Mathematics
Faculté des sciences d'Orsay

Sheet 3

Groups and geometry
M2 AAG 2026–2027

Lie groups and Lie algebra correspondence, coverings

Exercice 1 - Classify up to isomorphism :

1. all connected Lie groups with $\mathfrak{aff}(\mathbb{R})$ as their Lie algebra ;
2. all connected Lie groups of dimension 2 ;
3. all connected Lie groups with $\mathfrak{heis}(3)$ as their Lie algebra.

Solution. In all cases we know the simply connected Lie groups G corresponding to these Lie algebras. Other connected Lie groups are then given by quotiented Lie groups G/Γ for discrete subgroups Γ of $Z(G)$ (the center of G).

1. $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the simply connected Lie group $\mathrm{Aff}^+(\mathbb{R}) = \{f_{a,b} : x \mapsto ax + b \mid (a, b) \in \mathbb{R}^{+*} \times \mathbb{R}\}$ (see exercice 1 sheet 1). Since the center of this group is trivial, this is the only possibility of connected Lie group with this Lie algebra.
   Note that $\mathrm{Aff}(\mathbb{R})$ itself is not connected.
2. From exercice 1 sheet 1, a two dimensional Lie algebra is isomorphic, either to $\mathfrak{aff}(\mathbb{R})$, or to $\mathbb{R}^2$ with the trivial Lie bracket.
   In the case the 2-dimensional connected Lie group is not isomorphic to $\mathrm{Aff}^+(\mathbb{R})$ it has to be abelian, and is the quotient of $\mathbb{R}^2$ by a discrete group Γ.
   Additive discrete subgoups of $\mathbb{R}^2$ can be trivial, of type $\Gamma = v\mathbb{Z}$ (with $v \in \mathbb{R}^2 \setminus \{0\}$), or a lattice $\Gamma = v_1\mathbb{Z} + v_2\mathbb{Z}$, with (v_1, v_2) a basis of $\mathbb{R}^2$. This gives that the connected Lie groups with Lie algebra $\mathbb{R}^2$ are isomorphic to $\mathbb{R}^2$, $\mathbb{R} \times \mathbb{T}^1$ or $\mathbb{T}^2$, for one or two dimensional torus (circle or product of circles).
3. For $\mathfrak{heis}_3$, $Z(\mathrm{Heis}(3)) \simeq \mathbb{R}$ is the one parameter subgroup generated by $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (see sheet 1). Then Γ is of type $\exp(naZ)$ for a real and $n \in \mathbb{Z}$.

Actions of $\mathrm{SL}(2, \mathbb{R})$ and representations of $\mathfrak{sl}(2, \mathbb{R})$

Exercice 2 - We are interested in the actions of $\mathrm{SL}_2(\mathbb{R})$ on (very) low-dimensional manifolds.

1. Let M be a connected smooth manifold of dimension 1 admitting a smooth and transitive action of the Lie group $\mathrm{SL}_2(\mathbb{R})$.
   (a) Prove that such an action implies the existence of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_2(\mathbb{R})$ isomorphic to $\mathfrak{aff}(\mathbb{R})$.
   (b) By studying the eigenvalues of $\mathrm{ad}(X)$ for $X \in \mathfrak{h}$, show that $\mathfrak{h}$ can be identified to the Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ generated by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
   (c) Deduce that M is C^∞-diffeomorphic to the circle $\mathbb{S}^1$.
2. Deduce that any smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the circle is either trivial or transitive.
3. Give some concrete example of a smooth transitive action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{S}^1$.

4. Prove that there is no nontrivial smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the real line.
5. Consider a smooth action of $G = \mathrm{SL}_2(\mathbb{R})$ on the sphere $\mathbb{S}^2$. For every $x \in \mathbb{S}^2$, let G_x be the stabiliser of x.

   (a) Prove that there exist two points x and $y \in \mathbb{S}^2$ such that G_x contains an hyperbolic one-parameter subgroup and G_y a parabolic one.
   Hint. For every $X \in \mathfrak{sl}_2(\mathbb{R})$, one can consider the vector field

   $$X_{S^2}(x) = \frac{d}{dt}_{|t=0} \exp(tX)\, x$$

   induced by the action of $\exp(tX)$ on $\mathbb{S}^2$, and use the fact that any smooth vector field on $\mathbb{S}^2$ vanishes somewhere (since the Euler characteristic of $\mathbb{S}^2$ does not vanish).

   (b) Deduce that the action under consideration cannot be transitive.

Solution.

1. (a) Given $x \in M$, note $H \subset \mathrm{SL}_2(\mathbb{R})$ its stabiliser. This is a closed subgroup of a Lie group, hence a Lie subgroup. Let $\mathfrak{h} \subset \mathfrak{sl}_2(\mathbb{R})$ be its Lie algebra. The action of $\mathrm{SL}_2(\mathbb{R})$ on M being transitive, one has that $\mathrm{SL}_2(\mathbb{R})/H \simeq M$ is one dimensional, hence $\mathfrak{h}$ is two dimensional. One checks that $\mathfrak{sl}_2(\mathbb{R})$ does not contain $\mathbb{R}^2$, then $\mathfrak{h}$ is isomorphic to $\mathfrak{aff}(\mathbb{R})$.

   (b) Hence $\mathfrak{h}$ is generated by X, Y satisfying $[X, Y] = Y$. Therefore $1 \in \mathrm{Sp}(\mathrm{ad}(X))$. By exercice 4.1 sheet 2, this implies that X is hyperbolic, more precisely that X is conjugated to $H/2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   Since Y is in the eigenspace $\mathrm{ad}(X)$ for the eigenvalue 1, it is a multiple of $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   Then $\mathfrak{h}$ is conjugated to $\mathrm{Vect}(H, E)$.

   (c) From the previous question $\mathrm{SL}_2(\mathbb{R})$ contains, up to a change of basis, only one connected Lie subgroup whose Lie algebra is $\mathfrak{aff}(\mathbb{R})$, it is $H_0 = \exp(\mathfrak{h}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0,\ b \in \mathbb{R} \right\}$.

   Note that G/H_0 is a covering of $G/H \simeq M$ with H/H_0 as automorphism group.

   From the « $Q \times R$ » factorisation of $\mathrm{SL}_2(\mathbb{R})$ (with $Q = \mathrm{SO}(2)$ and $R = H_0$), it follows that $G/H_0 = \mathrm{SL}_2(\mathbb{R})/R \simeq \mathbb{S}^1$, so that the quotient $G/H \simeq M$ is a circle too.
2. All orbits of a continuous action of the connected group $\mathrm{SL}(2, \mathbb{R})$ are connected. Being also subsets of the circle here, they are circular arcs, and hence locally closed sets in $\mathbb{S}^1$. From the course, these orbits are then smooth manifolds. By the previous question they are either trivial or diffeomorphic to circles.
3. The group $\mathrm{SL}(2, \mathbb{R})$ acts smoothly and transitively on the projective space $\mathbb{R}P^1 = P(\mathbb{R}^2) \simeq \mathbb{S}^1$. In homogenous coordinates, this translates into the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R} \cup \{\infty\} \simeq \mathbb{S}^1$ by homographies : $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.x = \dfrac{ax + b}{cx + d}$.
4. Same argument than in question 2.
5. (a) Any $X \in \mathfrak{sl}_2(\mathbb{R})$ induces a vector field X_{S^2} on $\mathbb{S}^2$. One has $X_{\mathbb{S}^2}(x) = 0$ iff $X \in \mathfrak{g}_x$ the Lie algebra of G_x. Therefore, considering $X = H$ or E, one finds two points x and $y \in \mathbb{S}^2$ such that $H \in \mathfrak{g}_x$ and $E \in \mathfrak{g}_y$.

   (b) If the action were transitive, all stabilisers would be conjugated and $\mathfrak{g}_x$ would contain an hyperbolic and parabolic element. Therefore $\dim \mathfrak{g}_x \geq 2$, which is not possible because the orbit $\mathbb{S}^2 \simeq \mathrm{SL}(2, \mathbb{R})/G_x$ is two dimensional.

On the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$

Exercice 3 - The Minkowski space $\mathbb{R}^{1,n}$ of dimension $n+1$ is the vector space $\mathbb{R}^{n+1}$ endowed with the quadratic form q given by

$$q(x_0, x_1, \cdots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2 .$$

Let $\mathrm{O}(1,n) = \{g \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid q \circ g = q\}$ be the orthogonal group of q, and $\mathrm{SO}_0(1,n)$ its connected component of the identity. The real hyperbolic space of dimension n (or upper half–hyperboloid) is

$$\mathbb{H}^n_{\mathbb{R}} = \{x \in \mathbb{R}^{1,n} \mid q(x) = -1 \text{ and } x_0 > 0\}$$

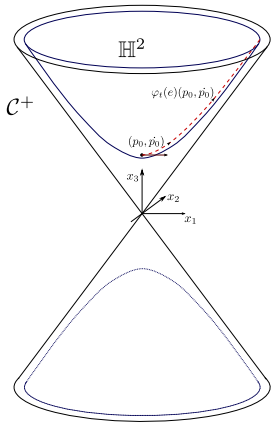
endowed with the $\mathrm{SO}_0(1,n)$ action we will study here.

1. For $n = 2$, draw the hypersurfaces with equations $q = 1$, $q = 0$, $q = -1$ and $\mathbb{H}^2_{\mathbb{R}}$.
2. Prove that $\mathrm{SO}_0(1,n)$ acts transitively on $\mathbb{H}^n_{\mathbb{R}}$.

   Hint. One may use (some adaptation of) the Lorentz transformations $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.
3. What is the stabiliser of a point in $\mathbb{H}^n_{\mathbb{R}}$ for this action ?
4. Construct a surjective group morphism $\varphi : \mathrm{O}(1,n) \to (\mathbb{Z}/2\mathbb{Z})^2$.
5. Deduce the number of connected components of $\mathrm{O}(1,n)$.

Solution.

1. Below, the light cone $\{q = 0\} = \mathcal{C}^+$ and $\{q = -1\}$ is a two-sheeted hyperbolic 'paraboloid'. The hyperbolic space $\mathbb{H}^2$ is the upper sheet. The undrawn set $\{q = -1\}$ corresponds to the one-sheeted hyperboloid (resembling a nuclear power plant cooling tower) asymptotic to the light-cone.



2. We show that $x = (x_0, \cdots, x_n) \in \mathbb{H}^n$ can be sent on $a = (1, 0, \cdots, 0)$ by some $M \in \mathrm{SO}_0(1,n)$.
   Firstly, since $\mathrm{SO}(n)$ acts transitively on $\mathbb{S}^n$, one can find some $A \in \mathrm{SO}(n)$ such that $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ send x to $x' = (x_0, y, 0, \cdots, 0)$. One has $M_1 \in \mathrm{SO}_0(1,n)$ by connexity of $\mathrm{SO}(n)$.
   Since $-x_0^2 + y^2 = q(x) = -1$ with $x_0 > 0$, one can find $t \in \mathbb{R}$ such that $y = \sinh t$ and $x_0 = \cosh t$.
   Then the Lorentz transform $\begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$ sends (x_0, y) to $(1, 0)$. Completing this matrix by a diagonal block I_{n-1} gives a matrix M_2. One finally finds a matrix $M = M_2 M_1$ sending x to $(1, 0, \cdots, 0)$. This M belongs to $\mathrm{SO}_0(1,n)$ as seen deforming t to 0.

3. The stabiliser of $(1, 0, \cdots, 0)$ is clearly $\mathrm{SO}(n)$ so that $\mathbb{H}^n = \mathrm{SO}_0(1,n)/\mathrm{SO}(n)$ is an homogenous space.
4. The quadric $\{q = -1\}$ is the union of two connected sets $\mathbb{H}^n$ and $-\mathbb{H}^n$. Hence $M \in \mathrm{O}(1,n)$ either preserves or permutes them. Set $\varphi(M) = (1, \det M)$ if M preserves $\mathbb{H}^n$ or $\varphi(M) = (-1, \det M)$ otherwise.
   One sees that $\varphi : \mathrm{SO}(1,n) \to (\mathbb{Z}/2\mathbb{Z})^2$ is a surjective group morphism using extensions of $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ by diagonal I_{n-1} blocks.
5. The previous matrices send $\mathrm{SO}_0(1,n)$ to the other connected components. Therefore $\mathrm{SO}(1,n)$ has four connected components.

Unimodular lattices

Exercice 4 - A *lattice* (or $\mathbb{Z}$-lattice) in $\mathbb{R}^n$ is a subgroup of the additive group $\mathbb{R}^n$ generated by an $\mathbb{R}$-basis of $\mathbb{R}^n$. Given a lattice $\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ where $(v_1, \cdots, v_n)$ is an $\mathbb{R}$-basis of $\mathbb{R}^n$, check that the volume of the parallelepiped determined by $(v_1, \cdots, v_n)$ is independent of the choice of $\mathbb{Z}$-basis of the $\mathbb{Z}$-module Γ. This is called the *covolume* of Γ. A lattice is *unimodular* if its covolume equals 1. Let $\mathcal{L}_n$ denotes the set of unimodular lattices of $\mathbb{R}^n$.

1. Prove that there exists on $\mathcal{L}_n$ a smooth manifold structure of dimension $n^2 - 1$ such that the linear action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathcal{L}_n$ is smooth and transitive.
   Prove that there exists a covering map $\mathrm{SL}_n(\mathbb{R}) \to \mathcal{L}_n$ with automorphism group $\mathrm{SL}_n(\mathbb{Z})$.
2. Prove that for every $\Gamma \in \mathcal{L}_n$, we have $\inf_{\gamma \in \Gamma \setminus \{0\}} \|\gamma\| > 0$.
3. Prove that $\mathcal{L}_n$ is not compact for $n \geq 2$.

Solution. A change of $\mathbb{Z}$-basis B of Γ, is given by $B \mapsto g(B)$ with $g \in \mathrm{GL}(n,\mathbb{Z})$ as well as its inverse g^{-1}. Therefore, one has that $\det g$ is an integer as its inverse, so that $|\det g| = 1$.
Hence the volume of a $\mathbb{Z}$-basis of Γ is independent of its choice since $\mathrm{vol}(g(B)) = |\det g| \,\mathrm{vol}(B)$.

1. Let B_{can} be the canonical basis of $\mathbb{R}^n$ and $\Gamma_{can} = \mathbb{Z}^n$ be its associated lattice. Every unimodular lattice Γ in $\mathbb{R}^n$ writes $\Gamma = g(\Gamma_{can})$ for some $g \in \mathrm{SL}(n,\mathbb{R})$, if one restricts to use oriented basis to represent Γ (achieved by an eventual change of v_1 into $-v_1$ in an initial basis of Γ).
   A change of oriented $\mathbb{Z}$-basis for a given $\Gamma = g(\mathbb{Z}^n)$ is obtained by $g \mapsto gh$ for $h \in \mathrm{SL}(n,\mathbb{Z})$. This shows that $\mathcal{L}_n$ identifies with the quotient space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.
   Since the group $\mathrm{SL}(n,\mathbb{Z})$ is discrete in $\mathrm{SL}(n,\mathbb{R})$, the map
   $$\mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}) \simeq \mathcal{L}_n$$
   is a covering and $\mathcal{L}_n$ has a smooth manifold structure of dimension $n^2 - 1$ locally given by the one of the Lie group $\mathrm{SL}(n,\mathbb{R})$. The (smooth) left linear action of $\mathrm{SL}(n,\mathbb{R})$ on itself goes to the quotient space.
2. Let $\Gamma = g(\mathbb{Z}^n)$ with $g \in \mathrm{SL}(n,\mathbb{R})$ be a lattice. Consider $\alpha = \inf_{\|v\|=1} \|g(v)\| > 0$ using the sup norm $\|v\| = \|v\|_\infty$ for instance.
   For any $v \in \mathbb{Z}^n \setminus \{0\}$ one has then $\|g(v)\| \geq \alpha\|v\| \geq \alpha$, hence $\|\gamma\| \geq \alpha$ for $\gamma \in \Gamma \setminus \{0\}$.
   This shows in particular that lattices are discrete in $\mathbb{R}^n$.
3. Consider the sequence of lattices $\Gamma_k = g_k(\mathbb{Z}^n) \in \mathcal{L}_n$ for $k \in \mathbb{N}^*$ associated to
   $$g_k = \begin{pmatrix} 1/k & 0 & (0) \\ 0 & k & (0) \\ (0) & (0) & I_{n-2} \end{pmatrix} \in \mathrm{SL}(n,\mathbb{R}).$$

We want to show that Γ_k has no subsequential limit in $\mathcal{L}_n$. Otherwise, suppose that a subsequence Γ_{k_p} converges to $\Gamma = g(\mathbb{Z}^n)$ in $\mathcal{L}_n$ when $p \to +\infty$. That means there exist $m_k \in \mathrm{SL}(n, \mathbb{Z})$ such that $g_k m_k$ converges to $g \in \mathrm{SL}(n, \mathbb{R})$ (let k_p becomes k in the sequel). In particular, the sequence of norms $\|(g_k m_k)^{-1}\|$ is bounded by some C. Therefore, $\forall v \in \mathbb{Z}^n$,

$$\|g_k m_k v\| \geq \frac{1}{C}\|v\| \geq 1/C.$$

This is not possible for $v = (m_k)^{-1}(1, 0, \cdots, 0)$ and k large since $g_k m_k v = \frac{1}{k}(1, 0, \cdots, 0)$ converges to 0