

About the exponential map

Exercise 1 - Let G be a real Lie group and e its identity element.

Prove that there exists a neighbourhood U of e in G such that if H is a subgroup of G contained in U , then $H = \{e\}$.

Exercise 2 - Let G and H be Lie groups. Assume that G is connected.

1. Consider $\varphi, \psi : G \rightarrow H$ two Lie group morphisms. Assume that there exists $g_0 \in G$ such that $\varphi(g_0) = \psi(g_0)$ and $T_{g_0}\varphi = T_{g_0}\psi$.
Prove that $T_e\varphi = T_e\psi$, then that φ and ψ coincide on a neighbourhood of g_0 . Finally, conclude that $\varphi = \psi$.
2. Let φ be a C^∞ diffeomorphism of G , such that for every left-invariant vector field X , we have $\varphi^*X = X$. Prove that there exists $g_0 \in G$ such that $\varphi = L_{g_0}$.

Universal cover of the group $\mathrm{SL}_2(\mathbb{R})$

Exercise 3 - Recall that $\mathrm{SL}_2(\mathbb{R})$ is the Lie group of 2×2 matrices of determinant $+1$ with real entries. Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be the basis of $\mathfrak{sl}_2(\mathbb{R})$ used in exercise 3 sheet 1.

1. Prove that every one-parameter subgroup $(g^t)_{t \in \mathbb{R}}$ of $\mathrm{SL}_2(\mathbb{R})$ is conjugated in $\mathrm{SL}_2(\mathbb{R})$ to one of the following three subgroups for some $a \in \mathbb{R}$:

$$\left(s^t = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} \right)_{t \in \mathbb{R}}, \quad \left(u^t = \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} \right)_{t \in \mathbb{R}}, \quad \left(r^t = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix} \right)_{t \in \mathbb{R}}.$$

If a is nonzero, $(g^t)_{t \in \mathbb{R}}$ is called *hyperbolic* in the first case, *parabolic* in the second case, *elliptic* otherwise.

2. If $g^t = \exp(tZ)$ with $Z \in \mathfrak{sl}_2(\mathbb{R})$ for every $t \in \mathbb{R}$, relate the hyperbolic, parabolic or elliptic nature of $(g^t)_{t \in \mathbb{R}}$ to the sign of $B(Z, Z)$, where B is the Killing form of $\mathfrak{sl}_2(\mathbb{R})$ defined for every $X, Y \in \mathfrak{sl}_2(\mathbb{R})$ by

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

3. Is the map $\exp : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathrm{SL}_2(\mathbb{R})$ surjective? If not, describe its image.
4. Prove that $\mathrm{SL}_2(\mathbb{R})$ is homeomorphic to the product $\mathbb{S}^1 \times \mathbb{R}^2$.

Denote by $\widetilde{\pi} : \widetilde{\mathrm{SL}_2(\mathbb{R})} \rightarrow \mathrm{SL}_2(\mathbb{R})$ the universal cover of $\mathrm{SL}_2(\mathbb{R})$, and by $\widetilde{\exp}$ the exponential map from $\mathfrak{sl}_2(\mathbb{R})$ to $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

5. What is the relationship between $\exp$, $\widetilde{\exp}$ and $\widetilde{\pi}$?
6. Describe the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
7. Show that if $B(Z, Z) < 0$, then $(\widetilde{\exp}(tZ))_{t \in \mathbb{R}}$ is an embedded Lie subgroup of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, isomorphic to $\mathbb{R}$ and containing the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
8. Are there any connected compact subgroups besides $\{e\}$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$?

On compact connected complex Lie groups

Exercise 4 - Let G be a connected compact complex Lie group. Let V be a finite-dimensional complex vector space. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a complex Lie group morphism.

1. Show that $\rho(G) = \{\mathrm{Id}\}$.

Hint : A bounded holomorphic map from $\mathbb{C}$ to $\mathbb{C}$ is constant.

2. Deduce that any connected compact complex Lie group is commutative.