GLOBAL MINIMALITY OF GENERIC MANIFOLDS AND HOLOMORPHIC EXTENDIBILITY OF CR FUNCTIONS

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Introduction.

Let M be a smooth generic submanifold of $\mathbf{C}^n$. Several authors have studied the property of CR functions on M to extend locally to manifolds with boundary attached to M and holomorphically to generic wedges with edge M (*cf.* [14], [67], [68]). In a recent work ([69]), Tumanov has showed that CR-extendibility of CR functions on M propagates along curves that run in complex tangential directions to M. His main result appears as a natural generalization of results by Trépreau on propagation of singularities of CR functions ([61]). Indeed, Theorem 5.1 in [69] states that the direction of CR-extendibility moves parallelly with respect to a certain differential geometric partial connection in a quotient bundle of the normal bundle to M, and this variation is dual to the one introduced by Trépreau, according to Proposition 7.3 in [69].

In this paper we give a new and simplified presentation of the connection introduced in Tumanov's work. Let M be a real manifold and N a submanifold of M, K a subbundle of TM with the property that $K|_N \subset TN$. Then by means of the Lie bracket, we can define a K-partial connection on the normal bundle of N in M (Proposition 1.1). In general, the parallel translation associated with that partial connection will be induced by the flow of K-tangent sections of TM (Proposition 1.2). When M is a generic submanifold of $\mathbf{C}^n$ containing a CR submanifold S with the same CR dimension we recover in section 2 the T^cS-partial connection constructed by Tumanov in [69].

Recall that the *CR-orbit* of a point $z \in M$ is the set of points that can be reached by piecewise smooth integral curves of complex tangent vector fields. We then say that M is *globally minimal* at a point $z \in M$ if the CR-orbit of z contains a neighborhood of z in M. Using previous results, we show that vector space generated by the directions of CR-extendibility of CR functions on M exchanges by the induced composed flow between two points in a same CR-orbit (Lemma 3.5). As an application, we prove the main result of this paper, conjectured by Trépreau in [61] : *for wedge extendibility of CR functions to hold at every point in the CR-orbit of $z \in M$ it is sufficient that M be globally minimal at z* (Theorem 3.4). Up till now we can only conjecture the converse (for a local result, see [6]).

I wish to thank J.-M. Trépreau for helpful critical and simplifying remarks.

Remark : After this paper was completed, we have received a preprint by B. Jöricke *Deformation of CR-manifolds, minimal points and CR-manifolds with the microlocal analytic extension property*, which contains also a proof of Theorem 3.4 and Theorem 3.6. Our proof seems quite different since we obtain these results relying on Tumanov's propagation theorems, the generic manifold M being fixed, whereas B. Jöricke works with conic perturbations of the base manifold so as to produce minimal points.

§1. Partial connections associated with a system of vector fields.

Let M be a real differentiable manifold of class C^2 of dimension n and $H \to M$ a r-dimensional vector bundle over M. Recall that a connection ∇ on the bundle $H \to M$ is a bilinear mapping which assigns to each pair of a vector field X with domain U and a section η of H over U a section $\nabla_X \eta$ of H over U and satisfy

$$\nabla_{\phi X} = \phi \nabla_X, \quad \nabla_X(\phi\eta) = \phi\nabla_X\eta + (X\phi)\eta, \qquad \phi \in C^1(M, \mathbf{R}).$$

When the covariant derivative $\nabla_X \eta$ can only be defined for vectors X that belong to a subbundle K of TM, we call the connection ∇ a *K-partial* connection (cf. [69]).

If N is a submanifold of M, let $T_N M$ be the *normal bundle of N in M, i.e.*

$$T_N M \;=\; TM|_N / TN.$$

Proposition 1.1. *Let M be a real manifold of class C^2, $N \subset M$ a submanifold of class C^2 too and let K be a C^1 subbundle of TM with the property that $K|_N \subset TN$. Then there exists a natural K-partial connection ∇ on the bundle $T_N M$ which is defined as follows. If $x \in N, X \in K[x]$ and η is a local section of $T_N M$ over a neighborhood of x, then take*

$$\nabla_X \eta \;=\; [\tilde{X}, \tilde{Y}](x) \;\; mod\ T_x N$$

where $\tilde{X}$ is a C^1 local section of K extending X and $\tilde{Y}$ is a lifting of η in TM in a neighborhood of x.

Proof. We first check that the definition is independent of the lifting $\tilde{Y}$. In fact, when $\tilde{Y}$ is tangent to N, as $\tilde{X}$ is tangent to N too, the Lie bracket $[\tilde{X}, \tilde{Y}]$ remains tangent to N hence is zero in the quotient bundle.

Next we have to check that the definition of ∇ is independent of the chosen section $\tilde{X}$ or, to rephrase, that if $\tilde{X}(x) = 0$ then $[\tilde{X}, \tilde{Y}](x)$ belongs to $T_x N$. Since K is a fiber bundle we can write

$$\tilde{X} \;=\; \sum_{j=1}^{r} f_j \tilde{X}_j \qquad f_j(0) = 0 \qquad j = 1, ..., r$$

where $r = rank\ K$, $(\tilde{X}_j)_{j=1,\ldots,r}$ is a frame for K near x and the f_j are C^1 real valued functions defined near x. Noting that

$$[f\tilde{X}, \tilde{Y}] \;\equiv\; f[\tilde{X}, \tilde{Y}] - (\tilde{Y} f)\tilde{X} \;\equiv\; f[\tilde{X}, \tilde{Y}] \qquad mod\ TN$$

the result follows and the mapping ∇ is well-defined. Moreover the preceding implies that if $\phi \in C^1(M, \mathbf{R})$

$$\nabla_{\phi X} \eta \;\equiv\; \phi \nabla_X \eta.$$

Last, we check that ∇_X is a derivation. Indeed

$$\nabla_X(\phi\eta) \;\equiv\; [\tilde{X}, \phi\tilde{Y}](x) \;\equiv\; (\tilde{X}.\phi)\tilde{Y} + \phi[\tilde{X}, \tilde{Y}] \;\equiv\; (X\phi)\eta + \phi\nabla_X\eta$$

and the proof is complete.

With the connection ∇ it is associated the *parallel translation* of fibers of $T_N M$ along smooth curves on the base N that run in directions tangent to K. Let $I \ni t$ be a subinterval of $\mathbf{R}$ and $\gamma : I \to N$ be a smooth curve with the property that $\dot\gamma(t) \in K[\gamma(t)]$, where $\dot\gamma = \frac{d}{dt}\gamma(t)$. A curve $\eta(t) \in T_N M[\gamma(t)]$ is a *horizontal lift* of γ if $\nabla_{\dot\gamma}\eta = 0$. Existence and uniqueness of horizontal lifts provide linear isomorphisms

$$\Phi_{t_0,t} \quad : T_N M[\gamma(t_0)] \;\to\; T_N M[\gamma(t)]$$

obtained by moving elements of K along horizontal lifts of γ.

Recall (cf. [58]) that the Lie bracket $[\tilde X, \tilde Y]$ is defined as the Lie derivative $L_{\tilde X}\tilde Y$ of $\tilde Y$ with respect to $\tilde X$

$$[\tilde X, \tilde Y](x) \;=\; L_{\tilde X}\tilde Y \;=\; \lim_{h\to 0}[\tilde Y(x) - d\tilde X_{-h}(\tilde Y(\tilde X_h(x)))]$$

where $\tilde X_t$ is the local flow on M generated in a neighborhood of x by the vector field $\tilde X$, and $d\tilde X_t$ denotes its differential. In the assumptions of Proposition 1.1, $\tilde X$ is of class C^1 so the mapping $x \to \tilde X_t(x)$ is of class C^1 and the differential is a well-defined continuous mapping. When $\tilde X$ is K-tangent its flow (and more generally any piecewise smooth composition of such flows) stabilizes the tangent bundle TN of the manifold N, hence its differential induces isomorphisms of fibers of $T_N M$, which we denote by dX_t. Assume moreover that the curve γ is an integral curve of a C^1 K-tangent vector field $\tilde X$, (which cannot be true for most general smooth curves γ but is sufficient enough for the applications) : $\gamma(0) = x$ and $\gamma(t) \;=\; \tilde X_t(x)$. Then we claim that the mapping

$$dX_t \quad : \quad T_N M[x] \;\to\; T_N M[\gamma(t)]$$

provides the parallel translation $\Phi_{0,t}$. Indeed let $\eta_0 \in T_N M[x]$ and take $\eta(t) = dX_t(\eta_0)$. Then by the definition of the partial connection ∇ and the definition of the Lie bracket we have

$$\nabla_{\dot\gamma}\eta(t) \;\;\equiv\;\; 0.$$

By uniqueness of solutions of linear differential equations of order one it must be that

$$\eta(t) \quad = \Phi_{0,t}(\eta_0).$$

Proposition 1.2. *Under the hypotheses of Proposition 1.1, let $\gamma(t) = X_t(x_1)$ be a smooth (piecewise smooth) integral curve of a K-tangent vector field X (a finite number of K-tangent vector fields) running from $x_1 \in N$ to $x_2 \in N$. Then the parallel translation along γ associated with the K-partial connection ∇ is induced by the differential of the flow of X (composed flow).*

In order to give an expression of the covariant derivatives induced by the partial connection ∇, we choose coordinates on M, $x = (x', x'') \in \mathbf{R}^l \times \mathbf{R}^m$ such that the base point corresponds to $x = 0$ and the submanifold N is defined by the equation $x'' = 0$. Let $(x, \eta) = (x', x'', \eta', \eta'')$ be the canonical coordinates on TM, and $(x', \eta'') \in \mathbf{R}^l \times \mathbf{R}^m$ the associated coordinates on $T_N M$.

If $X = \sum_{j=1}^{l+m} a_j(x)\, \frac{\partial}{\partial x_j}$ is a C^1 section of K, it must be tangent to N, so $a_j(x', 0) = 0,\ j = l+1, ..., l+m$. We choose a local section η of $T_N M$ over a neighborhood of 0 in N, in fact a section $\tilde Y$ of TM of the form

$$\tilde Y = \sum_{j=l+1}^{l+m} \eta_j(x')\, \frac{\partial}{\partial x_j},$$

Recalling Proposition 1.1 we have the following expression for the covariant derivative of η in the direction of X

$$\nabla_X \eta = \sum_{j=l+1}^{l+m} \sum_{k=1}^{l} a_k(x',0)\ \frac{\partial \eta_j}{\partial x_k}(x')\ \frac{\partial}{\partial x_j}\ -\ \sum_{j,k=l+1}^{l+m} \eta_k(x')\ \frac{\partial a_j}{\partial x_k}(x',0)\ \frac{\partial}{\partial x_j}.$$

Given an integral curve $\gamma(t) = (\gamma'(t), 0)$ of the field X, the equations for the horizontal lifts look like

$$X.\eta_j = \dot{\eta}_j(t) = \sum_{k=l+1}^{k=l+m} \frac{\partial a_j}{\partial x_k}(\gamma'(t),0)\eta_k(t) \qquad j = l+1, ..., l+m$$

so the curve $(\gamma'(t), \eta''(t))$ is the integral curve of the following vector field $\check{X}$ on $T_N M$

$$\check{X}(x', \eta'') = \sum_{j=1}^{l} a_j(x',0)\frac{\partial}{\partial x_j}\ +\ \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x',0)\eta_k(x')\frac{\partial}{\partial \eta_j}.$$

Alternately, the partial connection ∇ can be defined by the family of horizontal subspaces $H(\eta) \subset T_\eta(T_N M)$ generated by vectors of the form $\check{X}$.

Let us consider the *dual connection* ∇^* to the connection ∇ on the dual bundle $T^*_N M$. Recall that the conormal bundle of N in M, $T^*_N M$, consists of forms in T^*M that vanish on TN. It has fiber over a point $x \in N$

$$T^*_N M[x] = \{\phi \in T^*_x M;\ \phi|_{T_x N} = 0\}.$$

The dual connection ∇^* is defined by the following relation : if X is a K-tangent vector to N at x, η is any section of $T_N M$ near x and ϕ is any section of $T^*_N M$

$$X < \phi, \eta > = < \nabla^*_X \phi, \eta > + < \phi, \nabla_X \eta > .$$

It is easily checked that such a relation defines a K-partial connection on $T^*_N M$.

Along with the coordinates on $T_N M$ we introduced before we can introduce the canonical coordinates (x', ξ'') on the conormal bundle $T^*_N M$. These are dual to the coordinates (x', η'') for the canonical duality $<,>$ between $T_N M$ and $T^*_N M$:

$$< \sum_{j=l+1}^{l+m} \xi_j dx_j\ ,\ \sum_{j=l+1}^{l+m} \eta_j \frac{\partial}{\partial x_j} > = \sum_{j=l+1}^{l+m} \xi_j \eta_j.$$

Using the previous definition of the dual connection we can then compute the covariant derivative of a section $\sum \xi_j dx_j = \phi$ of $T^*_N M$. One easily shows

$$\nabla^*_X \phi = \sum_{j=l+1}^{l+m} (X.\xi_j + \sum_{k=l+1}^{l+m} \xi_k \frac{\partial a_k}{\partial x_j})\ dx_j.$$

Hence, under the assumption of Proposition 1.2, the parallel translation associated with the connection ∇^* is given by means of the integral curves of the following vector field on $T^*_N M$

$$\hat{X} = \sum_{j=1}^{l} a_j(x',0)\frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x',0)\xi_j \frac{\partial}{\partial \xi_k}.$$

There is another way of thinking the connection ∇^* dual to the partial connection ∇ which has been considered by Trépreau in [61].

To a general vector field X on M it is associated its symbol $\sigma(X)$ which is an invariantly defined function on the cotangent bundle T^*M of M. To a function f of class C^1 on T^*M it is associated its hamiltonian field H_f.

Let $X_j,\ j=1,...,r$ be a local basis of K-tangent sections of TM. Let Σ_K be the orthogonal complement of K in T^*M. If $X=\sum_{j=1}^r \phi_j X_j$ is a C^1 section of K we have

$$H_{\sigma(X)}|_{\Sigma_K} \ = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K} + \sum_{j=1}^r \sigma(X_j) H_{\phi_j}|_{\Sigma_K}.$$

Since $\sigma(X_j),\ j=1,...,r$ is zero on Σ_K, we deduce that the restricted hamiltonian field

$$H_{\sigma(X)}|_{\Sigma_K} \ = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K}$$

depends only on the value of X at the base point and not on the chosen section. If X is tangent to N, $H_{\sigma(X)}$ when restricted to $T^*_N M$ is tangent to $T^*_N M$. Hence we have constructed another vector field on $T^*_N M$ which is in fact the same as the one associated with the connection dual to the partial connection ∇.

Indeed, let as before (x',ξ'') be the canonical coordinates on the conormal bundle $T^*_N M$. Recall that the hamiltonian field of a function $f=f(x,\xi)$ just looks like

$$H_f = \sum_{j,k=1}^{j,k=l+1} \frac{\partial f}{\partial \xi_k}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_k}.$$

The symbol of the section

$$X = \sum_{j=1}^{l+m} a_j \frac{\partial}{\partial x_j} \qquad a_j(x',0)=0,\ j=l+1,...,l+m$$

of K being $\sigma(X)=\sum a_j\xi_j$ we can compute

$$H_{\sigma(X)}|_{T^*_N M} = \sum_{j=1}^{l} a_j(x',0)\frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x',0)\xi_j \frac{\partial}{\partial \xi_k}$$

and the last expression proves that $H_{\sigma(X)}$ is the same vector field on $T^*_N M$ as $\hat{X}$ computed previously, so the set of restricted hamiltonian fields $H_{\sigma(X)}|_{T^*_N M}$ defines the same family of horizontal subspaces for the partial connection ∇^*.

The next section is devoted to the application of the preceding results to the geometry of CR submanifolds of $\mathbf{C}^n$.

§2. Application to generic submanifolds of $\mathbf{C}^n$

In this section we apply results of section 1 in the context of differential geometry in the complex euclidean space $\mathbf{C}^n$. Afterwards we check that our definitions recover those of Trépreau [61] and Tumanov [62].

Let $T\mathbf{C}^n$ be the real tangent bundle of $\mathbf{C}^n$ and J be the standard complex structure operator on $T\mathbf{C}^n$. Let $T^*\mathbf{C}^n$ be the bundle of *holomorphic* ($\mathbf{C}$-linear) 1-forms on $\mathbf{C}^n$. In the canonical coordinates $z=(z_1,...,z_n)$ its fiber over a point z consists of (1,0)-forms $\omega=\sum_{j=1}^n \zeta_j dz_j,\ \zeta_j\in\mathbf{C},\ j=1,...,n$. Then

$T^*\mathbf{C}^n$ is a *complex* manifold. It can be (and it is usually) identified with the real dual bundle of $T\mathbf{C}^n$ introducing the real duality defined by

$$(\omega, X) \in T^*\mathbf{C}^n \times T\mathbf{C}^n \qquad (\omega, X) \longmapsto Im \ < \omega, X > .$$

In other words we identify real and holomorphic forms by $Im\ \omega \leftrightarrow \ \omega$.

Now, let M be a real submanifold of $\mathbf{C}^n$. In this identification, the conormal bundle $T^*_M\mathbf{C}^n$ is a subbundle of $T^*\mathbf{C}^n$ and it has fiber spaces

$$T^*_M\mathbf{C}^n[z] \ = \ \{\omega \in T^*\mathbf{C}^n; \ Im\ \omega \ |_{T_zM} = 0 \ \}.$$

Hence the bundles $T_M\mathbf{C}^n = T\mathbf{C}^n|_M/TM$ and $T^*_M\mathbf{C}^n$ are in duality by

$$(\omega, X \ mod \ TM) \ \longmapsto \ \mathrm{Im} \ < \omega, X > .$$

Assume moreover that M is generic (that is $TM + JTM = T\mathbf{C}^n|_M$) and let Σ_M be the orthogonal complement of the complex tangent bundle T^cM in the cotangent bundle T^*M. In the terminology of linear partial differential equations it is the *characteristic set* (and since T^cM is a fiber bundle, the characteristic *manifold*) of the system of CR vector fields. It is easily checked that Σ_M and TM/T^cM are in duality in the same way.

Since M is CR, Σ_M is a fiber bundle and there is a canonical bundle epimorphism

$$\theta \ : \ T^*_M\mathbf{C}^n \ \to \ \Sigma_M,$$

defined by $\theta(\omega) = \imath^*_M\omega$ where $\imath_M : \ M \ \to \ \mathbf{C}^n$ is the natural injection. Since M is generic, θ is an isomorphism.

On the other hand the complex structure J induces an isomorphism, still denoted by J

$$J \ : \ TM/T^cM \ \to \ T_M\mathbf{C}^n.$$

Lemma 2.1. *θ is the transposed of J, i.e.*

$$(\omega, JX) \ = \ (\theta(\omega), X)$$

for every $\omega, \ X$.

(Indeed $< \omega, JX >= i < \omega, X >$).

From now on we let $S \subset M$ be a CR submanifold of M with the property that $CRdim\ S = CRdim\ M$. Equivalently it is required that $T^cS = T^cM|_S$. By restriction analogous pairs of bundles remain isomorphic when TM/T^cM is replaced by T_SM, Σ_M is replaced by T^*_SM, $T_M\mathbf{C}^n$ is replaced by $T\mathbf{C}^n|_S \ / \ (TM|_S + JTS) = E$, and $T^*_M\mathbf{C}^n$ is replaced by $T^*_M\mathbf{C}^n \cap iT^*_N\mathbf{C}^n = E^*$, but now T^cS-partial connections can be defined by means of the isomorphisms J and θ on the two new bundles E and E^*. Note that the duplication essentialy deals with complex differential geometry.

First, the results of the previous section apply with $K = T^cM = TM \cap JTM$ and $N = S$ and produce a T^cS-partial connection ∇ on T_SM together with the dual connection ∇^* on T^*_SM. On the other hand, the push forward by J of ∇ defines a T^cS-partial connection Θ on E ; its action on a section ϑ of E in the direction of a complex tangent vector X is simply

$$\Theta_X\vartheta \ \ := \ \ J \ \nabla_X \ (J^{-1} \ \vartheta).$$

Similarly, the pull-back of the T^cS-partial connection ∇^* by θ defines a T^cS-partial connection Θ^* on E^*, and Θ^* is the connection dual to Θ since θ is the transposed of J (lemma 2.1).

Recall from section 1 that if X is a section of T^cM then $\hat{X} = H_{\sigma(X)}|_{T^*_SM}$ is tangent to T^*_SM. In [61], Trépreau showed that E^* is a CR manifold, using a lemma which states that given such a vector field $\hat{X}$ tangent to T^*_SM with horizontal part X complex tangent to M, there exists a unique vector field $\tilde{X}$ complex tangent to E^* with the same horizontal part X. Moreover Trépreau states that

$$\hat{X} \ = \ d\theta(\tilde{X})$$

Hence we deduce that the T^cS-partial connection $\Theta^* \ = \ \theta^*\nabla^*$ can alternately be given, as is originally done in [61], by means of the vector fields of the form $\tilde{X}$, i.e. horizontal subspaces of Θ^* are spanned by tangent vectors to integral curves of $\tilde{X}$. We then have checked that the parallel translation in E^* introduced by Trépreau with the assumption of Proposition 1.2 is the same as the one associated with the T^cS-partial connection Θ^* previously defined starting, as in section 1, with the partial connection associated with the bundle of complex tangents to M, $K = T^cM$. Moreover, since $\tilde{X}$ is complex tangent to E^*, we see that T^cE^* is the set of horizontal subspaces for the T^cS-partial connection Θ^* . This has been noticed in [69] and will be usefull in the next section when proving Theorem 3.4.

§3. Orbits and the extension of CR functions.

In this section, it is assumed that M is a generic submanifold of $\mathbf{C}^n$ of smoothness class C^2, and we let $\mathbf{X}$ be the set of C^1 sections over open subsets of M of T^cM. If $z \in M$, the subset of M consisting of points of M which can be reached by piecewise C^1-smooth integral curves of elements of $\mathbf{X}$, starting at z, is called the *CR-orbit* of z, and is denoted by $\mathcal{O}\ [z]$.

If U is an open subset of M, $\mathbf{X}|_U$ denotes the set of elements of $\mathbf{X}$ restricted to U. It is well-known (*cf.* [59], [6], [61]) that

$$\varinjlim_{\mathrm{U}} \quad \mathcal{O}(\mathbf{X}|_{\mathcal{U}}, \ddagger)$$

where U runs over the open neighborhoods of z in M defines the germ at z of the unique CR-submanifold of M with the same CR dimension as M of minimal dimension passing through z, which is called the *local CR-orbit* of z and is denoted by $\mathcal{O}^{loc}\ [z]$. When considering $\mathcal{O}^{loc}\ [z]$ in the following we shall mean such a submanifold of a neighborhood of z in M, i.e. an actual representative of the germ. It plays the crucial role in the study of automatic extendibility of CR functions (*cf.* Theorem 3.1 below).

Recall that a smooth complex-valued function on M is called a *CR function* if it is annihilated by every antiholomorphic tangent vector field on M. A continuous function can be thought CR in the sense of distribution theory. We denote by $CR(M)$ the set of all continous CR functions on M.

For completeness we recall definitions from [61] and [69]. We say that a manifold $\tilde{M}$ with boundary is *attached to M at (m,u)*, $m \in M$, $u \neq 0, u \in T_M\mathbf{C}^n[m]$ if $b\tilde{M} \cap U = M \cap U$ for some neighborhood U of m, and u is represented by a vector $u_1 \in T_m\tilde{M}$ directed inside $\tilde{M}$.

Let f be a CR function on M; we say that f is *CR-extendible* at (m, u) if it extends continuously to be CR on some $\tilde{M}$ attached to M at (m, u). When there is a CR submanifold S of M through m and a manifold $\tilde{M}$ attached to M at (m, u), $u \in T_M\mathbf{C}^n[m]$, we also say that $\tilde{M}$ is attached to M at (m, η), if u represents $\eta \in E_m$, $\eta \neq 0$ (E is the bundle defined in section 2). Similarly it makes sense to consider CR-extendibility at (m, η), $m \in S, \eta \in E_m$. But it should be noted that given $\eta \neq 0$ in E_m does not determine $\tilde{M}$ unambiguously unless S is complex

From now on we will require that M belong to the class $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$. This regularity assumption can be justified since it behaves well when proving the strongest local results on CR-extendibility (In fact, it behaves well through the so-called *Bishop equation*, [68], Theorem 1.), and constructing wedges with ribs and an egde having such a regularity (cf. [4]). Moreover, we need manifolds of class at least C^2 in order to apply Proposition 1.1. Since it will be of use in the proof of Theorem 3.4 we recall the following theorem due to Tumanov ([68])

THEOREM 3.1. (A. E. TUMANOV) *Let M be a generic submanifold of $\mathbf{C}^n, n = p+q$, with dim $M = 2p+q$, CRdim $M = p$, and of smoothness class $C^{k,\alpha}$ $(k \geq 2), 0 < \alpha < 1$. For every point $z \in M$ there exist $r = r(z) = dim\ \mathcal{O}^{loc}_{[z]} - 2\ CRdimM$ manifolds with boundary $\tilde{M}_1, ..., \tilde{M}_r$ attached to M at z, of class $C^{(k,\beta)}$ whenever $0 < \beta < \alpha$ such that*

(a) *Every CR function on M is CR-extendible to* $\tilde{M}_1, ..., \tilde{M}_r$

(b) $\sum_{j=1}^{r} T_{z'}\tilde{M}_j = T_{z'}M + JT_{z'}\mathcal{O}^{loc}_{[z]}$ *z' close to z in* $\mathcal{O}^{loc}_{[z]}$.

Moreover the manifold germ $\mathcal{O}^{loc}_{[z]}$ is of class $C^{(k,\beta)}$ whenever $0 < \beta < \alpha$.

Note that $\mathcal{O}^{loc}_{[z]}$ is at least of class C^2 so it can play the role of N in Propositions 1 and 2. Using the connections constructed in section 2 we can reinterpret the main result on propagation of analyticity for CR functions recently proved by Tumanov.

According to Tumanov ([69], Proposition 7.3), the connection dual to the one that is constructed during the paper has the property that its horizontal subspaces are exactly fibers of the complex tangent bundle T^cE^*, hence, concludes Tumanov, the induced parallel translation need be the same as the one introduced on E^* by Trépreau. We have shown in section 2 that our connection Θ has as a dual connection a connection Θ^* with the same property; so $\Theta = J_*\nabla$ coincides with the connection constructed by Tumanov.

Proposition 1.2 together with Theorem 5.1 in [69] leads to

THEOREM 3.2. *Let $M \subset \mathbf{C}^n$ be a generic manifold and $S \subset M$ a CR submanifold of M with the property that CRdim S = CRdim M. Let γ be a piecewise smooth integral curve of T^cM running from $z' \in S$ to $z'' \in S$ and let Φ_γ be the associated composed flow. Then for every $\epsilon > 0$, every $\eta' \in E_{z'}$ and every manifold $\tilde{M}'$ attached to M at (z', η'), there exists another manifold $\tilde{M}'$ attached to M at (z'', η''), $\eta'' \in E_{z''}$ such that*

(a) $|\eta'' - Jd\Phi_\gamma(z).J^{-1}\eta'| < \epsilon$

(b) *if a CR function on M extends to be CR on $\tilde{M}'$ it extends to be CR on* $\tilde{M}''$

(c) *if M, $\tilde{M}'$ belong to $C^{k,\alpha}$ $(k \geq 2), 0 < \gamma < \alpha < 1$ then there exists such a* $\tilde{M}'' \in C^{(k,\gamma)}$.

Theorem 3.2 shows that the so-called propagation of analyticity for CR functions is intrinsically related to the geometry of the base manifold M. Moreover, it fundamentally means that the study of extendibility for CR functions is closely related to the study of sections of the complex tangent space to M.

Following Sussmann ([59]), we begin with some adapted terminology and recalls. Let $X \in \mathbf{X}$ be a local section of T^cM. The C^1 integral curves $t \to \gamma(t)$ of X generate local diffeomorphisms of M where they are defined (the so-called *flow* of X) which we will denote by $z \to X_t z$. Composites of several maps of the form X_t can produce local diffeomorphisms of neighborhoods of points that are *far* from each other in a same CR-orbit. If $X = (X_1, ..., X_m)$ is an element of $\mathbf{X}^m$ such that for $t = (t_1, ..., t_m) \in \mathbf{R}^m$, the map $z \to X_{m,t_m} \cdots X_{1,t_1} z$ is well defined in a neighborhood of z, we will still denote it for convenience by X_t or Φ (*cf.* Proposition 1.2).

Let $\Delta_{\mathbf{X}}$ be the *distribution spanned by* $\mathbf{X}$, i.e. the mapping which to $z \in M$ assigns the linear hull of vectors $X(z)$ where X belongs to $\mathbf{X}$: it is just the distribution associated with the complex tangent bundle of M. We let $P_{\mathbf{X}}$ denote the smallest distribution which contains $\Delta_{\mathbf{X}}$ and is invariant under complex-flow diffeomorphisms, or for short the smallest $\mathbf{X}$-*invariant* distribution which contains $\Delta_{\mathbf{X}}$. Precisely, $P_{\mathbf{X}}(z)$ is the linear hull of vectors of the form $dX_t(v)$ where $v \in \Delta_{\mathbf{X}}(z')$ and $z = X_t z'$. A C^1 distribution P on M has the *maximal integral manifold property* if for every $z \in M$ there exists a submanifold S of M such that $z \in S$ and for every $z' \in S$, $T_{z'}S = P(z')$. Moreover, S is said to be a *maximal integral manifold* of P if S is an integral manifold of P such that every connected integral manifold of P which intersects S is an open submanifold of S.

Then the results of Sussmann, which extend to the C^2 case tell us that $\mathcal{O}\ [z]$ is a (connected) maximal integral submanifold of $P_{\mathbf{X}}$ (perhaps with a finer topology) and admits a unique differentiable structure making the injection $i : \mathcal{O}\ [z] \to M$ an immersion of class C^1.

We now introduce the following definitions.

Definition 3.3. Let M be a generic submanifold of $\mathbf{C}^n$ and $z \in M$. M is called *minimal at z* if $\mathcal{O}^{loc}\ [z]$ contains a neighborhood of z in M. It is called *globally minimal at z* if $\mathcal{O}\ [z]$ contains a neighborhood of z.

In view of the global results of Sussmann definition 3.3 means that the generic manifold M is globally minimal at a point z if and only if there exist a finite number of points $z_l^{'}$, $l = 1, ..., d$ in the CR-orbit of z and composed flow diffeomorphisms $\Phi_l^{'}$, $l = 1, ..., d$ of a neighborhood of $z_l^{'}$ in M on a neighborhood of z in M respectively such that

$$T_z M \ = \sum_{l=1}^{d} d\Phi_l(z_l^{'}).\ (T^c_{z_l^{'}} M).$$

We are now able to prove the theorem conjectured by Trépreau in [61] which is the natural generalization of a celebrated theorem of Tumanov ([67]). Here is the substance of this paper.

Theorem 3.4. *Let M be a generic submanifold of $\mathbf{C}^n$ of smoothness class $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$ which is* globally minimal *at a point $z \in M$. Then for every z' in the CR-orbit of z there exists a wedge $\mathcal{W}$ of edge M at z' such that*

$(*)$ every CR function on M extends holomorphically into $\mathcal{W}$.

Proof. We shall make use of the following abuse of langage : we will say that a CR-function u is CR-extendible in the direction $v \in TM/T^cM[z]$ if it is in fact CR-extendible in the direction of Jv. Let us consider the set

$$H_z = Vect \quad \{v \in T_z M/T^c_z M;\ \ u\ is\ \mathrm{CR-extendible}\ at\ (z,v)\ \}$$

and its preimage under the natural surjection $\pi : TM \to TM/T^cM$

$$\hat{H}_z = \pi^{-1}\ (H_z)\ \subset\ T_z M.$$

Lemma 3.5. *Let X be a C^1 section of T^cM over a neighborhood of $z \in M$ and let Φ_t be the flow of X and $\Phi = \Phi_t$ for some t. Then, if $v \in T_z M$,*

$$v\ \in\ \hat{H}_z \qquad \Longleftrightarrow \qquad d\Phi(z).v\ \in\ \hat{H}_{\Phi(z)}.$$

Proof. Since the statement is a symetric and a transitive one we can assume that z and z' are so close that $z' := \Phi(z)$ is contained in a CR submanifold S of M with $CRdimS = CRdimM$ which is minimal at z (for instance take for S the local CR-orbit of z) and such that z' belongs to the boundary of the manifolds whose existence comes from Theorem 3.1. Hence

$$(*) \qquad \hat{H}_{z'}\ \supset\ T_{z'}S.$$

So if v belongs to T_zS there is nothing to add. On the other hand, if $\xi = pr_{T_SM}v \neq 0$ we apply the propagation result Theorem 3.2 and obtain that for every $\epsilon > 0$ u is CR-extendible at (z', ξ''), where ξ'' is ϵ-close in euclidean norm to $\xi' = d\Phi(x).\xi$; so letting ϵ decrease to zero, since every finite-dimensional vector space is closed, we have $\xi' \in pr_{T_SM}\ (H_{z'})$. Because of $(*)$ the indetermination on the specific representative of ξ' is removed whence

$$d\Phi(z).v \ \in \ \hat{H}_{z'}$$

and the lemma is proved.

END OF PROOF OF THEOREM 3.4. The global lemma 3.5 and the condition of global minimality implie immediately that

$$\hat{H}_{z'} \quad = \quad T_{z'}M$$

for every z' in the (global) CR-orbit of z. The conclusion follows by the edge-of-the-wedge theorem and the proof is complete.

Theorem 4.1 admits an obvious generalization which involves the concept of $\mathcal{W}_r$-*wedges*. Recall that a $\mathcal{W}_r$-wedge at z with edge M is locally the general intersection of a wedge of edge M at z and a generic manifold containing M as a submanifold of codimension r.

THEOREM 3.6. *Let M be a generic submanifold of $\mathbf{C}^n$ of smoothness class $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$, and let $r = dim\mathcal{O}\ [z]\ -2CRdim\ M$. Then for every z' in the CR-orbit of z, every γ with $0 < \gamma < \alpha$, there exists a $\mathcal{W}_r$-wedge $\mathcal{W}$ of edge M at z' and of smoothness class $C^{k,\gamma}$ such that*

$(*)$ every CR function on M extends to be CR on $\mathcal{W}$.

Moreover, the tangent space to $\mathcal{W}$ at z' spans $T_{z'}M + JT_{z'}\ \mathcal{O}\ [z]$.

PROOF. The same argument runs in proving that $\hat{H}_{z'}$ contains $T_{z'}M + JT_{z'}\ \mathcal{O}\ [z]$ and the conclusion then follows by the edge-of-the wedge theorem of Ayrapetyan ([4]), in the classes $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$.

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