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Differential Geometry on C-C spaces and application to the Novikov-Shubin numbers of nilpotent Lie groups

Géométrie différentielle sur les espaces de Carnot et exposants de Novikov-Shubin des groupes de Lie nilpotents

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Abstract - We prove that the near cohomology [7] of a stratified nilpotent Lie group G concentrates on a sub-complex of the de Rham complex. This subcomplex is conjugated to a differential complex d_c acting on the Lie algebra cohomology of G. These constructions hold on any Carnot-Caratheodory space regular in some sense. The main result is that d_c exhibits hypoelliptic regularity. This allows to give bounds, and sometimes to compute some of the Novikov-Shubin numbers of G.

Résumé - Nous montrons que la presque cohomologie [7] d'un groupe de Lie nilpotent stratifié G se concentre sur un sous-complexe du complexe de de Rham. Celui-ci est lui même conjugué à un complexe d_c agissant sur la cohomologie de l'algèbre de Lie de G. Ces constructions s'appliquent à tous les espaces de Carnot-Caratheodory réguliers en un certain sens. Le résultat principal est que d_c possède une régularité hypoelliptique. Ceci permet de donner des bornes, et parfois de calculer, certains exposants de Novikov-Shubin de G.

Version française abrégée

Une structure de Carnot-Caratheodory équirégulière sur une variété M est la donnée d'une filtration de TM par des sous-fibrés H_i satisfaisant $H_{i+1} = [H, H_i]$ et $H_r = TM$ pour un certain entier r (cf [6] par ex). Le prototype (et le cône tangent) d'une telle structure s'obtient sur un groupe de Lie nilpotent stratifié G, c'est-à-dire un groupe dont l'algèbre de Lie $\mathfrak{g}$ est graduée et engendrée par les vecteurs de poids 1.

La filtration de TM induit une filtration H_p^* de Λ^*TM, et par dualité, une filtration descendante $F_p = \mathrm{Ann}(H_{p-1}) \subset \Lambda^*T^*M$. Le fait que cette dernière soit stable par différentiation est dual à la propriété $[H_p, H_q] \subset H_{p+q}$. Dans l'esprit de la théorie adiabatique, la suite spectrale naturellement associée à ce complexe filtré, qui converge vers le gradué de la cohomologie de M, doit aussi jouer un rôle dans l'étude de certains problèmes spectraux asymptotiques sur M ([12]). Nous nous intéressons ici aux vitesses de décroissance en grand temps de la chaleur sur les formes d'un groupe stratifié G.

Soit G un tel groupe, muni d'une métrique pour laquelle les composantes homogènes de $\mathfrak{g}$ sont orthogonales. On suppose que G est rationnel, c'est-à-dire possède un sous-groupe discret cocompact Γ. D'après [7], le spectre du Laplacien près de 0 se décrit géométriquement par la presque cohomologie. C'est la famille de cônes de Hilbert $C_\varepsilon = \{\alpha \in L^2(\Omega^p M)/\overline{\mathrm{im}\, d},\ ||d\alpha|| \leq \varepsilon||\alpha||\}$. En particulier la distribution spectrale de δd se lit dans C_ε par $F_{\delta d}(\varepsilon^2) = \dim_\Gamma(E_{\delta d}(\varepsilon^2)) = \sup_L(\dim_\Gamma L)$, où L décrit les sous-Hilbert Γ-invariants de C_ε. On retrouve ainsi les exposants de Novikov-Shubin $\alpha_p = \lim_{\varepsilon\to 0} \frac{\ln F_{\delta d}(\varepsilon)}{\ln \varepsilon}$ donnant la vitesse de décroissance de la chaleur, sur les formes cofermées de $\Omega^p G$ ([7]).

Le lien avec la suite spectrale introduite ci-dessus est que celle-ci décrit, au sens des séries formelles en ε, la structure des formes satisfaisant $d(h_\varepsilon^*\alpha_\varepsilon) = O(\varepsilon^p)$ à travers les dilatations h_ε de G. L'obstruction à produire de la presque cohomologie par dilatation provient de $d_0 = d$ agissant sur F_p/F_{p+1}. Sur une variété C-C générale, d_0 en x_0 est la différentielle de l'algèbre de Lie nilpotente tangente $\mathfrak{g}_{x_0}$. C'est un opérateur *algébrique* qui possède donc un inverse partiel $d_0^{-1} = (\delta_0 d_0)^{-1}\delta_0$ pour une métrique donnée.

On pose alors $E_0 = \ker d_0 \cap \ker d_0^{-1} \simeq \ker d_0 / \operatorname{im} d_0 = H^*(\mathfrak{g}_{x_0})$. On dit qu'une structure C-C est E_0-régulière en x_0 si E_0 est de dimension localement constante au voisinage de x_0, condition génériquement satisfaite. La suite spectrale considérée agit sur E_0. L'algébricité de d_0 permet de le faire plus géométriquement.

Théorème 1. - *Soit M une variété C-C E_0-régulière. Il existe deux sous-complexes (E,d) et (F,d) du complexe de de Rham tels que $\Omega^*M = E \oplus F$, et tels que la projection sur E soit une équivalence d'homotopie de la forme $\Pi_E = \mathrm{Id} - Qd - dQ$ où Q est un opérateur différentiel.*
D'autre part, (E,d) est lui-même conjugué par Π_{E_0} à un complexe différentiel (E_0, d_c).

Ceci entraîne un résultat de concentration de presque-cohomologie. On sait, d'après [7], qu'une équivalence d'homotopie bornée entre deux complexes Hilbertiens (H,d) et (H',d') induit une équivalence des presque-cohomologies au sens qu'il existe des applications bornées $f : C_\varepsilon \to C'_{K\varepsilon}$ et $g : C'_\varepsilon \to C_{K\varepsilon}$ inversibles sur leurs images. Ainsi, les presque-cohomologies de (Ω^*T^*M, d) et du complexe de de Rham spectralement tronqué $(E_{\Delta([0,1])}, d)$ sont équivalentes. Sur $M = G$ comme ci-dessus, on obtient alors :

Corollaire 2. - *La presque-cohomologie C_ε de $(E_{\Delta([0,1])}, d)$ se contracte sur E (c'est-à-dire que $||\Pi_F \lceil C_\varepsilon|| \leq K\varepsilon$).*
- Les complexes $(E_{\Delta([0,1])}, d)$, (E,d) et (E_0, d_c) ont des presque-cohomologies équivalentes.

Pour exploiter analytiquement cette construction, on introduit sur les variétés C-C une notion de régularité utilisant le calcul hypoelliptique développé dans [2]. Soit $|\nabla|$ un opérateur scalaire pseudo-différentiel d'ordre 1 inversible dans ce calcul (voir [2] thm 6.1). On l'étend diagonalement à Ω^*T^*M et on note N le degré homogène induit par la filtration F_p. On dit qu'un opérateur P sur Ω^*T^*M est C-C elliptique si $P^\nabla = |\nabla|^{-N} P |\nabla|^N$ a une paramétrix dans ce calcul.

Théorème 3. *Les complexes d et d_c sont C-C elliptiques.*

La preuve consiste à montrer que les complexes d^∇ et d_c^∇ sont exacts au niveau symbolique, c'est-à-dire à travers les représentations unitaires de G_{x_0} (voir ci-dessous).

Ce résultat permet d'utiliser le complexe d_c pour estimer certains exposants α_p sur les groupes considérés. On pose $\beta_p = \frac{N(G)}{2\alpha_p}$, où $N(G) = \sum i \dim(\mathfrak{g}_i/\mathfrak{g}_{i-1})$ est la dimension homogène de G. L'espace $E_0^* = H^*(\mathfrak{g})$ se décompose en composantes homogènes pour N. On dit que E_0^k est de poids pur p si $N = p$ sur cet espace. E_0^1 est toujours de poids pur 1 ici. Lorsque E_0^k est de poids pur p, on peut considérer les opérateurs tronqués $d_c^i : E_0^k \to E_0^{k+1}/F_{p+i}^{k+1}$, et le plus petit entier i, noté $r_k(d_c)$, pour lequel $d_c^i + \delta_c$ soit C-C elliptique. Cet entier s'interprète comme l'ordre de dégénérescence de la suite spectrale associée au complexe filtré d_c au niveau du calcul symbolique.

Théorème 4. *Soit G un groupe nilpotent, stratifié, rationnel. Si E_0^k est de poids pur p, on a*

$$\sup(N_{\min}(E_0^{k+1}) - p, 1) \leq \beta_k \leq r_k(d_c) \leq N_{\max}(E_0^{k+1}) - p.$$

En particulier, on a toujours $1 \leq \beta_1 \leq r$ où r est le rang de nilpotence de G. Les groupes G à présentation quadratique (voir [3,4]) sont précisément ceux tels que E_0^2 soit de poids pur 2 et ont donc $\beta_1 = 1$. Il en est ainsi des groupes avec $r = 2$ et possédant un plan legendrien Ω-isotrope régulier (cf [6]). A l'opposé, les quotients des algèbres de Lie libres par un idéal engendré par des éléments de degré $\geq r+1$ ont $r \leq \beta_1$. Enfin, il existe des groupes tels que $\beta_1 = r_1(d_c) = 1$ et $N_{\max}(E_0^2) = 3$.

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1. Filtered differential geometry on C-C spaces

An equiregular Carnot-Caratheodory structure on a manifold M consists in a filtration of TM by bundles H_i such that $H_{i+1} = [H, H_i]$ and $H_r = TM$ for some r (see [6]). The model for such a geometry is a stratified nilpotent Lie group G, whose Lie algebra is graded and generated by vectors of weight 1.

This filtration of TM induces a filtration $H_p^* = \operatorname{span}\{\wedge_{i=1}^r(\wedge^{k_i} H_i);\ \sum_{i=1}^r ik_i \leq p\}$ of Λ^*TM and a dual descending filtration $F_p = \operatorname{Ann}(H_{p-1}) \subset \Lambda^*T^*M$. This F_p is d-stable as comes from the dual property $[H_p, H_q] \subset H_{p+q}$. As in the adiabatic theory, the spectral sequence induced by this filtered complex should play a role in some asymptotic spectral problems on M (see [12]). We are concerned here with the rate of decay of heat on forms at large time on G. The relation is that, on one hand, one knows (see [7]) that the heat decay is encoded in the near-cohomology of G, that is in the family of Hilbert cones $C_\varepsilon = \{\alpha \in L^2(\Omega^p G)/\overline{\operatorname{im} d},\ ||d\alpha|| \leq \varepsilon||\alpha||\}$, and on the other hand, the above spectral sequence describes, in the sense of power series, the structure of forms α_ε such that $d(h_\varepsilon^*\alpha_\varepsilon) = O(\varepsilon^p)$ through the dilations h_ε of G. The obstruction to producing near-cohomology by dilations is $d_0 = d$ acting on F_p/F_{p+1}. On a general C-C space, d_0 at x_0 is the differential of the tangent Lie algebra $\mathfrak{g}_{x_0}$ whose structure comes from the bracket $[\,,]_0 : H_p/H_{p-1} \times H_q/H_{q-1} \to H_{p+q}/H_{p+q-1}$. It is an *algebraic* operator so that a partial inverse $d_0^{-1} = (\delta_0 d_0)^{-1}\delta_0$ can be chosen depending on a given metric.

Let $E_0 = \ker d_0 \cap \ker d_0^{-1} \simeq \ker d_0/\operatorname{im} d_0 = H^*(\mathfrak{g}_{x_0})$, and call a C-C structure E_0-regular at x_0 if $\dim E_0$ is locally constant near x_0. The spectral sequence we consider works algebraically on E_0 but the nature of d_0 here allows one to achieve this more geometrically *inside* the de Rham complex. Namely, the operator $r = \operatorname{Id} - d_0^{-1}d - dd_0^{-1}$ induces a homotopical equivalence whose zero order term is $r_0 = \Pi_{E_0}$. This map can be iterated in order to obtain representatives of forms with the smallest $\Pi_{E_0^\perp}$ part.

Theorem 1. - *Let M be a E_0-regular C-C space. The de Rham complex is the direct sum of two sub-complexes $E = \ker d_0^{-1} \cap \ker(d_0^{-1}d)$ and $F = \operatorname{im} d_0^{-1} + \operatorname{im}(dd_0^{-1})$. The projection Π_E on E along F is a homotopical equivalence of the form $\Pi_E = \operatorname{Id} - Qd - dQ$, where Q is a differential operator. The maps r^k converge to Π_E.*

- One has $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ and $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$, so that (E, d) is conjugated through Π_{E_0} and Π_E to the complex (E_0, d_c) with $d_c = \Pi_{E_0} d \Pi_E \Pi_{E_0}$.

The main result we need is :

Lemma. *The map $d_0^{-1}d$ induces an isomorphism on $\operatorname{im} d_0^{-1}$, whose inverse is a differential operator P. Define then $Q = Pd_0^{-1}$ for theorem 1.*

Proofs. - On $\operatorname{im} d_0^{-1}$, one can write $d_0^{-1}d = \operatorname{Id} + D$ where $D = d_0^{-1}(d - d_0)$ is a nilpotent differential operator since $D(F_p) \subset F_{p+1}$. One has then $P = (d_0^{-1}d)^{-1} = \operatorname{Id} - D + D^2 \cdots$.

- Let $Q = Pd_0^{-1}$ and consider $\Pi = Qd + dQ$. By construction $\operatorname{im} Q \subset \operatorname{im} P \subset \operatorname{im} d_0^{-1}$, so that, $\operatorname{im}\Pi \subset F = \operatorname{im}(d_0^{-1} + dd_0^{-1})$. Conversely, $Qd_0^{-1} = P(d_0^{-1})^2 = 0$ and $Qd = \mathrm{Id}$ on $\operatorname{im} d_0^{-1}$ so that $\Pi = \mathrm{Id}$ on F. Regarding $E = \ker d_0^{-1} \cap \ker(d_0^{-1}d)$, one has $\Pi = Pd_0^{-1}d + dPd_0^{-1} = 0$ on it. Moreover, $d_0^{-1}(\mathrm{Id} - \Pi) = d_0^{-1} - d_0^{-1}dPd_0^{-1} - d_0^{-1}Qd = 0$ and $d_0^{-1}d(\mathrm{Id} - \Pi) = d_0^{-1}d - d_0^{-1}dQd = 0$ so that $\operatorname{im}(\mathrm{Id} - \Pi) \subset E$. This shows that Π is the projection on F along E.
- The statement regarding r follows from the fact that first $r = \mathrm{Id} - d_0^{-1}d - dd_0^{-1} = \mathrm{Id}$ on E. Also r preserves and induces a nilpotent operator on F. Namely, on $\operatorname{im} d_0^{-1}$, one has $r = 1 - d_0^{-1}d = -d_0^{-1}(d - d_0) = -D$ (the above nilpotent operator), whereas on $\operatorname{im} dd_0^{-1}$, $r(dd_0^{-1}) = (1 - dd_0^{-1})dd_0^{-1} = d(1 - d_0^{-1}(d_0 + d - d_0))d_0^{-1} = -dd_0^{-1}(d - d_0)d_0^{-1}$. This shows that $r(dd_0^{-1}F_p) \subset dd_0^{-1}(F_{p+1})$ and the nilpotency holds.
- One has $\Pi_F = Qd + dQ$ with, by construction, $\operatorname{im} Q \subset \operatorname{im} d_0^{-1} \subset E_0^\perp$ and $E_0 \subset \ker d_0^{-1} \subset \ker Q$, so that we obtain $\Pi_{E_0}\Pi_F\Pi_{E_0} = 0$. Also, $\Pi_{E_0^\perp} = d_0^{-1}d_0 + d_0d_0^{-1}$ with $d_0^{-1}\Pi_E = \Pi_E d_0^{-1} = 0$ since $E \subset \ker d_0^{-1}$ and $\operatorname{im} d_0^{-1} \subset F = \ker \Pi_E$, and therefore $\Pi_E\Pi_{E_0^\perp}\Pi_E = 0$. □

Remarks. - Although this construction is obtained by elementary considerations, we have no doubt that it would also enter more general theories such as Spencer cohomology and related works of Bryant-Griffiths [1] and Vinogradov. We note also that in the case of two step distributions the spectral sequence we use turns out to be equivalent to the one defined by Julg in [9] using quite different methods. At last, we observe that the existence of d_c was suggested in [6] 4.11.C".

We have derived two complexes (E, d) and (E_0, d_c) that still compute cohomology but now *strictly* increase the filtration weight. On $M = G$ we can interpret them as near-cohomology complexes by showing that the de Rham near-cohomology concentrates in some sense on them. Namely, one knows by [7] that boundedly homotopically equivalent Hilbert complexes (H, d) and (H, d') have equivalent near-cohomologies in the sense that there exists bounded maps $f : C_\varepsilon \to C'_{K\varepsilon}$ and $g : C'_\varepsilon \to C_{K\varepsilon}$ invertible on their images. In particular, the de Rham complex and the spectrally cut-off one $(E_{\Delta([0,1])}, d)$ have equivalent near-cohomologies. Therefore we obtain the following results :

Corollary 2. - *The near-cohomology C_ε of $(E_{\Delta([0,1])}, d)$ contracts on E, that is*

$$||\Pi_F \lceil C_\varepsilon|| \leq K\varepsilon \quad \text{for some } K.$$

- *The complexes $(E_{\Delta([0,1])}, d)$, (E, d) and (E_0, d_c) have equivalent near-cohomologies.*

2. Analytic properties

We now address regularity questions. The first fact about the two geometrically equivalent complexes (E, d) and (E_0, d_c) is that E is a space defined by differential equations whereas E_0 is algebraic. Another feature of d_c is that it is Hodge $*$ self-dual. Indeed, one can show that $*$ preserves E_0 and $\delta_c = (-1)^k * d_c *$ on E_0^k.
Proof. - The choice of a metric gives an orthogonal splitting of $TM = \oplus V_i$ with $H_{i+1} = H_i \oplus^\perp V_i$. This fixes a weight N on Λ^*T^*M such that $F_p = F_{p+1} \oplus^\perp G_p$ and $G_p = \ker(N - p)$. One has then $*G_p = G_{N_0 - p}$ with $N_0 = N(\Lambda^{\max}T^*M)$. Identifying F_p/F_{p+1} with G_p, we now get $\delta_0 = (-1)^k * d_0 *$ as comes from $d_0\alpha \wedge *\beta + (-1)^k \alpha \wedge d_0 * \beta = d_0(\alpha \wedge *\beta) = 0$, since $d_0(\alpha \wedge *\beta)$ is a top form of weight $N_0 - 1$ here. This gives $*E_0 = E_0$.

- To prove $\delta_c = (-1)^k * d_c *$ we develop

$$d(\Pi_E\alpha \wedge \Pi_E * \beta) = \Pi_E d_c\alpha \wedge \Pi_E * \beta + (-1)^k \Pi_E\alpha \wedge \Pi_E d_c * \beta$$
$$= d_c\alpha \wedge *\beta + (-1)^k \alpha \wedge d_c * \beta$$

if we interpret these wedges as scalar products and observe that $\Pi_F(E_0) \subset \operatorname{im} d_0^{-1} = (\ker d_0)^\perp$. □

We consider now some regularity notion based on the pseudodifferential hypoelliptic calculus developed in [2]. Strictly speaking this calculus was developed there on groups but extends on equiregular C-C spaces (see [10]). This is because maximal hypoellipticity is a stable property so that it can be checked on the model space, the filtered Lie group here, the operators and the produced kernels of parabolic homogeneity being transplanted by Métivier's approximation map available in equiregular C-C geometry (see [8 (7.2),10,11]). Let $|\nabla|$ be a scalar first order peudodifferential operator invertible in this calculus. On a filtered group G, one can take $|\nabla| = (\Delta_H)^{\frac{1}{2}}$ with Δ_H the Kohn Laplacian (see [2,5]). We locally extend $|\nabla|$ as acting diagonally on Ω^*T^*M. An operator P on Ω^*T^*M is said C-C elliptic if $P^\nabla = |\nabla|^{-N}P|\nabla|^N$ has a parametrice in the calculus, where N is the filtration weight.

Theorem 3. *The complexes d and d_c are C-C elliptic on a E_0-regular C-C space.*

Proof. - We have to show that the complexes d^∇, d_c^∇ are acyclic at the symbolic level. Here the symbol consists in taking the images of the operator through all non-trivial unitary irreducible representations σ_ξ of G_{x_0}. They are parametrized by $\xi \neq 0$ in $\mathfrak{g}_{x_0}^*$. By nilpotency of $\mathfrak{g}_{x_0}$, one can find for each ξ a vector $X \in i\mathfrak{g}_{x_0}$ such that $\sigma_\xi(X) = \mathrm{Id}$. Consider then $\mathcal{L}_X = X.\mathrm{Id} + \mathrm{ad}(X)$ on G_{x_0}. Its C-C symbol is $\sigma_\xi(\mathcal{L}_X^\nabla) = \mathrm{Id} + \sigma_\xi(\mathrm{ad}(X)^\nabla)$ with $\mathrm{ad}(X)$ nilpotent, so that we get its inverse $P_X = \sum_{k\geq 0}(-1)^k \sigma_\xi(\mathrm{ad}(X)^\nabla)^k$. One has $[P, \sigma_\xi(d^\nabla)] = [P, \sigma_\xi(i_X^\nabla)] = 0$ by $[\mathcal{L}_X, d] = [\mathcal{L}_X, i_X] = 0$. Cartan's formula $\mathcal{L}_X = i_X d + d i_X$ gives then

$$\mathrm{Id} = (P_X\sigma(i_X^\nabla))\sigma_\xi(d^\nabla) + \sigma_\xi(d^\nabla)(P_X\sigma(i_X^\nabla))$$

which is the acyclicity of $\sigma_\xi(d^\nabla)$. The proof for d_c follows from the previous formula. We compose it on the left by $\sigma_\xi(\Pi_{E_0}\Pi_E)^\nabla$ and on the right by $\sigma_\xi(\Pi_E\Pi_{E_0})^\nabla$ to get on E_0

$$\mathrm{Id} = Q_X\sigma_\xi(d_c^\nabla) + \sigma_\xi(d_c^\nabla)Q_X$$

where $Q_X = \big(\sigma_\xi(\Pi_{E_0}\Pi_E)^\nabla\big)P_X\sigma(i_X^\nabla)\big(\sigma_\xi(\Pi_E\Pi_{E_0})^\nabla\big)$. □

3. Application to Novikov-Shubin numbers

We consider now nilpotent stratified Lie groups which moreover are rational, that is, possess a discrete cocompact subgroup Γ. Following [7], the spectral distribution of δd is interpreted as the Γ-linear thickness of the cones C_ε, which means that $F_{\delta d}(\varepsilon^2) = \dim_\Gamma(E_{\delta d}(\varepsilon^2)) = \sup_L(\dim_\Gamma L)$, where L describes the closed Γ-invariant linear subspaces of C_ε. The Novikov-Shubin invariants of G are then $\alpha_k = \lim_{\varepsilon\to 0} \frac{\ln F_{\delta d}(\varepsilon)}{\ln \varepsilon}$, giving the heat decay exponents at large time, on coclosed k-forms of Ω^*G ([7]). The space $E_0^* \simeq H^*(G)$ naturally splits under the weight N. When E_0^k is of pure weight p, one can consider the truncated operators $d_c^i : E_0^k \to E_0^{k+1}/F_{p+i}^{k+1}$, and the smallest integer i such that $d_c^i + \delta_c$ is C-C elliptic. This integer $r_k(d_c)$ is the degeneracy order at the symbolic level of the spectral sequence associated to the filtered complex d_c. Define at last $\beta_k = \frac{N_0}{2\alpha_k}$ with $N_0 = \sum i \dim(\mathfrak{g}_i/\mathfrak{g}_{i-1})$.

Theorem 4. *Let G be a stratified nilpotent rational Lie group with E_0^k of pure weight p, then*

$$\sup(N_{\min}(E_0^{k+1})-p,1)\leq\beta_k\leq r_k(d_c)\leq N_{\max}(E_0^{k+1})-p.$$

As E_0^1 is always of pure weight 1, one obtains $1\leq\beta_1\leq r$ for r-steps groups. Quadratically presented groups (see [3,4]) are precisely those whose E_0^2 is of pure weight 2, and thus have $\beta_1=1$. It is the case of two step groups with a regular Ω-isotropic Legendrian plane (see [6]). In the opposite direction, quotients of free Lie algebras by ideal generated by elements of degree $\geq r+1$ satisfy $r\leq\beta_1$. Furthermore, there exist groups with $\beta_1=r_1(d_c)=1$ and $N_{\max}(E_0^2)=3$. The proof of theorem 4 consists mainly in exploiting homogeneity and C-C regularity with the shift weight $N-p$.

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