

Exercice 1 - Consider the group of bijective affine transformations of $\mathbb{R}$

$$\text{Aff}(\mathbb{R}) = \{f_{a,b} : x \mapsto ax + b \mid (a,b) \in \mathbb{R}^* \times \mathbb{R}\}$$

endowed with the composition law.

1. Show that the group $\text{Aff}(\mathbb{R})$ has a Lie group structure.
2. Compute the Lie algebra of $\text{Aff}(\mathbb{R})$, that we denote by $\mathfrak{aff}(\mathbb{R})$.

Hint : One may write $\text{Aff}(\mathbb{R})$ as a subgroup of a matrix group.

Let V be a real vector space of dimension 2, and let $\{X, Y\}$ be a basis of V .

3. Show that any anticommutative bilinear map on V defines a Lie bracket.
4. Show that any 2-dimensional real Lie algebra is either commutative or isomorphic to $\mathfrak{aff}(\mathbb{R})$.

Exercice 2 - Define the lower central series $(C_n)_{n \in \mathbb{N}}$ and the derived series $(D_n)_{n \in \mathbb{N}}$ of a Lie algebra $\mathfrak{g}$ by setting $C_0 = D_0 = \mathfrak{g}$ and, for any integer $n \geq 0$, $C_{n+1} = [C_n, \mathfrak{g}]$ and $D_{n+1} = [D_n, D_n]$.

Call $\mathfrak{g}$ nilpotent if there exists n such that $C_n = 0$; solvable if there exists n such that $D_n = 0$.
For each group below :

- (a) show it is a Lie (sub)-group;
- (b) tell if it is connected or not;
- (c) compute if possible its Lie algebra $\mathfrak{g}$;
- (d) give a basis $\mathcal{B}$ for its Lie algebra $\mathfrak{g}$;
- (e) compute brackets between elements in $\mathcal{B}$;
- (f) is $\mathfrak{g}$ commutative (abelian)? nilpotent? solvable?

$$1. \text{O}_2(\mathbb{R}) = \{x \in \text{GL}_2(\mathbb{R}) \mid {}^t x x = I_2\} \text{ (also denoted by } \text{O}(2)).$$

$$2. \text{SL}_2(\mathbb{R}) = \{x \in \text{GL}_2(\mathbb{R}) \mid \det x = 1\}$$

$$3. \text{SU}(2) = \{x \in \text{GL}_2(\mathbb{C}) \mid {}^t \bar{x} x = I_2 \text{ \& } \det x = 1\}$$

$$4. \text{Heis}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\}.$$

$$5. \text{Sol}(3) = \mathbb{R} \ltimes \mathbb{R}^2, \text{ with the product manifold structure, where } t \in \mathbb{R} \text{ acts on } \mathbb{R}^2 \\ \text{via the matrix } \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \text{ (Realise } \text{Sol}(3) \text{ as an embedded Lie subgroup in } \text{SL}_3(\mathbb{R}).)$$

Exercice 3 - Isomorphisms $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ and $\mathfrak{sl}_2(\mathbb{R}) \simeq \mathfrak{so}(1, 2)$.

Recall the following definitions of classical Lie groups

$$\text{SO}(3) = \{x \in \text{GL}_3(\mathbb{R}) : {}^t x x = I_3 \text{ \& } \det x = 1\}$$

$$\text{SO}(2, 1) = \left\{ x \in \text{GL}_3(\mathbb{R}) : {}^t x \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix} x = \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix} \text{ \& } \det x = 1 \right\}$$

1. Check that $SU(2)$ consists in the matrices of the form $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, with $|a|^2 + |b|^2 = 1$.

Deduce that $SU(2)$ is diffeomorphic to a well-known 3-manifold.

2. Show that $E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $E_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $E_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

form a basis of its Lie algebra $\mathfrak{su}(2)$.

What are the bracket relations between E_1 , E_2 and E_3 .

On $\mathfrak{su}(2)$, consider the quadratic form $Q(M) = \det(M)$.

3. Let $\varphi : \mathfrak{su}(2) \rightarrow \mathbb{R}^3$ give the coordinates of $M \in \mathfrak{su}(2)$ in the basis $\{2E_1, 2E_2, 2E_3\}$.

What quadratic form on $\mathbb{R}^3$ corresponds to Q via φ ?

4. Deduce that the adjoint representation naturally defines a morphism $\rho : SU(2) \rightarrow SO(3)$.

5. Show that ρ is a submersion, then that it is a surjective morphism.

What is its kernel?

6. Deduce from the above that $SO(3)$ is diffeomorphic to a well-known 3-manifold and that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

7. Show that H , E and F form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

Compute the brackets between H , E , F .

Given a finite-dimensional real Lie algebra $\mathfrak{g}$, its Killing form is the symmetric bilinear form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ defined by $B_{\mathfrak{g}}(x, y) = \text{Tr}(\text{ad } x \circ \text{ad } y)$.

8. Show that a Lie algebra isomorphism sends Killing form to Killing form.

9. Denote by B the Killing form on $\mathfrak{sl}_2(\mathbb{R})$.

Find the matrix of B in the basis H , E , F .

What is its signature?

10. Determine the signature of the Killing form of $\mathfrak{su}(2)$.

Deduce that the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbb{R})$ are not isomorphic.

11. Adapt the above to show that the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(1, 2)$ are isomorphic.