

Exercice 1 - Consider the group of bijective affine transformations of $\mathbb{R}$

$$\text{Aff}(\mathbb{R}) = \{f_{a,b} : x \mapsto ax + b \mid (a, b) \in \mathbb{R}^* \times \mathbb{R}\}$$

endowed with the composition law.

1. Show that the group $\text{Aff}(\mathbb{R})$ has a Lie group structure.
2. Compute the Lie algebra of $\text{Aff}(\mathbb{R})$, that we denote by $\mathfrak{aff}(\mathbb{R})$.
Hint : One may write $\text{Aff}(\mathbb{R})$ as a subgroup of a matrix group.

Let V be a real vector space of dimension 2, and let $\{X, Y\}$ be a basis of V .

3. Show that any anticommutative bilinear map on V defines a Lie bracket.
4. Show that any 2-dimensional real Lie algebra is either commutative or isomorphic to $\mathfrak{aff}(\mathbb{R})$.

Solution.

1. The group $\text{Aff}(\mathbb{R})$ identifies with the smooth manifold $\mathbb{R}^* \times \mathbb{R}$ with smooth product and inverse expressions $f_{a,b} \circ f_{a',b'} = f_{aa',ab'+b}$ and $(f_{a,b})^{-1} = f_{a^{-1}, -b/a}$.
2. Following the hint, we find that $\text{Aff}(\mathbb{R})$ identifies, as a Lie group, with the matrix group which, by abuse of notation, we denote by

$$\text{Aff}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : (a, b) \in \mathbb{R}^* \times \mathbb{R} \right\}.$$

Its Lie algebra $\mathfrak{aff}(\mathbb{R})$ is its tangent space at the identity and is generated by the following matrices :

$$E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We check that $[E_{1,1}, E_{1,2}] = E_{1,2}$.

3. Recall what an antisymmetric bilinear form is :
 - $B(X, Y) = -B(Y, X)$ for all $X, Y \in V$;
 - $B(X, \cdot)$ and $B(\cdot, X)$ are linear for all $X \in V$.

It remains to prove the Jacobi identity : for all $X, Y, Z \in V$

$$B(X, B(Y, Z)) + B(Y, B(Z, X)) + B(Z, B(X, Y)) = 0.$$

We set $J(X, Y, Z) = B(X, B(Y, Z)) + B(Y, B(Z, X)) + B(Z, B(X, Y))$. This map is trilinear and alternating since it is cyclic and antisymmetric in X and Y . Since V is two dimensional, we get then that $J \in \Lambda^3 V^* \otimes V = \{0\}$.

4. If V is not abelian, there exist $A, B \in V$ such that $[A, B] \neq 0$. It follows that (A, B) is a basis of V , and we can write $[A, B] = \lambda A + \mu B$, assuming that $\lambda \neq 0$ (swapping A and B if necessary). We set $X = \frac{-1}{\lambda} B$ and $Y = \lambda A + \mu B$. We then find $[X, Y] = Y$.

Exercice 2 - Define the lower central series $(C_n)_{n \in \mathbb{N}}$ and the derived series $(D_n)_{n \in \mathbb{N}}$ of a Lie algebra $\mathfrak{g}$ by setting $C_0 = D_0 = \mathfrak{g}$ and, for any integer $n \geq 0$, $C_{n+1} = [C_n, \mathfrak{g}]$ and $D_{n+1} = [D_n, D_n]$.

Call $\mathfrak{g}$ nilpotent if there exists n such that $C_n = 0$; solvable if there exists n such that $D_n = 0$.
For each group below :

- (a) show it is a Lie (sub)-group ;
 - (b) tell if it is connected or not ;
 - (c) compute if possible its Lie algebra $\mathfrak{g}$;
 - (d) give a basis $\mathcal{B}$ for its Lie algebra $\mathfrak{g}$;
 - (e) compute brackets between elements in $\mathcal{B}$;
 - (f) is $\mathfrak{g}$ commutative (abelian) ? nilpotent ? solvable ?
1. $O_2(\mathbb{R}) = \{x \in GL_2(\mathbb{R}) \mid {}^t x x = I_2\}$ (also denoted by $O(2)$).
 2. $SL_2(\mathbb{R}) = \{x \in GL_2(\mathbb{R}) \mid \det x = 1\}$
 3. $SU(2) = \{x \in GL_2(\mathbb{C}) \mid {}^t \bar{x} x = I_2 \text{ \& } \det x = 1\}$
 4. $\text{Heis}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\}$.
 5. $\text{Sol}(3) = \mathbb{R} \ltimes \mathbb{R}^2$, with the product manifold structure, where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. (Realise $\text{Sol}(3)$ as an embedded Lie subgroup in $SL_3(\mathbb{R})$.)

Solution. Note that "commutative $\Rightarrow$ nilpotent" and "nilpotent $\Rightarrow$ solvable".

1. For the group of orthogonal matrices :

- (a) The group $O_2(\mathbb{R}) = O(2)$ is indeed a submanifold of $GL_2(\mathbb{R})$ of dimension one, thanks to the implicit function theorem. Indeed, we check that $\varphi : x \mapsto {}^t x x$ is a submersion from $GL_2(\mathbb{R})$ to the space of symmetric matrices, since $D_x \varphi(y) = {}^t ({}^t x y) + {}^t x y$. Moreover, the product and the inverse of matrices are smooth on $GL_2(\mathbb{R})$ and compatible with the equation ${}^t x x = I_2$.

- (b) This group is not connected since $\det : O_2(\mathbb{R}) \rightarrow \{-1, 1\}$ is surjective, as

$$\det \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1.$$

- (c) We differentiate the implicit equation and find the set of antisymmetric matrices :

$$\mathfrak{o}_2(\mathbb{R}) = \{x \in \mathcal{M}_2(\mathbb{R}) : {}^t x + x = 0\} = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

$$(d) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- (e), (f) This is a commutative Lie algebra.

2. For the special linear group :

- (a) $SL(2, \mathbb{R})$ is indeed a group since $\det$ is a multiplicative morphism. The Lie group structure follows from the implicit function theorem by noting that $D_x \det(y) = \det(x) \text{Tr}(x^{-1}y)$ is a nonzero linear form. Therefore $SL(2, \mathbb{R})$ has real dimension 3.
- (b) It is connected since every element of the group can be written as a product of transvection matrices of the form $I_2 + \lambda E_{i,j}$ with $i \neq j \in \{1, 2\}$ and $\lambda \in \mathbb{R}$. One can also use the polar decomposition of $GL_2(\mathbb{R})^+ \simeq SO_2(\mathbb{R}) \times \text{Sym}_2^{++}$ (with Sym_2^{++} the set of positive definite symmetric 2×2 matrices) or the « $Q \times R$ » decomposition (orthogonal $\times$ right triangular matrices).

(c) We differentiate the implicit equation at I_2 . We have $D_{I_2} \det = \text{Tr}$ and hence $\mathfrak{sl}(2, \mathbb{R}) = \{x \in \mathcal{M}_2(\mathbb{R}) : \text{Tr}(x) = 0\}$.

(d) A choice of basis is $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(e) We have $[H, E] = 2E$, then $[H, F] = -2F$ and $[E, F] = H$.

(f) $[\mathfrak{sl}(2, \mathbb{R}), \mathfrak{sl}(2, \mathbb{R})] = \mathfrak{sl}(2, \mathbb{R})$ hence none of the three.

3. For the special unitary group :

(a) This is a group since the equations are compatible with matrix multiplication and inversion. For the Lie group structure, we compute the differential of $\varphi : x \mapsto (x^*x, \det x)$. We find that the kernel of $D_x \varphi(y) = (x^*y + y^*x, \det(x) \text{Tr}(x^{-1}y))$ has constant dimension 3 (x^*y is anti-Hermitian with zero trace). By the implicit function theorem, $\text{SU}(2)$ is a Lie group of dimension 3.

(b) For connectedness, we show that

$$\text{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \text{ with } |a|^2 + |b|^2 = 1 \right\},$$

hence this Lie group is diffeomorphic to the real 3-dimensional sphere $\mathbb{S}^3$, which is indeed connected. Namely, $x^*(\begin{smallmatrix} = \\ t\bar{x} \end{smallmatrix}) = x^{-1}$ with $x = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ implies that

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \frac{1}{\det x} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Hence $d = \bar{a}$ and $c = -\bar{b}$, and then $\det x = |a|^2 + |b|^2 = 1$.

(c) We differentiate the implicit equations at Id ,

$$\mathfrak{su}(2) = \{M \in \mathcal{M}_2(\mathbb{C}) : M^* + M = 0, \text{Tr}(M) = 0\}.$$

Hence

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} it & -\bar{b} \\ b & -it \end{pmatrix} : (t, b) \in \mathbb{R} \times \mathbb{C} \right\}.$$

(d) We adopt the notation of the following exercise, which gives the basis

$$E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(e) We find that $[E_1, E_2] = E_3$ and $[E_3, E_1] = E_2$ and $[E_2, E_3] = E_1$.

(f) None of the three since $[\mathfrak{su}(2), \mathfrak{su}(2)] = \mathfrak{su}(2)$.

4. For the 3-dimensional Heisenberg group :

(a) This is indeed a real 3-dimensional matrix group, the whole chart being $\mathbb{R}^3$, so it is a manifold.

(b) Connected.

(c) $\mathfrak{heis}(3)$ is the set of upper triangular matrices with zero diagonal coefficients.

(d) Computing the partial derivatives at the identity with respect to x, y then z , we get for a basis

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ (0) & & 0 \end{pmatrix} \text{ and } E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ (0) & & 0 \end{pmatrix} \text{ and } E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ (0) & & 0 \end{pmatrix}.$$

(e) $[E_{12}, E_{23}] = E_{13}$ then $[E_{12}, E_{13}] = 0$ and $[E_{23}, E_{13}] = 0$.

(f) Hence $[\mathfrak{heis}(3), \mathfrak{heis}(3)] = \mathbb{R}E_{13}$ and $C_1 = D_1 = \mathbb{R}E_{13}$ then $D_2 = C_2 = 0$. This is a non-commutative Lie algebra, but nilpotent and solvable.

5. For $\text{Sol}(3)$:

(a) This is a real 3-dimensional manifold since

$$\text{Sol}(3) = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ (0) & & 1 \end{pmatrix} : t, x, y \in \mathbb{R} \right\}.$$

(b) Path-connected.

$$(c) \mathfrak{sol}(3) = \left\{ \begin{pmatrix} t & & x \\ & -t & y \\ (0) & & 0 \end{pmatrix} : t, x, y \in \mathbb{R} \right\}.$$

(d) We compute the partial derivatives with respect to the three variables t, x, y and set

$$T = \begin{pmatrix} 1 & & (0) \\ & -1 & \\ (0) & & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & \\ (0) & & 0 \end{pmatrix}, Y = \begin{pmatrix} (0) & 0 \\ & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(e) We have $[T, X] = X$ then $[T, Y] = -Y$ and $[X, Y] = 0$.

(f) So $[\mathfrak{sol}(3), \mathfrak{sol}(3)] = \mathbb{R}X \oplus \mathbb{R}Y = D_1 = C_1$. Then $C_2 = [C_1, \mathfrak{sol}(3)] = C_1$ and $[D_1, D_1] = 0$. So $\mathfrak{sol}(3)$ is solvable but not nilpotent.

Exercise 3 - Isomorphisms $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ and $\mathfrak{sl}_2(\mathbb{R}) \simeq \mathfrak{so}(1, 2)$.

Recall the following definitions of classical Lie groups

$$\begin{aligned} \text{SO}(3) &= \{x \in \text{GL}_3(\mathbb{R}) : {}^t x x = I_3 \text{ \& } \det x = 1\} \\ \text{SO}(2, 1) &= \left\{ x \in \text{GL}_3(\mathbb{R}) : {}^t x \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix} x = \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix} \text{ \& } \det x = 1 \right\} \end{aligned}$$

1. Check that $\text{SU}(2)$ consists in the matrices of the form $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, with $|a|^2 + |b|^2 = 1$.

Deduce that $\text{SU}(2)$ is diffeomorphic to a well-known 3-manifold.

2. Show that $E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $E_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $E_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

form a basis of its Lie algebra $\mathfrak{su}(2)$.

What are the bracket relations between E_1 , E_2 and E_3 .

On $\mathfrak{su}(2)$, consider the quadratic form $Q(M) = \det(M)$.

3. Let $\varphi : \mathfrak{su}(2) \rightarrow \mathbb{R}^3$ give the coordinates of $M \in \mathfrak{su}(2)$ in the basis $\{2E_1, 2E_2, 2E_3\}$.

What quadratic form on $\mathbb{R}^3$ corresponds to Q via φ ?

4. Deduce that the adjoint representation naturally defines a morphism $\rho : \text{SU}(2) \rightarrow \text{SO}(3)$.

5. Show that ρ is a submersion, then that it is a surjective morphism.

What is its kernel?

6. Deduce from the above that $\text{SO}(3)$ is diffeomorphic to a well-known 3-manifold and that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

7. Show that H , E and F form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.
Compute the brackets between H , E , F .

Given a finite-dimensional real Lie algebra $\mathfrak{g}$, its Killing form is the symmetric bilinear form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ defined by $B_{\mathfrak{g}}(x, y) = \text{Tr}(\text{ad } x \circ \text{ad } y)$.

8. Show that a Lie algebra isomorphism sends Killing form to Killing form.
9. Denote by B the Killing form on $\mathfrak{sl}_2(\mathbb{R})$.
Find the matrix of B in the basis H , E , F .
What is its signature?
10. Determine the signature of the Killing form of $\mathfrak{su}(2)$.
Deduce that the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbb{R})$ are not isomorphic.
11. Adapt the above to show that the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(2, 1)$ are isomorphic.

Solution.

1. We have $x^* = x^{-1}$ for $x = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ iff

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \frac{1}{\det x} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Hence $d = \bar{a}$ and $c = -\bar{b}$, and then $\det x = |a|^2 + |b|^2 = 1$. So this Lie group is diffeomorphic to the real 3-dimensional sphere $\mathbb{S}^3$.

2. By differentiating the implicit equations (see previous exercise), we obtain

$$\begin{aligned} \mathfrak{su}(2) &= \{X \in \mathcal{M}_2(\mathbb{C}) : {}^t\bar{X} + X = 0, \text{Tr}(X) = 0\} \\ &= \left\{ X = \begin{pmatrix} it & x + iy \\ -x + iy & -it \end{pmatrix}, (t, x, y) \in \mathbb{R}^3 \right\} \\ &= \{X = 2tE_1 + 2xE_2 + 2yE_3, (t, x, y) \in \mathbb{R}^3\}. \end{aligned}$$

Hence the free family E_1, E_2, E_3 is a basis of $\mathfrak{su}(2)$.

We find that $[E_1, E_2] = E_3$ and $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$.

3. From above, one has $Q(X) = \det X = t^2 + x^2 + y^2$ for $X = 2tE_1 + 2xE_2 + 2yE_3$. Thus, in the basis $B = (2E_1, 2E_2, 2E_3)$, we have that $\text{Mat}_B(Q) = I_3$. In other words, $Q \circ \varphi^{-1}(t, x, y) = t^2 + x^2 + y^2$.
4. For all $g \in \text{SU}(2)$ and $X \in \mathfrak{su}(2)$, we have

$$Q(\text{Ad}(g)X) = \det(gXg^{-1}) = \det(X) = Q(X).$$

So we have a morphism

$$\begin{aligned} \rho : \text{SU}(2) &\longrightarrow \text{SO}(Q) = \text{SO}(3) \\ g &\longmapsto \text{Ad}(g). \end{aligned}$$

Notice that $\text{Ad}(g)$ does belong to $\text{SO}(3)$ (and not only to $\text{O}(3)$) by connectedness of $\text{SU}(2)$.

5. $D_{I_2}\rho = \text{ad} : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ is an isomorphism by injectivity (that one can prove geometrically or using the explicit expressions in question 10 below) and since the real dimensions of $\mathfrak{su}(2)$

and $\mathfrak{so}(3)$ are the same. Since it is a group morphism, we deduce that $D_g\rho$ is an isomorphism for every $g \in \mathrm{SU}(2)$. The image of ρ is open, and every morphism of connected topological groups whose image is open is surjective (we can also argue here that the image of ρ is compact, hence closed). The kernel is $\ker \rho = \{\pm I_2\}$.

6. $D_e\rho$ is an isomorphism of Lie algebras and $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{S}^3/(-x \sim x) = \mathbb{P}\mathbb{R}^3$.
7. This is done in the previous exercise with $[H, E] = 2E$; $[H, F] = -2F$ and $[E, F] = H$.
8. We must compute $B_{\mathfrak{g}}(\varphi X, \varphi Y) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}} \varphi(X) \circ \mathrm{ad}_{\mathfrak{g}} \varphi(Y))$ for all $X, Y \in \mathfrak{h}$ and $\varphi : \mathfrak{h} \rightarrow \mathfrak{g}$. Now, by the Lie algebra morphism property, $\mathrm{ad}_{\mathfrak{g}} \varphi(X) = \varphi \circ \mathrm{ad}_{\mathfrak{h}}(X) \circ \varphi^{-1}$. Hence

$$B_{\mathfrak{g}}(\varphi X, \varphi Y) = \mathrm{Tr}(\varphi \circ \mathrm{ad}_{\mathfrak{h}} X \circ \mathrm{ad}_{\mathfrak{h}} Y \circ \varphi^{-1}) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{h}} X \circ \mathrm{ad}_{\mathfrak{h}} Y) = B_{\mathfrak{h}}(X, Y).$$

9. We find

$$\mathrm{ad}(H) = \begin{pmatrix} 0 & (0) \\ & 2 \\ (0) & & -2 \end{pmatrix}, \quad \mathrm{ad}(E) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(F) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

We represent these maps in the ordered basis (H, E, F) . The Killing form is therefore given by

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}.$$

It has signature $(+, +, -)$.

10. In the ordered basis (E_1, E_2, E_3) we have

$$\mathrm{ad}(E_1) = \begin{pmatrix} (0) & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}(E_2) = \begin{pmatrix} 0 & & 1 \\ & (0) & \\ -1 & & 0 \end{pmatrix}, \quad \mathrm{ad}(E_3) = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & (0) \end{pmatrix}.$$

We then check that $B(E_1, E_2) = B(E_1, E_3) = B(E_2, E_3) = 0$ and $B(E_1, E_1) = B(E_2, E_2) = B(E_3, E_3) = -2$. The Killing form is negative definite on $\mathfrak{su}(2)$. The Killing forms have different signatures, so by contraposition the Lie algebras are not isomorphic.

11. Set $\varphi : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathbb{R}^{2,1} = (\mathbb{R}^3, B)$ where we write the elements of $\mathfrak{sl}_2(\mathbb{R})$ in the basis H, E, F . The adjoint representation preserves the Killing form, so $\rho(g) = \varphi \circ \mathrm{Ad}(g) \circ \varphi^{-1} \in \mathrm{O}(1, 2)$ for all $g \in \mathrm{SL}(2, \mathbb{R})$. So ρ is a Lie group morphism, and $\mathfrak{sl}_2(\mathbb{R})$ is isomorphic to $\mathfrak{o}(2, 1)$. We must note that $\mathfrak{o}(2, 1) = \mathfrak{so}(2, 1)$, since the identity connected component of $\mathrm{O}(2, 1)$ is contained in $\mathrm{SO}(2, 1)$.