Sheet 3

Lie groups and Lie algebra correspondence, coverings

Exercice 1 - Classify up to isomorphism :

1. all connected Lie groups with $\mathfrak{aff}(\mathbb{R})$ as their Lie algebra ;
2. all connected Lie groups of dimension 2 ;
3. all connected Lie groups with $\mathfrak{heis}(3)$ as their Lie algebra.

Actions of $\mathrm{SL}(2,\mathbb{R})$ and representations of $\mathfrak{sl}(2,\mathbb{R})$

Exercice 2 - We are interested in the actions of $\mathrm{SL}_2(\mathbb{R})$ on (very) low-dimensional manifolds.

1. Let M be a connected smooth manifold of dimension 1 admitting a smooth and transitive action of the Lie group $\mathrm{SL}_2(\mathbb{R})$.
   (a) Prove that such an action implies the existence of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_2(\mathbb{R})$ isomorphic to $\mathfrak{aff}(\mathbb{R})$.
   (b) By studying the eigenvalues of $\mathrm{ad}(X)$ for $X \in \mathfrak{h}$, show that $\mathfrak{h}$ can be identified to the Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ generated by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
   (c) Deduce that M is C^∞-diffeomorphic to the circle $\mathbb{S}^1$.
2. Deduce that any smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the circle is either trivial or transitive.
3. Give some concrete example of a smooth transitive action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{S}^1$.
4. Prove that there is no nontrivial smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the real line.
5. Consider a smooth action of $G = \mathrm{SL}_2(\mathbb{R})$ on the sphere $\mathbb{S}^2$. For every $x \in \mathbb{S}^2$, let G_x be the stabiliser of x.
   (a) Prove that there exist two points x and $y \in \mathbb{S}^2$ such that G_x contains an hyperbolic one-parameter subgroup and G_y a parabolic one.
   Hint. For every $X \in \mathfrak{sl}_2(\mathbb{R})$, one can consider the vector field
   $$X_{S^2}(x) = \frac{d}{dt}_{|t=0} \exp(tX)\, x$$
   induced by the action of $\exp(tX)$ on $\mathbb{S}^2$, and use the fact that any smooth vector field on $\mathbb{S}^2$ vanishes somewhere (since the Euler characteristic of $\mathbb{S}^2$ does not vanish).
   (b) Deduce that the action under consideration cannot be transitive.

On the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$

Exercice 3 - The Minkowski space $\mathbb{R}^{1,n}$ of dimension $n+1$ is the vector space $\mathbb{R}^{n+1}$ endowed with the quadratic form q given by

$$q(x_0, x_1, \cdots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2 \,.$$

Let $\mathrm{O}(1,n) = \{g \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid q \circ g = q\}$ be the orthogonal group of q, and $\mathrm{SO}_0(1,n)$ its connected component of the identity. The real hyperbolic space of dimension n (or upper half–hyperboloid) is

$$\mathbb{H}^n_{\mathbb{R}} = \{x \in \mathbb{R}^{1,n} \mid q(x) = -1 \text{ and } x_0 > 0\}$$

endowed with the $\mathrm{SO}_0(1,n)$ action we will study here.

1. For $n = 2$, draw the hypersurfaces with equations $q = 1$, $q = 0$, $q = -1$ and $\mathbb{H}^2_{\mathbb{R}}$.
2. Prove that $\mathrm{SO}_0(1,n)$ acts transitively on $\mathbb{H}^n_{\mathbb{R}}$.

   Hint. One may use (some adaptation of) the Lorentz transformations $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.
3. What is the stabiliser of a point in $\mathbb{H}^n_{\mathbb{R}}$ for this action ?
4. Construct a surjective group morphism $\varphi : \mathrm{O}(1,n) \to (\mathbb{Z}/2\mathbb{Z})^2$.
5. Deduce the number of connected components of $\mathrm{O}(1,n)$.

Unimodular lattices

Exercice 4 - A *lattice* (or $\mathbb{Z}$-lattice) in $\mathbb{R}^n$ is a subgroup of the additive group $\mathbb{R}^n$ generated by an $\mathbb{R}$-basis of $\mathbb{R}^n$. Given a lattice $\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ where $(v_1, \cdots, v_n)$ is an $\mathbb{R}$-basis of $\mathbb{R}^n$, check that the volume of the parallelepiped determined by $(v_1, \cdots, v_n)$ is independent of the choice of $\mathbb{Z}$-basis of the $\mathbb{Z}$-module Γ. This is called the *covolume* of Γ. A lattice is *unimodular* if its covolume equals 1. Let $\mathcal{L}_n$ denotes the set of unimodular lattices of $\mathbb{R}^n$.

1. Prove that there exists on $\mathcal{L}_n$ a smooth manifold structure of dimension $n^2 - 1$ such that the linear action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathcal{L}_n$ is smooth and transitive.

   Prove that there exists a covering map $\mathrm{SL}_n(\mathbb{R}) \to \mathcal{L}_n$ with automorphism group $\mathrm{SL}_n(\mathbb{Z})$.
2. Prove that for every $\Gamma \in \mathcal{L}_n$, we have $\inf_{\gamma \in \Gamma \setminus \{0\}} \|\gamma\| > 0$.
3. Prove that $\mathcal{L}_n$ is not compact for $n \geq 2$.