Sheet 4

The Hopf fibration.

Exercice 1 - Consider the projective action of the subgroup $\mathrm{SU}(2)$ of $\mathrm{GL}(2,\mathbb{C})$ on the complex projective line $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$.

1. Prove that it induces a circle bundle (fibration by circles).
2. Is this fiber bundle trivialisable ?

Solution.

1. The projective action of the Lie group $\mathrm{SU}(2)$ on $\mathbf{P}(\mathbb{C}^2) : [v] \mapsto [Mv]$ is smooth, and transitive because $\mathrm{SU}(2)$ acts transitively on unit vectors of $\mathbb{C}^2$. Therefore $\mathrm{SU}(2)/G_x$ is diffeomorphic to $\mathbf{P}(\mathbb{C}^2)$, where G_x is the stabiliser of some complex line x in $\mathbb{C}^2$.
   The matrices $M \in \mathrm{SU}(2)$ that preserve the complex line generated by $e = (1,0)$ are of type $M = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, hence G_e is isomorphic to the Lie group $\mathrm{U}(1) = \mathbb{S}^1$ embedded into $\mathrm{SU}(2)$ using $i : \mathrm{U}(1) \to \mathrm{SU}(2)$ defined by $i(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. Thus we have an homogeneous space structure
   $$\mathbb{S}^1 = \mathrm{U}(1) \overset{i}{\hookrightarrow} \mathrm{SU}(2) \overset{\pi}{\to} \mathrm{SU}(2)/i(U(1)) \simeq \mathbb{CP}^1 .$$
   This gives a circle bundle structure since the product action of $i(\mathrm{U}(1))$ on $\mathrm{SU}(2)$ is proper, free and smooth.
2. If this fiber bundle were trivial, one would then have that $\mathrm{SU}(2) \simeq \mathbb{CP}^1 \times \mathbb{S}^1 \simeq \mathbb{S}^2 \times \mathbb{S}^1$, whose fundamental group is $\mathbb{Z}$, while $\mathrm{SU}(2) \simeq \mathbb{S}^3$ is simply connected, a contradiction.

A quotient of the Heisenberg group.

Exercice 2 - In the Heisenberg group $\mathrm{Heis}(3) = \left\{ M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,\ y,\ z \in \mathbb{R} \right\}$, we consider the subgroup Γ that consists in such matrices with x, y and z integers.

1. Considering the action of the center of $\mathrm{Heis}(3)$ on itself, prove that $M = \mathrm{Heis}(3)/\Gamma$ has a smooth circle bundle structure over some manifold and identify it.
2. Is this fiber bundle trivialisable ?

Solution.

1. Γ being a discrete subgroup of $\mathrm{Heis}(3)$, it is closed and the quotient $M = \mathrm{Heis}(3)/\Gamma$ is a smooth three dimensional manifold.
   If $M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ and $M' = \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(3)$, one finds that
   $$MM' = \begin{pmatrix} 1 & x+x' & z+z'+xy' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}.$$

One deduces that the center of Heis(3) is the one parameter Lie subgroup

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

generated by $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The left action of $Z(G)$ on Heis(3) induces a $Z(G)$ action on $M = \text{Heis}(3)/\Gamma$ by $g\Gamma \mapsto h(g\Gamma) = (hg)\Gamma$ for $h \in Z(G)$. It is not free. Indeed $hg\Gamma = g\Gamma$ for $h \in Z(G)$ iff $h\Gamma = \Gamma$ iff $h \in Z(G) \cap \Gamma = \{e^{nZ} \mid n \in \mathbb{Z}\}$.

Therefore this $Z(G)$ action factorises to a free action of $Z(G)/Z(G) \cap \Gamma \simeq \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$, hence also proper (and smooth).

Thus M is a circle bundle over $Z(G)\backslash M \simeq \mathbb{R}^2/\mathbb{Z}^2$, a torus.

Note that one can also observe that $Z(G)\backslash M \simeq \text{Heis}(3)/(Z(G) \times \Gamma)$ with

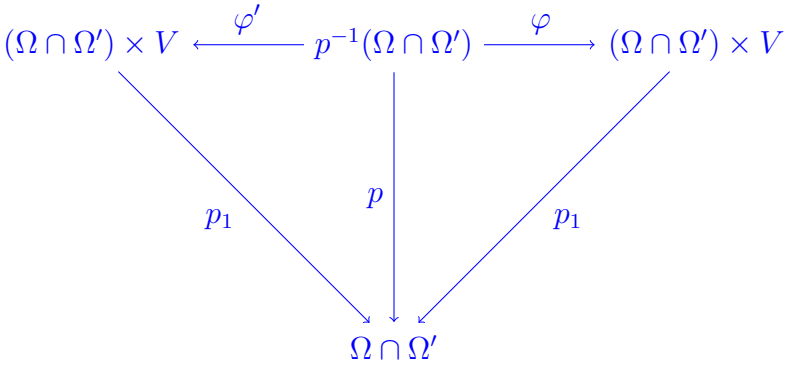
$$Z(G) \times \Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,\ y \in \mathbb{Z} \text{ and } z \in \mathbb{R} \right\},$$

which is a closed subgroup of Heis(3), hence with a smooth, proper and free right action on Heis(3).

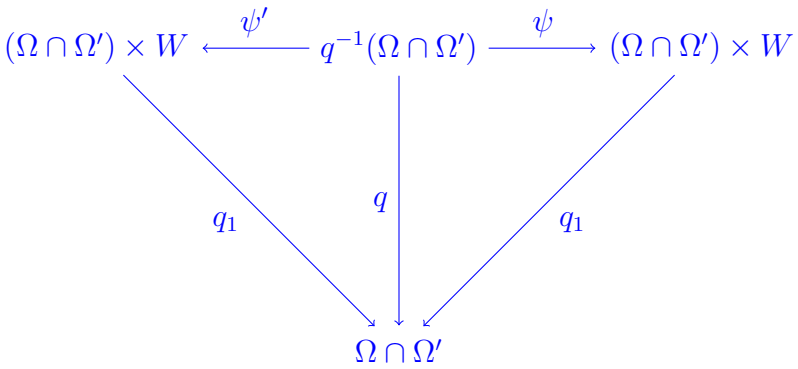
2. If this circle bundle were trivial, then M would be diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2/\mathbb{Z}^2$ whose fundamental group is the abelian group $\mathbb{Z}^3$. But $\pi^1(M) = \Gamma$, because Heis(3) is simply connected, and Γ is not abelian since $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ does not commute with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Exercice 3 - $\mathbf{L(E, F)}$.
Let M be a smooth manifold and E and F be two vector bundles over M. Describe the transitions maps between two local trivializations for the vector bundle $L(E, F)$.

Solution. Fix an atlas on M which contains only open trivialisation sets for both E and F.
Charts and transition map for $p : E \to M$ with fiber V



where $\varphi' \circ \varphi^{-1}(m, v) = (m, A_m v)$ with $A_m \in \text{GL}(V)$.
Charts and transition map for $q : F \to M$ with fiber W



where $\psi' \circ \psi^{-1}(m, w) = (m, B_m w)$ with $B_m \in \mathrm{GL}(W)$. For $L(E, F)$ we obtain for transition maps

$$\begin{array}{ccc} (\Omega \cap \Omega') \times L(V, W) & \longmapsto & (\Omega \cap \Omega') \times L(V, W) \\ (m, M) & \longmapsto & (m, B_m M A_m^{-1}). \end{array}$$

Triviality, or not, of some vector bundles.

Exercice 4 -

1. Let γ_n be the tautological vector bundle over the projective space $\mathbb{RP}^n = \mathbf{P}(\mathbb{R}^{n+1})$ (with $n \geq 1$). Recall that its total space is $\{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in x\}$.
   (a) Prove that if s is a nowhere-vanishing continuous section of γ_n, then there exists a continuous map $t : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^*$ such that $s(\mathbb{R}v) = t(v)v$.
   (b) Deduce that the vector bundle γ_n is not trivial.
2. Prove that the tangent bundle of $\mathbb{S}^3$ is trivialisable.
   Hint. Think about Lie groups.
   What about the tangent bundle of $\mathbb{S}^2$?
3. Let ξ be a real line bundle. Prove that the vector bundle $\mathrm{End}(\xi)$ of endomorphisms of ξ is always trivial.

Solution.

1. (a) Let $p : \gamma_n \to \mathbb{RP}^n$ be the tautological bundle. We assume we have $s : \mathbb{RP}^n \to \gamma_n$ such that $s(\mathbb{R}v) = (\mathbb{R}v, s_0(\mathbb{R}v))$ with $s_0(\mathbb{R}v) \neq 0$ for all $\mathbb{R}v \in \mathbb{RP}^n$. Let $t : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^*$ be defined by
$$t(v) = \frac{\langle s_0(\mathbb{R}v), v \rangle}{\langle v, v \rangle}.$$
   As $s_0(\mathbb{R}v) \in \mathbb{R}v$ (this is the definition of γ_n), by definition of t, $s_0(\mathbb{R}v) = t(v)v$. By construction t is continuous.
   (b) By contradiction, if the bundle is trivial, there exists a section s as above and a continuous function $t : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^*$. As $n \geq 1$, $\mathbb{R}^{n+1} \setminus \{0\}$ is connected and then $t(\mathbb{R}^{n+1} \setminus \{0\}) \subset (0, \infty)$ or $t(\mathbb{R}^{n+1} \setminus \{0\}) \subset (-\infty, 0)$.
   Yet, $s(\mathbb{R}(-v))) = -t(-v)v = s(\mathbb{R}v) = t(v)v$, so that $t(-v) = -t(-v)$, a contradiction.
2. The manifold $\mathbb{S}^3$ is a Lie group (it is diffeomorphic to $SU(2)$), therefore one can trivialise its tangent bundle using translations acting on a basis of $\mathfrak{su}(2)$.
   The tangent bundle of $\mathbb{S}^2$ is not trivial. It has no non vanishing smooth section as the Euler characteristic of $\mathbb{S}^2$ equals $2 \neq 0$. (See also exercice 6 for another proof of this question.)

3. Let $\xi \to M$ be a real vector bundle of rank 1. Therefore $\mathrm{End}(\xi)$ is a real vector bundle of rank 1. To prove $\mathrm{End}(\xi)$ is trivial, it is enough to find a nowhere vanishing section and $p \to \mathrm{Id}_{\xi(p)}$ is such a section.

Pull back of vector bundles.

Exercice 5 -

1. Let ξ be the tangent bundle to $\mathbb{S}^2$ and $f : \mathbb{S}^1 \to \mathbb{S}^2$ be the equatorial map $\theta \mapsto (\cos\theta, \sin\theta, 0)$. Show that $f^*\xi$ is a trivial vector bundle.
2. (Bonus) Let $E = \{(e^{i\theta}, w) \in \mathbb{S}^1 \times \mathbb{C}, w \in \mathbb{R}e^{i\theta/2}\}$. Let $p : E \to \mathbb{S}^1$ be the projection to the first factor.
   (a) Prove that p is the projection map of a line bundle over $\mathbb{S}^1$.
   (b) Identify E.
   (c) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $f(z) = z^2$. Describe f^*E.

Solution.

1. To prove that the the bundle ξ is trivial, we have to find two sections s_1 and s_2 such that for all $\theta \in \mathbb{S}^1$, $(s_1(\theta), s_2(\theta))$ is a basis of $f^*\xi_\theta$. Let $t_1 : \mathbb{S}^1 \to T\mathbb{S}^2$ be defined by
$$t_1(\theta) = (f(\theta), (-\sin\theta, \cos\theta, 0)).$$
Let $t_2 : \mathbb{S}^1 \to T\mathbb{S}^2$ be defined by
$$t_2(\theta) = (f(\theta), (0, 0, 1)).$$
You can draw these vector fields to see what happens!
The maps t_1 and t_2 are sections of $T\mathbb{S}^2$ and for all $\theta \in \mathbb{S}^1$, $(t_1(\theta), t_2(\theta))$ is a basis of $T_{f(\theta)}\mathbb{S}^2$. Moreover t_1 and t_2 induce sections s_1 and s_2 of $f^*\xi$ such that for all $\theta \in S^1$, $(s_1(\theta), s_2(\theta))$ is a basis of $f^*\xi_\theta$. Thus $f^*\xi$ is trivial.
2. (a) For all $k \in \mathbb{Z}$, we have $e^{i(\theta+2k\pi)/2} = e^{i\theta/2}e^{ik\pi}$. As $e^{ik\pi} = \pm 1$, we have $\mathbb{R}e^{i\theta/2} = \mathbb{R}e^{i(\theta+2k\pi)/2}$, so that E is well defined. Furthermore, given a local choice of argument θ in a neighbourhood U of a point of $\mathbb{S}^1$, the map $(e^{i\theta}, t) \in U \times \mathbb{R} \mapsto (e^{i\theta}, te^{i\theta/2}) \in E$ gives a local trivialisation of E over U, which is then a line bundle over $\mathbb{S}^1$.
   (b) It is a Möbius strip! Indeed, let $g : \mathbb{R} \to \mathbb{S}^1$ be defined by $g(\theta) = e^{i\theta}$. Then g^*E is trivial as $s : \mathbb{R} \to E$ defined by $s(\theta) = (e^{i\theta}, e^{i\theta/2})$ is a nowhere vanishing section which induces a nowhere vanishing section on g^*E. Now, E is obtained from g^*E using the identification $(\theta, v) \simeq (\theta + 2\pi, v)$. As $s(\theta + 2\pi) = (e^{i\theta}, -e^{i\theta/2})$. We obtain a Möbius strip.
   (c) This is a trivial bundle as the section s from the previous question induces a section of f^*E.

On the frame bundle of $\mathbb{S}^2$.

Exercice 6 - One considers the action of $\mathrm{SL}(3, \mathbb{R})$ on the sphere $\mathbb{S}^2$ induced from the linear action of $\mathrm{SL}(3, \mathbb{R})$ on the set of half-lines in $\mathbb{R}^3$. Considering the usual orientation of $\mathbb{S}^2 = \partial\mathbb{B}^3$, let $\mathcal{R}^+(\mathbb{S}^2)$ denote the fiber bundle of positive (direct) frames.

1. Prove that $\mathrm{SL}(3, \mathbb{R})$ acts smoothly and transitively on $\mathcal{R}^+(\mathbb{S}^2)$.

2. (Bonus) Deduce a computation of the fundamental group of $\mathcal{R}^+(\mathbb{S}^2)$.
   Hint. One may use the fact that a fibration $F \to E \to B$ with connected fibers induces an exact sequence $\pi^1(F) \to \pi^1(E) \to \pi^1(B) \to 0$.
3. (Bonus) Deduce by this way that the tangent bundle of $\mathbb{S}^2$ has no non vanishing section.

Solution.

1. We first identifies the set of half-lines of $\mathbb{R}^3$ with $\mathbb{S}^2 \simeq \mathbb{R}^{3*}/\mathbb{R}^{+*}$, using the action of (multiplicative) group $\mathbb{R}^{+*}$ on $\mathbb{R}^{3*}$ defined by $(\lambda, x) \in \mathbb{R}^{+*} \times \mathbb{R}^{3*} \mapsto \lambda x \in \mathbb{R}^{3*}$. The linear action of $\mathrm{SL}(3,\mathbb{R})$ on $\mathbb{R}^{3*}$ passes smoothly to the quotient.
   Any direct frame $B_x = (v_1, v_2)$ of $T_x\mathbb{S}^2$ at some point $x \in \mathbb{S}^2$ may be completed into a direct basis $B'_x = (v_1, v_2, x)$ of $\mathbb{R}^3$ (this is actually the way we define direct frames on $\mathbb{S}^2$). Reversely, a direct basis $B'_x = (v_1, v_2, x)$ of $\mathbb{R}^3$ is projected using $\pi : \mathbb{R}^{3*} \to \mathbb{R}^{3*}/\mathbb{R}^{+*} \simeq \mathbb{S}^2$ to a frame $B_{\pi(x)} = (T_x\pi(v_1), T_x\pi(v_2))$ in $T_{\pi(x)}\mathbb{S}^2$. Then up to a positive scaling, $\mathrm{SL}(3,\mathbb{R})$ acts smoothly and transitively on direct basis of $\mathbb{R}^3$. This goes down to a smooth and transitive action of $\mathrm{SL}(3,\mathbb{R})$ on $\mathcal{R}^+(\mathbb{S}^2)$.
   This shows that $\mathcal{R}^+(\mathbb{S}^2)$ is an homogeneous space with a $\mathrm{SL}(3,\mathbb{R})$ smooth transitive action. This is a fibration, with fibers F conjugated to a stabiliser G_{B_x} of any frame, since this closed subgroup of $\mathrm{SL}(3,\mathbb{R})$ acts on it freely, properly (general properties of the left or right actions of a closed subgroup on its ambient group) and smoothly.
2. We need to identify the stabiliser G_{B_x} of some frame B_x by the previous action of $\mathrm{SL}(3,\mathbb{R})$. From the previous identifications this stabiliser consists in matrices M of $\mathrm{SL}(3,\mathbb{R})$ sending $B'_x = (v_1, v_2, x)$ to some
   $$B'' = (\lambda v_1 + kx, \lambda v_2 + k'x, \lambda x)$$
   that will project to the original frame $B_x = (v_1, v_2)$ through the quotient map induces by $\pi : \mathbb{R}^{3*} \to \mathbb{R}^{3*}/\mathbb{R}^{+*}$. One has necessarily $\lambda^3 = \det B''/\det B'_x = \det M = 1$ so that $\lambda = 1$, and in the basis B'_x, M writes $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & k' & 1 \end{pmatrix}$. Hence the fiber $F_x = G_{B_x}$ over x is isomorphic to $\mathbb{R}^2$.
   Using the hint and $\pi^1(F) = \{0\}$, one gets that $\pi^1(\mathrm{SL}(3,\mathbb{R})) \simeq \pi^1(\mathcal{R}^+(\mathbb{S}^2))$.
   Moreover, one has by the "$Q \times R_1$" decomposition of $\mathrm{SL}(3,\mathbb{R})$ (with $Q = \mathrm{SO}(3,\mathbb{R})$ and R_1 the space of right triangular matrices with positive diagonal coefficients and determinant 1, a contractible space), that
   $$\pi^1(\mathrm{SL}(3,\mathbb{R})) \simeq \pi^1(\mathrm{SO}(3,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$$
   since $\mathrm{SO}(3,\mathbb{R}) \simeq \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ by Sheet 1.
   We get eventually that $\pi^1(\mathcal{R}^+(\mathbb{S}^2)) = \mathbb{Z}/2\mathbb{Z}$.
3. If $T\mathbb{S}^2$ has a non vanishing section v, we can complete it into a direct orthogonal frame (v, w) of $T_x\mathbb{S}^2$ at each point. This would produce a trivialisation of $T\mathbb{S}^2$ and then of the $\mathrm{GL}(2,\mathbb{R})^+$ bundle $\mathcal{R}^+(\mathbb{S}^2)$. This would imply that $\mathcal{R}^+(\mathbb{S}^2) \simeq \mathrm{GL}(2,\mathbb{R})^+ \times \mathbb{S}^2$ whose fundamental group is $\pi^1(\mathrm{GL}(2,\mathbb{R})^+) \simeq \pi^1(\mathrm{SO}(2,\mathbb{R})) = \mathbb{Z}$. This contradicts the previous question.

   Note that one can get this result more quickly considering the action of $\mathrm{SO}(3,\mathbb{R})$ on direct orthonormal frames of $\mathbb{S}^2$.