

About the exponential map

Exercise 1 - Let G be a real Lie group and e its identity element.

Prove that there exists a neighbourhood U of e in G such that if H is a subgroup of G contained in U , then $H = \{e\}$.

Solution. We consider a norm $\|\cdot\|$ on $\mathfrak{g}$ and $r > 0$ small enough such that $\exp$ realises a diffeomorphism from $B(0, 2r)$ to a neighbourhood V of e in G . We set $U = \exp(B(0, r))$.

Suppose that $H \subset U$ is a subgroup of G . For every $g \in H$, there exists $x \in B(0, r)$ such that $g = \exp(x)$. We then have $g^2 = \exp(2x) \in H \subset U = \exp(B(0, r))$ with $2x$ in $B(0, 2r)$. By injectivity of $\exp$ on $B(0, 2r)$, we then have $2x \in B(0, r)$. Hence $2^n x \in B(0, r)$ for every n . Therefore $x = 0$ and $g = e$.

Exercise 2 - Let G and H be Lie groups. Assume that G is connected.

1. Consider $\varphi, \psi : G \rightarrow H$ two Lie group morphisms. Assume that there exists $g_0 \in G$ such that $\varphi(g_0) = \psi(g_0)$ and $T_{g_0}\varphi = T_{g_0}\psi$.

Prove that $T_e\varphi = T_e\psi$, then that φ and ψ coincide on a neighbourhood of g_0 . Finally, conclude that $\varphi = \psi$.

Solution. By group morphisms,

$$\varphi(g) = \varphi(g_0)^{-1}\varphi(g_0g) \quad \text{and} \quad \psi(g) = \psi(g_0)^{-1}\psi(g_0g).$$

We differentiate at e using the chain rule

$$T_e\varphi = T_{\varphi(g_0)}L_{\varphi(g_0)^{-1}} \circ T_{g_0}\varphi \circ T_eL_{g_0} = T_{\varphi(g_0)}L_{\psi(g_0)^{-1}} \circ T_{g_0}\psi \circ T_eL_{g_0} = T_e\psi.$$

We choose neighbourhoods U_G of e in G and $B_{\mathfrak{g}}$ of 0 in $\mathfrak{g}$ such that $\exp : B_{\mathfrak{g}} \rightarrow U_G$ is a diffeomorphism onto its image. We do the same for $U_H \subset H$ and $B_{\mathfrak{h}} \subset \mathfrak{h}$. Since they are group morphisms, for every $X \in B_{\mathfrak{g}}$,

$$\varphi(\exp_G(X)) = \exp_H(T_e\varphi(X)) = \psi(\exp_G(X)).$$

So $\varphi|_{U_G} \equiv \psi|_{U_G}$. These are group morphisms, so they also coincide on $\langle U_G \rangle = G$ by connectedness.

We can also say that the set $\{g \in G \mid \varphi(g) = \psi(g)\}$ is open (this being true around e and then translates everywhere by group morphism), and it is also closed, so it equals G by connectedness.

2. Let φ be a C^∞ diffeomorphism of G , such that for every left-invariant vector field X , we have $\varphi^*X = X$. Prove that there exists $g_0 \in G$ such that $\varphi = L_{g_0}$.

Solution. Caution : for question 2, one can no longer use the compatibility of φ with the group laws. These two exercises are independent but involve the same objects. We replace the commutation argument with $\exp$ by a hypothesis on invariant vector fields.

Up to working with $\tilde{\varphi} = L_{\varphi(e)^{-1}} \circ \varphi$, we may assume that $\varphi(e) = e$. Indeed, for every G -invariant vector field $X \in \mathcal{X}(G)$, if $\varphi^*X = X$ then for $h = \varphi(e)^{-1}$,

$$(L_h \circ \varphi)^*X = \varphi^*L_h^*X = \varphi^*X = X.$$

Let $\overline{X}$ be a left invariant vector field. Since $\varphi^*(\overline{X}) = \overline{X}$, one has at any g , $(T_{\varphi(g)}\varphi^{-1})(\overline{X}(\varphi(g))) = \overline{X}(g)$, so that

$$\overline{X}(\varphi(g)) = (T_g\varphi)(\overline{X}(g)). \quad (1)$$

We want to prove that $\varphi \equiv id_G$. We will first show a commutation formula with the exponential.

For $X \in \mathfrak{g}$ and $\overline{X}$ the associated left invariant vector field, one has

$$\frac{d}{dt}\varphi(\exp(tX)) = T_{\varphi(\exp(tX))}(\overline{X}(\exp(tX))) = \overline{X}(\varphi(\exp(tX)))$$

by (1). Since $\varphi(e) = e$, one has by Cauchy-Lipschitz that $\varphi(\exp(tX)) = \exp(tX)$ for every $t \in \mathbb{R}$. We deduce that φ is the identity on a neighbourhood of e .

By considering $L_{g^{-1}} \circ \varphi \circ L_g$, we show that $\{g \in G | \varphi(g) = g\}$ is open. Since this set is closed and contains e , it equals G by connectedness.

Universal cover of the group $SL_2(\mathbb{R})$

Exercise 3 - Recall that $SL_2(\mathbb{R})$ is the Lie group of 2×2 matrices of determinant $+1$ with real entries. Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be the basis of $\mathfrak{sl}_2(\mathbb{R})$ used in exercise 3 sheet 1.

1. Prove that every one-parameter subgroup $(g^t)_{t \in \mathbb{R}}$ of $SL_2(\mathbb{R})$ is conjugated in $SL_2(\mathbb{R})$ to one of the following three subgroups for some $a \in \mathbb{R}$:

$$\left(s^t = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} \right)_{t \in \mathbb{R}}, \quad \left(u^t = \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} \right)_{t \in \mathbb{R}}, \quad \left(r^t = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix} \right)_{t \in \mathbb{R}}.$$

If a is nonzero, $(g^t)_{t \in \mathbb{R}}$ is called *hyperbolic* in the first case, *parabolic* in the second case, *elliptic* otherwise.

Solution.

We start by writing $g^t = \exp(tX)$ for $X \in \mathfrak{sl}_2(\mathbb{R})$. It thus suffices to show that there exists $a \in \mathbb{R}$ such that the zero trace X is conjugated in $SL_2(\mathbb{R})$ to $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$.

First case. If X has opposite real nonzero eigenvalues a and $-a$, then X is diagonalisable in $\mathbb{R}$, and there exists $g \in GL_2(\mathbb{R})$ such that $gXg^{-1} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. Then $h = \begin{pmatrix} 1/\det g & 0 \\ 0 & 1 \end{pmatrix} g \in$

$SL(2, \mathbb{R})$ and satisfies $hXh^{-1} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

Second case. X has a single (double) 0 eigenvalue and is then nilpotent. In this case, there exists $g \in GL_2(\mathbb{R})$ such that $gXg^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then taking h as above, one finds that

$$hXh^{-1} = \begin{pmatrix} 0 & \det g \\ 0 & 0 \end{pmatrix}.$$

Third case. X has opposite imaginary eigenvalues ia and $-ia$ with $a > 0$. Then $X^2 = -a^2 \text{id}$ and there exists $g \in GL_2(\mathbb{R})$ such that $gXg^{-1} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$.

If $\det g > 0$, $h = \frac{1}{\sqrt{\det g}}g \in SL_2(\mathbb{R})$ satisfies $hXh^{-1} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$. Otherwise, one first left multiplies g by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to get the result, up to replacing a into $-a$.

2. If $g^t = \exp(tZ)$ with $Z \in \mathfrak{sl}_2(\mathbb{R})$ for every $t \in \mathbb{R}$, relate the hyperbolic, parabolic or elliptic nature of $(g^t)_{t \in \mathbb{R}}$ to the sign of $B(Z, Z)$, where B is the Killing form of $\mathfrak{sl}_2(\mathbb{R})$ defined for every $X, Y \in \mathfrak{sl}_2(\mathbb{R})$ by

$$B(X, Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)).$$

Solution. Using the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{R})$, one obtains that $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. The computation of the Killing form B of $\mathfrak{sl}_2(\mathbb{R})$ then gives

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}.$$

In the case where $\exp(tX)$ is conjugate to s^t , then X is conjugated to aH , and one has

$$B(X, X) = B(aH, aH) = 8a^2 > 0.$$

If $\exp(tX)$ is conjugate to u^t , then X is conjugated to aE and

$$B(X, X) = a^2 B(E, E) = 0.$$

Finally, if $\exp(tX)$ is conjugate to r_t , then X is conjugated to $a(F - E)$, and

$$B(X, X) = a^2 B(F - E, F - E) = a^2 (B(F, F) + B(E, E) - 2B(F, E)) = -8a^2 < 0.$$

We can therefore distinguish the three situations by the sign of $B(X, X)$.

3. Is the map $\exp : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \text{SL}_2(\mathbb{R})$ surjective? If not, describe its image.

Solution. No, since every element in the image of the exponential belongs to a one-parameter subgroup, but $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not conjugate to any of the examples above. The same holds for all elements conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda < 0$. One sees that the image of $\exp$ consists in matrices $M \in \text{SL}_2(\mathbb{R})$ such that $\text{Tr}(M) > -2$ or $M = -\text{id}$.

Remark. Note that $\text{SL}_2(\mathbb{R})$ is generated by any neighbourhood of id . That means that you can reach any element using pieces of one parameter subgroups (by a kind of « zigzag » curve). But you can't reach matrices that are not in the image of exponential in « one shot » (in « straight line » $\exp(tX)$ for some $X \in \mathfrak{sl}_2(\mathbb{R})$).

4. Prove that $\text{SL}_2(\mathbb{R})$ is homeomorphic to the product $\mathbb{S}^1 \times \mathbb{R}^2$.

Solution. By the « $Q \times R$ » (smooth) decomposition of $\text{GL}(2, \mathbb{R})$ (coming for instance from Gram-Schmidt orthogonalisation method), one has $\text{GL}(2, \mathbb{R})^+ \simeq \text{SO}(2) \times R$ where R is the space of right 2×2 triangular matrices with strictly positive entries on the diagonal. Restricting to $\text{SL}_2(\mathbb{R})$ this homeomorphism only leads to matrices of type $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $a > 0$ and $b \in \mathbb{R}$. This produces the homeomorphism with $\mathbb{S}^1 \times \mathbb{R}^2$.

Denote by $\widetilde{\pi} : \widetilde{\text{SL}_2(\mathbb{R})} \rightarrow \text{SL}_2(\mathbb{R})$ the universal cover of $\text{SL}_2(\mathbb{R})$, and by $\widetilde{\exp}$ the exponential map from $\widetilde{\mathfrak{sl}_2(\mathbb{R})}$ to $\widetilde{\text{SL}_2(\mathbb{R})}$.

5. What is the relationship between $\exp$, $\widetilde{\exp}$ and $\tilde{\pi}$?

Solution. One has $\tilde{\pi} \circ \widetilde{\exp} = \exp$. We prove this using the fact that $\tilde{\pi} \circ \widetilde{\exp}(tZ)$ is a one-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$ (see course).

6. Describe the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Solution. We prove that $Z(\widetilde{\mathrm{SL}_2(\mathbb{R})}) = \tilde{\pi}^{-1}(\{\pm I_2\})$.

The inclusion $Z(\widetilde{\mathrm{SL}_2(\mathbb{R})}) \subset \pi^{-1}(\{\pm I_2\})$ comes from the surjectivity of π .

Let $g \in \pi^{-1}(\{\pm I_2\})$. For every $h \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$, we have $ghg^{-1}h^{-1} \in \ker \pi$. Since the map $h \mapsto ghg^{-1}h^{-1}$ is continuous and takes values in the discrete set $\ker \pi$, it follows from the connectedness of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ that it is constant, hence $gh = hg$ for every h , in other words $g \in Z(\widetilde{\mathrm{SL}_2(\mathbb{R})})$.

7. Show that if $B(Z, Z) < 0$, then $(\widetilde{\exp}(tZ))_{t \in \mathbb{R}}$ is an embedded Lie subgroup of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, isomorphic to $\mathbb{R}$ and containing the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Solution. The inclusion of the one-parameter subgroup $H = \{\exp(tZ) | t \in \mathbb{R}\}$ in $\mathrm{SL}(2, \mathbb{R})$ is a homotopy equivalence (since $\mathrm{SL}(2, \mathbb{R})$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ in a way that sends H to $\mathbb{S}^1 \times \{*\}$). It follows that the restriction of π to $\pi^{-1}(H)$ is the universal cover of H . Since H contains the centre of $\mathrm{SL}_2(\mathbb{R})$, the same holds for the universal cover, by the previous question.

8. Are there any connected compact subgroups besides $\{e\}$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$?

Solution. No, since no one-parameter subgroup is relatively compact.

On compact connected complex Lie groups

Exercise 4 - Let G be a connected compact complex Lie group. Let V be a finite-dimensional complex vector space. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a complex Lie group morphism.

1. Show that $\rho(G) = \{\mathrm{Id}\}$.

Hint : use the maximum modulus principle for (usual one variable) holomorphic maps.

2. Deduce that any connected compact complex Lie group is commutative.

Solution.

1. For any matrix coefficient, the map $\rho_{i,j} : g \mapsto \rho(g)_{i,j}$ is a complex map from G to $\mathbb{C}$. Since G is compact, $|\rho_{i,j}|$ reaches its maximum at some point g_0 . Pick a local complex coordinate map $\varphi : U \rightarrow V$ with $\varphi(0) = g_0$, for open neighborhoods U of 0 in $\mathbb{C}^n$ and V of g_0 in G . The local restriction of $\rho_{i,j} \circ \varphi$ to any complex line of $\mathbb{C}^n \cap U$ gives a (usual) single variable holomorphic map that reaches its maximum. These restrictions are locally constant by the maximum modulus principle for holomorphic functions.

One obtains that $\rho = \mathrm{id}$ near e and then everywhere on G by connectedness.

2. We apply the previous question to $\rho = \mathrm{Ad}$.

One has $\mathrm{Ad}(g) = \mathrm{id}$ for any $g \in G$. This implies that the Lie group morphism $h \mapsto ghg^{-1} = h$ on the neighborhood of e , and finally on the whole connected G .

Hence G is abelian.

Remarks. Examples of compact complex Lie groups are « complex torus » $\mathbb{C}^n/L$ where L is some lattice of $\mathbb{C}^n$, that is a discrete co-compact additive sub-group.

Note that for instance $SU(2)$ is a compact but *not* complex Lie group (it is three dimensional). If you complexify its Lie algebra, one gets

$$\mathfrak{su}(2) \otimes \mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C}) ,$$

which is the Lie algebra of $SL(2, \mathbb{C})$, now a complex, but not compact Lie group anymore!