta - \frac{2i}{3}\,\left(I_2\right)_{\overline{\zeta}}.
$$

The exterior derivative of Λ can be determined by taking the exterior derivative of the four equations (4), which leads to the so-called Bianchi-Cartan's identities. For example, taking the exterior derivative of the first equation of (4), one gets:

$$0 = \left[3\, d\Lambda + \left(\frac{I_2}{\mathsf{a}^3} + \frac{\overline{I_3}}{\mathsf{a}^3}\right) \rho\wedge\zeta + \left(\frac{\overline{I_2}}{\mathsf{a}^3} + \frac{I_3}{\mathsf{a}^3}\right) \rho\wedge\zeta\right] \wedge \sigma,$$

while taking the exterior derivative of the second equation gives:

$$0 = \left[2\, d\Lambda - i\frac{I_1}{\mathsf{a}^4}\, \sigma\wedge\overline{\zeta} + i\,\frac{\overline{I_1}}{\mathsf{a}^4}\, \sigma\wedge\zeta\right] \wedge \rho.$$

Eventually we get:
(5)

$$d\Lambda = \frac{i}{2}\frac{I_1}{\mathsf{a}^4}\,\sigma\wedge\overline{\zeta} - \frac{i}{2}\frac{\overline{I_1}}{\mathsf{a}^4}\,\sigma\wedge\zeta - \frac{1}{3}\left(\frac{I_2}{\mathsf{a}^3} + \frac{\overline{I_3}}{\mathsf{a}^3}\right)\rho\wedge\zeta - \frac{1}{3}\left(\frac{\overline{I_2}}{\mathsf{a}^3} + \frac{I_3}{\mathsf{a}^3}\right)\rho\wedge\zeta + \frac{I_0}{\mathsf{a}^4}\,\sigma\wedge\zeta\,,$$

where I_0 is given by:

$$I_0 := -\frac{1}{2\mathsf{a}^4}\,(I_1)_\zeta - \frac{1}{2\mathsf{a}^4}\,\left(\overline{I_1}\right)_{\overline{\zeta}}.$$

4. Cartan Connection

We recall that the model for CR-manifolds belonging to general class II is Beloshapka's cubic $\mathsf{B} \subset \mathbb{C}^3$, which is defined by the equations:

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,. \end{aligned}$$

Its Lie algebra of infinitesimal CR-automorphisms is given by the following theorem:

Theorem 2. [12]. *Beloshapka's cubic,*

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \end{aligned}$$

has a $\mathbf{5}$*-dimensional Lie algebra of CR-automorphisms* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{B})$. *A basis for the Maurer-Cartan forms of* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{B})$ *is provided by the* 5 *differential* 1*-forms* σ, ρ, ζ, $\overline{\zeta}$, α, *which satisfy the structure equations:*

$$\begin{aligned} d\sigma &= 3\,\alpha\wedge\sigma + \rho\wedge\zeta + \rho\wedge\overline{\zeta}\,, \\ d\rho &= 2\,\alpha\wedge\rho + i\,\zeta\wedge\overline{\zeta}\,, \\ d\zeta &= \alpha\wedge\zeta\,, \\ d\overline{\zeta} &= \alpha\wedge\overline{\zeta}\,, \\ d\alpha &= 0. \end{aligned}$$

Let us write $\mathfrak{g}$ instead of $\mathsf{aut}_{\mathsf{CR}}(\mathsf{B})$ for the Lie algebra of inifinitesimal automorphisms of Beloshapka's cubic, and let $\big(e_\alpha, e_\sigma, e_\rho, e_\zeta, e_{\overline{\zeta}}\big)$ be the dual basis of the basis of Maurer-Cartan 1-forms: $\big(\alpha, \sigma, \rho, \zeta, \overline{\zeta}\big)$ of $\mathfrak{g}$. From the above structure equations, the Lie brackets structure of $\mathfrak{g}$ is given by:

$$
\begin{aligned}
&[e_\alpha, e_\sigma] = -3\, e_\sigma, && [e_\alpha, e_\rho] = -2\, e_\rho, && [e_\alpha, e_\zeta] = -\, e_\zeta, \\
&[e_\alpha, e_{\overline{\zeta}}] = -\, e_{\overline{\zeta}}, && [e_\rho, e_\zeta] = -e_\sigma, && [e_\rho, e_{\overline{\zeta}}] = -e_\sigma, \\
&[e_\zeta, e_{\overline{\zeta}}] = -i\, e_\rho,
\end{aligned}
$$

the remaining brackets being equal to zero.

We refer to [5], p. 127-128, for the definition of a Cartan connection. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the subalgebra spanned by e_α, $\mathfrak{G}$ the connected, simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and $\mathfrak{G}_0$ the connected closed 1-dimensional subgroup of $\mathfrak{G}$ generated by $\mathfrak{g}_0$. We notice that $\mathfrak{G}_0 \cong G_4$, so that P^4 is a principal bundle over M with structure group $\mathfrak{G}_0$, and that $\dim \mathfrak{G}/\mathfrak{G}_0 = \dim M = 4$.

Let $\big(\Lambda, \sigma, \rho, \zeta, \overline{\zeta}\big)$ be the coframe of 1-forms on P^4 whose structure equation are given by (4) – (5) and ω the 1-form on P with values in $\mathfrak{g}$ defined by:

$$
\omega(X) := \Lambda(X)\, e_\alpha + \sigma(X)\, e_\sigma + \rho(X)\, e_\rho + \zeta(X)\, e_\zeta, + \overline{\zeta}(X)\, e_{\overline{\zeta}},
$$

for $X \in T_p P^4$. We have:

Theorem 3. *ω is a Cartan connection on P^4.*

Proof. We shall check that the following three conditions hold:

(1) $\omega(e_\alpha^*) = e_\alpha$, where e_α^* is the vertical vector field on P^4 generated by the action of e_α,
(2) $R_a^* \omega = \mathsf{Ad}(a^{-1})\, \omega$ for every $a \in \mathfrak{G}_0$,
(3) for each $p \in P^4$, ω_p is an isomorphism $T_p P^4 \xrightarrow{\omega_p} \mathfrak{g}$.

Condition (3) is trivially satisfied as $\big(\Lambda, \sigma, \rho, \zeta, \overline{\zeta}\big)$ is a coframe on P^4 and thus defines a basis of $T_p^* P^4$ at each point p.

Condition (1) follows simply from the fact that Λ is a modified-Maurer Cartan form on P^4:

$$
\Lambda = \frac{d\mathsf{a}}{\mathsf{a}} + \frac{W^\rho_{\sigma\rho}}{2}\, \rho - \frac{W^\sigma_{\sigma\rho}}{3}\, \sigma - \frac{W^\sigma_{\sigma\rho}}{3}\, \zeta - \frac{W^\sigma_{\sigma\overline{\zeta}}}{3}\, \overline{\zeta},
$$

so that

$$
\omega(e_\alpha^*) = \Lambda(e_{\alpha^*}) = e_\alpha,
$$

as

$$
\sigma(e_\alpha^*) = \rho(e_\alpha^*) = \zeta(e_\alpha^*) = \overline{\zeta}(e_\alpha^*) = 0, \qquad \frac{d\mathsf{a}}{\mathsf{a}}(e_\alpha^*) = 1,
$$

since e_α^* is a vertical vector field on P^4.

Condition (2) is equivalent to its infinitesimal counterpart:

$$\mathscr{L}_{e_\alpha^*}\,\omega = -\mathsf{ad}_{e_\alpha}\omega,$$

where $\mathscr{L}_{e_\alpha^*}\,\omega$ is the Lie derivative of ω by the vector field e_α^* and where ad_{e_α} is the linear map $\mathfrak{g} \to \mathfrak{g}$ defined by: $\mathsf{ad}_{e_\alpha}(X) = [e_\alpha, X]$. We determine $\mathscr{L}_{e_\alpha^*}\,\omega$ with the help of Cartan's formula:

$$\mathscr{L}_{e_\alpha^*}\,\omega = e_{\alpha^*} \lrcorner\, d\omega + d\left(e_\alpha^* \lrcorner\, \omega\right),$$

with

$$d\left(e_\alpha^* \lrcorner\, \omega\right) = 0$$

from condition (1). The structure equations (4)–(5) give:

$$e_{\alpha^*} \lrcorner\, d\omega = \begin{pmatrix} 0 \\ 3\,\sigma \\ 2\,\rho \\ \zeta \\ \overline{\zeta} \end{pmatrix},$$

which is easily seen being equal to $-\mathsf{ad}_{e_\alpha}\omega$ from the Lie bracket structure of $\mathfrak{g}$. □

From theorem 3, the structure equations (4) and (5), and the fact that the invariants I_0 and I_1 are expressed in terms of I_2, I_3, I_4, I_5, we have:

Theorem 4. *A CR-manifold M belonging to general class* II *is locally biholomorphic to Beloshapka's cubic* $\mathsf{B} \subset \mathbb{C}^3$ *if and only if the condition*

$$I_2 \equiv I_3 \equiv I_4 \equiv I_5 \equiv 0$$

holds locally on M.

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CANONICAL CARTAN CONNECTION FOR 5-DIMENSIONAL CR-MANIFOLDS BELONGING TO GENERAL CLASS III_2

SAMUEL POCCHIOLA

ABSTRACT

We study the equivalence problem for CR-manifolds belonging to general class III_2, i.e. the 5-dimensional CR-manifolds of CR-dimension 1 and codimension 3 whose CR-bundle satisfies a degeneracy condition which has been introduced in [9]. For such a CR-manifold M, we construct a canonical Cartan connection on a 6-dimensional principal bundle P on M. This provides a complete set of biholomorphic invariants for M.

1. Introduction

As highlighted by Henri Poincaré [14] in 1907, the (local) biholomorphic equivalence problem between two submanifolds M and M' of $\mathbb{C}^N$ is to determine whether or not there exists a (local) biholomorphism ϕ of $\mathbb{C}^N$ such that $\phi(M) = M'$. Elie Cartan [2, 3] solved this problem for hypersurfaces $M^3 \subset \mathbb{C}^2$ in 1932, as he constructed a "hyperspherical connection" on such hypersurfaces by using the powerful technique which is now referred to as Cartan's equivalence method.

Given a manifold M and some geometric data specified on M, which usually appears as a G-structure on M (i.e. a reduction of the bundle of coframes of M), Cartan's equivalence method seeks to provide a principal bundle P on M together with a coframe ω of 1-forms on P which is adapted to the geometric structure of M in the following sense: an isomorphism between two such geometric structures M and M' lifts to a unique isomorphism between P and P' which sends ω on ω'. The equivalence problem between M and M' is thus reduced to an equivalence problem between $\{e\}$-structures, which is well understood [10, 15].

We recall that a CR-manifold M is a real manifold endowed with a subbundle L of $\mathbb{C} \otimes TM$ of even rank $2n$ such that

(1) $L \cap \overline{L} = \{0\}$
(2) L is formally integrable, i.e. $[L, L] \subset L$.

The integer n is the CR-dimension of M and $k = \dim M - 2n$ is the codimension of M. In a recent attempt [9] to solve the equivalence problem for CR-manifolds up to dimension 5, it has been shown that one can restrict the study to six different general classes of CR-manifolds of dimension ≤ 5, which have been referred to as general classes I, II, III_1, III_2, IV_1 and IV_2. The aim of this paper is to provide a solution to the equivalence problem for CR-manifolds which belong to general class III_2, that is the CR-manifolds of dimension 5 and of CR-dimension 1 such that $\mathbb{C} \otimes TM$ is spanned by L, $\overline{L}$ and their Lie brackets up to order no less than 3. More precisely, the following rank conditions hold:

$$\begin{aligned}
3 &= \mathrm{rank}_{\mathbb{C}} \left(L + \overline{L} + [L,\, \overline{L}]\right), \\
4 &= \mathrm{rank}_{\mathbb{C}} \left(L + \overline{L} + [L,\, \overline{L}] + [L, [L,\, \overline{L}]]\right), \\
4 &= \mathrm{rank}_{\mathbb{C}} \left(L + \overline{L} + [L,\, \overline{L}] + [L, [L,\, \overline{L}]] + [\overline{L}, [L,\, \overline{L}]]\right), \\
5 &= \mathrm{rank}_{\mathbb{C}} \left(L + \overline{L} + [L,\, \overline{L}] + [L, [L,\, \overline{L}]] + [\overline{L}, [L,\, \overline{L}]] + [L, [L, [L,\, \overline{L}]]]\right),
\end{aligned}$$

the third one beeing an exceptional degeneracy assumption.

The main result of the present paper is the following:

Theorem 1. *Let M be a CR-manifold belonging to general class III_2. There exists a 6-dimensional subbundle P of the bundle of coframes $\mathbb{C} \otimes F(M)$ of M and a coframe $\omega := (\Lambda, \tau, \sigma, \rho, \zeta, \overline{\zeta})$ on P such that any CR-diffeomorphism h of M lifts to a bundle isomorphism h^* of P which satisfies $h^*(\omega) = \omega$. Moreover the structure equations of ω on P are of the form:*

$$d\tau = 4\,\Lambda \wedge \tau + \mathfrak{J}_1\, \tau \wedge \zeta - \mathfrak{J}_1\, \tau \wedge \overline{\zeta} + 3\,\mathfrak{J}_1\, \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}\,,$$

$$\begin{aligned}
d\sigma = 3\,\Lambda \wedge \sigma
&+ \mathfrak{J}_2\, \tau \wedge \rho + \mathfrak{J}_3\, \tau \wedge \zeta + \overline{\mathfrak{J}_3}\, \tau \wedge \overline{\zeta} + \mathfrak{J}_4\, \sigma \wedge \rho \\
&- \frac{\mathfrak{J}_1}{2}\, \sigma \wedge \zeta + \frac{\mathfrak{J}_1}{2}\, \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta},
\end{aligned}$$

$$\begin{aligned}
d\rho = 2\Lambda \wedge \rho
&+ \mathfrak{J}_5\, \tau \wedge \sigma + \mathfrak{J}_6\, \tau \wedge \rho + \mathfrak{J}_7\, \tau \wedge \zeta + \overline{\mathfrak{J}_7}\, \tau \wedge \overline{\zeta} + \mathfrak{J}_8\, \sigma \wedge \rho + \mathfrak{J}_9\, \sigma \wedge \zeta \\
&+ \overline{\mathfrak{J}_9}\, \sigma \wedge \overline{\zeta} - \frac{\mathfrak{J}_1}{2}\, \rho \wedge \zeta + \frac{\mathfrak{J}_1}{2}\, \rho \wedge \overline{\zeta} + i\; \zeta \wedge \overline{\zeta}\,,
\end{aligned}$$

$$\begin{aligned}
d\zeta = \Lambda \wedge \zeta
&+ \mathfrak{J}_{10}\, \tau \wedge \sigma + \mathfrak{J}_{11}\, \tau \wedge \rho + \mathfrak{J}_{12}\, \tau \wedge \zeta + \mathfrak{J}_{13}\, \tau \wedge \overline{\zeta} \\
&+ \mathfrak{J}_{14}\, \sigma \wedge \rho + \mathfrak{J}_{15}\, \sigma \wedge \zeta\,,
\end{aligned}$$

$$d\Lambda = \sum_{\nu\mu} X_{\nu\mu}\, \nu \wedge \mu\,, \qquad \nu,\, \mu = \tau,\, \sigma,\, \rho,\, \zeta, \overline{\zeta},$$

where $\mathfrak{J}_i$, $X_{\nu\mu}$, are functions on P.

The model manifold for this class is provided by the CR-manifold $\mathsf{N} \subset \mathbb{C}^3$ given by the equations:

$$\mathsf{N}: \quad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z + \overline{z}), \\ w_3 &= \overline{w_3} + 2i\,z\overline{z}\left(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2\right), \end{aligned}$$

Cartan's equivalence method has been applied to this model in [12], where it has been shown that the coframe $(\Lambda, \tau, \sigma, \rho, \zeta, \overline{\zeta})$ of theorem 1 satisfy the simplified structure equations:

$$\begin{aligned} d\tau &= 4\,\Lambda \wedge \tau \,+\, \sigma \wedge \zeta \,+\, \sigma \wedge \overline{\zeta}\;, \\ d\sigma &= 3\,\Lambda \wedge \sigma \,+\, \rho \wedge \zeta \,+\, \rho \wedge \overline{\zeta}\;, \\ d\rho &= 2\,\Lambda \wedge \rho + i\;\zeta \wedge \overline{\zeta}\;, \\ d\zeta &= \Lambda \wedge \zeta\;, \\ d\overline{\zeta} &= \Lambda \wedge \overline{\zeta}\;, \\ d\Lambda &= 0, \end{aligned}$$

corresponding to the case where the biholomorphic invariants $\mathfrak{J}_i$ vanish identically. This result, together with the Lie algebra structure of the infinitesimal CR-automorphisms of the model, implies the existence of a Cartan connection on M, which we construct in section 4.

We start in section 2 with the construction of a canonical G-structure P^1 on M, (e.g. a subbundle of the bundle of coframes of M), which encodes the equivalence problem for M under CR-automorphisms in the following sense: a diffeomorphism

$$h : M \longrightarrow M$$

is a CR-automorphism of M if and only if

$$h^* : P^1 \longrightarrow P^1$$

is a G-structure isomorphism of P^1. We refer to [9, 6, 7] for details on the results summarized in this section and to [15] for an introduction to G-structures. Section 3 is devoted to reduce successively P^1 to four subbundles:

$$P^5 \subset P^4 \subset P^3 \subset P^2 \subset P^1,$$

which are still adapted to the biholomorphic equivalence problem for M. We use Cartan equivalence method, for which we refer to [10]. Eventually a Cartan connection is constructed on P^5 in section 4.

2. INITIAL G-STRUCTURE

Let M be a CR-manifold belonging to general class III_2 and $\mathscr{L}$ be a local generator of the CR-bundle L of M. As M belongs to general class III_2, the three vector fields $\mathscr{T}$, $\mathscr{S}$, $\mathscr{R}$, defined by:

$$\mathscr{T} := i\,[\mathscr{L}, \overline{\mathscr{L}}],$$
$$\mathscr{S} := [\mathscr{L}, \mathscr{T}],$$
$$\mathscr{R} := [\mathscr{L}, \mathscr{T}],$$

are such that the following biholomorphic invariant conditions hold:

$$3 = \text{rank}_{\mathbb{C}} \left(\mathscr{L}, \overline{\mathscr{L}}, \mathscr{T}\right), \qquad 4 = \text{rank}_{\mathbb{C}} \left(\mathscr{L}, \overline{\mathscr{L}}, \mathscr{T}, \mathscr{S}\right),$$
$$4 = \text{rank}_{\mathbb{C}} \left(\mathscr{L}, \overline{\mathscr{L}}, \mathscr{T}, \mathscr{S}, \overline{\mathscr{S}}\right), \qquad 5 = \text{rank}_{\mathbb{C}} \left(\mathscr{L}, \overline{\mathscr{L}}, \mathscr{T}, \mathscr{S}, \mathscr{R}\right).$$

As a result there exist two functions A and B such that:

$$\overline{\mathscr{S}} = A \cdot \mathscr{T} + B \cdot \mathscr{S}.$$

From the fact that $\overline{\overline{\mathscr{S}}} = \mathscr{S}$, the functions A and B satisfy the relations:

$$B\overline{B} = 1,$$
$$\overline{A} + \overline{B}A = 0.$$

There also exist three functions E, F, G, such that:

$$[\mathscr{L}, \mathscr{R}] = E \cdot \mathscr{T} + F \cdot \mathscr{S} + G \cdot \mathscr{R}.$$

The five functions A, B, E, F, G appear to be fundamental as all other Lie brackets between the vector fields $\mathscr{L}$, $\overline{\mathscr{L}}$, $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{R}$ can be expressed in terms of these five functions and their $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivatives.

In the case of an embedded CR-manifold $M \subset \mathbb{C}^4$, we can give an explicit formula for the fundamental vector field $\mathscr{L}$, and hence for the functions A, B, P, Q, in terms of a graphing function of M. We refer to [8] for details on this question. Let us just mention that the submanifold $M \subset \mathbb{C}^4$ is represented in local coordinates:

$$(z, w_1, w_2, w_3) = (x + i\,y,\; u_1 + i\,v_1,\; u_2 + i\,v_2,\; u_3 + i\,v_3),$$

as a graph:

$$v_1 = \phi_1(x, y, u_1, u_2, u_3),$$
$$v_2 = \phi_2(x, y, u_1, u_2, u_3),$$
$$v_3 = \phi_3(x, y, u_1, u_2, u_3).$$

There exists a unique local generator $\mathscr{L}$ of $T^{1,0}M$ of the form:

$$\mathscr{L} = \frac{\partial}{\partial z} + A^1 \frac{\partial}{\partial u_1} + A^2 \frac{\partial}{\partial u_2} + A^3 \frac{\partial}{\partial u_3},$$

having conjugate:

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \bar{z}} + \overline{A^1}\,\frac{\partial}{\partial u_1} + \overline{A^2}\,\frac{\partial}{\partial u_2} + \overline{A^3}\,\frac{\partial}{\partial u_3},$$

which is a generator of $T^{0,1}M$. The explicit expressions of the functions A^1, A^2 and A^3 in terms of ϕ can be found in [8].

Returning to the general case of abstract CR-manifolds, let

$$\omega_0 := \big(\tau_0, \sigma_0, \rho_0, \zeta_0, \overline{\zeta}_0\big)$$

be the dual coframe of $\big(\mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}\big)$. We have:

Lemma 1. [7]. *The structure equations enjoyed by ω_0 are of the form:*

$$\begin{aligned}
d\tau_0 &= T\,\tau_0\wedge\sigma_0 + Q\,\tau_0\wedge\rho_0 + K\,\tau_0\wedge\zeta_0 + G\,\tau_0\wedge\zeta_0 \\
&\quad + N\,\sigma_0\wedge\rho_0 + \sigma_0\wedge\zeta_0 + B\,\sigma_0\wedge\overline{\zeta}_0\,, \\
d\sigma_0 &= S\,\tau_0\wedge\sigma_0 + P\,\tau_0\wedge\rho_0 + F\,\tau_0\wedge\zeta_0 + J\,\tau_0\wedge\overline{\zeta}_0 + M\,\sigma_0\wedge\rho_0 \\
&\quad + (\mathscr{L}(B) + A)\,\sigma_0\wedge\overline{\zeta}_0 + B\,\rho_0\wedge\overline{\zeta}_0 + \rho_0\wedge\zeta_0\,, \\
d\rho_0 &= R\,\tau_0\wedge\sigma_0 + O\,\tau_0\wedge\rho_0 + H\,\tau_0\wedge\zeta_0 + E\,\tau_0\wedge\zeta_0 \\
&\quad + L\,\sigma_0\wedge\rho_0 + \mathscr{L}(A)\,\sigma_0\wedge\overline{\zeta}_0 + A\,\rho_0\wedge\overline{\zeta}_0 + i\,\zeta_0\wedge\overline{\zeta}_0\,, \\
d\zeta_0 &= 0, \\
d\overline{\zeta}_0 &= 0,
\end{aligned}$$

where the twelve functions:

$$H,\ J,\ K,\ L,\ M,\ N,\ O,\ P,\ Q,\ R,\ S,\ T,$$

can be expressed in terms of the five fundamental functions:

$$A,\ B,\ E,\ F,\ G,$$

and their $\{\mathscr{L}, \overline{\mathscr{L}}\}$-derivatives.

Let $h:\ M \longrightarrow M$ be a CR-automorphism of M. As we have

$$h_*\,(L) = L,$$

there exists a non-vanishing complex-valued function a on M such that:

$$h_*\,(\mathscr{L}) = \mathsf{a}\,\mathscr{L}.$$

From the definition of $\mathscr{T}$, $\mathscr{S}$, $\mathscr{R}$ and the invariance

$$h_*\,([X,Y]) = [h_*(X), h_*(Y)]$$

for any vector fields X, Y on M, we easily get the existence of eight functions

$$\mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k} : M \longrightarrow \mathbb{C},$$

such that

$$h_* \begin{pmatrix} \mathscr{L} \\ \overline{\mathscr{L}} \\ \mathscr{T} \\ \mathscr{S} \\ \mathscr{R} \end{pmatrix} = \begin{pmatrix} \mathsf{a} & 0 & 0 & 0 & 0 \\ 0 & \overline{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{b} & \overline{\mathsf{b}} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \mathsf{e} & \mathsf{d} & \mathsf{c} & \mathsf{a}^2\overline{\mathsf{a}} & 0 \\ \mathsf{k} & \mathsf{h} & \mathsf{g} & \mathsf{f} & \mathsf{a}^3\overline{\mathsf{a}} \end{pmatrix} \cdot \begin{pmatrix} \mathscr{L} \\ \overline{\mathscr{L}} \\ \mathscr{T} \\ \mathscr{S} \\ \mathscr{R} \end{pmatrix}.$$

This is summarized in the following lemma:

Lemma 2. [6]. *Let $h: M \longrightarrow M$ a CR-automorphism of M and let G_1 be the subgroup of $\mathsf{GL}_5(\mathbb{C})$:*

$$G_1 := \left\{ \begin{pmatrix} \mathsf{a}^3\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ \mathsf{f} & \mathsf{a}^2\overline{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{g} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \mathsf{h} & \mathsf{d} & \mathsf{b} & \mathsf{a} & 0 \\ \mathsf{k} & \mathsf{e} & \overline{\mathsf{b}} & 0 & \overline{\mathsf{a}} \end{pmatrix}, \ \mathsf{a} \in \mathbb{C} \setminus \{0\}, \ \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k} \in \mathbb{C} \right\}.$$

Then the pullback ω of ω_0 by h, $\omega := h^*\omega_0$, satisfies:

$$\omega = g \cdot \omega_0,$$

where g is smooth (locally defined) function $M \xrightarrow{g} G_1$.

Let P^1 be the G_1-structure on M defined by the coframes ω of the form

$$\omega := g \cdot \omega_0, \qquad g \in G_1$$

The next section is devoted to construct four subgroups of G_1:

$$G_5 \subset G_4 \subset G_3 \subset G_2 \subset G_1,$$

and four G_i-structures on M:

$$P^5 \subset P^4 \subset P^3 \subset P^2 \subset P^1,$$

which are adapted to the biholomorphic equivalence problem for M in the sense that a diffeomorphism h of M is a CR-automorphism if and only if h^* is a G_i-structure isomorphism of P^i.

3. Reductions of P^1

The coframe ω_0 gives a natural (local) trivialisation $P^1 \xrightarrow{tr} M \times G_1$ from which we may consider any differential form on M (resp. G_1) as a differential form on P^1 through the pullback by the first (resp. the second) component of tr. With this identification, the structure equations of P^1 are naturally obtained by the formula:

$$dw = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0. \tag{1}$$

The term $g \cdot d\omega_0$ contains the so-called torsion coefficients of P^1. A 1-form $\widetilde{\alpha}$ on P^1 is called a modified Maurer-Cartan form if its restriction to any fiber of P^1 is a Maurer-Cartan form of G_1, or equivalently, if it is of the form:

$$\widetilde{\alpha} := \alpha - x_\tau\, \tau - x_\sigma\, \sigma - x_\rho\, \rho - x_\zeta\, \zeta\, - x_{\overline{\zeta}}\, \overline{\zeta},$$

where x_σ, x_ρ, x_ζ, $x_{\overline{\zeta}}$, are arbitrary complex-valued functions on M and where α is a Maurer-Cartan form of G_1.

A basis for the Maurer-Cartan forms of G_1 is given by the following 1-forms:

$$\alpha^1 := \frac{d\mathsf{a}}{\mathsf{a}},$$

$$\alpha^2 := -\frac{\mathsf{b}\,d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}},$$

$$\alpha^3 := -\frac{\mathsf{c}\,d\mathsf{a}}{\overline{\mathsf{a}}\,\mathsf{a}^3} - \frac{\mathsf{c}\,d\overline{\mathsf{a}}}{\overline{\mathsf{a}}^2\mathsf{a}^2} + \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}},$$

$$\alpha^4 := -\frac{(\mathsf{d}\mathsf{a}\overline{\mathsf{a}} - \mathsf{b}\mathsf{c})\,d\mathsf{a}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{c}\,d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}},$$

$$\alpha^5 := -\frac{\big(\mathsf{e}\mathsf{a}\overline{\mathsf{a}} - \overline{\mathsf{b}}\,\mathsf{c}\big)\,d\overline{\mathsf{a}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\mathsf{c}\,d\overline{\mathsf{b}}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},$$

$$\alpha^6 := -2\,\frac{\mathsf{f}\,d\mathsf{a}}{\overline{\mathsf{a}}\,\mathsf{a}^4} - \frac{\mathsf{f}\,d\overline{\mathsf{a}}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{f}}{\overline{\mathsf{a}}\,\mathsf{a}^3},$$

$$\alpha^7 := -\frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\mathsf{a}}{\overline{\mathsf{a}}^2\mathsf{a}^6} - \frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\overline{\mathsf{a}}}{\overline{\mathsf{a}}^3\mathsf{a}^5} - \frac{\mathsf{f}\,d\mathsf{c}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{g}}{\overline{\mathsf{a}}\,\mathsf{a}^3},$$

$$\alpha^8 := -\frac{\big(\mathsf{h}\mathsf{a}^3\overline{\mathsf{a}}^2 - \mathsf{d}\mathsf{f}\mathsf{a}\overline{\mathsf{a}} - \mathsf{b}\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} + \mathsf{b}\mathsf{c}\mathsf{f}\big)\,d\mathsf{a}}{\mathsf{a}^7\overline{\mathsf{a}}^3} - \frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\mathsf{b}}{\mathsf{a}^6\overline{\mathsf{a}}^3} - \frac{\mathsf{f}\,d\mathsf{d}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{h}}{\overline{\mathsf{a}}\,\mathsf{a}^3},$$

$$\alpha^9 := -\frac{\big(\mathsf{k}\mathsf{a}^3\overline{\mathsf{a}}^2 - \mathsf{e}\mathsf{f}\mathsf{a}\overline{\mathsf{a}} - \overline{\mathsf{b}}\,\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} + \overline{\mathsf{b}}\,\mathsf{c}\mathsf{f}\big)\,d\overline{\mathsf{a}}}{\mathsf{a}^6\overline{\mathsf{a}}^4} - \frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\overline{\mathsf{b}}}{\mathsf{a}^6\overline{\mathsf{a}}^3} - \frac{\mathsf{f}\,d\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{k}}{\overline{\mathsf{a}}\,\mathsf{a}^3},$$

together with their conjugates.

We derive the structure equations of P^1 from the relations (1). The expression of $d\tau$ is:

$$\begin{aligned} d\tau = 3\,\alpha^1 \wedge \tau + \overline{\alpha^1} \wedge \tau \\ + T^\tau_{\tau\sigma}\,\tau\wedge\sigma + T^\tau_{\tau\rho}\,\tau\wedge\rho + T^\tau_{\tau\zeta}\,\tau\wedge\zeta \\ + T^\tau_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta} + T^\tau_{\sigma\rho}\,\sigma\wedge\rho + \sigma\wedge\zeta - \frac{\mathsf{a}}{\overline{\mathsf{a}}}\,B\;\sigma\wedge\overline{\zeta} \end{aligned}$$

The coefficient

$$\frac{\mathsf{a}}{\overline{\mathsf{a}}}\,B,$$

which can not be absorbed for any choice of the modified Maurer-Cartan form $\widetilde{\alpha}^1$, is referred to as an essential torsion coefficient. From standard results on Cartan theory (see [10, 15]), a diffeomorphism of M is an isomorphism of the G_1-structure P^1 if and only if it is an isomorphism of the reduced bundle $P^2 \subset P^1$ consisting of those coframes ω on M such that

$$\frac{\mathsf{a}}{\overline{\mathsf{a}}}\, B = 1.$$

This is equivalent to the normalization:

$$\overline{\mathsf{a}} = \mathsf{a} B.$$

A coframe $\omega \in P^2$ is related to the coframe ω_0 by the relations:

$$\begin{aligned}
&\tau = \mathsf{a}^4\, B\, \tau_0, && \sigma = \mathsf{f}\, \tau_0 + \mathsf{a}^3\, B\, \sigma_0,\\
&\rho = \mathsf{g}\, \tau_0 + \mathsf{c}\, \sigma_0 + \mathsf{a}^2\, B\, \rho_0, && \zeta = \mathsf{h}\, \tau_0 + \mathsf{d}\, \sigma_0 + \mathsf{b}\, \rho_0 + \mathsf{a}\, \zeta_0,\\
&\overline{\zeta} = \mathsf{k}\, \tau_0 + \mathsf{e}\, \sigma_0 + \overline{\mathsf{b}}\, \rho_0 + \mathsf{a}\, B\, \overline{\zeta}_0,
\end{aligned}$$

which are equivalent to:

$$\begin{aligned}
&\tau = \mathsf{a}'^{\,4}\, \tau_1, && \sigma = \mathsf{f}'\, \tau_1 + \mathsf{a}'^{\,3}\, \sigma_1,\\
&\rho = \mathsf{g}'\, \tau_1 + \mathsf{c}'\, \sigma_1 + \mathsf{a}'^{\,2}\, \rho_1, && \zeta = \mathsf{h}'\, \tau_1 + \mathsf{d}'\, \sigma_1 + \mathsf{b}\, \rho_1 + \mathsf{a}'\, \zeta_1,\\
&\overline{\zeta} = \mathsf{k}'\, \tau_1 + \mathsf{e}'\, \sigma_1 + \overline{\mathsf{b}}\, \rho_1 + \mathsf{a}'\, \overline{\zeta}_1,
\end{aligned}$$

where:

$$\tau_1 := \frac{\tau_0}{B}, \qquad \sigma_1 := \frac{\sigma_0}{B^{\frac{1}{2}}}, \qquad \rho_1 = \rho_0, \qquad \zeta_1 := \frac{\zeta_0}{B^{\frac{1}{2}}},$$

and

$$x' := \begin{cases} x \cdot B^{\frac{1}{2}}, & \text{for} \quad x = \mathsf{a},\, \mathsf{c},\, \mathsf{d},\, \mathsf{e},\\ x \cdot B, & \text{for} \quad x = \mathsf{f},\, \mathsf{g},\, \mathsf{h},\, \mathsf{k}. \end{cases}$$

We notice that a' is a real parameter, and that τ_1 is a real 1-form. Let ω_1 be the coframe $\omega_1 := \big(\tau_1, \sigma_1, \rho_1, \zeta_1, \overline{\zeta}_1\big)$, and G_2 be the subgroup of G_1:

$$G_2 := \left\{ \begin{pmatrix} \mathsf{a}^4 & 0 & 0 & 0 & 0\\ \mathsf{f} & \mathsf{a}^3 & 0 & 0 & 0\\ \mathsf{g} & \mathsf{c} & \mathsf{a}^2 & 0 & 0\\ \mathsf{h} & \mathsf{d} & \mathsf{b} & \mathsf{a} & 0\\ \mathsf{k} & \mathsf{e} & \overline{\mathsf{b}} & 0 & \mathsf{a} \end{pmatrix},\ \mathsf{a} \in \mathbb{R} \setminus \{0\},\ \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k} \in \mathbb{C} \right\}.$$

A coframe ω on M belongs to P^2 if and only if there is a local function $g : M \xrightarrow{g} G_2$ such that $\omega = g \cdot \omega_1$, namely P^2 is a G_2 structure on M.

The Maurer-Cartan forms of G_2 are given by:

$$\beta^1 := \frac{d\mathsf{a}}{\mathsf{a}},$$
$$\beta^2 := -\frac{\mathsf{b}d\mathsf{a}}{\mathsf{a}^3} + \frac{d\mathsf{b}}{\mathsf{a}^2},$$
$$\beta^3 := -2\,\frac{\mathsf{c}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{c}}{\mathsf{a}^3},$$
$$\beta^4 = -\frac{(\mathsf{da}^2 - \mathsf{bc})\,d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\mathsf{b}}{\mathsf{a}^5} + \frac{d\mathsf{d}}{\mathsf{a}^3},$$
$$\beta^5 = -\frac{\left(\mathsf{ea}^2 - \overline{\mathsf{b}}\,\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\overline{\mathsf{b}}}{\mathsf{a}^5} + \frac{d\mathsf{e}}{\mathsf{a}^3},$$
$$\beta^6 = -3\,\frac{\mathsf{f}d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{f}}{\mathsf{a}^4},$$
$$\beta^7 = -2\,\frac{(\mathsf{ga}^3 - \mathsf{cf})\,d\mathsf{a}}{\mathsf{a}^8} - \frac{\mathsf{f}d\mathsf{c}}{\mathsf{a}^7} + \frac{d\mathsf{g}}{\mathsf{a}^4},$$
$$\beta^8 = -\frac{(\mathsf{ha}^5 - \mathsf{dfa}^2 - \mathsf{bga}^3 + \mathsf{bcf})\,d\mathsf{a}}{\mathsf{a}^{10}} - \frac{(\mathsf{ga}^3 - \mathsf{cf})\,d\mathsf{b}}{\mathsf{a}^9} - \frac{\mathsf{f}d\mathsf{d}}{\mathsf{a}^7} + \frac{d\mathsf{h}}{\mathsf{a}^4},$$
$$\beta^9 = -\frac{\left(\mathsf{ka}^5 - \mathsf{efa}^2 - \overline{\mathsf{b}}\,\mathsf{ga}^3 + \overline{\mathsf{b}}\,\mathsf{cf}\right)d\mathsf{a}}{\mathsf{a}^{10}} - \frac{(\mathsf{ga}^3 - \mathsf{cf})\,d\overline{\mathsf{b}}}{\mathsf{a}^9} - \frac{\mathsf{f}d\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{k}}{\mathsf{a}^4},$$

together with $\overline{\beta^i}$, $\;i = 2\ldots 9$.

Using formula (1), we get the structure equations of P^2:

$$\begin{aligned} d\tau = 4\,\beta^1 \wedge \tau & \\ & + U^\tau_{\tau\sigma}\,\tau\wedge\sigma + U^\tau_{\tau\rho}\,\tau\wedge\rho + U^\tau_{\tau\zeta}\,\tau\wedge\zeta + U^\tau_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta} \\ & + U^\tau_{\sigma\rho}\,\sigma\wedge\rho + \sigma\wedge\zeta + \sigma\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\sigma = 3\,\beta^1\wedge\sigma + \beta^6\wedge\tau & \\ & + U^\sigma_{\tau\sigma}\,\tau\wedge\sigma + U^\sigma_{\tau\rho}\,\tau\wedge\rho + U^\sigma_{\tau\zeta}\,\tau\wedge\zeta \\ & + U^\sigma_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta} + U^\sigma_{\sigma\rho}\,\sigma\wedge\rho + U^\sigma_{\sigma\zeta}\,\sigma\wedge\zeta \\ & + U^\sigma_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta} \end{aligned}$$

$$\begin{aligned} d\rho = 2\beta^1\wedge\rho + \beta^3\wedge\sigma + \beta^7\wedge\tau & \\ & + U^\rho_{\tau\sigma}\,\tau\wedge\sigma + U^\rho_{\tau\rho}\,\tau\wedge\rho + U^\rho_{\tau\zeta}\,\tau\wedge\zeta + U^\rho_{\tau\overline{\zeta}}\,\rho\wedge\overline{\zeta} + U^\rho_{\sigma\rho}\,\sigma\wedge\rho \\ & + U^\rho_{\sigma\zeta}\,\sigma\wedge\zeta + U^\rho_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + U^\rho_{\rho\zeta}\,\rho\wedge\zeta + U^\rho_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\zeta = \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma + \beta^8 \wedge \tau \\ + U^{\zeta}_{\tau\sigma}\, \tau \wedge \sigma + U^{\zeta}_{\tau\rho}\, \tau \wedge \rho + U^{\zeta}_{\tau\zeta}\, \tau \wedge \zeta + U^{\zeta}_{\tau\overline{\zeta}}\, \tau \wedge \overline{\zeta} \\ + U^{\zeta}_{\sigma\rho}\, \sigma \wedge \rho + U^{\zeta}_{\sigma\zeta}\, \sigma \wedge \zeta + U^{\zeta}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + U^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta \\ + U^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + U^{\zeta}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}. \end{aligned}$$

Introducing the modified Maurer-Cartan forms:

$$\widetilde{\beta}^i = \beta^i - y^i_\tau\, \tau - y_\sigma\, \sigma - y^i_\rho\, \rho - y^i_\zeta\, \zeta \, - y^i_{\overline{\zeta}}\, \overline{\zeta},$$

the structure equations rewrite:

$$\begin{aligned} d\tau = 4\,\widetilde{\beta}^1 \wedge \tau \\ + \left(U^{\tau}_{\tau\sigma} - 4\, y^1_\sigma\right)\ \tau \wedge \sigma + \left(U^{\tau}_{\tau\rho} - 4\, y^1_\rho\right)\ \tau \wedge \rho \\ + \left(U^{\tau}_{\tau\zeta} - 4\, y^1_\zeta\right)\ \tau \wedge \zeta + \left(U^{\tau}_{\tau\overline{\zeta}} - 4\, y^1_{\overline{\zeta}}\right)\ \tau \wedge \overline{\zeta} \\ + U^{\tau}_{\sigma\rho}\ \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}\ , \end{aligned}$$

$$\begin{aligned} d\sigma = 3\ \widetilde{\beta}^1 \wedge \sigma + \widetilde{\beta}^6 \wedge \tau \\ + \left(U^{\sigma}_{\tau\sigma} + 3\, y^1_\tau - y^6_\sigma\right)\ \tau \wedge \sigma + \left(U^{\sigma}_{\tau\rho} - y^6_\rho\right)\ \tau \wedge \rho \\ + \left(U^{\sigma}_{\tau\zeta} - y^6_\zeta\right)\ \tau \wedge \zeta + \left(U^{\sigma}_{\tau\overline{\zeta}} - y^6_{\overline{\zeta}}\right)\ \tau \wedge \overline{\zeta} \\ + \left(U^{\sigma}_{\sigma\rho} - 3\, y^1_\rho\right)\ \sigma \wedge \rho + \left(U^{\sigma}_{\sigma\zeta} - 3\, y^1_\zeta\right)\ \sigma \wedge \zeta \\ + \left(U^{\sigma}_{\sigma\overline{\zeta}} - 3\, y^1_{\overline{\zeta}}\right)\ \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta} \end{aligned}$$

$$\begin{aligned} d\rho = 2\widetilde{\beta}^1 \wedge \rho + \widetilde{\beta}^3 \wedge \sigma + \widetilde{\beta}^7 \wedge \tau \\ + \left(U^{\rho}_{\tau\sigma} + y^3_\tau - y^7_\sigma\right)\ \tau \wedge \sigma + \left(U^{\rho}_{\tau\rho} + 2\, y^1_\tau - y^7_\rho\right)\ \tau \wedge \rho \\ + \left(U^{\rho}_{\tau\zeta} - y^7_\zeta\right)\ \tau \wedge \zeta + \left(U^{\rho}_{\tau\overline{\zeta}} - y^7_{\overline{\zeta}}\right)\ \rho \wedge \overline{\zeta} \\ + \left(U^{\rho}_{\sigma\rho} + 2\, y^1_\sigma - y^3_\rho\right)\ \sigma \wedge \rho + \left(U^{\rho}_{\sigma\zeta} - y^3_\zeta\right)\ \sigma \wedge \zeta \\ + \left(U^{\rho}_{\sigma\overline{\zeta}} - y^3_{\overline{\zeta}}\right)\ \sigma \wedge \overline{\zeta} + \left(U^{\rho}_{\rho\zeta} - 2\, y^1_\zeta\right)\ \rho \wedge \zeta \\ + \left(U^{\rho}_{\rho\overline{\zeta}} - 2\, y^1_{\overline{\zeta}}\right)\ \rho \wedge \overline{\zeta} + i\ \zeta \wedge \overline{\zeta}\ , \end{aligned}$$

$$
\begin{aligned}
d\zeta = \widetilde{\beta^1} \wedge \zeta + \widetilde{\beta^2} \wedge \rho + \widetilde{\beta^4} \wedge \sigma + \widetilde{\beta^8} \wedge \tau \\
+ \left(U^{\zeta}_{\tau\sigma} + y^4_{\tau} - y^8_{\sigma}\right)\ \tau \wedge \sigma + \left(U^{\zeta}_{\tau\rho} + y^2_{\tau} - y^8_{\rho}\right)\ \tau \wedge \rho \\
+ \left(U^{\zeta}_{\tau\zeta} + y^1_{\tau} - y^8_{\zeta}\right)\ \tau \wedge \zeta + \left(U^{\zeta}_{\tau\overline{\zeta}} - y^8_{\overline{\zeta}}\right)\ \tau \wedge \overline{\zeta} \\
+ \left(U^{\zeta}_{\sigma\rho} + y^2_{\sigma} - y^4_{\rho}\right)\ \sigma \wedge \rho + \left(U^{\zeta}_{\sigma\zeta} + y^1_{\sigma} - y^4_{\zeta}\right)\ \sigma \wedge \zeta \\
+ \left(U^{\zeta}_{\sigma\overline{\zeta}} - y^4_{\overline{\zeta}}\right)\ \sigma \wedge \overline{\zeta} + \left(U^{\zeta}_{\rho\zeta} + y^1_{\rho} - y^2_{\zeta}\right)\ \rho \wedge \zeta \\
+ \left(U^{\zeta}_{\rho\overline{\zeta}} - y^2_{\overline{\zeta}}\right)\ \rho \wedge \overline{\zeta} + \left(U^{\zeta}_{\zeta\overline{\zeta}} - y^1_{\overline{\zeta}}\right)\ \zeta \wedge \overline{\zeta}\ .
\end{aligned}
$$

We get the following absorbtion equations:

$$
\begin{aligned}
4\,y^1_{\sigma} &= U^{\tau}_{\tau\sigma}, & 4\,y^1_{\rho} &= U^{\tau}_{\tau\rho}, & 4\,y^1_{\zeta} &= U^{\tau}_{\tau\zeta}, \\
4\,y^1_{\overline{\zeta}} &= U^{\tau}_{\tau\overline{\zeta}}, & -3\,y^1_{\tau} + y^6_{\sigma} &= U^{\sigma}_{\tau\sigma}, & y^6_{\rho} &= U^{\sigma}_{\tau\rho}, \\
y^6_{\zeta} &= U^{\sigma}_{\tau\zeta}, & y^6_{\overline{\zeta}} &= U^{\sigma}_{\tau\overline{\zeta}}, & 3\,y^1_{\rho} &= U^{\sigma}_{\sigma\rho}, \\
3\,y^1_{\zeta} &= U^{\sigma}_{\sigma\zeta}, & 3\,y^1_{\overline{\zeta}} &= U^{\sigma}_{\sigma\overline{\zeta}}, & -y^3_{\tau} + y^7_{\sigma} &= U^{\rho}_{\tau\sigma}, \\
-2\,y^1_{\tau} + y^7_{\rho} &= U^{\rho}_{\tau\rho}, & y^7_{\zeta} &= U^{\rho}_{\tau\zeta}, & y^7_{\overline{\zeta}} &= U^{\rho}_{\tau\overline{\zeta}}, \\
-2\,y^1_{\sigma} + y^3_{\rho} &= U^{\rho}_{\sigma\rho}, & y^3_{\zeta} &= U^{\rho}_{\sigma\zeta}, & y^3_{\overline{\zeta}} &= U^{\rho}_{\sigma\overline{\zeta}}, \\
2\,y^1_{\zeta} &= U^{\rho}_{\rho\zeta}, & 2\,y^1_{\overline{\zeta}} &= U^{\rho}_{\rho\overline{\zeta}}, & -y^4_{\tau} + y^8_{\sigma} &= U^{\zeta}_{\tau\sigma}, \\
-y^2_{\tau} + y^8_{\rho} &= U^{\zeta}_{\tau\rho}, & -y^1_{\tau} + y^8_{\zeta} &= U^{\zeta}_{\tau\zeta}, & y^8_{\overline{\zeta}} &= U^{\zeta}_{\tau\overline{\zeta}}, \\
-y^2_{\sigma} + y^4_{\rho} &= U^{\zeta}_{\sigma\rho}, & -y^1_{\sigma} + y^4_{\zeta} &= U^{\zeta}_{\sigma\zeta}, & y^4_{\overline{\zeta}} &= U^{\zeta}_{\sigma\overline{\zeta}}, \\
-y^1_{\rho} + y^2_{\zeta} &= U^{\zeta}_{\rho\zeta}, & y^2_{\overline{\zeta}} &= U^{\zeta}_{\rho\overline{\zeta}}, & y^1_{\overline{\zeta}} &= U^{\zeta}_{\zeta\overline{\zeta}}.
\end{aligned}
$$

Eliminating $y^1_{\overline{zeta}}$ and y^1_{ζ} among the previous equations leads to the normalizations:

$$
\begin{aligned}
\mathsf{b} &= \mathsf{a}\,\mathbf{B}_0, \\
\mathsf{c} &= \mathsf{a}^2\,\mathbf{C}_0, \\
\mathsf{f} &= \mathsf{a}^3\,\mathbf{F}_0,
\end{aligned}
$$

where the functions $\mathbf{B}_0$, $\mathbf{C}_0$ and $\mathbf{F}_0$ are defined by:

$$\begin{aligned}
\mathbf{B}_0 &:= \frac{3i}{10}\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}} - \frac{i}{5}\frac{A}{B^{\frac{1}{2}}} - \frac{i}{10}\frac{K}{B^{\frac{1}{2}}} - \frac{i}{10}\frac{\mathscr{L}(B)}{B^{\frac{1}{2}}},\\
\mathbf{C}_0 &:= \frac{11}{20}\frac{\mathscr{L}(B)}{B^{\frac{1}{2}}} + \frac{3}{20}B^{\frac{1}{2}}G + \frac{1}{20}\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}} - \frac{1}{5}\frac{A}{B^{\frac{1}{2}}} + \frac{3}{20}\frac{K}{B^{\frac{1}{2}}},\\
\mathbf{F}_0 &:= \frac{1}{10}\frac{\mathscr{L}(B)}{B} + \frac{3}{10}B^{\frac{1}{2}}G + \frac{1}{10}\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}} - \frac{2}{5}\frac{A}{B^{\frac{1}{2}}} + \frac{3}{10}\frac{K}{B^{\frac{1}{2}}}.
\end{aligned}$$

The absorbed structure equations take the form:

$$\begin{aligned}
d\tau &= 4\,\widetilde{\beta}^1\wedge\tau + \frac{\mathfrak{I}_1}{\mathsf{a}}\,\tau\wedge\zeta - \frac{\mathfrak{I}_1}{\mathsf{a}}\,\tau\wedge\overline{\zeta} + 3\,\frac{\mathfrak{I}_1}{\mathsf{a}}\,\sigma\wedge\rho + \sigma\wedge\zeta + \sigma\wedge\overline{\zeta}\,,\\
d\sigma &= 3\;\widetilde{\beta}^1\wedge\sigma + \widetilde{\beta}^6\wedge\tau - \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\sigma\wedge\zeta + \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta},\\
d\rho &= 2\widetilde{\beta}^1\wedge\rho + \widetilde{\beta}^3\wedge\sigma + \widetilde{\beta}^7\wedge\tau - \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\rho\wedge\zeta + \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\rho\wedge\overline{\zeta} + i\;\zeta\wedge\overline{\zeta}\,,\\
d\zeta &= \widetilde{\beta^1}\wedge\zeta + \widetilde{\beta^2}\wedge\rho + \widetilde{\beta^4}\wedge\sigma + \widetilde{\beta^8}\wedge\tau,
\end{aligned}$$

where the function $\mathfrak{I}_1$ is a biholomorphic invariant of M and is given by:

$$\mathfrak{I}_1 := \frac{1}{2}\frac{\mathscr{L}(B)}{B} + \frac{3}{10}B^{\frac{1}{2}}G - \frac{1}{10}\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}} + \frac{2}{5}\frac{A}{B^{\frac{1}{2}}} - \frac{3}{10}\frac{K}{B^{\frac{1}{2}}}.$$

We introduce the coframe $\omega_2 := \big(\tau_2, \sigma_2, \rho_2, \zeta_2, \overline{\zeta}_2\big)$ on M, defined by:

$$\begin{cases}
\tau_2 := \tau_1\\
\sigma_2 := \mathbf{F}_0\,\tau_1 + \sigma_1,\\
\rho_2 := \rho_1 + \mathbf{C}_0\,\sigma_1,\\
\zeta_2 := \zeta_1 + \mathbf{B}_0\,\rho_1,
\end{cases}$$

and the subgroup $G_3 \subset G_2$:

$$G_3 := \left\{ \begin{pmatrix} \mathsf{a}^4 & 0 & 0 & 0 & 0\\ 0 & \mathsf{a}^3 & 0 & 0 & 0\\ \mathsf{g} & 0 & \mathsf{a}^2 & 0 & 0\\ \mathsf{h} & \mathsf{d} & 0 & \mathsf{a} & 0\\ \mathsf{k} & \mathsf{e} & 0 & 0 & \mathsf{a} \end{pmatrix},\ \mathsf{a}\in\mathbb{R}\setminus\{0\},\ \mathsf{d},\ \mathsf{e},\ \mathsf{g},\ \mathsf{h},\ \mathsf{k},\in\mathbb{C} \right\}.$$

We notice that σ_2 is a real one-form. The normalizations:

$$\mathsf{b} := \mathsf{a}\,\mathbf{B}_0, \qquad \mathsf{c} := \mathsf{a}^2\,\mathbf{C}_0, \qquad \mathsf{f} := \mathsf{a}^3\,\mathbf{F}_0,$$

amount to consider the subbundle $P^3 \subset P^2$ consisting of those coframes ω of the form

$$\omega := g\cdot\omega_2, \qquad \text{where } g \text{ is a function } \; g : M \xrightarrow{\;g\;} G_3.$$

A basis of the Maurer Cartan forms of G_3 is given by:

$$\begin{aligned}
\gamma^1 &:= \frac{d\mathsf{a}}{\mathsf{a}}, \\
\gamma^2 &:= -\frac{\mathsf{d}\,d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{d}}{\mathsf{a}^3}, \\
\gamma^3 &:= -\frac{\mathsf{e}\,d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{e}}{\mathsf{a}^3}, \\
\gamma^4 &:= -2\,\frac{\mathsf{g}\,d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{g}}{\mathsf{a}^4}, \\
\gamma^5 &:= -\frac{\mathsf{h}\,d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{h}}{\mathsf{a}^4}, \\
\gamma^6 &:= -\frac{\mathsf{k}\,d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{k}}{\mathsf{a}^4}.
\end{aligned}$$

We get the following absorbed structure equations for P^3:

$$d\tau = 4\,\widetilde{\gamma}^1\wedge\tau + \frac{\mathfrak{I}_1}{\mathsf{a}}\,\tau\wedge\zeta - \frac{\mathfrak{I}_1}{\mathsf{a}}\,\tau\wedge\overline{\zeta} + 3\,\frac{\mathfrak{I}_1}{\mathsf{a}}\,\sigma\wedge\rho + \sigma\wedge\zeta + \sigma\wedge\overline{\zeta}\,,$$

$$\begin{aligned}
d\sigma = 3\,\widetilde{\gamma}^1\wedge\sigma &\\
&+ V^\sigma_{\tau\rho}\,\tau\wedge\rho + V^\sigma_{\tau\zeta}\,\tau\wedge\zeta + V^\sigma_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta} + V^\sigma_{\sigma\rho}\,\sigma\wedge\rho \\
&- \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\sigma\wedge\zeta + \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta}\,,
\end{aligned}$$

$$\begin{aligned}
d\rho = 2\,\widetilde{\gamma}^1\wedge\rho + \widetilde{\gamma}^4\wedge\tau &\\
&+ V^\rho_{\sigma\rho}\,\sigma\wedge\rho + V^\rho_{\sigma\zeta}\,\sigma\wedge\zeta + V^\rho_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} \\
&+ \frac{\mathfrak{I}_1}{2\mathsf{a}} + \rho\wedge\zeta + \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\rho\wedge\overline{\zeta} + i\;\zeta\wedge\overline{\zeta}\,,
\end{aligned}$$

$$d\zeta \quad = \quad \widetilde{\gamma}^1\wedge\zeta \;+\; \widetilde{\gamma}^2\wedge\sigma \;+\; \widetilde{\gamma}^5\wedge\tau \;+\; V^\zeta_{\rho\zeta}\,\rho\wedge\zeta \;+\; V^\zeta_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta}\,,$$

From the essential torsion coefficients $V^\sigma_{\tau\zeta}$, $V^\sigma_{\tau\overline{\zeta}}$ and $V^\zeta_{\rho\overline{\zeta}}$, we obtain the normalizations:

$$\mathsf{d} := \mathsf{a}\,\mathbf{D_0}, \qquad\qquad \mathsf{g} := \mathsf{a}^2\,\mathbf{G_0},$$

where

$$\mathbf{D_0} := i\,\mathbf{B}_0^2 - \frac{A\mathbf{B}_0}{B^{\frac{1}{2}}} + \frac{\overline{\mathscr{L}}\,(\mathbf{B}_0)}{B^{\frac{1}{2}}} + \frac{1}{2}\,\frac{\overline{\mathscr{L}}(B)\mathbf{B}_0}{B^{\frac{3}{2}}},$$

and

$$\mathbf{G_0} := -\frac{1}{4}\,\frac{\mathscr{L}(B)}{B^{\frac{1}{2}}}\mathbf{F}_0 - \mathbf{F}_0^2 + \frac{1}{2}\,B^{\frac{1}{2}}G\mathbf{F}_0 - \frac{1}{2}\,B^{\frac{1}{2}}\mathscr{L}\left(\mathbf{F}_0\right) + \mathbf{C}_0\mathbf{F}_0$$
$$+\frac{1}{2}\,FB + \frac{1}{4}\,\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}}\mathbf{F}_0 + \frac{1}{2}\,\frac{K}{B^{\frac{1}{2}}}\mathbf{F}_0 - \frac{1}{2}\,\frac{\overline{\mathscr{L}}\left(\mathbf{F}_0\right)}{B^{\frac{1}{2}}} + \frac{J}{2} - \frac{1}{2}\,\frac{A}{B^{\frac{1}{2}}}\mathbf{F}_0.$$

We introduce the coframe $\omega_3 := \left(\tau_3, \sigma_3, \rho_3, \zeta_3, \overline{\zeta}_3\right)$ on M, defined by:

$$\begin{cases} \tau_3 := \tau_2 \\ \sigma_3 := \sigma_2 \\ \rho_3 := \rho_2 + \mathbf{C}_0\,\tau_2, \\ \zeta_3 := \zeta_2 + \mathbf{D}_0\,\sigma_2, \end{cases}$$

and the subgroup $G_4 \subset G_3$:

$$G_4 := \left\{ \begin{pmatrix} \mathsf{a}^4 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}^2 & 0 & 0 \\ \mathsf{h} & 0 & 0 & \mathsf{a} & 0 \\ \overline{\mathsf{h}} & 0 & 0 & 0 & \mathsf{a} \end{pmatrix},\ \mathsf{a} \in \mathbb{R}\setminus\{0\},\ \mathsf{h} \in \mathbb{C} \right\}$$

The normalizations:

$$\mathsf{d} := \mathsf{a}\,\mathbf{D_0}, \qquad\qquad \mathsf{g} := \mathsf{a}^2\,\mathbf{G_0},$$

amount to consider the subbundle $P^4 \subset P^3$ consisting of those coframes ω of the form

$$\omega := g\cdot\omega_3, \qquad \text{where } g \text{ is a function } \; g : M \xrightarrow{\;g\;} G_4.$$

A basis of the Maurer-Cartan forms is given by:

$$\delta^1 := \frac{d\mathsf{a}}{\mathsf{a}},$$
$$\delta^2 := -\frac{\mathsf{h}d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{h}}{\mathsf{a}^4},$$

together with $\overline{\delta}^2$.

As for the previous step, we determine the structure equations of P^4 using formula (1). We just write here the expression of $d\zeta$, as it provides a normalization of h:

$$d\zeta = \widetilde{\delta}^1\wedge\zeta + \widetilde{\delta}^2\wedge\tau + W^\zeta_{\sigma\rho}\,\sigma\wedge\rho + W^\zeta_{\sigma\zeta}\,\sigma\wedge\zeta + W^\zeta_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta}\;,$$

for some modified Maurer-Cartan forms $\widetilde{\delta}^1$, $\widetilde{\delta}^2$.

The essential torsion coefficient $W^\zeta_{\sigma\zeta}$ can be normalized to 0, which is equivalent to the normalization:

$$\mathsf{h} := \mathsf{a}\,\mathbf{H}_0,$$

where

$$\mathbf{H}_0 := -\mathbf{D}_0\,\mathbf{F}_0 + \mathbf{C}_0\,\mathbf{D}_0 - \frac{\mathscr{L}(B)}{B^{\frac{1}{2}}}\mathbf{D}_0 - \frac{A}{B^{\frac{1}{2}}}\mathbf{D}_0 + \overline{\mathscr{L}}\,(\mathbf{D}_0)B^{\frac{1}{2}} + i\,\mathbf{B}_0\,\mathbf{D}_0$$
$$-\,i\,\mathbf{B}_0^2\,\mathbf{C}_0 + \frac{A}{B^{\frac{1}{2}}}\,\mathbf{B}_0\,\mathbf{C}_0 - \mathscr{L}(A)\,\mathbf{B}_0 - \frac{\overline{\mathscr{L}}\,(\mathbf{B}_0)}{B^{\frac{1}{2}}}\,\mathbf{C}_0 - \frac{1}{2}\,\frac{\overline{\mathscr{L}}(B)}{B^{\frac{3}{2}}}\mathbf{B}_0\,\mathbf{C}_0.$$

Let G_5 be the 1-dimensional Lie subgroup of G_4 whose elements g are of the form:

$$g := \begin{pmatrix} \mathsf{a}^4 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}^2 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{a} & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a} \end{pmatrix}, \qquad \mathsf{a} \in \mathbb{R} \setminus \{0\},$$

and let $\omega_4 := \big(\tau_4, \sigma_4, \rho_4, \zeta_4, \overline{\zeta}_4\big)$ be the coframe defined on M by:

$$\sigma_4 := \sigma_3, \qquad\qquad \rho_4 := \rho_3, \qquad\qquad \zeta_4 := \zeta_3 + \mathbf{H}_0\,\tau_3.$$

The normalization of h is equivalent to the reduction of P^4 to a subbundle P^5 consisting of those coframes ω on M such that:

$$\omega := g \cdot \omega_3, \qquad \text{where } g \text{ is a function } \; g : M \xrightarrow{\;g\;} G_4.$$

The Maurer-Cartan forms of G_5 are spanned by:

$$\alpha := \frac{d\mathsf{a}}{\mathsf{a}}.$$

Proceeding as in the previous steps, we determine the structure equations of P^4 which take the absorbed form:

$$d\tau = 4\,\Lambda \wedge \tau + \frac{\mathfrak{I}_1}{\mathsf{a}}\,\tau \wedge \zeta - \frac{\mathfrak{I}_1}{\mathsf{a}}\,\tau \wedge \overline{\zeta} + 3\,\frac{\mathfrak{I}_1}{\mathsf{a}}\,\sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}\,,$$

$$\begin{aligned} d\sigma = 3\,\Lambda \wedge \sigma & \\ & + \frac{\mathfrak{I}_2}{\mathsf{a}^3}\,\tau \wedge \rho + \frac{\mathfrak{I}_3}{\mathsf{a}^2}\,\tau \wedge \zeta + \frac{\overline{\mathfrak{I}}_3}{\mathsf{a}^2}\,\tau \wedge \overline{\zeta} + \frac{\mathfrak{I}_4}{\mathsf{a}^2}\,\sigma \wedge \rho \\ & - \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\sigma \wedge \zeta + \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\rho = 2\Lambda \wedge \rho & \\ & + \frac{\mathfrak{I}_5}{\mathsf{a}^5}\,\tau \wedge \sigma + \frac{\mathfrak{I}_6}{\mathsf{a}^4}\,\tau \wedge \rho + \frac{\mathfrak{I}_7}{\mathsf{a}^3}\,\tau \wedge \zeta + \frac{\overline{\mathfrak{I}}_7}{\mathsf{a}^3}\,\tau \wedge \overline{\zeta} + \frac{\mathfrak{I}_8}{\mathsf{a}^3}\,\sigma \wedge \rho \\ & + \frac{\mathfrak{I}_9}{\mathsf{a}^2}\,\sigma \wedge \zeta + \frac{\overline{\mathfrak{I}}_9}{\mathsf{a}^2}\,\sigma \wedge \overline{\zeta} - \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\rho \wedge \zeta + \frac{\mathfrak{I}_1}{2\mathsf{a}}\,\rho \wedge \overline{\zeta} + i\;\zeta \wedge \overline{\zeta}\,, \end{aligned}$$

$$d\zeta = \Lambda \wedge \zeta + \frac{\mathfrak{I}_{10}}{\mathsf{a}^6}\,\tau\wedge\sigma + \frac{\mathfrak{I}_{11}}{\mathsf{a}^5}\,\tau\wedge\rho + \frac{\mathfrak{I}_{12}}{\mathsf{a}^4}\,\tau\wedge\zeta + \frac{\mathfrak{I}_{13}}{\mathsf{a}^4}\,\tau\wedge\overline{\zeta} + \frac{\mathfrak{I}_{14}}{\mathsf{a}^4}\,\sigma\wedge\rho + \frac{\mathfrak{I}_{15}}{\mathsf{a}^3}\,\sigma\wedge\zeta\,,$$

(2)

where Λ is a modified-Maurer Cartan form:

$$\Lambda := \frac{d\mathsf{a}}{\mathsf{a}} - X_\tau\,\tau - X_\sigma\,\sigma - X_\rho\,\rho - X_\zeta\,\zeta - X_{\overline{\zeta}}\,\overline{\zeta},$$

and where

$$\mathfrak{I}_i, \quad i = 1\ldots 15,$$

are biholomorphic invariants of M.

The exterior derivative of Λ can be determined by taking the exterior derivative of the four previous equations which leads to the so-called Bianchi-Cartan's identities. We obtain the fact that $d\Lambda$ does not contain any 2-form involving the 1-form Λ, namely:

$$d\Lambda = \sum_{\nu\mu} X_{\nu\mu}\,\nu\wedge\mu\,, \qquad \nu,\,\mu = \tau,\,\sigma,\,\rho,\,\zeta,\overline{\zeta}. \tag{3}$$

4. Cartan Connection

We recall that the model for CR-manifolds belonging to general class III_2 is the CR-manifold defined by the equations:

$$\mathsf{N}: \quad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \\ w_3 &= \overline{w_3} + 2i\,z\overline{z}\left(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2\right). \end{aligned}$$

Its Lie algebra of infinitesimal CR-automorphisms is given by the following theorem:

Theorem 2. [12]. *The model of the class* III_2:

$$\mathsf{N}: \quad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \\ w_3 &= \overline{w_3} + 2i\,z\overline{z}\left(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2\right), \end{aligned}$$

has a **6**-*dimensional Lie algebra of CR-automorphisms* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{N})$. *A basis for the Maurer-Cartan forms of* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{N})$ *is provided by the* 6 *differential*

1-*forms* τ, σ, ρ, ζ, $\overline{\zeta}$, α, *which satisfy the Maurer-Cartan equations:*

$$\begin{aligned}
d\tau &= 4\,\alpha\wedge\tau + \sigma\wedge\zeta + \sigma\wedge\overline{\zeta}\ ,\\
d\sigma &= 3\,\alpha\wedge\sigma + \rho\wedge\zeta + \rho\wedge\overline{\zeta}\ ,\\
d\rho &= 2\,\alpha\wedge\rho + i\ \zeta\wedge\overline{\zeta}\ ,\\
d\zeta &= \alpha\wedge\zeta\ ,\\
d\overline{\zeta} &= \alpha\wedge\overline{\zeta}\ ,\\
d\alpha &= 0.
\end{aligned}$$

Let us write $\mathfrak{g}$ instead of $\mathsf{aut}_{\mathsf{CR}}(\mathsf{N})$ for the Lie algebra of inifinitesimal automorphisms of N and let $\big(e_\alpha, e_\tau, e_\sigma, e_\rho, e_\zeta, e_{\overline{\zeta}}\big)$ be the dual basis of the basis of Maurer-Cartan 1-forms: $\big(\alpha,\tau,\sigma,\rho,\zeta,\overline{\zeta}\big)$. From the above structure equations, the Lie brackets structure of $\mathfrak{g}$ is given by:

$$\begin{aligned}
&[e_\alpha, e_\tau] = -4\,e_\tau, && [e_\sigma, e_\zeta] = -\,e_\tau, && [e_\sigma, e_{\overline{\zeta}}] = -\,e_\tau,\\
&[e_\alpha, e_\sigma] = -3\,e_\sigma, && [e_\alpha, e_\rho] = -2\,e_\rho, && [e_\alpha, e_\zeta] = -\,e_\zeta,\\
&[e_\alpha, e_{\overline{\zeta}}] = -\,e_{\overline{\zeta}}, && [e_\rho, e_\zeta] = -e_\sigma, && [e_\rho, e_{\overline{\zeta}}] = -e_\sigma,\\
&[e_\zeta, e_{\overline{\zeta}}] = -i\,e_\rho,
\end{aligned}$$

the remaining brackets being equal to zero.

We refer to [5], p. 127-128, for the definition of a Cartan connection. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the subalgebra spanned by e_α, $\mathfrak{G}$ the connected, simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and $\mathfrak{G}_0$ the closed 1-dimensional subgroup of $\mathfrak{G}$ generated by $\mathfrak{g}_0$. We notice that $\mathfrak{G}_0 \cong G_5$, so that P^5 is a principal bundle over M with structure group $\mathfrak{G}_0$, and that $\dim \mathfrak{G}/\mathfrak{G}_0 = \dim M = 5$.

Let $\big(\Lambda,\tau,\sigma,\rho,\zeta,\overline{\zeta}\big)$ be the coframe of 1-forms on P^5 whose structure equation are given by (2) – (3) and ω the 1-form on P with values in $\mathfrak{g}$ defined by:

$$\omega(X) := \Lambda(X)\,e_\alpha + \tau(X)\,e_\tau + \sigma(X)\,e_\sigma + \rho(X)\,e_\rho + \zeta(X)\,e_\zeta, + \overline{\zeta}(X)\,e_{\overline{\zeta}},$$

for $X \in T_pP^5$. We have:

Theorem 3. *ω is a Cartan connection on P^5.*

Proof. We shall check that the following three conditions hold:

(1) $\omega(e^*_\alpha) = e_\alpha$, where e^*_α is the vertical vector field on P^4 generated by the action of e_α,

(2) $R^*_a\,\omega = \mathsf{Ad}(a^{-1})\,\omega$ for every $a \in \mathfrak{G}_0$,

(3) for each $p \in P^5$, ω_p is an isomorphism $T_pP^5 \xrightarrow{\omega_p} \mathfrak{g}$.

Condition (3) is trivially satisfied as $\big(\Lambda,\tau,\sigma,\rho,\zeta,\overline{\zeta}\big)$ is a coframe on P^5 and thus defines a basis of $T^*_pP^5$ at each point p.

Condition (1) follows simply from the fact that Λ is a modified-Maurer Cartan form on P^5:

$$\Lambda := \frac{d\mathsf{a}}{\mathsf{a}} - X_\tau\,\tau - X_\sigma\,\sigma - X_\rho\,\rho - X_\zeta\,\zeta\, - X_{\overline{\zeta}}\,\overline{\zeta},$$

so that

$$\omega(e^*_\alpha) = \Lambda(e_{\alpha^*}) = e_\alpha,$$

as

$$\tau(e_\alpha) = \sigma(e^*_\alpha) = \rho(e^*_\alpha) = \zeta(e^*_\alpha) = \overline{\zeta}(e^*_\alpha) = 0, \qquad \frac{d\mathsf{a}}{\mathsf{a}}(e^*_\alpha) = 1,$$

since e^*_α is a vertical vector field on P^5.

Condition (2) is equivalent to its infinitesimal counterpart:

$$\mathscr{L}_{e^*_\alpha}\,\omega = -\mathsf{ad}_{e_\alpha}\omega,$$

where $\mathscr{L}_{e^*_\alpha}\,\omega$ is the Lie derivative of ω by the vector field e^*_α and where ad_{e_α} is the linear map $\mathfrak{g} \to \mathfrak{g}$ defined by: $\mathsf{ad}_{e_\alpha}(X) = [e_\alpha, X]$. We determine $\mathscr{L}_{e^*_\alpha}\,\omega$ with the help of Cartan's formula:

$$\mathscr{L}_{e^*_\alpha}\,\omega = e_{\alpha^*} \lrcorner\, d\omega + d\left(e^*_\alpha \lrcorner\, \omega\right),$$

with

$$d\left(e^*_\alpha \lrcorner\, \omega\right) = 0$$

from condition (1). The structure equations (2)–(3) give:

$$e_{\alpha^*} \lrcorner\, d\omega = \begin{pmatrix} 0 \\ 4\,\tau \\ 3\,\sigma \\ 2\,\rho \\ \zeta \\ \overline{\zeta} \end{pmatrix},$$

which is easily seen being equal to $-\mathsf{ad}_{e_\alpha}\omega$ from the Lie bracket structure of $\mathfrak{g}$. □

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LIE ALGEBRAS OF INFINITESIMAL AUTOMORPHISMS FOR THE MODEL MANIFOLDS OF GENERAL CLASSES II, III_2 AND IV_2

SAMUEL POCCHIOLA

ABSTRACT

We determine the Lie algebra of infinitesimal CR-automorphisms of the models of general classes II, III_2 and IV_2 through Cartan's equivalence method.

1. INTRODUCTION

The classification of CR-manifolds up to dimension 5 has highlighted the existence of 6 non-trivial classes of CR-manifolds, which have been referred to as general classes I, II, III_1, III_2, IV_1 and IV_2 [9]. Each of these classes entails a distinguished manifold, the model, whose Lie algebra of infinitesimal CR-automorphisms is of maximal dimension. It plays a special role, as CR-manifolds belonging to the same class can be viewed as its deformations, generally by the way of Cartan connection. The aim of this paper is to determine the Lie algebra of infinitesimal CR-automorphisms of the models for general classes II, III_2 and IV_2. This is already known [1, 5] for general classes II (Engel manifolds) and IV_2 (2-nondegenerate, 5-dimensional CR-manifolds of constant Levi rank 1), but is unknown, to our knowledge, in the case of general class III_2. In our view, the main interest of this paper is to provide a unified treatment for the 3 classes through the use of Cartan's equivalence method, in the spirit of [10]. Cartan's equivalence method has indeed been employed recently to solve the equivalence problem for general classes II, III_2 and IV_2 [11, 12, 13]. For each of these classes, the solution to the equivalence problem for the model has been of a great help for the treatment of the general case, as a similar structure of normalizations of the group parameters occurs in both cases.

For general class II, the model is provided by Beloshapka's cubic in $\mathbb{C}^3$, which is the CR-manifold defined by the equations:

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,. \end{aligned}$$

For general class III_2, the model is the 5-dimensional submanifold $\mathsf{N} \subset \mathbb{C}^4$ defined by:

$$\mathsf{N}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \\ w_3 &= \overline{w_3} + 2i\,z\overline{z}(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2). \end{aligned}$$

For general class IV_2, the model is provided by the tube over the future light cone, $\mathsf{LC} \subset \mathbb{C}^3$, defined by:

$$\mathsf{LC}: \qquad (\mathsf{Re}\, z_1)^2 - (\mathsf{Re}\, z_2)^2 - (\mathsf{Re}\, z_3)^2 = 0, \qquad \mathsf{Re}\, z_1 > 0.$$

A Cartan connection has been constructed for CR-manifolds belonging to general class II [1, 12] and III_2 [13]. The equivalence problem for manifolds belonging to general class IV_2 has been solved either by the determination of an absolute parallelism [4, 11], or the construction of a Cartan connection [7]. We use Cartan's equivalence method for which we refer to [10] as a standard reference.

2. Class II

This section is devoted to the determination of the Lie algebra of CR-automorphisms of Beloshapka's cubic in $\mathbb{C}^3$, which is the CR-manifold defined by the equations:

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,. \end{aligned}$$

It is the model manifold for generic 4-dimensional CR-manifolds of CR dimension 1 and real codimension 2, i.e. CR-manifolds belonging to class II, in the sense that any such manifold might be viewed as a deformation of Beloshapka's cubic by the way of a Cartan connection [1, 12]. The main result of this section is:

Theorem 1. *Beloshapka's cubic,*

$$\mathsf{B}: \qquad \begin{aligned} w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\ w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \end{aligned}$$

has a $\mathbf{5}$*-dimensional Lie algebra of CR-automorphisms. A basis for the Maurer-Cartan forms of* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{B})$ *is provided by the* 5 *differential* 1*-forms*

σ, ρ, ζ, $\overline{\zeta}$, α, which satisfy the structure equations:

$$\begin{aligned}
d\sigma &= 3\,\alpha\wedge\sigma + \rho\wedge\zeta + \rho\wedge\overline{\zeta}\,,\\
d\rho &= 2\,\alpha\wedge\rho + i\;\zeta\wedge\overline{\zeta}\,,\\
d\zeta &= \alpha\wedge\zeta\,,\\
d\overline{\zeta} &= \alpha\wedge\overline{\zeta}\,,\\
d\alpha &= 0.
\end{aligned}$$

2.1. **Initial G-structure.** The vectors field $\mathscr{L}_1$ defined by:

$$\mathscr{L}_1 := \frac{\partial}{\partial z} + i\,\overline{z}\,\frac{\partial}{\partial u_1} + i\,\left(2z\overline{z}+\overline{z}^2\right)\frac{\partial}{\partial u_2},$$

together with its conjugate:

$$\overline{\mathscr{L}_1} := \frac{\partial}{\partial \overline{z}} - i\,z\,\frac{\partial}{\partial u_1} - i\,\left(2z\overline{z}+z^2\right)\frac{\partial}{\partial u_2},$$

constitute a basis of $T_p^{1,0}\mathsf{B}$ at each point p of B. Moreover the vector fields $\mathscr{T}$ and $\mathscr{S}$ defined by:

$$\mathscr{T} := i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}],$$

and

$$\mathscr{S} := [\mathscr{L}_1, \mathscr{T}],$$

complete a frame on B:

$$\{\mathscr{S},\, \mathscr{T},\, \mathscr{L},\, \overline{\mathscr{L}}\}.$$

The expressions of $\mathscr{T}$ and $\mathscr{S}$ are:

$$\begin{aligned}
\mathscr{T} &= 2\,\frac{\partial}{\partial u_1} + (4z+4\overline{z})\,\frac{\partial}{\partial u_2},\\
\mathscr{S} &= 4\,\frac{\partial}{\partial u_2}.
\end{aligned}$$

The dual coframe $(\sigma_0,\, \rho_0,\, \zeta_0,\, \overline{\zeta_0})$ is thus given by:

$$\begin{aligned}
\sigma_0 &= \frac{i}{4}\,\overline{z}^2\,dz - \frac{i}{4}\,z^2\,d\overline{z} - \left(\frac{1}{2}\,z + \frac{1}{2}\,\overline{z}\right)du_1 + \frac{1}{4}\,du_2,\\
\rho_0 &= -\frac{i}{2}\,\overline{z}\,dz + \frac{i}{2}\,z\,d\overline{z} + \frac{1}{2}\,du_1,\\
\zeta_0 &= dz,\\
\overline{\zeta_0} &= d\overline{z}.
\end{aligned}$$

We deduce the structure equations enjoyed by $(\sigma_0,\, \rho_0,\, \zeta_0,\, \overline{\zeta_0},)$:

$$
\begin{aligned}
d\sigma_0 &= \rho_0 \wedge \zeta_0 + \rho_0 \wedge \overline{\zeta_0}, \\
d\rho_0 &= i\, \zeta_0 \wedge \overline{\zeta_0}, \\
d\zeta_0 &= 0, \\
d\overline{\zeta_0} &= 0.
\end{aligned}
\tag{1}
$$

As the torsion coefficients of these structure equations are constants, we have the following result:

Lemma 1. *Beloshapka's cubic is locally isomorphic to a Lie group whose Maurer-Cartan forms satisfy the structure equations* (1).

The matrix Lie group which encodes suitably the equivalence problem for Beloshapka's cubic (see [12]) is the 10-dimensional Lie group G_1 whose elements g are of the form:

$$
g := \begin{pmatrix} \mathsf{a}^2\overline{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \mathsf{d} & \mathsf{b} & \mathsf{a} & 0 \\ \mathsf{e} & \overline{\mathsf{b}} & 0 & \overline{\mathsf{a}} \end{pmatrix}.
$$

With the notations:

$$
\omega_0 := \begin{pmatrix} \sigma_0 \\ \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix}, \qquad \omega := \begin{pmatrix} \sigma \\ \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix},
$$

we introduce the G_1-structure P^1 on B constituted by the coframes ω which satisfy the relation:

$$
\omega := g \cdot \omega_0.
$$

The proof of theorem (1) relies on successive reductions of P^1 through Cartan's equivalence method.

2.2. **Normalization of a.** The structure equations for the lifted coframe ω are related to those of the base coframe ω_0 by the relation:

$$
d\omega = dg \cdot g^{-1} \wedge \omega + g \cdot d\omega_0. \tag{2}
$$

The term $dg \cdot g^{-1} \wedge \omega$ depends only on the structure equations of G_1 and is expressed through its Maurer-Cartan forms. The term $g \cdot d\omega_0$ contains the so-called torsion coefficients of the G_1-structure. We can compute it easily in terms of the forms σ, ρ, ζ, $\overline{\zeta}$, by a simple multiplication by g in the formulae (1) and a linear change of variables. The Maurer-Cartan forms for the group G_1 are given by the linearly independent entries of the matrix $dg \cdot g^{-1}$, which are:

$$\begin{aligned}
\alpha^1 &:= \frac{d\mathsf{a}}{\mathsf{a}},\\
\alpha^2 &:= -\frac{\mathsf{b}d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}},\\
\alpha^3 &:= -\frac{\mathsf{c}d\mathsf{a}}{\overline{\mathsf{a}}\,\mathsf{a}^3} - \frac{\mathsf{c}d\overline{\mathsf{a}}}{\overline{\mathsf{a}}^2\mathsf{a}^2} + \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
\alpha^4 &:= -\frac{\left(\mathsf{d}\mathsf{a}\overline{\mathsf{a}} - \mathsf{b}\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{c}d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
\alpha^5 &:= -\frac{\left(\mathsf{e}\mathsf{a}\overline{\mathsf{a}} - \overline{\mathsf{b}}\,\mathsf{c}\right)d\overline{\mathsf{a}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\mathsf{c}d\overline{\mathsf{b}}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},
\end{aligned}$$

together with their conjugates.

The first structure equation is given by:

$$d\sigma = 2\,\alpha^1 \wedge \sigma + \overline{\alpha^1} \wedge \sigma + \left(\frac{\mathsf{e}}{\mathsf{a}^2} + \frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}}\right)\sigma \wedge \rho - \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}}\;\sigma \wedge \zeta - \frac{\mathsf{c}}{\mathsf{a}\overline{\mathsf{a}}^2}\;\sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \frac{\mathsf{a}}{\overline{\mathsf{a}}}\;\rho \wedge \overline{\zeta}\,.$$

from which we immediately deduce that $\frac{\mathsf{a}}{\overline{\mathsf{a}}}$ is an essential torsion coefficient which might be normalised to 1 by setting:

$$\mathsf{a} = \overline{\mathsf{a}}.$$

2.3. **Normalizations of b and c.** We have thus reduced the G_1 equivalence problem on B to a G_2 equivalence problem, where G_2 is the 9 dimensional real matrix Lie group whose elements are of the form

$$g := \begin{pmatrix} \mathsf{a}^3 & 0 & 0 & 0\\ \mathsf{c} & \mathsf{a}^2 & 0 & 0\\ \mathsf{d} & \mathsf{b} & \mathsf{a} & 0\\ \mathsf{e} & \overline{\mathsf{b}} & 0 & \mathsf{a}\end{pmatrix}, \qquad\qquad \mathsf{a} \in \mathbb{R}.$$

The Maurer-Cartan forms of G_2 are given by:

$$\begin{aligned}
\beta^1 &:= \frac{d\mathsf{a}}{\mathsf{a}},\\
\beta^2 &:= -\frac{\mathsf{b}d\mathsf{a}}{\mathsf{a}^3} + \frac{d\mathsf{b}}{\mathsf{a}^2},\\
\beta^3 &:= -2\,\frac{\mathsf{c}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{c}}{\mathsf{a}^3},\\
\beta^4 &:= -\frac{\left(\mathsf{d}\mathsf{a}^2 - \mathsf{b}\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\mathsf{b}}{\mathsf{a}^5} + \frac{d\mathsf{d}}{\mathsf{a}^3},\\
\beta^5 &:= -\frac{\left(\mathsf{e}\mathsf{a}^2 - \overline{\mathsf{b}}\,\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\overline{\mathsf{b}}}{\mathsf{a}^5} + \frac{d\mathsf{e}}{\mathsf{a}^3},
\end{aligned}$$

together with $\overline{\beta^2}$, $\overline{\beta^3}$, $\overline{\beta^4}$, $\overline{\beta^5}$. Using formula (2), we get the structure equations for the lifted coframe $(\sigma, \rho, \zeta, \overline{\zeta})$ from those of the base coframe $(\sigma_0, \rho_0, \zeta_0, \overline{\zeta_0})$ by a matrix multiplication and a linear change of coordinates, as in the first step:

$$\begin{aligned} d\sigma &= 3\,\beta^1 \wedge \sigma \\ &\quad + U^\sigma_{\sigma\rho}\,\sigma\wedge\rho + U^\sigma_{\sigma\zeta}\,\sigma\wedge\zeta + U^\sigma_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\rho &= 2\beta^1\wedge\rho + \beta^3\wedge\sigma \\ &\quad + U^\rho_{\sigma\rho}\,\sigma\wedge\rho + U^\rho_{\sigma\zeta}\,\sigma\wedge\zeta + U^\rho_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} \\ &\quad + U^\rho_{\rho\zeta}\,\rho\wedge\zeta + U^\rho_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\,\zeta\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\zeta &= \beta^1\wedge\zeta + \beta^2\wedge\rho + \beta^4\wedge\sigma \\ &\quad + U^\zeta_{\sigma\rho}\,\sigma\wedge\rho + U^\zeta_{\sigma\zeta}\,\sigma\wedge\zeta + U^\zeta_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + U^\zeta_{\rho\zeta}\,\rho\wedge\zeta \\ &\quad + U^\zeta_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + U^\zeta_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}. \end{aligned}$$

We now proceed with the absorption phase. We introduce the modified Maurer-Cartan forms:

$$\widetilde{\beta}^i = \beta^i - y_\sigma\,\sigma - y^i_\rho\,\rho - y^i_\zeta\,\zeta - y^i_{\overline{\zeta}}\,\overline{\zeta},$$

such that the structure equations rewrite:

$$\begin{aligned} d\sigma &= 3\,\widetilde{\beta}^1\wedge\sigma \\ &\quad + \left(U^\sigma_{\sigma\rho} - 3\,y^1_\rho\right)\,\sigma\wedge\rho + \left(U^\sigma_{\sigma\zeta} - 3\,y^1_\zeta\right)\,\sigma\wedge\zeta \\ &\quad + \left(U^\sigma_{\sigma\overline{\zeta}} - 3\,y^1_{\overline{\zeta}}\right)\,\sigma\wedge\overline{\zeta}\ \rho\wedge\zeta + \rho\wedge\overline{\zeta}, \end{aligned}$$

$$\begin{aligned} d\rho &= 2\widetilde{\beta}^1\wedge\rho + \widetilde{\beta}^3\wedge\sigma \\ &\quad + \left(U^\rho_{\sigma\rho} + 2\,y^1_\sigma - y^3_\rho\right)\,\sigma\wedge\rho + \left(U^\rho_{\sigma\zeta} - y^3_\zeta\right)\,\sigma\wedge\zeta \\ &\quad + \left(U^\rho_{\sigma\overline{\zeta}} - y^3_{\overline{\zeta}}\right)\,\sigma\wedge\overline{\zeta} + \left(U^\rho_{\rho\zeta} - 2\,y^1_\zeta\right)\,\rho\wedge\zeta \\ &\quad + \left(U^\rho_{\rho\overline{\zeta}} - 2\,y^1_{\overline{\zeta}}\right)\,\rho\wedge\overline{\zeta} + i\ \zeta\wedge\overline{\zeta}\,, \end{aligned}$$

$$
\begin{aligned}
d\zeta = \widetilde{\beta^1} \wedge \zeta + \widetilde{\beta^2} \wedge \rho + \widetilde{\beta^4} \wedge \sigma
& + \left(U^{\zeta}_{\sigma\rho} + y^2_{\sigma} - y^4_{\rho}\right)\, \sigma \wedge \rho + \left(U^{\zeta}_{\sigma\zeta} + y^1_{\sigma} - y^4_{\zeta}\right)\, \sigma \wedge \zeta \\
& + \left(U^{\zeta}_{\sigma\overline{\zeta}} - y^4_{\overline{\zeta}}\right)\, \sigma \wedge \overline{\zeta} + \left(U^{\zeta}_{\rho\zeta} + y^1_{\rho} - y^2_{\zeta}\right)\, \rho \wedge \zeta \\
& + \left(U^{\zeta}_{\rho\overline{\zeta}} - y^2_{\overline{\zeta}}\right)\, \rho \wedge \overline{\zeta} + \left(U^{\zeta}_{\zeta\overline{\zeta}} - y^1_{\overline{\zeta}}\right)\, \zeta \wedge \overline{\zeta}\, .
\end{aligned}
$$

We get the following absorbtion equations:

$$
\begin{aligned}
3\, y^1_{\rho} &= U^{\sigma}_{\sigma\rho}, & 3\, y^1_{\zeta} &= U^{\sigma}_{\sigma\zeta}, & 3\, y^1_{\overline{\zeta}} &= U^{\sigma}_{\sigma\overline{\zeta}}, \\
-2\, y^1_{\sigma} + y^3_{\rho} &= U^{\rho}_{\sigma\rho}, & y^3_{\zeta} &= U^{\rho}_{\sigma\zeta}, & y^3_{\overline{\zeta}} &= U^{\rho}_{\sigma\overline{\zeta}}, \\
2\, y^1_{\zeta} &= U^{\rho}_{\rho\zeta}, & 2\, y^1_{\overline{\zeta}} &= U^{\rho}_{\rho\overline{\zeta}}, & -y^2_{\sigma} + y^4_{\rho} &= U^{\zeta}_{\sigma\rho}, \\
-y^1_{\sigma} + y^4_{\zeta} &= U^{\zeta}_{\sigma\zeta}, & y^4_{\overline{\zeta}} &= U^{\zeta}_{\sigma\overline{\zeta}}, & -y^1_{\rho} + y^2_{\zeta} &= U^{\zeta}_{\rho\zeta}, \\
y^2_{\overline{\zeta}} &= U^{\zeta}_{\rho\overline{\zeta}}, & y^1_{\overline{\zeta}} &= U^{\zeta}_{\zeta\overline{\zeta}}.
\end{aligned}
$$

Eliminating $y^1_{\overline{\zeta}}$ among the previous equations leads to:

$$
U^{\zeta}_{\zeta\overline{\zeta}} = \frac{1}{2}\, U^{\rho}_{\rho\overline{\zeta}} = \frac{1}{3}\, U^{\sigma}_{\sigma\overline{\zeta}},
$$

that is:

$$
\frac{i\mathsf{b}}{\mathsf{a}^2} = \frac{1}{2}\left(\frac{\mathsf{c}}{\mathsf{a}^3} - \frac{i\mathsf{b}}{\mathsf{a}^2}\right) = -\frac{1}{3}\,\frac{\mathsf{c}}{\mathsf{a}^3},
$$

from which we easily deduce that

$$
\mathsf{b} = \mathsf{c} = 0.
$$

2.4. **Normalizations of** d **and** e**.** We have thus reduced the group G_2 to a new group G_3, whose elements are of the form

$$
g := \begin{pmatrix} \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & \mathsf{a}^2 & 0 & 0 \\ \mathsf{d} & 0 & \mathsf{a} & 0 \\ \mathsf{e} & 0 & 0 & \mathsf{a} \end{pmatrix}.
$$

The Maurer Cartan forms of G_3 are:

$$\gamma^1 := \frac{d\mathsf{a}}{\mathsf{a}},$$
$$\gamma^2 := -\frac{\mathsf{d}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{d}}{\mathsf{a}^3},$$
$$\gamma^3 := -\frac{\mathsf{e}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{e}}{\mathsf{a}^3}.$$

The third loop of Cartan's method is straightforward. We get the following structure equations:

$$d\sigma = 3\,\gamma^1 \wedge \sigma + \frac{\mathsf{d}+\mathsf{e}}{\mathsf{a}^4}\,\sigma \wedge \rho + \rho \wedge \zeta + \rho \wedge \overline{\zeta}\,,$$
$$d\rho = 2\,\gamma^1 \wedge \rho + i\,\frac{\mathsf{e}}{\mathsf{a}^3}\,\sigma \wedge \zeta - i\,\frac{\mathsf{d}}{\mathsf{a}^3}\,\sigma \wedge \overline{\zeta} + i\,\zeta \wedge \overline{\zeta}\,,$$
$$d\zeta = \gamma^1 \wedge \zeta + \gamma^2 \wedge \sigma + \frac{\mathsf{d}\,(\mathsf{d}+\mathsf{e})}{\mathsf{a}^6}\,\sigma \wedge \rho + \frac{\mathsf{d}}{\mathsf{a}^3}\,\rho \wedge \zeta + \frac{\mathsf{d}}{\mathsf{a}^3}\,\rho \wedge \overline{\zeta}\,,$$
$$d\overline{\zeta} = \gamma^1 \wedge \overline{\zeta} + \gamma^3 \wedge \sigma + \frac{\mathsf{e}\,(\mathsf{d}+\mathsf{e})}{\mathsf{a}^6}\,\sigma \wedge \rho + \frac{\mathsf{e}}{\mathsf{a}^3}\,\rho \wedge \zeta + \frac{\mathsf{e}}{\mathsf{a}^3}\,\rho \wedge \overline{\zeta}\,,$$

from which we deduce that we can perform the normalizations:

$$\mathsf{e} = \mathsf{d} = 0.$$

With the 1-dimensional group G_4 whose elements g are of the form:

$$g := \begin{pmatrix} \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & \mathsf{a}^2 & 0 & 0 \\ 0 & 0 & \mathsf{a} & 0 \\ 0 & 0 & 0 & \mathsf{a} \end{pmatrix},$$

and whose Maurer-Cartan form is given by

$$\alpha := \frac{d\mathsf{a}}{\mathsf{a}},$$

we get the following structure equations:

$$d\sigma = 3\,\alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}\,,$$
$$d\rho = 2\,\alpha \wedge \rho + i\,\zeta \wedge \overline{\zeta}\,,$$
$$d\zeta = \alpha \wedge \zeta\,,$$
$$d\overline{\zeta} = \alpha \wedge \overline{\zeta}\,.$$

No more normalizations are allowed at this stage. We thus just perform a prolongation by adjoining the form α to the structure equations, whose exterior derivative is given by:

$$d\alpha = 0.$$

This completes the proof of Theorem 1.

3. Class III_2

This section is devoted to the determination of the Lie algebra of CR-automorphisms of the model manifold of class III_2 which is defined by the equations:

$$
\mathsf{N}: \quad
\begin{aligned}
w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\
w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \\
w_3 &= \overline{w_3} + 2i\,z\overline{z}(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2).
\end{aligned}
$$

It is the model manifold for CR-manifolds belonging to class III_2, in the sense that any such manifold might be viewed as a deformation of N by the way of a Cartan connection ([13]). The main result of this section is the following:

Theorem 2. *The model of the class* III_2:

$$
\mathsf{N}: \quad
\begin{aligned}
w_1 &= \overline{w_1} + 2\,i\,z\overline{z}, \\
w_2 &= \overline{w_2} + 2\,i\,z\overline{z}\,(z+\overline{z})\,, \\
w_3 &= \overline{w_3} + 2i\,z\overline{z}(z^2 + \frac{3}{2}\,z\overline{z} + \overline{z}^2),
\end{aligned}
$$

has a **6***-dimensional Lie algebra of CR-automorphisms. A basis for the Maurer-Cartan forms of* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{N})$ *is provided by the* 6 *differential* 1*-forms* τ, σ, ρ, ζ, $\overline{\zeta}$, α*, which satisfy the structure equations:*

$$
\begin{aligned}
d\tau &= 4\,\alpha\wedge\tau \,+\, \sigma\wedge\zeta \,+\, \sigma\wedge\overline{\zeta}\,, \\
d\sigma &= 3\,\alpha\wedge\sigma \,+\, \rho\wedge\zeta \,+\, \rho\wedge\overline{\zeta}\,, \\
d\rho &= 2\,\alpha\wedge\rho + i\,\ \zeta\wedge\overline{\zeta}\,, \\
d\zeta &= \alpha\wedge\zeta\,, \\
d\overline{\zeta} &= \alpha\wedge\overline{\zeta}\,, \\
d\alpha &= 0.
\end{aligned}
$$

3.1. **Initial G-structure.** The vector fields :

$$
\mathscr{L} := \frac{\partial}{\partial z} + i\overline{z}\,\frac{\partial}{\partial u_1} + i(2z\overline{z}+\overline{z}^2)\,\frac{\partial}{\partial u_2} + i(3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3)\,\frac{\partial}{\partial u_3},
$$

with its conjugate:

$$
\overline{\mathscr{L}} := \frac{\partial}{\partial \overline{z}} - iz\,\frac{\partial}{\partial u_1} - i(2z\overline{z}+z^2)\,\frac{\partial}{\partial u_2} - i(3z\overline{z}^2 + 3z^2\overline{z} + z^3)\,\frac{\partial}{\partial u_3},
$$

constitute a basis of $T_p^{1,0}\mathsf{N}$ and of $T_p^{0,1}\mathsf{N}$ at each point p of N. Moreover the vector fields $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{R}$ defined by:

$$\mathscr{T} := i\,[\mathscr{L}, \overline{\mathscr{L}_1}],$$

$$\mathscr{S} := [\mathscr{L}_1, \mathscr{T}],$$

and

$$\mathscr{R} := [\mathscr{L}_1, \mathscr{S}],$$

complete a frame on N:

$$\{\mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}\}.$$

The expressions of $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{R}$ are:

$$\begin{aligned}
\mathscr{T} &:= 2\,\frac{\partial}{\partial u_1} + (4z + 4\overline{z})\frac{\partial}{\partial u_2} + (6z^2 + 12z\overline{z} + 6\overline{z}^2)\,\frac{\partial}{\partial u_3},\\
\mathscr{S} &:= 4\,\frac{\partial}{\partial u_2} + (12z + 12\overline{z})\,\frac{\partial}{\partial u_3},\\
\mathscr{R} &:= 12\,\frac{\partial}{\partial u_3}.
\end{aligned}$$

The dual coframe $\{\tau_0,\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\}$ is thus given by:

$$\begin{aligned}
\tau_0 &= -\frac{i}{12}\,\overline{z}^3\,dz + \frac{i}{12}\,z^3\,d\overline{z} + \left(\frac{1}{4}\,z^2 + \frac{1}{2}\,z\overline{z} + \frac{1}{4}\,\overline{z}^2\right) du_1 - \left(\frac{1}{4}\,z + \frac{1}{4}\,\overline{z}\right) du_2 + \frac{1}{12}\,du_3,\\
\sigma_0 &= \frac{i}{4}\,\overline{z}^2\,dz - \frac{i}{4}\,z^2\,d\overline{z} - \left(\frac{1}{2}\,z + \frac{1}{2}\,\overline{z}\right) du_1 + \frac{1}{4}\,du_2,\\
\rho_0 &= -\frac{i}{2}\,\overline{z}\,dz + \frac{i}{2}\,z\,d\overline{z} + \frac{1}{2}\,du_1,\\
\zeta_0 &= dz,\\
\overline{\zeta_0} &= d\overline{z}.
\end{aligned}$$

We deduce the structure equations enjoyed by the base coframe $\{\tau_0,\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\}$:

$$\begin{aligned}
d\tau_0 &= \sigma_0 \wedge \zeta_0 + \sigma_0 \wedge \overline{\zeta_0},\\
d\sigma_0 &= \rho_0 \wedge \zeta_0 + \rho_0 \wedge \overline{\zeta_0},\\
d\rho_0 &= i\,\zeta_0 \wedge \overline{\zeta_0},\\
d\zeta_0 &= 0,\\
d\overline{\zeta_0} &= 0.
\end{aligned} \tag{3}$$

As the torsion coefficients of these structure equations are constants, we have the following result:

Lemma 2. *The model of the class* III_2 *is locally isomorphic to a Lie group whose Maurer-Cartan forms satisfy the structure equations* (3).

The matrix Lie group which encodes suitably the equivalence problem for the model of class III_2 (see [13]) is the 18-dimensional Lie group G_1 whose elements g are of the form:

$$g := \begin{pmatrix} \mathsf{a}^3\bar{\mathsf{a}} & 0 & 0 & 0 & 0 \\ \mathsf{f} & \mathsf{a}^2\bar{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{g} & \mathsf{c} & \mathsf{a}\bar{\mathsf{a}} & 0 & 0 \\ \mathsf{h} & \mathsf{d} & \mathsf{b} & \mathsf{a} & 0 \\ \mathsf{k} & \mathsf{e} & \bar{\mathsf{b}} & 0 & \bar{\mathsf{a}} \end{pmatrix}.$$

With the notations:

$$\omega_0 := \begin{pmatrix} \tau_0 \\ \sigma_0 \\ \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix}, \qquad \omega := \begin{pmatrix} \tau \\ \sigma \\ \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix},$$

we introduce the G_1-structure P^1 on N constituted by the coframes ω which satisfy the relation:

$$\omega := g \cdot \omega_0.$$

As in the case of Beloshapka's cubic, the proof of theorem (2) relies on successive reductions of P^1 through Cartan's equivalence method.

3.2. **Normalization of a.** The Maurer-Cartan forms of G_1 are given by:

$$
\begin{aligned}
\alpha^1 &:= \frac{d\mathsf{a}}{\mathsf{a}},\\
\alpha^2 &:= -\frac{\mathsf{b}\,d\mathsf{a}}{\mathsf{a}^2\overline{\mathsf{a}}} + \frac{d\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}},\\
\alpha^3 &:= -\frac{\mathsf{c}\,d\mathsf{a}}{\overline{\mathsf{a}}\,\mathsf{a}^3} - \frac{\mathsf{c}\,d\overline{\mathsf{a}}}{\overline{\mathsf{a}}^2\mathsf{a}^2} + \frac{d\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
\alpha^4 &:= -\frac{(\mathsf{d}\mathsf{a}\overline{\mathsf{a}} - \mathsf{b}\mathsf{c})\,d\mathsf{a}}{\mathsf{a}^4\overline{\mathsf{a}}^2} - \frac{\mathsf{c}\,d\mathsf{b}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
\alpha^5 &:= -\frac{\big(\mathsf{e}\mathsf{a}\overline{\mathsf{a}} - \overline{\mathsf{b}}\,\mathsf{c}\big)\,d\overline{\mathsf{a}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\mathsf{c}\,d\overline{\mathsf{b}}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}},\\
\alpha^6 &:= -2\,\frac{\mathsf{f}\,d\mathsf{a}}{\overline{\mathsf{a}}\,\mathsf{a}^4} - \frac{\mathsf{f}\,d\overline{\mathsf{a}}}{\mathsf{a}^3\overline{\mathsf{a}}^2} + \frac{d\mathsf{f}}{\overline{\mathsf{a}}\,\mathsf{a}^3},\\
\alpha^7 &:= -\frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\mathsf{a}}{\overline{\mathsf{a}}^2\mathsf{a}^6} - \frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\overline{\mathsf{a}}}{\overline{\mathsf{a}}^3\mathsf{a}^5} - \frac{\mathsf{f}\,d\mathsf{c}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{g}}{\overline{\mathsf{a}}\,\mathsf{a}^3},\\
\alpha^8 &:= -\frac{\big(\mathsf{h}\mathsf{a}^3\overline{\mathsf{a}}^2 - \mathsf{d}\mathsf{f}\mathsf{a}\overline{\mathsf{a}} - \mathsf{b}\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} + \mathsf{b}\mathsf{c}\mathsf{f}\big)\,d\mathsf{a}}{\mathsf{a}^7\overline{\mathsf{a}}^3} - \frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\mathsf{b}}{\mathsf{a}^6\overline{\mathsf{a}}^3} - \frac{\mathsf{f}\,d\mathsf{d}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{h}}{\overline{\mathsf{a}}\,\mathsf{a}^3},\\
\alpha^9 &:= -\frac{\big(\mathsf{k}\mathsf{a}^3\overline{\mathsf{a}}^2 - \mathsf{e}\mathsf{f}\mathsf{a}\overline{\mathsf{a}} - \overline{\mathsf{b}}\,\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} + \overline{\mathsf{b}}\,\mathsf{c}\mathsf{f}\big)\,d\overline{\mathsf{a}}}{\mathsf{a}^6\overline{\mathsf{a}}^4} - \frac{(\mathsf{g}\mathsf{a}^2\overline{\mathsf{a}} - \mathsf{c}\mathsf{f})\,d\overline{\mathsf{b}}}{\mathsf{a}^6\overline{\mathsf{a}}^3} - \frac{\mathsf{f}\,d\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{k}}{\overline{\mathsf{a}}\,\mathsf{a}^3},
\end{aligned}
$$

together with their conjugates.

The first structure equation is given by:

$$
\begin{aligned}
d\tau = 3\,\alpha^1 \wedge \tau + \overline{\alpha^1} \wedge \tau
&+ T^{\tau}_{\tau\sigma}\,\tau\wedge\sigma + T^{\tau}_{\tau\rho}\,\tau\wedge\rho + T^{\tau}_{\tau\zeta}\,\tau\wedge\zeta\\
&+ T^{\tau}_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta} + T^{\tau}_{\sigma\rho}\,\sigma\wedge\rho + \sigma\wedge\zeta - \frac{\mathsf{a}}{\overline{\mathsf{a}}}\;\sigma\wedge\overline{\zeta}\,,
\end{aligned}
$$

from which we immediately deduce that $\frac{\mathsf{a}}{\overline{\mathsf{a}}}$ is an essential torsion coefficient which shall be normalized to 1 by setting:

$$\mathsf{a} = \overline{\mathsf{a}}.$$

We thus have reduced the G_1 equivalence problem to a G_2 equivalence problem, where G_2 is the 10 dimensional real matrix Lie group whose elements are of the form

$$
g = \begin{pmatrix}
\mathsf{a}^4 & 0 & 0 & 0 & 0\\
\mathsf{f} & \mathsf{a}^3 & 0 & 0 & 0\\
\mathsf{g} & \mathsf{c} & \mathsf{a}^2 & 0 & 0\\
\mathsf{h} & \mathsf{d} & \mathsf{b} & \mathsf{a} & 0\\
\mathsf{k} & \mathsf{e} & \overline{\mathsf{b}} & 0 & \mathsf{a}
\end{pmatrix},
$$

3.3. **Normalizations of f, b and c.** The Maurer-Cartan forms of G_2 are given by:

$$
\begin{aligned}
\beta^1 &:= \frac{d\mathsf{a}}{\mathsf{a}},\\
\beta^2 &:= -\frac{\mathsf{b}d\mathsf{a}}{\mathsf{a}^3} + \frac{d\mathsf{b}}{\mathsf{a}^2},\\
\beta^3 &:= -2\,\frac{\mathsf{c}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{c}}{\mathsf{a}^3},\\
\beta^4 &:= -\frac{(\mathsf{d}\mathsf{a}^2 - \mathsf{b}\mathsf{c})\,d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\mathsf{b}}{\mathsf{a}^5} + \frac{d\mathsf{d}}{\mathsf{a}^3},\\
\beta^5 &:= -\frac{\left(\mathsf{e}\mathsf{a}^2 - \overline{\mathsf{b}}\,\mathsf{c}\right)d\mathsf{a}}{\mathsf{a}^6} - \frac{\mathsf{c}d\overline{\mathsf{b}}}{\mathsf{a}^5} + \frac{d\mathsf{e}}{\mathsf{a}^3},\\
\beta^6 &:= -3\,\frac{\mathsf{f}d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{f}}{\mathsf{a}^4},\\
\beta^7 &:= -2\,\frac{(\mathsf{g}\mathsf{a}^3 - \mathsf{c}\mathsf{f})\,d\mathsf{a}}{\mathsf{a}^8} - \frac{\mathsf{f}d\mathsf{c}}{\mathsf{a}^7} + \frac{d\mathsf{g}}{\mathsf{a}^4},\\
\beta^8 &:= -\frac{(\mathsf{h}\mathsf{a}^5 - \mathsf{d}\mathsf{f}\mathsf{a}^2 - \mathsf{b}\mathsf{g}\mathsf{a}^3 + \mathsf{b}\mathsf{c}\mathsf{f})\,d\mathsf{a}}{\mathsf{a}^{10}} - \frac{(\mathsf{g}\mathsf{a}^3 - \mathsf{c}\mathsf{f})\,d\mathsf{b}}{\mathsf{a}^9} - \frac{\mathsf{f}d\mathsf{d}}{\mathsf{a}^7} + \frac{d\mathsf{h}}{\mathsf{a}^4},\\
\beta^9 &:= -\frac{\left(\mathsf{k}\mathsf{a}^5 - \mathsf{e}\mathsf{f}\mathsf{a}^2 - \overline{\mathsf{b}}\,\mathsf{g}\mathsf{a}^3 + \overline{\mathsf{b}}\,\mathsf{c}\mathsf{f}\right)d\mathsf{a}}{\mathsf{a}^{10}} - \frac{(\mathsf{g}\mathsf{a}^3 - \mathsf{c}\mathsf{f})\,d\overline{\mathsf{b}}}{\mathsf{a}^9} - \frac{\mathsf{f}d\mathsf{e}}{\mathsf{a}^5\overline{\mathsf{a}}^2} + \frac{d\mathsf{k}}{\mathsf{a}^4},
\end{aligned}
$$

together with $\overline{\beta^i}$, $\;i = 2\ldots 9$.

Using formula (2), we get the structure equations for the lifted coframe $(\tau, \sigma, \rho, \zeta, \overline{\zeta})$ from those of the base coframe $(\tau_0, \sigma_0, \rho_0, \hat{\zeta}_0, \overline{\hat{\zeta}_0})$ by a matrix multiplication and a linear change of coordinates, as in the first step:

$$
\begin{aligned}
d\tau = 4\,\beta^1 \wedge \tau &\\
&+ U^{\tau}_{\tau\sigma}\,\tau\wedge\sigma + U^{\tau}_{\tau\rho}\,\tau\wedge\rho + U^{\tau}_{\tau\zeta}\,\tau\wedge\zeta + U^{\tau}_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta}\\
&\qquad + U^{\tau}_{\sigma\rho}\,\sigma\wedge\rho + \sigma\wedge\zeta + \sigma\wedge\overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\sigma = 3\,\beta^1 \wedge \sigma + \beta^6 \wedge \tau &\\
&+ U^{\sigma}_{\tau\sigma}\,\tau\wedge\sigma + U^{\sigma}_{\tau\rho}\,\tau\wedge\rho + U^{\sigma}_{\tau\zeta}\,\tau\wedge\zeta\\
&+ U^{\sigma}_{\tau\overline{\zeta}}\,\tau\wedge\overline{\zeta} + U^{\sigma}_{\sigma\rho}\,\sigma\wedge\rho + U^{\sigma}_{\sigma\zeta}\,\sigma\wedge\zeta\\
&\qquad + U^{\sigma}_{\sigma\overline{\zeta}}\,\sigma\wedge\overline{\zeta} + \rho\wedge\zeta + \rho\wedge\overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\rho = {} & 2\beta^1 \wedge \rho + \beta^3 \wedge \sigma + \beta^7 \wedge \tau \\
& + U^\rho_{\tau\sigma}\, \tau \wedge \sigma + U^\rho_{\tau\rho}\, \tau \wedge \rho + U^\rho_{\tau\zeta}\, \tau \wedge \zeta + U^\rho_{\tau\overline{\zeta}}\, \rho \wedge \overline{\zeta} + U^\rho_{\sigma\rho}\, \sigma \wedge \rho \\
& + U^\rho_{\sigma\zeta}\, \sigma \wedge \zeta + U^\rho_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + U^\rho_{\rho\zeta}\, \rho \wedge \zeta + U^\rho_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \zeta \wedge \overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = {} & \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma + \beta^8 \wedge \tau \\
& + U^\zeta_{\tau\sigma}\, \tau \wedge \sigma + U^\zeta_{\tau\rho}\, \tau \wedge \rho + U^\zeta_{\tau\zeta}\, \tau \wedge \zeta + U^\zeta_{\tau\overline{\zeta}}\, \tau \wedge \overline{\zeta} \\
& + U^\zeta_{\sigma\rho}\, \sigma \wedge \rho + U^\zeta_{\sigma\zeta}\, \sigma \wedge \zeta + U^\zeta_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + U^\zeta_{\rho\zeta}\, \rho \wedge \zeta \\
& + U^\zeta_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + U^\zeta_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

We now proceed with the absorption phase. We introduce the modified Maurer-Cartan forms:

$$
\widetilde{\beta}^i = \beta^i - y^i_\tau\, \tau - y_\sigma\, \sigma - y^i_\rho\, \rho - y^i_\zeta\, \zeta - y^i_{\overline{\zeta}}\, \overline{\zeta}.
$$

The structure equations rewrite:

$$
\begin{aligned}
d\tau = {} & 4\, \widetilde{\beta}^1 \wedge \tau \\
& + \left(U^\tau_{\tau\sigma} - 4\, y^1_\sigma\right)\, \tau \wedge \sigma + \left(U^\tau_{\tau\rho} - 4\, y^1_\rho\right)\, \tau \wedge \rho \\
& + \left(U^\tau_{\tau\zeta} - 4\, y^1_\zeta\right)\, \tau \wedge \zeta + \left(U^\tau_{\tau\overline{\zeta}} - 4\, y^1_{\overline{\zeta}}\right)\, \tau \wedge \overline{\zeta} \\
& + U^\tau_{\sigma\rho}\, \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}\,,
\end{aligned}
$$

$$
\begin{aligned}
d\sigma = {} & 3\, \widetilde{\beta}^1 \wedge \sigma + \widetilde{\beta}^6 \wedge \tau \\
& + \left(U^\sigma_{\tau\sigma} + 3\, y^1_\tau - y^6_\sigma\right)\, \tau \wedge \sigma + \left(U^\sigma_{\tau\rho} - y^6_\rho\right)\, \tau \wedge \rho \\
& + \left(U^\sigma_{\tau\zeta} - y^6_\zeta\right)\, \tau \wedge \zeta + \left(U^\sigma_{\tau\overline{\zeta}} - y^6_{\overline{\zeta}}\right)\, \tau \wedge \overline{\zeta} \\
& + \left(U^\sigma_{\sigma\rho} - 3\, y^1_\rho\right)\, \sigma \wedge \rho + \left(U^\sigma_{\sigma\zeta} - 3\, y^1_\zeta\right)\, \sigma \wedge \zeta \\
& + \left(U^\sigma_{\sigma\overline{\zeta}} - 3\, y^1_{\overline{\zeta}}\right)\, \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\rho = 2\widetilde{\beta}^1 \wedge \rho + \widetilde{\beta}^3 \wedge \sigma + \widetilde{\beta}^7 \wedge \tau
&+ \left(U^\rho_{\tau\sigma} + y^3_\tau - y^7_\sigma\right)\ \tau \wedge \sigma + \left(U^\rho_{\tau\rho} + 2\, y^1_\tau - y^7_\rho\right)\ \tau \wedge \rho \\
&+ \left(U^\rho_{\tau\zeta} - y^7_\zeta\right)\ \tau \wedge \zeta + \left(U^\rho_{\tau\overline{\zeta}} - y^7_{\overline{\zeta}}\right)\ \rho \wedge \overline{\zeta} \\
&+ \left(U^\rho_{\sigma\rho} + 2\, y^1_\sigma - y^3_\rho\right)\ \sigma \wedge \rho + \left(U^\rho_{\sigma\zeta} - y^3_\zeta\right)\ \sigma \wedge \zeta \\
&+ \left(U^\rho_{\sigma\overline{\zeta}} - y^3_{\overline{\zeta}}\right)\ \sigma \wedge \overline{\zeta} + \left(U^\rho_{\rho\zeta} - 2\, y^1_\zeta\right)\ \rho \wedge \zeta \\
&+ \left(U^\rho_{\rho\overline{\zeta}} - 2\, y^1_{\overline{\zeta}}\right)\ \rho \wedge \overline{\zeta} + i\ \zeta \wedge \overline{\zeta}\,,
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = \widetilde{\beta^1} \wedge \zeta + \widetilde{\beta^2} \wedge \rho + \widetilde{\beta^4} \wedge \sigma + \widetilde{\beta^8} \wedge \tau
&+ \left(U^\zeta_{\tau\sigma} + y^4_\tau - y^8_\sigma\right)\ \tau \wedge \sigma + \left(U^\zeta_{\tau\rho} + y^2_\tau - y^8_\rho\right)\ \tau \wedge \rho \\
&+ \left(U^\zeta_{\tau\zeta} + y^1_\tau - y^8_\zeta\right)\ \tau \wedge \zeta + \left(U^\zeta_{\tau\overline{\zeta}} - y^8_{\overline{\zeta}}\right)\ \tau \wedge \overline{\zeta} \\
&+ \left(U^\zeta_{\sigma\rho} + y^2_\sigma - y^4_\rho\right)\ \sigma \wedge \rho + \left(U^\zeta_{\sigma\zeta} + y^1_\sigma - y^4_\zeta\right)\ \sigma \wedge \zeta \\
&+ \left(U^\zeta_{\sigma\overline{\zeta}} - y^4_{\overline{\zeta}}\right)\ \sigma \wedge \overline{\zeta} + \left(U^\zeta_{\rho\zeta} + y^1_\rho - y^2_\zeta\right)\ \rho \wedge \zeta \\
&+ \left(U^\zeta_{\rho\overline{\zeta}} - y^2_{\overline{\zeta}}\right)\ \rho \wedge \overline{\zeta} + \left(U^\zeta_{\zeta\overline{\zeta}} - y^1_{\overline{\zeta}}\right)\ \zeta \wedge \overline{\zeta}\,.
\end{aligned}
$$

We get the following absorbtion equations:

$$
\begin{array}{rrr}
4\, y^1_\sigma = U^\tau_{\tau\sigma}, & 4\, y^1_\rho = U^\tau_{\tau\rho}, & 4\, y^1_\zeta = U^\tau_{\tau\zeta}, \\
4\, y^1_{\overline{\zeta}} = U^\tau_{\tau\overline{\zeta}}, & -3\, y^1_\tau + y^6_\sigma = U^\sigma_{\tau\sigma}, & y^6_\rho = U^\sigma_{\tau\rho}, \\
y^6_\zeta = U^\sigma_{\tau\zeta}, & y^6_{\overline{\zeta}} = U^\sigma_{\tau\overline{\zeta}}, & 3\, y^1_\rho = U^\sigma_{\sigma\rho}, \\
3\, y^1_\zeta = U^\sigma_{\sigma\zeta}, & 3\, y^1_{\overline{\zeta}} = U^\sigma_{\sigma\overline{\zeta}}, & -y^3_\tau + y^7_\sigma = U^\rho_{\tau\sigma}, \\
-2\, y^1_\tau + y^7_\rho = U^\rho_{\tau\rho}, & y^7_\zeta = U^\rho_{\tau\zeta}, & y^7_{\overline{\zeta}} = U^\rho_{\tau\overline{\zeta}}, \\
-2\, y^1_\sigma + y^3_\rho = U^\rho_{\sigma\rho}, & y^3_\zeta = U^\rho_{\sigma\zeta}, & y^3_{\overline{\zeta}} = U^\rho_{\sigma\overline{\zeta}}, \\
2\, y^1_\zeta = U^\rho_{\rho\zeta}, & 2\, y^1_{\overline{\zeta}} = U^\rho_{\rho\overline{\zeta}}, & -y^4_\tau + y^8_\sigma = U^\zeta_{\tau\sigma}, \\
-y^2_\tau + y^8_\rho = U^\zeta_{\tau\rho}, & -y^1_\tau + y^8_\zeta = U^\zeta_{\tau\zeta}, & y^8_{\overline{\zeta}} = U^\zeta_{\tau\overline{\zeta}}, \\
-y^2_\sigma + y^4_\rho = U^\zeta_{\sigma\rho}, & -y^1_\sigma + y^4_\zeta = U^\zeta_{\sigma\zeta}, & y^4_{\overline{\zeta}} = U^\zeta_{\sigma\overline{\zeta}}, \\
-y^1_\rho + y^2_\zeta = U^\zeta_{\rho\zeta}, & y^2_{\overline{\zeta}} = U^\zeta_{\rho\overline{\zeta}}, & y^1_{\overline{\zeta}} = U^\zeta_{\zeta\overline{\zeta}}.
\end{array}
$$

Eliminating $y^1_{\overline{\zeta}}$ among the previous equations leads to:

$$U^{\zeta}_{\zeta\overline{\zeta}} = \frac{1}{2}\,U^{\rho}_{\rho\overline{\zeta}} = \frac{1}{3}\,U^{\sigma}_{\sigma\overline{\zeta}} = \frac{1}{4}\,U^{\tau}_{\tau\overline{\zeta}},$$

that is:

$$\frac{i\mathsf{b}}{\mathsf{a}^2} = \frac{1}{2}\,\left(\frac{\mathsf{c}}{\mathsf{a}^3} - \frac{i\mathsf{b}}{\mathsf{a}^2}\right) = -\frac{1}{3}\,\left(\frac{\mathsf{c}}{\mathsf{a}^3} + \frac{\mathsf{f}}{\mathsf{a}^4}\right) = -\frac{1}{4}\,\frac{\mathsf{f}}{\mathsf{a}^4},$$

from which we easily deduce that

$$\mathsf{b} = \mathsf{c} = \mathsf{f} = 0.$$

We have thus reduced the group G_2 to a new group G_3, whose elements are of the form

$$g := \begin{pmatrix} \mathsf{a}^4 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}^3 & 0 & 0 & 0 \\ \mathsf{g} & 0 & \mathsf{a}^2 & 0 & 0 \\ \mathsf{h} & \mathsf{d} & 0 & \mathsf{a} & 0 \\ \mathsf{k} & \mathsf{e} & 0 & 0 & \mathsf{a} \end{pmatrix}.$$

3.4. **Normalization of** g**,** d **and** e**.** The Maurer Cartan forms of G_3 are:

$$\begin{aligned}
\gamma^1 &:= \frac{d\mathsf{a}}{\mathsf{a}}, \\
\gamma^2 &:= -\frac{\mathsf{d}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{d}}{\mathsf{a}^3}, \\
\gamma^3 &:= -\frac{\mathsf{e}d\mathsf{a}}{\mathsf{a}^4} + \frac{d\mathsf{e}}{\mathsf{a}^3}, \\
\gamma^4 &:= -2\,\frac{\mathsf{g}d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{g}}{\mathsf{a}^4}, \\
\gamma^5 &:= -\frac{\mathsf{h}d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{h}}{\mathsf{a}^4}, \\
\gamma^6 &:= -\frac{\mathsf{k}d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{k}}{\mathsf{a}^4}.
\end{aligned}$$

We get the following structure equations:

$$d\tau = 4\,\gamma^1 \wedge \tau + V^{\tau}_{\tau\sigma}\;\tau \wedge \sigma + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta},$$

$$d\sigma = 3\,\gamma^1 \wedge \sigma + V^{\sigma}_{\tau\rho}\,\tau \wedge \rho + V^{\sigma}_{\tau\zeta}\,\tau \wedge \zeta + V^{\sigma}_{\tau\overline{\zeta}}\,\tau \wedge \overline{\zeta} + V^{\sigma}_{\sigma\rho}\,\sigma \wedge \rho + \rho \wedge \zeta + \rho \wedge \overline{\zeta}\;,$$

$$d\rho = 2\,\gamma^1 \wedge \rho + \gamma^4 \wedge \tau + V^{\rho}_{\tau\sigma}\,\tau \wedge \sigma + V^{\rho}_{\tau\zeta}\,\tau \wedge \zeta + V^{\rho}_{\tau\overline{\zeta}}\,\rho \wedge \overline{\zeta} + V^{\rho}_{\sigma\zeta}\,\sigma \wedge \zeta + V^{\rho}_{\sigma\overline{\zeta}}\,\sigma \wedge \overline{\zeta} + i\;\zeta \wedge \overline{\zeta}\;,$$

$$\begin{aligned} d\zeta = \gamma^1 \wedge \zeta + \gamma^2 \wedge \sigma + \gamma^5 \wedge \tau \\ + V^{\zeta}_{\tau\sigma}\, \tau \wedge \sigma + V^{\zeta}_{\tau\rho}\, \tau \wedge \rho + V^{\zeta}_{\tau\zeta}\, \tau \wedge \zeta + V^{\zeta}_{\tau\overline{\zeta}}\, \tau \wedge \overline{\zeta} \\ + V^{\zeta}_{\sigma\rho}\, \sigma \wedge \rho + V^{\zeta}_{\sigma\zeta}\, \sigma \wedge \zeta + V^{\zeta}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + V^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta \\ + V^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta}, \end{aligned}$$

and

$$\begin{aligned} d\overline{\zeta} = \gamma^1 \wedge \overline{\zeta} + \gamma^3 \wedge \sigma + \gamma^6 \wedge \tau \\ + V^{\overline{\zeta}}_{\tau\sigma}\, \tau \wedge \sigma + V^{\overline{\zeta}}_{\tau\rho}\, \tau \wedge \rho + V^{\overline{\zeta}}_{\tau\zeta}\, \tau \wedge \zeta + V^{\overline{\zeta}}_{\tau\overline{\zeta}}\, \tau \wedge \overline{\zeta} \\ + V^{\overline{\zeta}}_{\sigma\rho}\, \sigma \wedge \rho + V^{\overline{\zeta}}_{\sigma\zeta}\, \sigma \wedge \zeta + V^{\overline{\zeta}}_{\sigma\overline{\zeta}}\, \sigma \wedge \overline{\zeta} + V^{\overline{\zeta}}_{\rho\zeta}\, \rho \wedge \zeta \\ + V^{\overline{\zeta}}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta}. \end{aligned}$$

From these equations, we immediately see that $V^{\sigma}_{\tau\zeta}$, $V^{\zeta}_{\rho\overline{\zeta}}$ and $V^{\overline{\zeta}}_{\rho\zeta}$ are essential torsion coefficients. As we have:

$$V^{\sigma}_{\tau\zeta} = -\frac{\mathsf{g}}{\mathsf{a}^4}, \qquad V^{\zeta}_{\rho\overline{\zeta}} = \frac{\mathsf{d}}{\mathsf{a}^3}, \qquad V^{\overline{\zeta}}_{\rho\zeta} = \frac{\mathsf{e}}{\mathsf{a}^3},$$

we obtain the new normalizations:

$$\mathsf{d} = \mathsf{e} = \mathsf{g} = 0.$$

The reduced group G_4 is of the form:

$$g := \begin{pmatrix} \mathsf{a}^4 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}^3 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{a}^2 & 0 & 0 \\ \mathsf{h} & 0 & 0 & \mathsf{a} & 0 \\ \mathsf{k} & 0 & 0 & 0 & \mathsf{a} \end{pmatrix}.$$

Its Maurer-Cartan forms are given by:

$$\begin{aligned} \delta^1 &:= \frac{d\mathsf{a}}{\mathsf{a}}, \\ \delta^2 &:= -\frac{\mathsf{h}\, d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{h}}{\mathsf{a}^4}, \\ \delta^3 &:= -\frac{\mathsf{k}\, d\mathsf{a}}{\mathsf{a}^5} + \frac{d\mathsf{k}}{\mathsf{a}^4}. \end{aligned}$$

The structure equations are easily computed as:

$$
\begin{aligned}
d\tau &= 4\,\delta^1 \wedge \tau + \frac{\mathsf{h}+\mathsf{k}}{\mathsf{a}^4}\,\tau \wedge \sigma + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}\,,\\
d\sigma &= 3\,\delta^1 \wedge \sigma + \frac{\mathsf{h}+\mathsf{k}}{\mathsf{a}^4}\,\tau \wedge \rho + \rho \wedge \zeta + \rho \wedge \overline{\zeta}\,,\\
d\rho &= 2\,\delta^1 \wedge \rho + i\,\frac{\mathsf{k}}{\mathsf{a}^4}\,\tau \wedge \zeta - i\,\frac{\mathsf{h}}{\mathsf{a}^4}\,\tau \wedge \overline{\zeta} + i\,\zeta \wedge \overline{\zeta}\,,\\
d\zeta &= \delta^1 \wedge \zeta + \delta^2 \wedge \tau + \frac{\mathsf{h}\,(\mathsf{h}+\mathsf{k})}{\mathsf{a}^8}\,\tau \wedge \sigma + \frac{\mathsf{h}}{\mathsf{a}^4}\,\sigma \wedge \zeta + \frac{\mathsf{h}}{\mathsf{a}^4}\,\sigma \wedge \overline{\zeta}\,,\\
d\overline{\zeta} &= \delta^1 \wedge \overline{\zeta} + \delta^3 \wedge \tau + \frac{\mathsf{k}\,(\mathsf{h}+\mathsf{k})}{\mathsf{a}^8}\,\tau \wedge \sigma + \frac{\mathsf{k}}{\mathsf{a}^4}\,\sigma \wedge \zeta + \frac{\mathsf{k}}{\mathsf{a}^4}\,\sigma \wedge \overline{\zeta}\,.
\end{aligned}
$$

We deduce from these equations that we can perform the normalization:

$$\mathsf{h} = \mathsf{k} = 0.$$

With the 1-dimensional group G_5 of the form:

$$
g := \begin{pmatrix}
\mathsf{a}^4 & 0 & 0 & 0 & 0\\
0 & \mathsf{a}^3 & 0 & 0 & 0\\
0 & 0 & \mathsf{a}^2 & 0 & 0\\
0 & 0 & 0 & \mathsf{a} & 0\\
0 & 0 & 0 & 0 & \mathsf{a}
\end{pmatrix},
$$

whose Maurer-Cartan form is given by

$$\alpha := \frac{d\mathsf{a}}{\mathsf{a}},$$

we get the following structure equations:

$$
\begin{aligned}
d\tau &= 4\,\alpha \wedge \tau + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta}\,,\\
d\sigma &= 3\,\alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}\,,\\
d\rho &= 2\,\alpha \wedge \rho + i\,\zeta \wedge \overline{\zeta}\,,\\
d\zeta &= \alpha \wedge \zeta\,,\\
d\overline{\zeta} &= \alpha \wedge \overline{\zeta}\,.
\end{aligned}
$$

No more normalizations are allowed at this stage. We thus just perform a prolongation by adjoining the form α to the structure equations, whose exterior derivative is given by:

$$d\alpha = 0.$$

This completes the proof of Theorem 2 .

4. CLASS IV_2

Class IV_2 is constituted by the 5-dimensional real hypersurfaces $M^5 \subset \mathbb{C}^3$ which are of CR-dimension 2, whose Levi form is of constant rank 1 and which are 2-nondegenerate, i.e. their Freeman forms are non-zero. The most symmetric manifold of this class is the tube over the future light cone, which is defined by the equation:

$$\mathsf{LC}: \qquad (\mathsf{Re}\, z_1)^2 - (\mathsf{Re}\, z_2)^2 - (\mathsf{Re}\, z_3)^2 = 0, \qquad \mathsf{Re}\, z_1 > 0.$$

This section is devoted to the determination of the Lie algebra $\mathsf{aut}_{\mathsf{CR}}(\mathsf{LC})$ of infinitesimal CR-automorphisms of LC. This has been done before by Kaup and Zaitsev [5]. We prove the following result:

Theorem 3. *The tube over the future light cone:*

$$\mathsf{LC}: \qquad (\mathsf{Re}\, z_1)^2 - (\mathsf{Re}\, z_2)^2 - (\mathsf{Re}\, z_3)^2 = 0, \qquad \mathsf{Re}\, z_1 > 0.$$

has a **10**-*dimensional Lie algebra of CR-automorphisms. A basis for the Maurer-Cartan forms of* $\mathsf{aut}_{\mathsf{CR}}(\mathsf{LC})$ *is provided by the* 10 *differential* 1-*forms* ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, π^1, π^2, $\overline{\pi^1}$, $\overline{\pi^2}$, Λ, *which satisfy the Maurer-Cartan equations:*

$$\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
d\zeta &= i\, \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta, \\
d\overline{\kappa} &= \overline{\pi^1} \wedge \overline{\kappa} + \overline{\pi^2} \wedge \rho - \kappa \wedge \overline{\zeta}, \\
d\overline{\zeta} &= -i\, \overline{\pi^2} \wedge \overline{\kappa} + \overline{\pi^1} \wedge \overline{\zeta} - \pi^1 \wedge \overline{\zeta}, \\
d\pi^1 &= \Lambda \wedge \rho + i\, \kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta}, \\
d\pi^2 &= \Lambda \wedge \kappa + \zeta \wedge \overline{\pi^2} + \pi^2 \wedge \overline{\pi^1}, \\
d\overline{\pi^1} &= \Lambda \wedge \rho - i\, \overline{\kappa} \wedge \pi^2 - \zeta \wedge \overline{\zeta}, \\
d\overline{\pi^2} &= \Lambda \wedge \overline{\kappa} + \overline{\zeta} \wedge \pi^2 - \pi^1 \wedge \overline{\pi^2}, \\
d\Lambda &= -\pi^1 \wedge \Lambda + i\, \pi^2 \wedge \overline{\pi^2} - \pi^1 \wedge \overline{\Lambda}.
\end{aligned} \tag{4}$$

4.1. **Geometric set-up.** In order to motivate our subsequent notations, it is convenient to introduce some general results on CR-manifolds belonging to class IV_2, for which we refer to [11] for a proof.

Let $M \subset \mathbb{C}^3$ be a smooth hypersurface locally represented as a graph over the 5-dimensional real hyperplane $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{R}_v$:

$$u = F(z_1, z_2, \overline{z_1}, \overline{z_2}, v),$$

where F is a local smooth function depending on 5 arguments. We assume that M is a CR-submanifold of CR dimension 2 which is 2-nondegenerate and whose Levi form is of constant rank 1. The two vector fields $\mathscr{L}_1$ and $\mathscr{L}_2$ defined by:

$$\mathscr{L}_j = \frac{\partial}{\partial z_j} + A^j \frac{\partial}{\partial v}, \qquad A^j := -i\, \frac{F_{z_j}}{1 + i\, F_v}, \qquad\qquad j = 1, 2,$$

constitute a basis of $T_p^{1,0}M$ at each point p of M and thus provide an identification of $T_p^{1,0}M$ with $\mathbb{C}^2$ at each point. Moreover, the real 1-form σ defined by:

$$\sigma := dv - A^1\, dz_1 - A^2\, dz_2 - \overline{A^1}\, d\overline{z_1} - \overline{A^2}\, d\overline{z_2},$$

satisfies

$$\{\sigma = 0\} = T^{1,0}M \oplus T^{0,1}M,$$

and thus provides an identication of the projection

$$\mathbb{C} \otimes T_pM \longrightarrow \mathbb{C} \otimes T_pM \;/\; \big(T_p^{1,0}M \oplus T_p^{0,1}M\big)$$

with the map $\sigma_p\colon\;\; \mathbb{C} \otimes T_pM \longrightarrow \mathbb{C}$. With these two identifications, the Levi form LF can be viewed at each point p as a skew hermitian form on $\mathbb{C}^2$ represented by the matrix:

$$LF = \begin{pmatrix} \sigma_p\big(i\,[\mathscr{L}_1, \overline{\mathscr{L}_1}]\big) & \sigma_p\big(i\,[\mathscr{L}_2, \overline{\mathscr{L}_1}]\big) \\ \sigma_p\big(i\,[\mathscr{L}_1, \overline{\mathscr{L}_2}]\big) & \sigma_p\big(i\,[\mathscr{L}_2, \overline{\mathscr{L}_2}]\big) \end{pmatrix}.$$

The fact that LF is supposed to be of constant rank 1 ensures the existence of a certain function k such that the vector field

$$\mathscr{K} := k\, \mathscr{L}_1 + \mathscr{L}_2$$

lies in the kernel of LF. Here are the expressions of $\mathscr{K}$ and k in terms of the graphing function F:

$$\mathscr{K} = k\, \partial_{z_1} + \partial_{z_2} - \frac{i}{1 + i\, F_v}\, (k\, F_{z_1} + F_{z_2})\, \partial_v,$$

$$k = -\frac{F_{z_2,\overline{z_1}} + F_{z_2,\overline{z_1}}\, F_v^2 - i\, F_{\overline{z_1}}\, F_{z_2,v} - F_{\overline{z_1}}\, F_v\, F_{v,z_2} + i\, F_{z_2}\, F_{\overline{z_1}}\, F_{v,v} - F_{z_2}\, F_v\, F_{v,\overline{z_1}}}{F_{z_1,\overline{z_1}} + F_{z_1,\overline{z_1}}\, F_v^2 - i\, F_{\overline{z_1}}\, F_{z_1,v} - F_{\overline{z_1}}\, F_v\, F_{z_1,v} + i\, F_{z_1}\, F_{\overline{z_1},v} + F_{z_1}\, F_{\overline{z_1}}\, F_{v,v} - F_{z_1}\, F_v\, F_{v,\overline{z_1}}}.$$

From the above construction, the four vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ constitute a basis of $T_p^{1,0}M \oplus T_p^{0,1}M$ at each point p of M. It turns out that the vector field $\mathscr{T}$ defined by:

$$\mathscr{T} := i\, [\mathscr{L}_1, \overline{\mathscr{L}_1}]$$

is linearly independant from $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$.

It is well known (see [3, 8]) that the tube over the future light cone is locally biholomorphic to the graphed hypersurface:

$$u = \frac{z_1\overline{z_1} + \frac{1}{2}z_1^2\overline{z_2} + \frac{1}{2}\overline{z_1}^2 z_2}{1 - z_2\overline{z_2}}.$$

The five vector fields $\mathscr{L}_1$, $\mathscr{K}$, $\overline{\mathscr{L}_1}$, $\overline{\mathscr{K}}$ and $\mathscr{T}$, wich constitute a local frame on LC, have thus the following expressions:

$$\mathscr{L}_1 := \frac{\partial}{\partial z_1} - i\frac{\overline{z_1} + z_1\overline{z_2}}{1 - z_2\overline{z_2}}\frac{\partial}{\partial v},$$

$$\mathscr{K} := -\frac{\overline{z_1} + z_1\overline{z_2}}{1 - z_2\overline{z_2}}\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{i}{2}\frac{\overline{z_1}^2 + 2z_1\overline{z_1}\overline{z_2} + z_1^2\overline{z_2}^2}{(1 - z_2\overline{z_2})^2}\frac{\partial}{\partial v},$$

and

$$\mathscr{T} := -\frac{2}{1 - z_2\overline{z_2}}\frac{\partial}{\partial v}.$$

Moreover the function k is given by

$$k := -\frac{\overline{z_1} + z_1\overline{z_2}}{1 - z_2\overline{z_2}}.$$

Let $(\rho_0, \kappa_0, \zeta_0, \overline{\kappa_0}, \overline{\zeta}_0)$ be the dual coframe of $(\mathscr{T}, \mathscr{L}_1, \mathscr{K}, \overline{\mathscr{L}_1}, \overline{\mathscr{K}})$. We have:

$$\begin{aligned}
\rho_0 = &-\frac{i}{2}\left(\overline{z_1} + z_1\overline{z_2}\right) dz_1 - \frac{i}{4}\frac{\overline{z_1}^2 + 2z_1\overline{z_1}\overline{z_2} + z_1^2\overline{z_2}^2}{1 - z_2\overline{z_2}} dz_2 + \frac{i}{2}\left(z_1 + \overline{z_1}z_2\right) d\overline{z_1} \\
&+ \frac{i}{4}\frac{z_1^2 + 2z_1z_2\overline{z_1} + \overline{z_1}^2 z_2^2}{1 - z_2\overline{z_2}} dz_2 + \frac{1}{2}\left(-1 + z_2\overline{z_2}\right) dv\,, \\
\kappa_0 = &\; dz_1 + \frac{\overline{z_1} + z_1\overline{z_2}}{1 - z_2\overline{z_2}} dz_2\,, \\
\zeta_0 = &\; dz_2\,, \\
\overline{\kappa_0} = &\; d\overline{z_1} + \frac{z_1 + \overline{z_1}z_2}{1 - z_2\overline{z_2}} d\overline{z_2}\,, \\
\overline{\zeta_0} = &\; d\overline{z_2}\,.
\end{aligned}$$

A direct computation gives the structure equations enjoyed by the coframe $(\rho_0, \kappa_0, \zeta_0, \overline{\kappa_0}, \overline{\zeta}_0)$:

$$
(5)\qquad
\begin{aligned}
d\rho_0 &= \frac{\overline{z_2}}{1-z_2\overline{z_2}}\,\rho_0\wedge\zeta_0 + \frac{z_2}{1-z_2\overline{z_2}}\,\rho_0\wedge\overline{\zeta_0} + i\,\kappa_0\wedge\overline{\kappa_0}\,,\\
d\kappa_0 &= \frac{\overline{z_2}}{1-z_2\overline{z_2}}\,\kappa_0\wedge\zeta_0 - \frac{1}{1-z_2\overline{z_2}}\,\zeta_0\wedge\overline{\kappa}_0\,,\\
d\zeta_0 &= 0,\\
d\overline{\kappa}_0 &= \frac{1}{1-z_2\overline{z_2}}\,\kappa_0\wedge\overline{\zeta_0} + \frac{z_2}{1-z_2\overline{z_2}}\,\overline{\kappa_0}\wedge\overline{\zeta_0}\,,\\
d\overline{\zeta_0} &= 0.
\end{aligned}
$$

The matrix Lie group which encodes the equivalence problem for LC is the 10 dimensional Lie group G_1 whose elements are of the form:

$$
g := \begin{pmatrix}
\mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0\\
\mathsf{b} & \mathsf{c} & 0 & 0 & 0\\
\mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0\\
\overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0\\
\overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}}
\end{pmatrix},
$$

where c and f are non-zero complex numbers whereas b, d and e are arbitrary complex numbers (see [11, 9]). We introduce the 5 new one-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$ by the relation:

$$
\begin{pmatrix}\rho\\ \kappa\\ \zeta\\ \overline{\kappa}\\ \overline{\zeta}\end{pmatrix} := g\cdot\begin{pmatrix}\rho_0\\ \kappa_0\\ \zeta_0\\ \overline{\kappa_0}\\ \overline{\zeta_0}\end{pmatrix},
$$

which we abbreviate as:

$$
\omega := g\cdot\omega_0.
$$

The coframes ω define a G_1 structure P^1 on LC. The rest of this section is devoted to reduce P^1 to an absolute parallelism on LC through Cartan equivalence method.

4.2. **Normalization of f.** The Maurer Cartan forms of G_1 are the following:

$$\begin{aligned}
\alpha^1 &:= \frac{d\mathsf{c}}{\mathsf{c}}, \\
\alpha^2 &:= \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\, d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}}, \\
\alpha^3 &:= \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}\, d\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{(-\mathsf{dc}+\mathsf{eb})\, d\mathsf{f}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}, \\
\alpha^4 &:= \frac{d\mathsf{e}}{\mathsf{c}} - \frac{\mathsf{e}\, d\mathsf{f}}{\mathsf{cf}}, \\
\alpha^5 &:= \frac{d\mathsf{f}}{\mathsf{f}}.
\end{aligned}$$

The structure equations read as:

$$\begin{aligned}
d\rho &= \alpha^1 \wedge \rho + \overline{\alpha^1} \wedge \rho \\
&\quad + T^{\rho}_{\rho\kappa}\, \rho \wedge \kappa + T^{\rho}_{\rho\zeta}\, \rho \wedge \zeta + T^{\rho}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + T^{\rho}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \alpha^1 \wedge \kappa + \alpha^2 \wedge \rho \\
&\quad + T^{\kappa}_{\rho\kappa}\, \rho \wedge \kappa + T^{\kappa}_{\rho\zeta}\, \rho \wedge \zeta + T^{\kappa}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\
&\quad + T^{\kappa}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + T^{\kappa}_{\kappa\zeta}\, \kappa \wedge \zeta + T^{\kappa}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + T^{\kappa}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa}, \\
d\zeta &= \alpha^3 \wedge \rho + \alpha^4 \wedge \kappa + \alpha^5 \wedge \zeta \\
&\quad + T^{\zeta}_{\rho\kappa}\, \rho \wedge \kappa + T^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta + T^{\zeta}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\
&\quad + T^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + T^{\zeta}_{\kappa\zeta}\, \kappa \wedge \zeta + T^{\zeta}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + T^{\zeta}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa},
\end{aligned}$$

where the expressions of the torsion coefficients $T^{\bullet}_{\bullet\bullet}$ are given in the appendix.

We now proceed with the absorption step of Cartan's method. We introduce the modified Maurer-Cartan forms $\widetilde{\alpha}^i$, which are a related to the 1-forms α^i by the relations:

$$\widetilde{\alpha}^i := \alpha^i - x^i_{\rho}\, \rho - x^i_{\kappa}\, \kappa - x^i_{\zeta}\, \zeta - x^i_{\overline{\kappa}}\, \overline{\kappa} - x^i_{\overline{\zeta}}\, \overline{\zeta},$$

where x^1, x^2, x^3, x^4 and x^5 are arbitrary complex-valued functions. The previously written structure equations take the new form:

$$\begin{aligned}
d\rho &= \widetilde{\alpha}^1 \wedge \rho + \overline{\widetilde{\alpha}^1} \wedge \rho \\
&\quad + \left(T^{\rho}_{\rho\kappa} - x^1_{\kappa} - x^1_{\overline{\kappa}}\right) \rho \wedge \kappa + \left(T^{\rho}_{\rho\zeta} - x^1_{\kappa} - \overline{x^1_{\overline{\zeta}}}\right) \rho \wedge \zeta \\
&\quad + \left(T^{\rho}_{\rho\overline{\kappa}} - x^1_{\overline{\kappa}} - \overline{x^1_{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(T^{\rho}_{\rho\overline{\zeta}} - x^1_{\zeta} - x^1_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} \\
&\quad + i\, \kappa \wedge \overline{\kappa},
\end{aligned}$$

$$
\begin{aligned}
d\kappa = \widetilde{\alpha}^1 \wedge \kappa + \widetilde{\alpha}^2 \wedge \rho \\
&+ \left(T^{\kappa}_{\rho\kappa} - x^2_{\kappa} + x^1_{\rho}\right) \rho \wedge \kappa + \left(T^{\kappa}_{\rho\zeta} - x^2_{\kappa}\right) \rho \wedge \zeta \\
&+ \left(T^{\kappa}_{\rho\overline{\kappa}} - x^2_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(T^{\kappa}_{\rho\overline{\zeta}} - x^2_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} \\
&+ \left(T^{\kappa}_{\kappa\zeta} + x^1_{\zeta}\right) \kappa \wedge \zeta + \left(T^{\kappa}_{\kappa\overline{\kappa}} - x^1_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} \\
&+ T^{\kappa}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa} + \left(T^{1}_{\kappa\overline{\zeta}} - x^1_{\kappa\overline{\zeta}}\right) \kappa \wedge \zeta,
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = \widetilde{\alpha}^3 \wedge \rho + \widetilde{\alpha}^4 \wedge \kappa + \widetilde{\alpha}^5 \wedge \zeta \\
&+ \left(T^{\zeta}_{\rho\kappa} - x^3_{\kappa} + x^4_{\rho}\right) \rho \wedge \kappa + \left(T^{\zeta}_{\rho\zeta} - x^3_{\zeta} + x^5_{\rho}\right) \rho \wedge \zeta \\
&+ \left(T^{\zeta}_{\rho\kappa} - x^3_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(T^{\zeta}_{\rho\overline{\zeta}} - x^3_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} \\
&+ \left(T^{\zeta}_{\kappa\overline{\kappa}} - x^4_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} + \left(T^{\zeta}_{\zeta\overline{\kappa}} - x^5_{\overline{\kappa}}\right) \zeta \wedge \overline{\kappa} \\
&+ \left(x^5_{\kappa} - x^4_{\zeta}\right) \kappa \wedge \zeta - x^4_{\overline{\kappa}}\, \kappa \wedge \overline{\kappa} \\
&+ \left(x^5_{\overline{\kappa}} - x^4_{\overline{\zeta}}\right) \overline{\kappa} \wedge \zeta - x^5_{\overline{\zeta}}\, \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

We then choose x^1, x^2, x^3, x^4 and x^5 in a way that eliminates as many torsion coefficients as possible. We easily see that the only coefficient which can not be absorbed is the one in front of $\zeta \wedge \overline{\kappa}$ in $d\kappa$, because it does not depend on the x^i's. We choose the normalization

$$T^{\kappa}_{\zeta\overline{\kappa}} = 1,$$

which yields to :

$$\mathsf{f} = -\frac{\mathsf{c}}{\overline{\mathsf{c}}}\, \frac{1}{1 - \mathsf{z}_2 \overline{\mathsf{z}_2}}.$$

We notice that the absorbed structure equations take the form:

$$
\begin{aligned}
d\rho &= \widetilde{\alpha}^1 \wedge \rho + \overline{\widetilde{\alpha}^1} \wedge \rho + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \widetilde{\alpha}^1 \wedge \kappa + \widetilde{\alpha}^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \widetilde{\alpha}^3 \wedge \rho + \widetilde{\alpha}^4 \wedge \kappa + \widetilde{\alpha}^5 \wedge \zeta.
\end{aligned}
$$

The normalization of f gives the new relation :

$$
\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \end{pmatrix}
=
\begin{pmatrix}
\mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\
\mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\
\mathsf{d} & \mathsf{e} & \frac{c}{\overline{c}}\, \frac{1}{-1 + z_2\overline{z_2}} & 0 & 0 \\
\overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\
0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}}\, \frac{1}{-1 + z_2\overline{z_2}}
\end{pmatrix}
\cdot
\begin{pmatrix} \rho_0 \\ \kappa_0 \\ \zeta_0 \\ \overline{\kappa}_0 \\ \overline{\zeta}_0 \end{pmatrix}.
$$

We thus introduce the new one-form

$$\hat{\zeta}_0 = -\frac{1}{1 - z_2\overline{z_2}} \cdot \zeta_0,$$

such that the previous equation rewrites :

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \end{pmatrix} = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix} \cdot \begin{pmatrix} \rho_0 \\ \kappa_0 \\ \hat{\zeta}_0 \\ \overline{\kappa_0} \\ \overline{\hat{\zeta}_0} \end{pmatrix}.$$

We have reduced the G_1 equivalence problem to a G_2 equivalence problem, where G_2 is the 8 dimensional real matrix Lie group whose elements are of the form

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ 0 & 0 & \overline{\mathsf{d}} & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

We determine the new structure equations enjoyed by the base coframe $(\rho_0, \kappa_0, \hat{\zeta}_0, \kappa_0, \overline{\hat{\zeta}_0})$. We get :

$$\begin{aligned} d\rho_0 &= -\overline{z_2}\,\rho_0 \wedge \hat{\zeta}_0 - z_2\,\rho_0 \wedge \overline{\hat{\zeta}_0} + i\,\kappa_0 \wedge \overline{\kappa_0}, \\ d\kappa_0 &= -\overline{z_2}\,\kappa_0 \wedge \hat{\zeta}_0 + \hat{\zeta}_0 \wedge \overline{\kappa_0}, \\ d\hat{\zeta}_0 &= z_2\,\hat{\zeta}_0 \wedge \overline{\hat{\zeta}_0}\,. \end{aligned}$$

4.3. **Normalization of** b**.** The Maurer forms of the G_2 are given by the independant entries of the matrix $dg \cdot g^{-1}$. We have:

$$dg \cdot g^{-1} = \begin{pmatrix} \beta^1 + \overline{\beta^1} & 0 & 0 & 0 & 0 \\ \beta^2 & \beta^1 & 0 & 0 & 0 \\ \beta^3 & \beta^4 & \beta^1 - \overline{\beta^1} & 0 & 0 \\ \overline{\beta^2} & 0 & 0 & \overline{\beta^1} & 0 \\ \overline{\beta^3} & 0 & 0 & \overline{\beta^4} & -\beta^1 + \overline{\beta^1} \end{pmatrix},$$

where the forms β^1, β^2, β^3 and β^4 are defined by

$$\beta^1 := \frac{d\mathsf{c}}{\mathsf{c}},$$
$$\beta^2 := \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}},$$
$$\beta^3 := \frac{(-\mathsf{dc}+\mathsf{eb})\,d\mathsf{c}}{\mathsf{c}^3\overline{\mathsf{c}}} - \frac{(-\mathsf{dc}+\mathsf{eb})\,d\overline{\mathsf{c}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}},$$
$$\beta^4 := -\frac{\mathsf{e}d\mathsf{c}}{\mathsf{c}^2} + \frac{\mathsf{e}d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\mathsf{c}} + \frac{d\mathsf{e}}{\mathsf{c}}.$$

Using formula (2), we get the structure equations for the lifted coframe $(\rho,\kappa,\zeta,\overline{\kappa},\overline{\zeta})$ from those of the base coframe $(\rho_0,\kappa_0,\hat{\zeta}_0,\overline{\kappa_0},\overline{\hat{\zeta}_0})$:

$$\begin{aligned} d\rho = {} & \beta^1\wedge\rho + \overline{\beta^1}\wedge\rho \\ & + U^\rho_{\rho\kappa}\,\rho\wedge\kappa + U^\rho_{\rho\zeta}\,\rho\wedge\zeta + U^\rho_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} \\ & + U^\rho_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa}, \end{aligned}$$

$$\begin{aligned} d\kappa = {} & \beta^1\wedge\kappa + \beta^2\wedge\rho \\ & + U^\kappa_{\rho\kappa}\,\rho\wedge\kappa + U^\kappa_{\rho\zeta}\,\rho\wedge\zeta + U^\kappa_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + U^\kappa_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} \\ & + U^\kappa_{\kappa\zeta}\,\kappa\wedge\zeta + U^\kappa_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + \zeta\wedge\overline{\kappa}, \end{aligned}$$

$$\begin{aligned} d\zeta = {} & \beta^3\wedge\rho + \beta^4\wedge\kappa + \beta^1\wedge\zeta - \overline{\beta^1}\wedge\zeta \\ & + U^\zeta_{\rho\kappa}\,\rho\wedge\kappa + U^\zeta_{\rho\zeta}\,\rho\wedge\zeta + U^\zeta_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} \\ & + U^\zeta_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + U^\zeta_{\kappa\zeta}\,\kappa\wedge\zeta + U^\zeta_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} \\ & + U^\zeta_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + U^\zeta_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + U^\zeta_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}. \end{aligned}$$

We introduce the modified Maurer-Cartan forms $\widetilde{\beta}^i$ which differ from the β^i by a linear combination of the 1-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, i.e. that is :

$$\widetilde{\beta}^i = \beta^i - y^i_\rho\,\rho - y^i_\kappa\,\kappa - y^i_\zeta\,\zeta - y^i_{\overline{\kappa}}\,\overline{\kappa} - y^i_{\overline{\zeta}}\,\overline{\zeta}.$$

The structure equations rewrite:

$$\begin{aligned} d\rho = {} & \widetilde{\beta}^1\wedge\rho + \overline{\widetilde{\beta}^1}\wedge\rho \\ & + \left(U^\rho_{\rho\kappa} - y^1_\kappa - \overline{y}^1_{\overline{\kappa}}\right)\rho\wedge\kappa + \left(U^\rho_{\rho\zeta} - y^1_\zeta - \overline{y}^1_{\overline{\zeta}}\right)\rho\wedge\zeta \\ & + \left(U^\rho_{\rho\overline{\kappa}} - y^1_{\overline{\kappa}} - \overline{y}^1_\kappa\right)\rho\wedge\overline{\kappa} + \left(U^\rho_{\rho\zeta} - y^1_{\overline{\zeta}} - \overline{y}^1_\zeta\right)\rho\wedge\zeta + i\,\kappa\wedge\overline{\kappa}, \end{aligned}$$

$$
\begin{aligned}
d\kappa = {} & \widetilde{\beta}^1 \wedge \kappa + \widetilde{\beta}^2 \wedge \rho \\
& + \left(U^{\kappa}_{\rho\kappa} + y^1_{\rho} - y^2_{\kappa}\right) \rho \wedge \kappa + \left(U^{\kappa}_{\rho\zeta} - y^2_{\zeta}\right) \rho \wedge \zeta + \left(U^{\kappa}_{\rho\overline{\kappa}} - y^2_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} \\
& + \left(U^{\kappa}_{\rho\overline{\zeta}} - y^2_{\overline{\zeta}}\right) \rho \wedge \overline{\zeta} + \left(U^{\kappa}_{\kappa\zeta} - y^1_{\zeta}\right) \kappa \wedge \zeta \\
& + \left(U^{\kappa}_{\kappa\overline{\kappa}} - y^1_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} - y^1_{\overline{\zeta}}\, \kappa \wedge \overline{\zeta} + \zeta \wedge \overline{\kappa}, \\
d\zeta = {} & \widetilde{\beta}^3 \wedge \rho + \widetilde{\beta}^4 \wedge \kappa + \widetilde{\beta}^1 \wedge \zeta - \overline{\widetilde{\beta}^1} \wedge \zeta \\
& + \left(U^{\zeta}_{\rho\kappa} - y^3_{\kappa} + y^4_{\rho}\right) \rho \wedge \kappa + \left(U^{\zeta}_{\rho\zeta} - y^3_{\zeta} + y^1_{\rho} - \overline{y}^1_{\rho}\right) \rho \wedge \zeta \\
& + \left(U^{\zeta}_{\rho\overline{\kappa}} - y^3_{\overline{\kappa}}\right) \rho \wedge \overline{\kappa} + \left(U^{\zeta}_{\kappa\zeta} - y^4_{\zeta} + y^1_{\kappa} - \overline{y}^1_{\kappa}\right) \kappa \wedge \zeta + \left(U^{\zeta}_{\kappa\overline{\kappa}} - y^4_{\overline{\kappa}}\right) \kappa \wedge \overline{\kappa} \\
& + \left(U^{\zeta}_{\kappa\overline{\zeta}} - y^4_{\overline{\zeta}}\right) \kappa \wedge \overline{\zeta} + \left(U^{\zeta}_{\zeta\overline{\kappa}} - y^1_{\overline{\kappa}} + \overline{y}^1_{\kappa}\right) \zeta \wedge \overline{\kappa} + \left(U^{\zeta}_{\zeta\overline{\zeta}} - y^1_{\overline{\zeta}} + \overline{y}^1_{\zeta}\right) \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

We get the following absorbtion equations:

$$
\begin{aligned}
y^1_{\kappa} + \overline{y}^1_{\overline{\kappa}} &= U^{\rho}_{\rho\kappa}, & y^1_{\zeta} + \overline{y}^1_{\overline{\zeta}} &= U^{\rho}_{\rho\zeta}, & y^1_{\overline{\kappa}} + \overline{y}^1_{\kappa} &= U^{\rho}_{\rho\overline{\kappa}}, \\
y^1_{\overline{\zeta}} + \overline{y}^1_{\zeta} &= U^{\rho}_{\rho\overline{\zeta}}, & -y^1_{\rho} + y^2_{\kappa} &= U^{\kappa}_{\rho\kappa}, & y^2_{\zeta} &= U^{\kappa}_{\rho\zeta}, \\
y^2_{\overline{\kappa}} &= U^{\kappa}_{\rho\overline{\kappa}}, & y^2_{\overline{\zeta}} &= U^{\kappa}_{\rho\overline{\zeta}}, & y^1_{\zeta} &= U^{\kappa}_{\kappa\zeta}, \\
y^1_{\overline{\kappa}} &= U^{\kappa}_{\kappa\overline{\kappa}}, & y^1_{\overline{\zeta}} &= 0, & y^3_{\kappa} - y^4_{\rho} &= U^{\zeta}_{\rho\kappa}, \\
y^3_{\zeta} - y^1_{\rho} + \overline{y}^1_{\rho} &= U^{\zeta}_{\rho\zeta}, & y^3_{\overline{\kappa}} &= U^{\zeta}_{\rho\overline{\kappa}}, & y^4_{\zeta} - y^1_{\kappa} + \overline{y}^1_{\kappa} &= U^{\zeta}_{\kappa\zeta}, \\
y^4_{\overline{\kappa}} &= U^{\zeta}_{\kappa\overline{\kappa}}, & y^4_{\overline{\zeta}} &= U^{\zeta}_{\kappa\overline{\zeta}}, & y^1_{\overline{\kappa}} - \overline{y}^1_{\kappa} &= U^{\zeta}_{\zeta\overline{\kappa}}, \\
y^1_{\overline{\zeta}} - \overline{y}^1_{\zeta} &= U^{\zeta}_{\zeta\overline{\zeta}}.
\end{aligned}
$$

Eliminating the $y^{\bullet}_{\bullet}$ among theses equations leads to the following relations between the torsion coefficients :

$$
\begin{aligned}
U^{\rho}_{\rho\overline{\kappa}} &= \overline{U^{\rho}_{\rho\kappa}}, \\
U^{\rho}_{\rho\overline{\zeta}} &= \overline{U^{\rho}_{\rho\zeta}}, \\
U^{\rho}_{\rho\zeta} &= U^{\kappa}_{\kappa\zeta}, \\
U^{\zeta}_{\zeta\overline{\zeta}} &= -U^{\rho}_{\rho\overline{\zeta}}, \\
2\, U^{\kappa}_{\kappa\overline{\kappa}} &= U^{\zeta}_{\zeta\overline{\kappa}} + U^{\rho}_{\rho\overline{\kappa}}.
\end{aligned}
$$

We verify easily that the first four equations do not depend on the group coefficients and are already satisfied. However, the last one does depend on the group coefficients. It gives us the normalization of b as it rewrites :

$$\mathsf{b} = -i\, \overline{\mathsf{c}}\mathsf{e}.$$

The absorbed structure equations rewrite:

$$
\begin{aligned}
d\rho &= \widetilde{\beta}^1 \wedge \rho + \overline{\widetilde{\beta}^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa}, \\
d\kappa &= \widetilde{\beta}^1 \wedge \kappa + \widetilde{\beta}^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \widetilde{\beta}^3 \wedge \rho + \widetilde{\beta}^4 \wedge \kappa + \widetilde{\beta}^1 \wedge \zeta - \overline{\widetilde{\beta}^1} \wedge \zeta + \left(U^{\zeta}_{\zeta\overline{\kappa}} + U^{\rho}_{\rho\overline{\kappa}} - 2\, U^{\kappa}_{\kappa\overline{\kappa}} \right) \zeta \wedge \overline{\kappa}.
\end{aligned}
$$

4.4. **Normalization of** d**.** We have thus reduced the group G_2 to a new group G_3, whose elements are of the form

$$
\mathsf{g} = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.
$$

It is a six-dimensional real Lie group. We compute its Maurer Cartan forms with the usual formula

$$
d\mathsf{g} \cdot \mathsf{g}^{-1} = \begin{pmatrix} \gamma^1 + \overline{\gamma}^1 & 0 & 0 & 0 & 0 \\ \gamma^2 & \gamma^1 & 0 & 0 & 0 \\ \gamma^3 & i\,\gamma^2 & \gamma^1 - \overline{\gamma}^1 & 0 & 0 \\ \overline{\gamma}^2 & 0 & 0 & \overline{\gamma}^1 & 0 \\ -\gamma^3 & 0 & 0 & -i\,\overline{\gamma}^2 & -\gamma^1 + \overline{\gamma}^1 \end{pmatrix},
$$

where

$$
\gamma^1 := \frac{d\mathsf{c}}{\mathsf{c}},
$$

$$
\gamma^2 := i\,\mathsf{e}\frac{d\mathsf{c}}{\mathsf{c}^2} - i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{d\mathsf{e}}{\mathsf{c}},
$$

and

$$
\gamma^3 := \left(\frac{\mathsf{dc} + i\,\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}^2\overline{\mathsf{c}}} \right) \left(\frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}} - \frac{d\mathsf{c}}{\mathsf{c}} \right) + \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{\mathsf{e}d\mathsf{e}}{\mathsf{c}^2}.
$$

As the normalization of b does not depend on the base variables, the third loop of Cartan's method is straightforward. We get the following structure equations:

$$
\begin{aligned}
d\rho = \gamma^1 \wedge \rho + \overline{\gamma^1} \wedge \rho
& + V^{\rho}_{\rho\kappa}\,\rho \wedge \kappa + V^{\rho}_{\rho\zeta}\,\rho \wedge \zeta + V^{\rho}_{\rho\overline{\kappa}}\,\rho \wedge \overline{\kappa} \\
& + V^{\rho}_{\rho\overline{\zeta}}\,\rho \wedge \overline{\zeta} + i\,\kappa \wedge \overline{\kappa},
\end{aligned}
$$

$$
\begin{aligned}
d\kappa = {} & \gamma^1 \wedge \kappa + \gamma^2 \wedge \rho \\
& + V^{\kappa}_{\rho\kappa}\, \rho \wedge \kappa + V^{\kappa}_{\rho\zeta}\, \rho \wedge \zeta + V^{\kappa}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} \\
& + V^{\kappa}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} + V^{\kappa}_{\kappa\zeta}\, \kappa \wedge \zeta + V^{\kappa}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa},
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = {} & \gamma^3 \wedge \rho + i\, \gamma^2 \wedge \kappa + \gamma^1 \wedge \zeta - \overline{\gamma^1} \wedge \zeta \\
& + V^{\zeta}_{\rho\kappa}\, \rho \wedge \kappa + V^{\zeta}_{\rho\zeta}\, \rho \wedge \zeta + V^{\zeta}_{\rho\overline{\kappa}}\, \rho \wedge \overline{\kappa} + V^{\zeta}_{\rho\overline{\zeta}}\, \rho \wedge \overline{\zeta} \\
& + V^{\zeta}_{\kappa\zeta}\, \kappa \wedge \zeta + V^{\zeta}_{\kappa\overline{\kappa}}\, \kappa \wedge \overline{\kappa} + V^{\zeta}_{\kappa\overline{\zeta}}\, \kappa \wedge \overline{\zeta} + V^{\zeta}_{\zeta\overline{\kappa}}\, \zeta \wedge \overline{\kappa} \\
& + V^{\zeta}_{\zeta\overline{\zeta}}\, \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

We now start the absorption step. We introduce:

$$
\widetilde{\gamma}^i := \gamma^i - z^i_{\rho}\, \rho - z^i_{\kappa}\, \kappa - z^i_{\zeta}\, \zeta - z^i_{\overline{\kappa}}\, \overline{\kappa} - z^i_{\overline{\zeta}}\, \overline{\zeta}.
$$

The structure equations are modified accordingly:

$$
\begin{aligned}
d\rho = {} & \widetilde{\gamma}^1 \wedge \rho + \overline{\widetilde{\gamma}^1} \wedge \rho \\
& + \left(V^{\rho}_{\rho\kappa} - z^1_{\kappa} - \overline{z^1_{\overline{\kappa}}} \right) \rho \wedge \kappa + \left(V^{\rho}_{\rho\zeta} - z^1_{\zeta} - \overline{z^1_{\overline{\zeta}}} \right) \rho \wedge \zeta \\
& + \left(V_{\rho\overline{\kappa}} - z^1_{\overline{\kappa}} - \overline{z^1_{\kappa}} \right) \rho \wedge \overline{\kappa} + \left(V^{\rho}_{\rho\overline{\zeta}} - \overline{z^1_{\zeta}} - z^1_{\overline{\zeta}} \right) \rho \wedge \overline{\zeta}\ , \\
d\kappa = {} & \widetilde{\gamma}^1 \wedge \kappa + \widetilde{\gamma}^2 \wedge \rho \\
& + \left(V^{\kappa}_{\rho\kappa} - z^2_{\kappa} + z^{\kappa}_{\rho} \right) \rho \wedge \kappa + \left(V^{\kappa}_{\rho\zeta} - z^2_{\zeta} \right) \rho \wedge \zeta \\
& + \left(V^{\kappa}_{\rho\overline{\kappa}} - z^2_{\overline{\kappa}} \right) \rho \wedge \overline{\kappa} + \left(V_{\rho\overline{\zeta}} - z^2_{\overline{\zeta}} \right) \rho \wedge \overline{\zeta} + \left(V^{\kappa}_{\kappa\zeta} - z^1_{\zeta} \right) \kappa \wedge \zeta \\
& + \left(V^{\kappa}_{\kappa\overline{\kappa}} - z^1_{\overline{\kappa}} \right) \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa} - z^1_{\overline{\zeta}}\, \kappa \wedge \overline{\zeta}\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
d\zeta = {} & \widetilde{\gamma}^3 \wedge \rho + i\, \widetilde{\gamma}^2 \wedge \kappa + \widetilde{\gamma}^1 \wedge \zeta - \overline{\widetilde{\gamma}^1} \wedge \zeta \\
& + \left(V^{\zeta}_{\rho\kappa} - z^3_{\kappa} + i\, z^2_{\rho} \right) \rho \wedge \kappa + \left(V^{\zeta}_{\rho\zeta} + z^1_{\rho} - z^3_{\zeta} - \overline{\zeta^1_{\rho}} \right) \rho \wedge \zeta \\
& + \left(V^{\zeta}_{\rho\overline{\kappa}} - z^3_{\overline{\kappa}} \right) \rho \wedge \overline{\kappa} + \left(V^{\zeta}_{\rho\overline{\zeta}} - z^3_{\overline{\zeta}} \right) \rho \wedge \overline{\zeta} + \left(V^{\zeta}_{\kappa\overline{\kappa}} - i\, z^2_{\overline{\kappa}} \right) \kappa \wedge \overline{\kappa} \\
& + \left(V^{\zeta}_{\kappa\overline{\zeta}} - i\, z^2_{\overline{\zeta}} \right) \kappa \wedge \overline{\zeta} + \left(V^{\zeta}_{\zeta\overline{\kappa}} - z^1_{\overline{\kappa}} + \overline{z^1_{\kappa}} \right) \zeta \wedge \overline{\zeta}\ .
\end{aligned}
$$

We thus want to solve the system of linear equations :

$$
\begin{aligned}
z^1_\kappa + \overline{z^1_{\overline{\kappa}}} &= V^\rho_{\rho\kappa}, & z^1_{\overline{\kappa}} + \overline{z^1_{\kappa}} &= V^\rho_{\rho\overline{\kappa}}, & z^1_\zeta + \overline{z^1_{\overline{\zeta}}} &= V^\rho_{\rho\zeta}, \\
\overline{z^1_\zeta} + z^1_{\overline{\zeta}} &= V^\rho_{\rho\overline{\zeta}}, & z^2_\kappa - z^1_\rho &= V^\kappa_{\rho\zeta}, & z^2_{\overline{\kappa}} &= V^\kappa_{\rho\overline{\kappa}}, \\
z^2_\zeta &= V^\kappa_{\rho\zeta}, & z^2_{\overline{\zeta}} &= V^\kappa_{\rho\overline{\zeta}}, & z^1_\zeta &= V^\kappa_{\kappa\zeta}, \\
z^1_{\overline{\zeta}} &= 0, & z^1_{\overline{\kappa}} &= V^\kappa_{\kappa\overline{\kappa}}, & z^3_\kappa - i\, z^2_\rho &= V^\zeta_{\rho\kappa}, \\
-z^1_\rho + \overline{z^1_\rho} + z^3_\zeta &= V^\zeta_{\rho\zeta}, & z^1_\kappa - \overline{z^1_{\overline{\kappa}}} - i\, z^2_\zeta &= -V^\zeta_{\kappa\zeta}, & i\, z^2_{\overline{\kappa}} &= V^\zeta_{\kappa\overline{\kappa}}, \\
z^3_{\overline{\kappa}} &= V^\zeta_{\rho\overline{\kappa}}, & z^3_{\overline{\zeta}} &= V^\zeta_{\rho\overline{\zeta}}, & i\, z^2_{\overline{\zeta}} &= V^\zeta_{\kappa\overline{\zeta}}, \\
z^1_\kappa - \overline{z^1_\kappa} &= V^\zeta_{\zeta\overline{\kappa}}, & z^1_{\overline{\zeta}} - \overline{z^1_\zeta} &= V^\zeta_{\zeta\overline{\zeta}}. & &
\end{aligned}
$$

This is easily done as:

$$
\begin{cases}
z^1_\kappa = \overline{V^\rho_{\rho\overline{\kappa}}}, \\
z^1_{\overline{\kappa}} = V^\kappa_{\kappa\overline{\kappa}}, \\
z^1_\zeta = V^\rho_{\rho\zeta}, \\
z^1_{\overline{\zeta}} = 0, \\
z^2_{\overline{\kappa}} = V^\kappa_{\rho\overline{\kappa}}, \\
z^2_{\overline{\zeta}} = V^\kappa_{\rho\overline{\zeta}}, \\
z^2_\zeta = V^\kappa_{\rho\zeta}, \\
z^3_{\overline{\kappa}} = V^\zeta_{\rho\overline{\kappa}}, \\
z^3_{\overline{\zeta}} = V^\zeta_{\rho\overline{\zeta}}, \\
z^3_\zeta = V^\zeta_{\rho\zeta} + z^1_\rho - z^1_\rho, \\
z^3_\kappa = V^\zeta_{\rho\kappa} + i\, z^2_\rho, \\
z^2_\kappa = V^\kappa_{\rho\zeta} + z^1_\rho,
\end{cases}
$$

where z^1_ρ and z^2_ρ may be choosen freely. Eliminating the $z^{\bullet}_{\bullet}$ we get the following additional conditions on the $V^{\bullet}_{\bullet\bullet}$:

$$\left\{\begin{aligned} V^\rho_{\rho\overline{\kappa}} &= \overline{V^\rho_{\rho\kappa}}, \\ V^\rho_{\rho\overline{\zeta}} &= \overline{V^\rho_{\rho\zeta}}, \\ V^\rho_{\rho\zeta} &= V^\kappa_{\kappa\zeta}, \\ i\, V^\kappa_{\rho\overline{\zeta}} &= V^\zeta_{\kappa\overline{\zeta}}, \\ V^\rho_{\rho\zeta} &= -\overline{V^\zeta_{\zeta\overline{\zeta}}}, \\ 2\, V^\kappa_{\kappa\overline{\kappa}} &= V^\rho_{\rho\overline{\kappa}} + V^\zeta_{\zeta\overline{\kappa}}, \end{aligned}\right. \tag{6}$$

and

$$\left\{\begin{aligned} i\, V^\kappa_{\rho\overline{\kappa}} &= V^\zeta_{\kappa\overline{\kappa}}, \\ V^{\overline{\zeta}}_{\kappa\overline{\zeta}} + V^\zeta_{\kappa\zeta} &= i\, V^\kappa_{\rho\zeta}. \end{aligned}\right. \tag{7}$$

We easily verify that the equations (6) are indeed satisfied. However the remaining two equations are not and they provide two essential torsion coefficients, namely $i\, V^\kappa_{\rho\overline{\kappa}} - V^\zeta_{\kappa\overline{\kappa}}$ and $V^{\overline{\zeta}}_{\kappa\overline{\zeta}} + V^\zeta_{\kappa\zeta} - i\, V^\kappa_{\rho\zeta}$, which will give us at least one new normalization of the group coefficients. Indeed we have

$$i\, V^\kappa_{\rho\overline{\kappa}} - V^\zeta_{\kappa\overline{\kappa}} = -2i\, \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}^2}{\mathsf{c}^2}.$$

Setting this expression to 0, we get the normalization of the parameter d:

$$\mathsf{d} = -i\, \frac{1}{2}\, \frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}}.$$

4.5. **Prologation of the G_4 structure.** We have reduced the previous G_3-structure to a G_4-structure, where G_4 is the four dimensional matrix Lie group whose elements are of the form :

$$\begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ -i\,\mathsf{e}\overline{\mathsf{c}} & \mathsf{c} & 0 & 0 & 0 \\ -\frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} & \mathsf{e} & \frac{\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 \\ i\,\overline{\mathsf{e}}\mathsf{c} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \frac{i}{2}\,\frac{\overline{\mathsf{e}}^2\mathsf{c}}{\overline{\mathsf{c}}} & 0 & 0 & \overline{\mathsf{e}} & \frac{\overline{\mathsf{c}}}{\mathsf{c}} \end{pmatrix}.$$

The basis for the Maurer-Cartan forms of G_4 is provided by the four forms

$$\delta^1 := \frac{d\mathsf{c}}{\mathsf{c}} \quad , \quad \delta^2 := i\,\mathsf{e}\frac{d\mathsf{c}}{\mathsf{c}^2} - i\,\frac{\mathsf{e}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}} - i\,\frac{d\mathsf{e}}{\mathsf{c}} \quad , \quad \overline{\delta^1} \quad , \quad \overline{\delta^2}.$$

Now we just substitute the parameter d by its normalization in the structure equations at the third step. We have to take into account the fact that $d\mathsf{d}$ is modified accordingly. Indeed we have:

$$d\mathsf{d} = -i\mathsf{e}\frac{\overline{\mathsf{c}}}{\mathsf{c}} - \frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}}\left(\frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}} - \frac{d\mathsf{c}}{\mathsf{c}}\right).$$

The forms γ^1 and γ^2 are not modified as they do not involve terms in $d\mathsf{d}$, but this is not the case for γ^3 which is transformed into:

$$\begin{aligned}\gamma^3 &= \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + i\,\frac{\mathsf{e}}{\mathsf{c}^2} - \frac{\mathsf{d}\,d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - i\,\mathsf{e}^2\frac{d\mathsf{c}}{\mathsf{c}^3} + \frac{\mathsf{d}\,d\overline{\mathsf{c}}}{\mathsf{c}\overline{\mathsf{c}}^2} + i\,\frac{\mathsf{e}^2\,d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\mathsf{c}^2}\\ &= 0.\end{aligned}$$

The expressions of $d\rho$ and $d\kappa$ are thus unchanged from the expressions given by the structure equations at the third step, except the fact that we shall replace d by $-\frac{i}{2}\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} + i\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,H$ in the expression of each torsion coefficient $V_{\bullet\bullet}^{\bullet}$, which we rename $W_{\bullet\bullet}^{\bullet}$, and the fact that the forms γ^1 and γ^1 shall be replaced by the forms δ^1 and δ^2, that is:

$$\begin{aligned}d\rho &= \delta^1 \wedge \rho + \overline{\delta^1} \wedge \rho\\ &\quad + W_{\rho\kappa}^{\rho}\,\rho\wedge\kappa + W_{\rho\zeta}^{\rho}\,\rho\wedge\zeta + W_{\rho\overline{\kappa}}^{\rho}\,\rho\wedge\overline{\kappa} + W_{\rho\overline{\zeta}}^{\rho}\,\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa},\end{aligned}$$

and

$$\begin{aligned}d\kappa &= \delta^1\wedge\kappa + \delta^2\wedge\rho\\ &\quad + W_{\rho\kappa}^{\kappa}\,\rho\wedge\kappa + W_{\rho\zeta}^{\kappa}\,\rho\wedge\zeta + W_{\rho\overline{\kappa}}^{\kappa}\,\rho\wedge\overline{\kappa}\\ &\quad + W_{\rho\overline{\zeta}}^{\kappa}\,\rho\wedge\overline{\zeta} + W_{\kappa\zeta}^{\kappa}\,\kappa\wedge\zeta + W_{\kappa\overline{\kappa}}^{\kappa}\,\kappa\wedge\overline{\kappa} + \zeta\wedge\overline{\kappa}.\end{aligned}$$

The expression of $d\zeta$ is obtained in the same way, setting γ_3 to zero, and renaming $W_{\bullet\bullet}^{\bullet}$ the coefficients $V_{\bullet\bullet}^{\bullet}$ in which one performs the substitution $d = -i\frac{1}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}}$:

$$\begin{aligned}d\zeta &= i\,\delta_2\wedge\kappa + \delta_1\wedge\zeta - \overline{\delta_1}\wedge\zeta\\ &\quad + W_{\rho\kappa}^{\zeta}\,\rho\wedge\kappa + W_{\rho\zeta}^{\zeta}\,\rho\wedge\zeta + W_{\rho\overline{\kappa}}^{\zeta}\,\rho\wedge\overline{\kappa} + W_{\rho\overline{\zeta}}^{\zeta}\,\rho\wedge\overline{\zeta} + W_{\kappa\zeta}^{\zeta}\,\kappa\wedge\zeta\\ &\quad + W_{\kappa\overline{\kappa}}^{\zeta}\,\kappa\wedge\overline{\kappa} + W_{\kappa\overline{\zeta}}^{\zeta}\,\kappa\wedge\overline{\zeta} + W_{\zeta\overline{\kappa}}^{\zeta}\,\zeta\wedge\overline{\kappa} + W_{\zeta\overline{\zeta}}^{\zeta}\,\zeta\wedge\overline{\zeta}.\end{aligned}$$

Let us now proceed with the absorption phase. We make the two substitutions:

$$\begin{aligned}\delta^1 &:= \widetilde{\delta}^1 + w_\rho^1\,\rho + w_\kappa^1\,\kappa + w_\zeta^1\,\zeta + w_{\overline{\kappa}}^1\,\overline{\kappa} + w_{\overline{\zeta}}^1\,\overline{\zeta},\\ \delta^2 &:= \widetilde{\delta}^2 + w_\rho^2\,\rho + w_\kappa^2\,\kappa + w_\zeta^2\,\zeta + w_{\overline{\kappa}}^2\,\overline{\kappa} + w_{\overline{\zeta}}^2\,\overline{\zeta},\end{aligned}$$

in the previous equations. We get:

$$
\begin{aligned}
d\rho = {} & \widetilde{\delta}^1 \wedge \rho + \overline{\widetilde{\delta}^1} \wedge \rho \\
& + \left(W^{\rho}_{\rho\kappa} - w^1_{\kappa} - \overline{w^1_{\overline{\kappa}}} \right) \rho \wedge \kappa + \left(W^{\rho}_{\rho\zeta} - w^1_{\zeta} - \overline{w^1_{\overline{\zeta}}} \right) \rho \wedge \zeta \\
& + \left(W_{\rho\overline{\kappa}} - w^1_{\overline{\kappa}} - \overline{w^1_{\kappa}} \right) \rho \wedge \overline{\kappa} + \left(W^{\rho}_{\rho\overline{\zeta}} - \overline{w^1_{\zeta}} - w^1_{\overline{\zeta}} \right) \rho \wedge \overline{\zeta} \, ,
\end{aligned}
$$

$$
\begin{aligned}
d\kappa = {} & \widetilde{\delta}^1 \wedge \kappa + \widetilde{\delta}^2 \wedge \rho \\
& + \left(W^{\kappa}_{\rho\kappa} - w^2_{\kappa} + w^1_{\rho} \right) \rho \wedge \kappa + \left(W^{\kappa}_{\rho\zeta} - w^2_{\zeta} \right) \rho \wedge \zeta \\
& + \left(W^{\kappa}_{\rho\overline{\kappa}} - w^2_{\overline{\kappa}} \right) \rho \wedge \overline{\kappa} + \left(W_{\rho\overline{\zeta}} - w^2_{\overline{\zeta}} \right) \rho \wedge \overline{\zeta} \\
& + \left(W^{\kappa}_{\kappa\zeta} - w^1_{\zeta} \right) \kappa \wedge \zeta + \left(W^{\kappa}_{\kappa\overline{\kappa}} - w^1_{\overline{\kappa}} \right) \kappa \wedge \overline{\kappa} + \zeta \wedge \overline{\kappa} - w^1_{\overline{\zeta}} \, \kappa \wedge \overline{\zeta} \, ,
\end{aligned}
$$

and

$$
\begin{aligned}
d\zeta = {} & i \, \widetilde{\delta}_2 \wedge \kappa + \widetilde{\delta}_1 \wedge \zeta - \overline{\widetilde{\delta}_1} \wedge \zeta \\
& + \left(W^{\zeta}_{\rho\kappa} + i \, w^2_{\rho} \right) \rho \wedge \kappa + \left(W^{\zeta}_{\rho\zeta} + w^1_{\rho} - \overline{w^1_{\rho}} \right) \rho \wedge \zeta \\
& + W^{\zeta}_{\rho\overline{\kappa}} \, \rho \wedge \overline{\kappa} + W^{\zeta}_{\rho\overline{\zeta}} \, \rho \wedge \overline{\zeta} + \left(W^{\zeta}_{\kappa\overline{\kappa}} - i \, w^2_{\overline{\kappa}} \right) \kappa \wedge \overline{\kappa} \\
& + \left(W^{\zeta}_{\kappa\overline{\zeta}} - i \, w^2_{\overline{\zeta}} \right) \kappa \wedge \overline{\zeta} + \left(W^{\zeta}_{\zeta\overline{\kappa}} - w^1_{\overline{\kappa}} + \overline{w^1_{\kappa}} \right) \zeta \wedge \overline{\zeta} \, .
\end{aligned}
$$

From the last equation, we immediately see that $W^{\zeta}_{\rho\overline{\kappa}}$ and $W^{\zeta}_{\rho\overline{\zeta}}$ are two new essential torsion coefficients. We find the remaining ones by solving the set of equations:

$$
\begin{aligned}
w^1_{\kappa} + \overline{w^1_{\overline{\kappa}}} &= W^{\rho}_{\rho\kappa}, & w^1_{\overline{\kappa}} + \overline{w^1_{\kappa}} &= W^{\rho}_{\rho\overline{\kappa}}, & w^1_{\zeta} + \overline{w^1_{\overline{\zeta}}} &= W^{\rho}_{\rho\zeta}, \\
\overline{w^1_{\zeta}} + w^1_{\overline{\zeta}} &= W^{\rho}_{\rho\overline{\zeta}}, & w^2_{\kappa} - w^1_{\rho} &= W^{\kappa}_{\rho\kappa}, & w^2_{\overline{\kappa}} &= W^{\kappa}_{\rho\overline{\kappa}}, \\
w^2_{\zeta} &= W^{\kappa}_{\rho\zeta}, & w^2_{\overline{\zeta}} &= W^{\kappa}_{\rho\overline{\zeta}}, & w^1_{\zeta} &= W^{\kappa}_{\kappa\zeta}, \\
w^1_{\overline{\zeta}} &= 0, & w^1_{\overline{\kappa}} &= W^{\kappa}_{\kappa\overline{\kappa}}, & -i \, w^2_{\rho} &= W^{\zeta}_{\rho\kappa}, \\
-w^1_{\rho} + \overline{w^1_{\rho}} &= W^{\zeta}_{\rho\zeta}, & w^1_{\kappa} - \overline{w^1_{\overline{\kappa}}} - i \, w^2_{\zeta} &= -W^{\zeta}_{\kappa\zeta}, & i \, w^2_{\overline{\kappa}} &= W^{\zeta}_{\kappa\overline{\kappa}}, \\
w^1_{\kappa} - \overline{w^1_{\kappa}} &= W^{\zeta}_{\zeta\overline{\kappa}}, & i \, w^2_{\overline{\zeta}} &= W^{\zeta}_{\kappa\overline{\zeta}}, & w^1_{\overline{\zeta}} - \overline{w^1_{\zeta}} &= W^{\zeta}_{\zeta\overline{\zeta}},
\end{aligned}
$$

which lead easily as before to:

$$
(8)\qquad
\begin{cases}
w^1_\kappa = \overline{W^\rho_{\rho\overline{\kappa}}},\\
w^1_{\overline{\kappa}} = W^\kappa_{\kappa\overline{\kappa}},\\
w^1_\zeta = W^\rho_{\rho\zeta},\\
w^1_{\overline{\zeta}} = 0,\\
w^2_{\overline{\kappa}} = W^\kappa_{\rho\overline{\kappa}},\\
w^2_{\overline{\zeta}} = W^\kappa_{\rho\overline{\zeta}},\\
w^2_\zeta = W^\kappa_{\rho\zeta},\\
w^2_\kappa = W^\kappa_{\rho\kappa} + w^1_\rho,\\
w^2_\rho = W^\zeta_{\rho\kappa},\\
-w^1_\rho + \overline{w^1_\rho} = W^\zeta_{\rho,\zeta}.
\end{cases}
$$

Eliminating the $w_{\bullet}^{\bullet}$ from (8), we get one additionnal condition on the $W_{\bullet\bullet}^{\bullet}$ which has not yet been checked, namely that $W^\zeta_{\rho,\zeta}$ shall be purely imaginary. The computation of $W^\zeta_{\rho,\zeta}$, $W^\zeta_{\rho\overline{\kappa}}$ and $W^\zeta_{\rho\overline{\zeta}}$ gives:

$$
W^\zeta_{\rho,\zeta} = i\,\frac{\mathsf{e}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{i}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}^3}\,\overline{z_2} - \frac{i}{2}\,\frac{\overline{\mathsf{e}}^2\mathsf{c}}{\overline{\mathsf{c}}^3}\,z_2,
$$

$$
W^\zeta_{\rho\overline{\kappa}} = 0,
$$

and

$$
W^\zeta_{\rho\overline{\zeta}} = 0,
$$

which indicates that no further normalizations are allowed at this stage and that we must perform a prolongation of the problem. Let us introduce the modified Maurer Cartan forms of the group G_4, namely :

$$
\begin{cases}
\hat{\delta}^1 := \delta^1 - w^1_\rho\,\rho - w^1_\kappa\,\kappa - w^1_\zeta\,\zeta - w^1_{\overline{\kappa}}\,\overline{\kappa} - w^1_{\overline{\zeta}}\,\overline{\zeta},\\
\hat{\delta}^2 := \delta^2 - w^2_\rho\,\rho - w^2_\kappa\,\kappa - w^2_\zeta\,\zeta - w^2_{\overline{\kappa}}\,\overline{\kappa} - w^2_{\overline{\zeta}}\,\overline{\zeta},
\end{cases}
$$

where w^i_ρ, w^i_κ, w^i_ζ, $w^i_{\overline{\kappa}}$, $w^i_{\overline{\zeta}}$, $i = 1, 2$, are the solutions of the system of equations (8) corresponding to $w^1_\rho + \overline{w^1_\rho} = 0$, that is :

$$
(9)\qquad
\begin{cases}
\hat{\delta}^1 := \delta^1 + \frac{1}{2}\,V^\zeta_{\rho\zeta}\,\rho - \overline{V^\rho_{\rho\kappa}}\,\kappa - V^\rho_{\rho\zeta}\,\zeta - V^\kappa_{\kappa\overline{\kappa}}\,\overline{\kappa},\\
\hat{\delta}^2 := \delta^2 - V^\zeta_{\rho\kappa}\,\rho - \left(V^\kappa_{\rho\kappa} - \frac{1}{2}V^\zeta_{\rho\zeta}\right)\kappa - V^\kappa_{\rho\zeta}\,\zeta - V^\kappa_{\rho\overline{\kappa}}\,\overline{\kappa} - V^\kappa_{\rho\overline{\zeta}}\,\overline{\zeta}.
\end{cases}
$$

We also introduce the modified Maurer Cartan forms which correspond to solutions of the system (8) when $\mathsf{Re}(w^1_\rho)$ is not necessarily set to zero, namely :

$$\begin{cases} \pi^1 := \hat{\delta}^1 - \Re(w^1_\rho)\,\rho, \\ \pi^2 := \hat{\delta}^2 - \Re(w^1_\rho)\,\kappa. \end{cases} \tag{10}$$

Let P^9 be the nine dimensional G_4-structure constituted by the set of all coframes of the form $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta})$ on M^5. The initial coframe $(\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0)$ gives a natural trivialisation $P^9 \overset{p}{\to} M^5 \times G_4$ which allows us to consider any differential form on M^5 or G^4 as a differential form on P^9. If ω is a differential form on M^5 for example, we just consider $p^*(pr_1^*(\omega))$, where pr_1 is the projection on the first component $M^5 \times G_4 \overset{pr_1}{\to} M^5$. We still denote this form by ω in the sequel. Following [10], we introduce the two coframes $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \delta^1, \delta^2, \overline{\delta^1}, \overline{\delta^2})$ and $(\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \pi^1, \pi^2, \overline{\pi^1}, \overline{\pi^2})$ on P^9. Setting $\mathsf{t} := -\Re(w^1_\rho)$, they relate to each other by the relation:

$$\begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \\ \pi^1 \\ \pi^2 \\ \overline{\pi^1} \\ \overline{\pi^2} \end{pmatrix} = g_{\mathsf{t}} \cdot \begin{pmatrix} \rho \\ \kappa \\ \zeta \\ \overline{\kappa} \\ \overline{\zeta} \\ \delta^1 \\ \delta^2 \\ \overline{\delta^1} \\ \overline{\delta^2} \end{pmatrix},$$

where g_{t} is defined by

$$g_{\mathsf{t}} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The set $\{g_{\mathsf{t}}, \mathsf{t} \in \mathbb{R}\}$ defines a one- dimensional Lie group, whose Maurer Cartan form is given by $d\mathsf{t}$, which we rename Λ in the sequel. We now start the reduction step in the equivalence problem on P^9. From the definition of π^1 and π^2 as the solutions of the absorption equations (8), the five first structure equations read as

$$
\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi^1} \wedge \rho + i\,\kappa \wedge \overline{\kappa},\\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa},\\
d\zeta &= i\,\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi^1} \wedge \zeta,\\
d\overline{\kappa} &= \overline{\pi^1} \wedge \overline{\kappa} + \overline{\pi^2} \wedge \rho - \kappa \wedge \overline{\zeta},\\
d\overline{\zeta} &= -i\,\overline{\pi^2} \wedge \overline{\kappa} + \overline{\pi^1} \wedge \overline{\zeta} - \pi^1 \wedge \overline{\zeta}.
\end{aligned}
\tag{11}
$$

We could obtain the expressions of $d\pi^1$ and $d\pi^2$ by taking the exterior derivative of the previous five equations. But for now, as we have explicit expressions of π^1 and π^2 given by formulae (9) and (10), we can perform an actual computation:

$$
\begin{aligned}
d\pi^1 = d\mathsf{t} \wedge \rho
&+ X^1_{\rho\kappa}\,\rho \wedge \kappa + X^1_{\rho\zeta}\,\rho \wedge \zeta + X^1_{\rho\overline{\kappa}}\,\rho \wedge \overline{\kappa} + X^1_{\rho\overline{\zeta}}\,\rho \wedge \overline{\zeta}\\
&+ X^1_{\rho\pi^1}\,\rho \wedge \pi^1 + X^1_{\rho\pi^2}\,\rho \wedge \pi^2 + X^1_{\rho\overline{\pi^1}}\,\rho \wedge \overline{\pi^1}\\
&+ X^1_{\rho\overline{\pi^2}}\,\rho \wedge \overline{\pi^2} + i\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta},
\end{aligned}
$$

and

$$
\begin{aligned}
d\pi^2 = d\mathsf{t} \wedge \kappa
&+ X^2_{\rho\kappa}\,\rho \wedge \kappa + X^2_{\kappa\zeta}\,\kappa \wedge \zeta + X^2_{\kappa\overline{\kappa}}\,\kappa \wedge \overline{\kappa} + X^2_{\kappa\overline{\zeta}}\,\kappa \wedge \overline{\zeta}\\
&+ X^2_{\kappa\pi^1}\,\kappa \wedge \pi^1 + X^2_{\kappa\pi^2}\,\kappa \wedge \pi^2 + X^2_{\kappa\overline{\pi^1}}\,\kappa \wedge \overline{\pi^1}\\
&+ X^2_{\kappa\overline{\pi^2}}\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\pi^2} + \pi^2 \wedge \overline{\pi^1}.
\end{aligned}
$$

From these equations, we see that the absorption is straightforward and that there remain no nonconstant essential torsion term. Indeed if we define the absorbed form Λ by:

$$
\Lambda = d\mathsf{t} - X^2_{\rho\kappa}\,\rho - X^1_{\rho\kappa}\,\kappa - \sum_{\nu=\zeta,\pi^1,\cdots,\overline{\pi^2}} X^1_{\rho\nu}\,\nu,
$$

the previous two equations become:

$$
d\pi^1 = \Lambda \wedge \rho + i\,\kappa \wedge \overline{\pi^2} + \zeta \wedge \overline{\zeta},
$$

and

$$
d\pi^2 = \Lambda \wedge \kappa + \zeta \wedge \overline{\pi^2} + \pi^2 \wedge \overline{\pi^1}.
$$

A straightforward computation gives the expression of $d\Lambda$:

$$
d\Lambda = -\pi^1 \wedge \Lambda + i\,\pi^2 \wedge \overline{\pi^2} - \pi^1 \wedge \overline{\Lambda}.
$$

Let us summarize the results that we have obtained so far: The ten 1-forms ρ, κ, ζ, $\overline{\kappa}$, $\overline{\zeta}$, π^1, π^2, $\overline{\pi^1}$, $\overline{\pi^2}$, Λ satisfies the structure equations given by (4). This completes the proof ofh Theorem 3.

APPENDIX A. TORSION COEFFICIENTS FOR THE G-STRUCTURES ON B

A.1. Coefficients $U_{\bullet\bullet}^{\bullet}$.

$$U_{\sigma\rho}^{\sigma} = \frac{\mathsf{e}}{\mathsf{a}^3} + \frac{\mathsf{d}}{\mathsf{a}^3},$$

$$U_{\sigma\zeta}^{\sigma} = -\frac{\mathsf{c}}{\mathsf{a}^3},$$

$$U_{\sigma\overline{\zeta}}^{\sigma} = -\frac{\mathsf{c}}{\mathsf{a}^3},$$

$$U_{\sigma\rho}^{\rho} = \frac{\mathsf{ce}}{\mathsf{a}^6} + \frac{\mathsf{cd}}{\mathsf{a}^6} - \frac{i\mathsf{be}}{\mathsf{a}^5} + \frac{i\mathsf{d}\overline{\mathsf{b}}}{\mathsf{a}^5},$$

$$U_{\sigma\zeta}^{\rho} = \frac{i\mathsf{e}}{\mathsf{a}^3} - \frac{i\overline{\mathsf{b}}\,\mathsf{c}}{\mathsf{a}^5} - \frac{\mathsf{c}^2}{\mathsf{a}^6},$$

$$U_{\sigma\overline{\zeta}}^{\rho} = \frac{i\mathsf{bc}}{\mathsf{a}^5} - \frac{\mathsf{c}^2}{\mathsf{a}^6} - \frac{i\mathsf{d}}{\mathsf{a}^3},$$

$$U_{\rho\zeta}^{\rho} = \frac{\mathsf{c}}{\mathsf{a}^3} + \frac{i\overline{\mathsf{b}}}{\mathsf{a}^2},$$

$$U_{\rho\overline{\zeta}}^{\rho} = \frac{\mathsf{c}}{\mathsf{a}^3} - \frac{i\mathsf{b}}{\mathsf{a}^2},$$

$$U_{\sigma\rho}^{\zeta} = \frac{\mathsf{d}^2}{\mathsf{a}^6} + \frac{i\overline{\mathsf{b}}\,\mathsf{db}}{\mathsf{a}^7} - \frac{i\mathsf{eb}^2}{\mathsf{a}^7} + \frac{\mathsf{de}}{\mathsf{a}^6},$$

$$U_{\sigma\zeta}^{\zeta} = \frac{i\mathsf{be}}{\mathsf{a}^5} - \frac{i\overline{\mathsf{b}}\,\mathsf{cb}}{\mathsf{a}^7} - \frac{\mathsf{cd}}{\mathsf{a}^6},$$

$$U_{\sigma\overline{\zeta}}^{\zeta} = -\frac{\mathsf{cd}}{\mathsf{a}^6} - \frac{i\mathsf{db}}{\mathsf{a}^5} + \frac{i\mathsf{b}^2\mathsf{c}}{\mathsf{a}^7},$$

$$U_{\rho\zeta}^{\zeta} = \frac{\mathsf{d}}{\mathsf{a}^3} + \frac{i\overline{\mathsf{b}}\,\mathsf{b}}{\mathsf{a}^4},$$

$$U_{\rho\overline{\zeta}}^{\zeta} = \frac{\mathsf{d}}{\mathsf{a}^3} - \frac{i\mathsf{b}^2}{\mathsf{a}^4},$$

$$U_{\zeta\overline{\zeta}}^{\zeta} = \frac{i\mathsf{b}}{\mathsf{a}^2}.$$

B.1. Coefficients $U^{\bullet}_{\bullet\bullet}$.

$$U^{\tau}_{\tau\sigma} = \frac{\mathsf{h}}{\mathsf{a}^4} - \frac{\bar{\mathsf{b}}\mathsf{g}}{\mathsf{a}^6} - \frac{\mathsf{bg}}{\mathsf{a}^6} + \frac{\mathsf{k}}{\mathsf{a}^6},$$

$$U^{\tau}_{\tau\rho} = \frac{\mathsf{bf}}{\mathsf{a}^6} + \frac{\bar{\mathsf{b}}\,\mathsf{f}}{\mathsf{a}^6},$$

$$U^{\tau}_{\tau\zeta} = -\frac{\mathsf{f}}{\mathsf{a}^4},$$

$$U^{\tau}_{\tau\bar{\zeta}} = -\frac{\mathsf{f}}{\mathsf{a}^4},$$

$$U^{\tau}_{\sigma\rho} = -\frac{\mathsf{b}}{\mathsf{a}^2} - \frac{\bar{\mathsf{b}}}{\mathsf{a}^2},$$

$$U^{\sigma}_{\tau\sigma} = \frac{\mathsf{ge}}{\mathsf{a}^7} - \frac{\mathsf{hc}}{\mathsf{a}^7} - \frac{\mathsf{kc}}{\mathsf{a}^7} + \frac{\mathsf{gd}}{\mathsf{a}^7} + \frac{\mathsf{fk}}{\mathsf{a}^8} + \frac{\mathsf{fh}}{\mathsf{a}^8} - \frac{\mathsf{f}\bar{\mathsf{b}}\,\mathsf{g}}{\mathsf{a}^{10}} - \frac{\mathsf{fbg}}{\mathsf{a}^{10}},$$

$$U^{\sigma}_{\tau\rho} = \frac{\bar{\mathsf{b}}\,\mathsf{f}^2}{\mathsf{a}^{10}} + \frac{\mathsf{bf}^2}{\mathsf{a}^{10}} - \frac{\mathsf{fe}}{\mathsf{a}^7} - \frac{\mathsf{fd}}{\mathsf{a}^7} + \frac{\mathsf{k}}{\mathsf{a}^4} + \frac{\mathsf{h}}{\mathsf{a}^4},$$

$$U^{\sigma}_{\tau\zeta} = -\frac{\mathsf{g}}{\mathsf{a}^4} + \frac{\mathsf{cf}}{\mathsf{a}^7} - \frac{\mathsf{f}^2}{\mathsf{a}^8},$$

$$U^{\sigma}_{\tau\bar{\zeta}} = -\frac{\mathsf{g}}{\mathsf{a}^4} + \frac{\mathsf{cf}}{\mathsf{a}^7} - \frac{\mathsf{f}^2}{\mathsf{a}^8},$$

$$U^{\sigma}_{\sigma\rho} = \frac{\mathsf{e}}{\mathsf{a}^3} + \frac{\mathsf{d}}{\mathsf{a}^3} - \frac{\mathsf{bf}}{\mathsf{a}^6} - \frac{\bar{\mathsf{b}}\,\mathsf{f}}{\mathsf{a}^6},$$

$$U^{\sigma}_{\sigma\zeta} = -\frac{\mathsf{c}}{\mathsf{a}^3} + \frac{\mathsf{f}}{\mathsf{a}^4},$$

$$U^{\sigma}_{\sigma\bar{\zeta}} = -\frac{\mathsf{c}}{\mathsf{a}^3} + \frac{\mathsf{f}}{\mathsf{a}^4},$$

$$U^{\rho}_{\tau\sigma} = \frac{-i\mathsf{ebg}}{\mathsf{a}^9} - \frac{i\bar{\mathsf{b}}\,\mathsf{ch}}{\mathsf{a}^9} + \frac{i\mathsf{d}\bar{\mathsf{b}}\,\mathsf{g}}{\mathsf{a}^9} + \frac{i\mathsf{bck}}{\mathsf{a}^9} + \frac{\mathsf{egc}}{\mathsf{a}^{10}} + \frac{\mathsf{dgc}}{\mathsf{a}^{10}} - \frac{i\mathsf{dk}}{\mathsf{a}^7} + \frac{i\mathsf{eh}}{\mathsf{a}^7} - \frac{\mathsf{c}^2\mathsf{h}}{\mathsf{a}^{10}} - \frac{\mathsf{c}^2\mathsf{k}}{\mathsf{a}^{10}} - \frac{\mathsf{g}^2\mathsf{b}}{\mathsf{a}^{10}} + \frac{\mathsf{gk}}{\mathsf{a}^8} + \frac{\mathsf{gh}}{\mathsf{a}^8} - \frac{\mathsf{g}^2\bar{\mathsf{b}}}{\mathsf{a}^{10}},$$

$$U^{\rho}_{\tau\rho} = -\frac{\mathsf{cdf}}{\mathsf{a}^{10}} + \frac{\mathsf{fbg}}{\mathsf{a}^{10}} + \frac{\mathsf{f}\bar{\mathsf{b}}\,\mathsf{g}}{\mathsf{a}^{10}} - \frac{\mathsf{cef}}{\mathsf{a}^{10}} + \frac{\mathsf{hc}}{\mathsf{a}^7} + \frac{\mathsf{kc}}{\mathsf{a}^7} - \frac{i\mathsf{bk}}{\mathsf{a}^6} + \frac{i\mathsf{bef}}{\mathsf{a}^9} - \frac{i\mathsf{d}\bar{\mathsf{b}}\,\mathsf{f}}{\mathsf{a}^9} + \frac{i\bar{\mathsf{b}}\,\mathsf{h}}{\mathsf{a}^6},$$

$$U^{\rho}_{\tau\zeta} = \frac{i\bar{\mathsf{b}}\,\mathsf{cf}}{\mathsf{a}^9} - \frac{i\mathsf{ef}}{\mathsf{a}^7} - \frac{i\bar{\mathsf{b}}\,\mathsf{g}}{\mathsf{a}^6} + \frac{\mathsf{c}^2\mathsf{f}}{\mathsf{a}^{10}} - \frac{\mathsf{gc}}{\mathsf{a}^7} + \frac{i\mathsf{k}}{\mathsf{a}^4} - \frac{\mathsf{gf}}{\mathsf{a}^8},$$

$$U^{\rho}_{\tau\bar{\zeta}} = \frac{-i\mathsf{bcf}}{\mathsf{a}^9} + \frac{i\mathsf{df}}{\mathsf{a}^7} + \frac{i\mathsf{bg}}{\mathsf{a}^6} + \frac{\mathsf{c}^2\mathsf{f}}{\mathsf{a}^{10}} - \frac{\mathsf{gc}}{\mathsf{a}^7} - \frac{\mathsf{gf}}{\mathsf{a}^8} - \frac{i\mathsf{h}}{\mathsf{a}^4},$$

$$U^{\rho}_{\sigma\rho} = \frac{\mathsf{ce}}{\mathsf{a}^6} + \frac{\mathsf{cd}}{\mathsf{a}^6} - \frac{\mathsf{gb}}{\mathsf{a}^6} - \frac{\mathsf{g}\bar{\mathsf{b}}}{\mathsf{a}^6} - \frac{i\mathsf{be}}{\mathsf{a}^5} + \frac{i\mathsf{d}\bar{\mathsf{b}}}{\mathsf{a}^5},$$

$$U^{\rho}_{\sigma\zeta} = \frac{i\mathsf{e}}{\mathsf{a}^3} - \frac{i\bar{\mathsf{b}}\,\mathsf{c}}{\mathsf{a}^5} - \frac{\mathsf{c}^2}{\mathsf{a}^6} + \frac{\mathsf{g}}{\mathsf{a}^4},$$

$$U^{\rho}_{\sigma\bar{\zeta}} = \frac{i\mathsf{bc}}{\mathsf{a}^5} - \frac{\mathsf{c}^2}{\mathsf{a}^6} - \frac{i\mathsf{d}}{\mathsf{a}^3} + \frac{\mathsf{g}}{\mathsf{a}^4},$$

$$U^{\rho}_{\rho\zeta} = \frac{\mathsf{c}}{\mathsf{a}^3} + \frac{i\bar{\mathsf{b}}}{\mathsf{a}^2},$$

$$U^{\rho}_{\rho\bar{\zeta}} = \frac{\mathsf{c}}{\mathsf{a}^3} - \frac{i\mathsf{b}}{\mathsf{a}^2},$$

$$U^{\zeta}_{\tau\sigma} = \frac{-i\mathsf{eb}^2\mathsf{g}}{\mathsf{a}^{11}} - \frac{i\mathsf{dkb}}{\mathsf{a}^9} + \frac{i\mathsf{ehb}}{\mathsf{a}^9} + \frac{i\mathsf{b}^2\mathsf{ck}}{\mathsf{a}^{11}} + \frac{\mathsf{hk}}{\mathsf{a}^8} + \frac{\mathsf{d}^2\mathsf{g}}{\mathsf{a}^{10}} - \frac{i\bar{\mathsf{b}}\,\mathsf{chb}}{\mathsf{a}^{11}} + \frac{i\mathsf{d}\bar{\mathsf{b}}\,\mathsf{gb}}{\mathsf{a}^{11}} - \frac{\mathsf{cdk}}{\mathsf{a}^{10}} - \frac{\mathsf{cdh}}{\mathsf{a}^{10}} + \frac{\mathsf{ged}}{\mathsf{a}^{10}} - \frac{\mathsf{h}\bar{\mathsf{b}}\,\mathsf{g}}{\mathsf{a}^{10}} - \frac{\mathsf{hbg}}{\mathsf{a}^{10}} + \frac{\mathsf{h}^2}{\mathsf{a}^8},$$

$$U^{\zeta}_{\tau\rho} = \frac{\mathsf{kd}}{\mathsf{a}^7} - \frac{\mathsf{def}}{\mathsf{a}^{10}} + \frac{\mathsf{h}\bar{\mathsf{b}}\,\mathsf{f}}{\mathsf{a}^{10}} + \frac{\mathsf{hbf}}{\mathsf{a}^{10}} - \frac{i\mathsf{kb}^2}{\mathsf{a}^8} - \frac{i\bar{\mathsf{b}}\,\mathsf{dfb}}{\mathsf{a}^{11}} + \frac{i\bar{\mathsf{b}}\,\mathsf{hb}}{\mathsf{a}^8} + \frac{i\mathsf{efb}^2}{\mathsf{a}^{11}} + \frac{\mathsf{dh}}{\mathsf{a}^7} - \frac{\mathsf{d}^2\mathsf{f}}{\mathsf{a}^{10}},$$

$$U^{\zeta}_{\tau\zeta} = -\frac{\mathsf{gd}}{\mathsf{a}^7} - \frac{\mathsf{fh}}{\mathsf{a}^8} - \frac{i\mathsf{bef}}{\mathsf{a}^9} - \frac{i\bar{\mathsf{b}}\,\mathsf{gb}}{\mathsf{a}^8} + \frac{i\bar{\mathsf{b}}\,\mathsf{cfb}}{\mathsf{a}^{11}} + \frac{\mathsf{cdf}}{\mathsf{a}^{10}} + \frac{i\mathsf{bk}}{\mathsf{a}^6},$$

$$U^{\zeta}_{\tau\bar{\zeta}} = -\frac{\mathsf{gd}}{\mathsf{a}^7} - \frac{\mathsf{fh}}{\mathsf{a}^8} + \frac{i\mathsf{dfb}}{\mathsf{a}^9} + \frac{\mathsf{cdf}}{\mathsf{a}^{10}} - \frac{i\mathsf{hb}}{\mathsf{a}^6} + \frac{i\mathsf{b}^2\mathsf{g}}{\mathsf{a}^8} - \frac{i\mathsf{b}^2\mathsf{cf}}{\mathsf{a}^{11}},$$

$$U^{\zeta}_{\sigma\rho} = \frac{\mathsf{d}^2}{\mathsf{a}^6} - \frac{\mathsf{h}\bar{\mathsf{b}}}{\mathsf{a}^6} + \frac{i\bar{\mathsf{b}}\,\mathsf{db}}{\mathsf{a}^7} - \frac{i\mathsf{eb}^2}{\mathsf{a}^7} + \frac{\mathsf{de}}{\mathsf{a}^6} - \frac{\mathsf{hb}}{\mathsf{a}^6},$$

$$U^{\zeta}_{\sigma\zeta} = \frac{i\mathsf{be}}{\mathsf{a}^5} - \frac{i\bar{\mathsf{b}}\,\mathsf{cb}}{\mathsf{a}^7} - \frac{\mathsf{cd}}{\mathsf{a}^6} + \frac{\mathsf{h}}{\mathsf{a}^4},$$

$$U^{\zeta}_{\sigma\bar{\zeta}} = -\frac{\mathsf{cd}}{\mathsf{a}^6} - \frac{i\mathsf{db}}{\mathsf{a}^5} + \frac{\mathsf{h}}{\mathsf{a}^4} + \frac{i\mathsf{b}^2\mathsf{c}}{\mathsf{a}^7},$$

$$U^{\zeta}_{\rho\zeta} = \frac{\mathsf{d}}{\mathsf{a}^3} + \frac{i\bar{\mathsf{b}}\,\mathsf{b}}{\mathsf{a}^4},$$

$$U^{\zeta}_{\rho\bar{\zeta}} = \frac{\mathsf{d}}{\mathsf{a}^3} - \frac{i\mathsf{b}^2}{\mathsf{a}^4},$$

$$U^{\zeta}_{\zeta\bar{\zeta}} = \frac{i\mathsf{b}}{\mathsf{a}^2}.$$

APPENDIX C. TORSION COEFFICIENTS FOR THE G-STRUCTURES ON LC

C.1. Coefficients $T^{\bullet}_{\bullet\bullet}$.

$$T^{\rho}_{\rho\kappa} = i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{e}}{\mathsf{c}\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}},$$

$$T^{\rho}_{\rho\zeta} = \frac{1}{\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}},$$

$$T^{\rho}_{\rho\overline{\kappa}} = -i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\overline{\mathsf{e}}}{\overline{\mathsf{c}}\overline{\mathsf{f}}}\, - \frac{z_2}{1-z_2\overline{z_2}},$$

$$T^{\rho}_{\rho\overline{\zeta}} = \frac{1}{\overline{\mathsf{f}}}\,\frac{z_2}{1-z_2\overline{z_2}},$$

$$T^{\kappa}_{\rho\kappa} = \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}} + i\,\frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}}$$

$$T^{\kappa}_{\rho\zeta} = -\frac{\overline{\mathsf{b}}}{\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}},$$

$$T^{\kappa}_{\rho\overline{\kappa}} = \frac{\mathsf{d}}{\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} - \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} - i\,\frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}^2\overline{\mathsf{f}}}\, - \frac{z_2}{1-z_2\overline{z_2}},$$

$$T^{\kappa}_{\rho\overline{\zeta}} = \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\frac{z_2}{1-z_2\overline{z_2}},$$

$$T^{\kappa}_{\kappa\zeta} = \frac{1}{\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}},$$

$$T^{\kappa}_{\kappa\overline{\kappa}} = \frac{\mathsf{e}}{\overline{\mathsf{c}}\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} + i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$T^{\kappa}_{\zeta\overline{\kappa}} = -\frac{\mathsf{c}}{\overline{\mathsf{c}}\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}},$$

$$T^{\zeta}_{\rho\kappa} = \frac{\mathsf{e}^2\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} + i\,\frac{\mathsf{d}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2},$$

$$T^{\zeta}_{\rho\zeta} = -\frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} - \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}} + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}},$$

$$T^{\zeta}_{\rho\overline{\kappa}} = \frac{\mathsf{e}\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} - \frac{\mathsf{e}^2\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2\mathsf{f}}\,\frac{1}{1-z_2\overline{z_2}} - i\,\frac{\mathsf{b}\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}^2} - \frac{\mathsf{d}\overline{\mathsf{e}}}{\mathsf{c}\overline{\mathsf{c}}^2\overline{\mathsf{f}}}\,\frac{z_2}{1-z_2\overline{z_2}},$$

$$T^{\zeta}_{\rho\overline{\zeta}} = \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}\overline{\mathsf{f}}}\,\frac{z_2}{1-z_2\overline{z_2}},$$

$$T^{\zeta}_{\kappa\zeta} = +\frac{\mathsf{e}}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}}\,\frac{\overline{z_2}}{1-z_2\overline{z_2}},$$

$$T^{\zeta}_{\kappa\overline{\kappa}} = \frac{\mathsf{e}^2}{\mathsf{c}\overline{\mathsf{c}}\mathsf{f}} \, \frac{1}{1 - z_2\overline{z_2}} + i \, \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$T^{\zeta}_{\zeta\overline{\kappa}} = -\frac{\mathsf{e}}{\overline{\mathsf{c}}\mathsf{f}} \, \frac{1}{1 - z_2\overline{z_2}}.$$

C.2. **Coefficients $U^{\bullet}_{\bullet\bullet}$.**

$$U^{\rho}_{\rho\kappa} = i \, \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2} \, \overline{z_2},$$

$$U^{\rho}_{\rho\zeta} = -\frac{\overline{\mathsf{c}}}{\mathsf{c}} \, \overline{z_2},$$

$$U^{\rho}_{\rho\overline{\kappa}} = -i \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2} \, z_2,$$

$$U^{\rho}_{\rho\overline{\zeta}} = -\frac{\mathsf{c}}{\overline{\mathsf{c}}} \, z_2,$$

$$U^{\kappa}_{\rho\kappa} = -\frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}} - \frac{\mathsf{d}}{\mathsf{c}^2} \, \overline{z_2} + i \, \frac{\mathsf{b}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^3} \, \overline{z_2},$$

$$U^{\kappa}_{\rho\zeta} = \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$U^{\kappa}_{\rho\overline{\kappa}} = -\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}} - i \, \frac{\mathsf{b}^2}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{b}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^3} \, z_2,$$

$$U^{\kappa}_{\rho\overline{\zeta}} = -\frac{\mathsf{b}}{\overline{\mathsf{c}}^2} \, z_2,$$

$$U^{\kappa}_{\kappa\zeta} = -\frac{\overline{\mathsf{c}}}{\mathsf{c}} \, \overline{z_2},$$

$$U^{\kappa}_{\kappa\overline{\kappa}} = -\frac{\mathsf{e}}{\mathsf{c}} + i \, \frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$U^{\zeta}_{\rho\kappa} = -\frac{\mathsf{e}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}^2} \, z_2 + \frac{\overline{\mathsf{b}}\mathsf{e}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^3\mathsf{c}} \, z_2 - \frac{\mathsf{e}^2\overline{\mathsf{b}}}{\overline{\mathsf{c}}\mathsf{c}^3} + i \, \frac{\mathsf{d}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2},$$

$$U^{\zeta}_{\rho\zeta} = \frac{\overline{\mathsf{d}}}{\overline{\mathsf{c}}^2} \, z_2 - \frac{\overline{\mathsf{e}}\overline{\mathsf{b}}}{\overline{\mathsf{c}}^3} \, z_2 + \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^3} \, \overline{z_2} - \frac{\mathsf{d}}{\mathsf{c}^2} \, \overline{z_2},$$

$$U^{\zeta}_{\rho\overline{\kappa}} = 2 \, \frac{\overline{\mathsf{e}}\mathsf{d}}{\overline{\mathsf{c}}^3} \, z_2 - \frac{\mathsf{e}\overline{\mathsf{e}}\mathsf{b}}{\overline{\mathsf{c}}^3\mathsf{c}} \, z_2 - \frac{\mathsf{e}\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{\mathsf{e}^2\mathsf{b}}{\overline{\mathsf{c}}\mathsf{c}^3} - i \, \frac{\mathsf{d}\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}^2},$$

$$U^{\zeta}_{\rho\overline{\zeta}} = -2 \, \frac{\mathsf{d}}{\overline{\mathsf{c}}^2} \, z_2 + \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}^2} \, z_2,$$

$$U^{\zeta}_{\kappa\zeta} = -\frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2} \, \overline{z_2},$$

$$U^{\zeta}_{\kappa\bar{\kappa}} = \frac{\mathsf{e}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2}\, z_2 - \frac{\mathsf{e}^2}{\mathsf{c}^2} + i\, \frac{\mathsf{d}}{\mathsf{c}\bar{\mathsf{c}}},$$

$$U^{\zeta}_{\kappa\bar{\zeta}} = -\frac{\mathsf{e}}{\bar{\mathsf{c}}}\, z_2,$$

$$U^{\zeta}_{\zeta\bar{\kappa}} = -\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2}\, z_2 + \frac{\mathsf{e}}{\mathsf{c}},$$

$$U^{\zeta}_{\zeta\bar{\zeta}} = \frac{\mathsf{c}}{\bar{\mathsf{c}}}\, z_2.$$

C.3. **Coefficients $V^{\bullet}_{\bullet\bullet}$.**

$$V^{\rho}_{\rho\kappa} = -\frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}} + \frac{\mathsf{e}\bar{\mathsf{c}}}{\mathsf{c}^2}\, \overline{z_2},$$

$$V^{\rho}_{\rho\zeta} = -\frac{\bar{\mathsf{c}}}{\mathsf{c}}\, \overline{z_2},$$

$$V^{\rho}_{\rho\bar{\kappa}} = -\frac{\mathsf{e}}{\mathsf{c}} + \frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2}\, z_2,$$

$$V^{\rho}_{\rho\bar{\zeta}} = -\frac{\mathsf{c}}{\bar{\mathsf{c}}}\, z_2,$$

$$V^{\kappa}_{\rho\kappa} = -\frac{\mathsf{d}}{\mathsf{c}^2}\, \overline{z_2} - i\, \frac{\mathsf{e}^2\bar{\mathsf{c}}}{\mathsf{c}^3}\, \overline{z_2}$$

$$V^{\kappa}_{\rho\zeta} = i\, \frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}},$$

$$V^{\kappa}_{\rho\bar{\kappa}} = -\frac{\mathsf{d}}{\mathsf{c}\bar{\mathsf{c}}} - i\, \frac{\mathsf{e}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2}\, z_2,$$

$$V^{\kappa}_{\rho\bar{\zeta}} = i\, \frac{\mathsf{e}}{\bar{\mathsf{c}}}\, z_2,$$

$$V^{\kappa}_{\kappa\zeta} = -\frac{\bar{\mathsf{c}}}{\mathsf{c}}\, \overline{z_2},$$

$$V^{\kappa}_{\kappa\bar{\kappa}} = 0,$$

$$V^{\zeta}_{\rho\kappa} = -\frac{\bar{\mathsf{d}}\mathsf{e}}{\mathsf{c}\bar{\mathsf{c}}^2}\, z_2 - \frac{\mathsf{d}\bar{\mathsf{e}}}{\mathsf{c}\bar{\mathsf{c}}^2} + i\, \frac{\mathsf{e}\bar{\mathsf{e}}^2}{\bar{\mathsf{c}}^3}\, z_2 - i\, \frac{\mathsf{e}^2\bar{\mathsf{e}}}{\mathsf{c}^2\bar{\mathsf{c}}},$$

$$V^{\zeta}_{\rho\zeta} = -\frac{\bar{\mathsf{d}}}{\bar{\mathsf{c}}^2}\, z_2 + i\, \frac{\mathsf{e}\,\bar{\mathsf{e}}}{\mathsf{c}\,\bar{\mathsf{c}}} - i\, \frac{\bar{\mathsf{c}}\mathsf{e}^2}{\mathsf{c}^3}\, \overline{z_2} - i\, \frac{\mathsf{c}\bar{\mathsf{e}}^2}{\bar{\mathsf{c}}^3}\, z_2 - \frac{\mathsf{d}}{\mathsf{c}^2}\, \overline{z_2},$$

$$V^{\zeta}_{\rho\bar{\kappa}} = 2\, \frac{\mathsf{d}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^3}\, z_2 + i\, \frac{\bar{\mathsf{e}}\mathsf{e}^2}{\bar{\mathsf{c}}^2\mathsf{c}}\, z_2 - 2\, \frac{\mathsf{e}\mathsf{d}}{\bar{\mathsf{c}}\mathsf{c}^2} - i\, \frac{\mathsf{e}^3}{\mathsf{c}^3},$$

$$V^{\zeta}_{\rho\bar{\zeta}} = -2\, \frac{\mathsf{d}}{\bar{\mathsf{c}}^2}\, z_2 - i\, \frac{\mathsf{e}^2}{\mathsf{c}\,\bar{\mathsf{c}}}\, z_2,$$

$$V^{\zeta}_{\kappa\zeta} = -\frac{\mathsf{e}\,\bar{\mathsf{c}}}{\mathsf{c}^2}\,\overline{z_2},$$

$$V^{\zeta}_{\kappa\bar\kappa} = \frac{\mathsf{e}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^2}\,z_2 - \frac{\mathsf{e}^2}{\mathsf{c}^2} + i\,\frac{\mathsf{d}}{\mathsf{c}\,\bar{\mathsf{c}}},$$

$$V^{\zeta}_{\kappa\bar\zeta} = -\frac{\mathsf{e}}{\bar{\mathsf{c}}}\,z_2,$$

$$V^{\zeta}_{\zeta\bar\kappa} = -\frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2}\,z_2 + \frac{\mathsf{e}}{\mathsf{c}},$$

$$V^{\zeta}_{\zeta\bar\zeta} = \frac{\mathsf{c}}{\bar{\mathsf{c}}}\,z_2.$$

C.4. **Coefficients $X^{\bullet}_{\bullet\bullet}$.**

$$X^1_{\rho\kappa} = -\frac{1}{2}\,\mathsf{t}\,\frac{\bar{\mathsf{c}}\mathsf{e}}{\mathsf{c}}\,\overline{z_2} - \frac{3}{8}\,i\,\frac{\mathsf{e}^2\bar{\mathsf{e}}}{\mathsf{c}^3}\,\overline{z_2} + \frac{1}{2}\,\mathsf{t}\frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}} + \frac{1}{8}\,i\,\frac{\mathsf{e}\bar{\mathsf{e}}^2}{\mathsf{c}\bar{\mathsf{c}}^2}\,z_2\overline{z_2} + \frac{1}{8}\,i\,\frac{\mathsf{e}^3\bar{\mathsf{c}}^2}{\mathsf{c}^5}\,\overline{z_2}^2 + \frac{1}{4}\,i\,\frac{\bar{\mathsf{e}}^2\mathsf{e}}{\mathsf{c}\bar{\mathsf{c}}^2},$$

$$X^1_{\rho\zeta} = -\frac{1}{4}\,i\,\frac{\bar{\mathsf{e}}^2}{\bar{\mathsf{c}}^2} + \frac{1}{2}\,i\,\frac{\mathsf{e}\bar{\mathsf{e}}}{\mathsf{c}^2}\,\overline{z_2} - \frac{1}{4}\,i\,\frac{\bar{\mathsf{c}}^2\mathsf{e}^2}{\mathsf{c}^4}\,\overline{z_2}^2,$$

$$X^1_{\rho\bar\kappa} = \overline{X^1_{\rho\kappa}},$$

$$X^1_{\rho\bar\zeta} = \overline{X^1_{\rho\zeta}},$$

$$X^1_{\rho\pi^1} = -\mathsf{t},$$

$$X^1_{\rho\pi^2} = \frac{1}{2}\,\frac{\bar{\mathsf{e}}}{\bar{\mathsf{c}}} + \frac{1}{2}\,\frac{\mathsf{e}\bar{\mathsf{c}}}{\mathsf{c}^2}\,\overline{z_2},$$

$$X^1_{\rho\overline{\pi^1}} = -\mathsf{t},$$

$$X^1_{\rho\overline{\pi^2}} = \overline{X^1_{\rho\pi^2}},$$

$$X^2_{\rho\kappa} = -\frac{1}{4}\,\frac{\mathsf{e}\bar{\mathsf{e}}^3}{\bar{\mathsf{c}}^4}\,z_2 - \frac{1}{4}\,\frac{\mathsf{e}^3\bar{\mathsf{e}}}{\bar{\mathsf{c}}^4}\,\overline{z_2} + \frac{1}{8}\,\frac{\mathsf{e}^2\bar{\mathsf{e}}^2}{\mathsf{c}^2\bar{\mathsf{c}}^2}\,\overline{z_2} + \frac{1}{4}\,\frac{\mathsf{e}^2\bar{\mathsf{e}}^2}{\mathsf{c}^2\bar{\mathsf{c}}^2} + \frac{1}{16}\,\frac{\bar{\mathsf{e}}^4\mathsf{c}^2}{\bar{\mathsf{c}}^6}\,z_2^2 + \frac{1}{16}\,\frac{\mathsf{e}^4\bar{\mathsf{c}}^2}{\mathsf{c}^6}\,\overline{z_2}^2 - \mathsf{t}^2,$$

$$X^2_{\kappa\nu} = X^1_{\rho\nu} \qquad \text{for } \nu = \zeta, \pi^1, \cdots, \overline{\pi^2}.$$

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