DIABATIC LIMIT, ETA INVARIANTS AND CAUCHY-RIEMANN MANIFOLDS OF DIMENSION 3

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Abstract. We relate a recently introduced non-local invariant of compact strictly pseudoconvex Cauchy-Riemann (CR) manifolds of dimension 3 to various η-invariants: on the one hand a renormalized η-invariant appearing when considering a sequence of metrics converging to the CR structure by expanding the size of the Reeb field; on the other hand the η-invariant of the middle degree operator of the contact complex. We then provide explicit computations for transverse circle invariant CR structures on Seifert manifolds. This yields obstructions to filling a CR manifold by complex hyperbolic, Kähler-Einstein, or Einstein manifolds.

Résumé. Nous relions un nouvel invariant non local des variétés Cauchy-Riemann (CR) strictement pseudoconvexes et compactes de dimension 3 à d'autres invariants de type η en géométrie CR : d'une part celui obtenu en considérant une suite de métriques riemanniennes adaptées à la structure CR et en faisant tendre vers l'infini la longueur du champ de Reeb, d'autre part l'invariant η de l'opérateur apparaissant en degré moitié dans le complexe de contact. Nous les calculons ensuite sur les variétés de Seifert admettant une structure CR invariante par l'action transverse d'un cercle. Les résultats fournissent des obstructions au remplissage d'une variété CR par une variété hyperbolique complexe, Kähler-Einstein ou d'Einstein.

Contents

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1. Introduction

In [11] the first two authors of this paper introduced a new invariant, called the ν-invariant, of strictly pseudoconvex Cauchy-Riemann (CR) compact 3-manifolds. This invariant was obtained by taking the limit of the η-invariants of an adequately defined (but quite complicated) sequence of Riemannian metrics approximating the CR structure, after cancellation of the possibly diverging terms by adding well-chosen local contributions. We claimed in [11] that this invariant may have an analogous role in CR geometry of as the η-invariant has in conformal geometry. However, its rather abstract definition makes it difficult to compute explicit expressions for it or to get a further understanding of its properties. The goal of this paper is then to provide links between ν and other natural η-invariants in CR geometry.

In a first step, we introduce a renormalized η-invariant that takes into account the fact that CR geometry can be seen as a limit of a sequence of conformal structures that diverges outside the contact distribution. If a compatible contact form θ is fixed on the CR manifold M, one considers the family of metrics

$$h_\varepsilon = \varepsilon^{-1}\theta^2 + \gamma, \tag{1}$$

where $\gamma = d\theta(\cdot, J\cdot)$ and J is the underlying complex structure on the contact distribution. *When* ε *goes to* 0 the metrics h_ε blow up except in the contact distribution, and therefore the metric geometry of h_ε converges to the Carnot-Carathéodory metric associated to the CR structure and the contact form (this is one of the main motivation for considering this kind of sequences). A natural object one can consider is the constant term η_0 in an asymptotic expansion for $(\eta(h_\varepsilon))$ in powers of ε, *when* ε *goes to* 0. This always exists, as we shall see, and we shall call it the *renormalized* η*-invariant of the pseudohermitian manifold* (M, θ, J). This invariant is of course much more easily studied than the ν-invariant, because it is built from the sequence (1) of metrics that is much simpler than the one used to build ν in [11]. Note however that it is a pseudohermitian invariant, *i.e.* it depends on the choice of θ and J, contrarily to ν which depends only on the choice of the contact distribution H (and of course on J).

In the other direction, *i.e. when* ε *goes to* ∞, one can also obtain another natural invariant in case the Tanaka-Webster torsion of (M, θ) vanishes, that is when the action of the Reeb vector field is isometric. In this case, $\eta(h_\varepsilon)$ converges and its limit η_{ad} is the so called *adiabatic limit*. It has attracted much attention in the past few years, see [12, 23] for instance. We shall call the reverse process of taking a limit when ε goes to 0 a *diabatic* limit. When torsion vanishes, it turns out that the diabatic η_0 equals the adiabatic η_{ad}.

Our first result shows that the difference between the CR invariant ν and the pseudohermitian η_0 is an integral of a local contribution involving the square of the Tanaka-Webster curvature.

1.1. **Theorem.** *For any compact strictly pseudoconvex Cauchy-Riemann* 3*-manifold* M*, and any choice* θ *of contact form, one has*

$$\nu(M) \;=\; -3\,\eta_0(M,\theta) + \frac{1}{16\pi^2}\int_M R^2\theta\wedge d\theta \;, \tag{2}$$

where R *is the Tanaka-Webster curvature of* (M,θ)*.*

This yields a new definition of the ν-invariant, see Remark 4.2, together with some explicit computations: they can be done on manifolds on which η_0 is computable. We are then able to apply this to transverse $\mathbb{S}^1$-invariant CR structures on Seifert manifolds. The CR manifolds we are interested in come with a locally free action of $\mathbb{S}^1$ that is transverse to the contact distribution, and preserves both the contact and the complex structures. We shall call them *Cauchy-Riemann-Seifert manifolds* (in short CR Seifert). We refer to [31] for more information on the more general class of $\mathbb{S}^1$-invariant CR structures. CR Seifert manifolds can be efficiently described as orbifold $\mathbb{S}^1$-bundles over 2-dimensional orbifolds. At each orbifold point on the base, the orbifold bundle data consists of the following: if the local fundamental group is $\mathbb{Z}/\alpha\mathbb{Z}$ ($\alpha\in\mathbb{N}^*$), a generator acts on a local chart around p on the basis manifold as $\mathrm{e}^{i\frac{2\pi}{\alpha}}$ and on the fiber as $\mathrm{e}^{i\frac{2\pi\beta}{\alpha}}$ with β prime to α. The orbifold $\mathbb{S}^1$-bundles are topologically classified by their degrees (first Chern numbers), which are in this case rational numbers. One then endows the manifold with an invariant strictly pseudoconvex CR structure as follows: the underlying contact structure is provided by an equivariant connection 1-form on the bundle, whereas the complex structure is induced from the basis (orbifold) Riemann surface; the strict pseudoconvexity condition constrains the degrees d of these $\mathbb{S}^1$-bundles to be negative.

Building on computations done by Ouyang [41] and Komuro [33], we get the following:

1.2. **Theorem.** *Let* M *be a compact strictly pseudoconvex CR Seifert* 3*-manifold, of degree* d *over the orbifold surface* Σ*, and with* $\mathbb{S}^1$*-action generated by the Reeb field of a contact form* θ*. If* R *is the Tanaka-Webster curvature of* (M,θ)*, then*

$$\nu(M) = -d - 3 - 12\sum_{j=1}^{p} s(\alpha_j,1,\beta_j) + \frac{1}{8\pi}\int_\Sigma R^2 d\theta \;, \tag{3}$$

where $s(\alpha,\rho,\beta)$ *is the Rademacher-Dedekind sum* $\frac{1}{4\alpha}\sum_{k=1}^{\alpha-1}\cot\left(\frac{k\rho\pi}{\alpha}\right)\cot\left(\frac{k\beta\pi}{\alpha}\right)$ *.*

The Tanaka-Webster curvature R of such an (M,θ) actually coincides with Riemannian curvature of the base Σ, if it is endowed with the metric $\gamma = d\theta(\cdot, J\cdot)$. When this curvature is constant, (3) specializes into the following interesting formula, which shows that the ν-invariant is a topological invariant in this case:

1.3. **Corollary.** *Let M be a CR Seifert manifold as above, with constant Tanaka-Webster curvature. Let χ be the rational Euler characteristic of Σ. Then,*

$$\nu(M) = -d - 3 - \frac{\chi^2}{4d} - 12 \sum_{j=1}^{p} s(\alpha_j, 1, \beta_j)\ . \tag{4}$$

Similar formulas in the case of a smooth circle bundle over a smooth Riemann surface have been computed by Burns-Epstein for their μ-invariant [14]. The μ-invariant also is a CR invariant, but it is only defined on compact 3-dimensional strictly pseudoconvex CR manifolds whose holomorphic tangent bundle's first Chern class is a torsion element in homology with integer coefficients. In some sense, our ν-invariant appears as the general Atiyah-Patodi-Singer invariant for 3-dimensional CR geometry that reduces to the Chern-Simons invariant μ in the case some characteristic tangent bundle is trivial (up to a finite covering).

However, Theorem 1.1 is not entirely satisfactory, as it provides a link between the CR invariant ν and the diabatic invariant η_0; one would instead prefer a relationship between ν and invariants defined directly in terms of the CR or pseudohermitian geometry. One such object is the contact-de Rham complex [46], and especially the η-invariant of the middle degree operator appearing there.

The relevant operator (denoted by $D*$ henceforth) is the analogue in this setting of the *boundary operator for the signature* $\pm(d * - * d)$ that gives rise to the η-invariant on 3-dimensional Riemannian manifolds. It is known that the spectrum of the operator $D*$ appears in the rescaled limit of the collapsing spectrum of $P_\varepsilon = \pm(d *_\varepsilon - *_\varepsilon d)$ for the metrics h_ε of (1), when performing the diabatic limit [47]. However, this limit is not uniform enough to yield a direct relation between the η-invariants. In this paper, we prove a general relation between ν and $\eta(D*)$ in the special case provided by our CR Seifert manifolds. In effect, we show that $\eta(D*)$ and ν differ only by a simple local term in the Tanaka-Webster curvature of any chosen pseudohermitian structure. Our second main set of results then reads:

1.4. **Theorem.** *Let M be a compact strictly pseudoconvex CR Seifert 3-manifold, with $\mathbb{S}^1$-action generated by the Reeb field of an $\mathbb{S}^1$-invariant contact form θ. If R is the Tanaka-Webster curvature of (M,θ) and D is the middle operator of the contact complex, then*

$$\eta_0(M,\theta) = \eta(D*) + \frac{1}{512}\int_M R^2\theta\wedge d\theta. \tag{5}$$

1.5. **Corollary.** *Let M be a CR Seifert 3-manifold as above, then for the $\mathbb{S}^1$-invariant contact form θ one has:*

$$\nu(M) \ = \ -3\,\eta(D*) \ + \ \left(\frac{1}{16\pi^2} - \frac{3}{512}\right)\int_M R^2\,\theta\wedge d\theta\ . \tag{6}$$

The philosophy underlying our results is indeed the following: whereas ν is easily related to η_0, $\eta(D*)$ compares itself more easily with η_0 rather than to ν. This somehow "explains" the quite strange combination of constants appearing in front of the curvature term in (6) in Theorem 1.5: it is a sum of diabatic contribution stemming from Theorem 1.1 and a purely spectral term linking $\eta(D*)$ and η_0, as will be apparent from Section 7.

For general CR manifolds, we expect that when we take the diabatic limit $\varepsilon \to 0$, the collapsing spectrum of P_ε gives the contribution $\eta(D*)$ in the limit, while the remaining part of the spectrum, after renormalization, gives only an integral of local terms. This leads to the following conjecture.

1.6. **Conjecture.** *There exists a constant C such that, for any compact strictly pseudoconvex Cauchy-Riemann 3-manifold M and any choice θ of contact form, one has*

$$\nu(M) \;=\; -\,3\,\eta(D*) \;+\; \left(\frac{1}{16\pi^2} - \frac{3}{512}\right) \int_M R^2 \theta \wedge d\theta \;+\; C \int_M |\tau|^2\, \theta \wedge d\theta \;, \tag{7}$$

with R and τ the Tanaka-Webster curvature and torsion of (M, θ).

As a first indication for the conjecture, we shall give in Theorem 9.4 an abstract argument that shows that there exists a CR invariant of the form $\eta(D*) + C_1 \int R^2 + C_2 \int |\tau|^2$. Unfortunately, we are unable to calculate the constants completely, see Remark 9.6.

It is known that the η-invariant of the boundary operator for signature is conformally invariant. If the conjecture is true, then this is no more the case for $\eta(D*)$, which is a priori an invariant of the pseudohermitian structure only: it depends on the choice of a metric in the conformal class adapted to the CR structure. (Note that formula (6) alone is not enough to conclude that $\eta(D*)$ is not a CR invariant, since it is *a priori* true only for a specific choice of contact form on CR Seifert manifold.)

A third goal of this paper is to provide some geometric applications on CR Seifert manifolds, mainly with constant curvature. They are spherical (locally isomorphic to the standard CR sphere $\mathbb{S}^3$), hence are the boundary at infinity of a complex hyperbolic metric defined in a neighbourhood $(0, \varepsilon) \times M$ of M (in the case of the 3-sphere we can of course extend the metric globally to get the Bergman metric on the 4-ball). From [11, Theorem 1.2] and Theorem 1.3, we get the following obstruction for this neighbourhood to have a global extension to a smooth complex hyperbolic surface (with only one end):

1.7. **Corollary.** *If a CR Seifert manifold M^3 is the boundary at infinity of a complex hyperbolic metric defined on the interior of a smooth compact manifold N^4 with boundary M, then one has necessarily $\nu(M) = -\chi(N) + 3\tau(N)$, where $\chi(N)$ and $\tau(N)$ denote the Euler characteristic and signature of N. In particular, $\nu(M)$, as provided by the formula* (3)*, is an integer.*

This is a topological constraint on a filling, which we can restate in the smooth case (no orbifold singularities):

1.8. **Corollary.** *Let M be a $\mathbb{S}^1$-bundle of degree d over a Riemann surface Σ of Euler characteristic χ, with a $\mathbb{S}^1$-invariant spherical CR structure. If $\frac{\chi^2}{4d}$ is not an integer then M is not the boundary at infinity of a complex hyperbolic metric.*

The case $d = \frac{\chi}{2}$ yields an integer, and indeed, if Σ is hyperbolic, N can be taken to be the disk bundle of a square root of the tangent bundle of Σ, which is well known to carry a complex hyperbolic metric issued from a representation of $\pi_1(\Sigma)$ in $SU(1,1) \subset SU(1,2)$. Our obstruction then gives an interesting hint on whether a spherical CR Seifert 3-manifold may appear as the quotient of the complement of the limit set in the 3-sphere of some discrete fixed point-free subgroup of $SU(1,2)$ [1]. Similar results were proved by Burns-Epstein with their μ-invariant [15]; since both invariants, although sharing some properties, are truly different, it turns out that our results are more precise and forbid existence of a complex hyperbolic filling in cases that are not obstructed by Burns-Epstein's μ-invariant. A detailed study of this point is done in Section 10 below.

More generally, the calculation in Theorem 1.2 gives an obstruction for M to be the boundary at infinity of a Kähler-Einstein or Einstein metric. The manifolds considered in this paper are known to bound a complex Stein space with at most a finite number of singular points [27] and one may wish to endow it with a Kähler-Einstein metric as in Cheng-Yau [20]. The type of metric to be considered has the same kind of asymptotic expansion near the boundary M as the Bergman metric [10]; we called them "asymptotically complex hyperbolic" (ACH) in [11]. If no singular points are present and if the Cheng-Yau metric exists, one gets from the Miyaoka-Yau inequality proved in [45] the following:

1.9. **Corollary.** *Let M be as in Theorem 1.2. If M is the boundary at infinity of an ACH Einstein metric on N, such that a Kronheimer-Mrowka invariant of (N, M) is non-zero (in particular, if M is the boundary at infinity of a Kähler-Einstein metric on N), then*

$$\chi(N) - 3\tau(N) \geqslant -\nu(M) = d + 3 + 12 \sum_{j=1}^{p} s(\alpha_j, 1, \beta_j) - \frac{1}{8\pi} \int_{\Sigma} R^2 \, d\theta \ .$$

For more information on Stein fillings, see [38, 52]. The Kronheimer-Mrowka invariants are Seiberg-Witten type invariants defined for a compact 4-manifold with contact boundary; in particular, they do not vanish if N carries a symplectic form compatible with the contact structure on the boundary, and this implies the Miyaoka-Yau inequality [45]. This inequality can of course be obtained directly for Kähler-Einstein metrics.

The paper is organized as follows. After recalling the definition of the ν-invariant in Section 2, we define the renormalized η-invariant η_0 and compare it to ν in Sections 3 and 4. The proof relies on relatively simple considerations on η-invariants and Chern-Simons theory, that prove that the difference between $\nu + 3\,\eta_0$ is necessarily of the expected form: an integral term in the square of the curvature and the squared norm of the torsion. The constants in front of these local terms are then computed by considering sufficiently many examples: left invariant structures on the 3-sphere.

The reader will then find in Section 5 the explicit computations of ν on CR Seifert manifolds.

Taking one step further, Sections 6 to 8 lead to the relation between η_0 and $\eta(D*)$ in the case of transverse $\mathbb{S}^1$-invariant CR structures. The proof of Theorem 1.5 relies on an explicit study of the spectra of the $D*$ operator and the boundary operator for the signature $\pm(d_\varepsilon * - * d_\varepsilon)$ on closed 2-forms for the sequence of Riemannian metrics h_ε that performs the diabatic limit in (1). This can be done only for $\mathbb{S}^1$-invariant structures and index theory shows once again that a relation of the expected type must exist. One then has again to evaluate the constant in front of the integral term by looking at explicit computations of both $\eta(D*)$ and ν on the standard sphere.

The existence of a CR invariant of the form $\eta(D*) + C_1 \int R^2 + C_2 \int |\tau|^2$ is considered in Section 9. We also present a proof of the existence of $\eta(D*)$ on any compact strictly pseudoconvex CR manifold of dimension 3, a fact certainly known to specialists but whose proof seems to have never been published so far.

The paper ends with a short Section 10 devoted to the proof of the corollaries and to some generalizations, and also to a comparison with the results one can get in the Kähler-Einstein case using the μ-invariant of Burns and Epstein [14].

2. The ν-invariant

Cauchy-Riemann and pseudohermitian geometry. Let M be a 3-dimensional compact strictly pseudoconvex CR manifold, *i.e.* a compact manifold M endowed with a complex structure J defined on a contact distribution H in TM. A pseudohermitian structure on M consists in the additional choice of a contact form θ. It induces a metric $\gamma = d\theta(\cdot, J\cdot)$ on H. The CR structure is said to be strictly pseudoconvex if this metric is definite (positive or negative) for some choice of contact form (and, hence, any choice). The choice of a pseudo-hermitian structure also induces a splitting of both TM and T^*M by means of the Reeb vector field T defined by $\theta(T) = 1$ and $\iota_T d\theta = 0$. The Tanaka-Webster connection is then defined by working in a local coframe $(\theta, \theta^1, \theta^{\bar{1}})$ such that $d\theta = i\theta^1 \wedge \theta^{\bar{1}}$: the connection form is a purely imaginary 1-form ω_1^1, and the torsion τ^1 is a $(0,1)$-form such that

$$d\theta^1 = \theta^1 \wedge \omega_1^1 + \theta \wedge \tau^1,$$

and the curvature R is defined by

$$d\omega_1^1 = -iR\, d\theta + (\tau^{\bar{1}}_{,\bar{1}} - \tau^1_{,1}) \wedge \theta.$$

In more invariant terms, it is the only metric and complex compatible connection ∇ on H such that the torsion $\tau = T^\nabla(T, \cdot)_{|H}$ anticommutes with J.

Fillings of 3-dimensional CR manifolds. The ν-invariant was first defined in [11] by considering *Einstein asymptotically complex hyperbolic* (*ACHE*, in short) fillings of 3-dimensional strictly pseudoconvex compact CR manifolds. We now review the necessary elements for this definition.

Given a pseudohermitian manifold (M, θ), we first consider the product space $N = \mathbb{R}_+ \times M$. One can then define a metric g_0 on N by

$$g_0 = dr^2 + h_0(r), \qquad \text{with} \quad h_0(r) = \mathrm{e}^{2r}\,\theta^2 + \mathrm{e}^r\,\gamma. \tag{8}$$

To understand properly what is done here, one must think of the initial M as a boundary of N *at infinity* (*i.e.* when r goes to infinity); when conformally changing θ into $\theta' = f\theta$, one gets a metric $g_0' = (dr')^2 + \mathrm{e}^{2r'} f^2\theta^2 + \mathrm{e}^{r'} f\gamma$, and the difference $g_0' - g_0$ goes to zero at infinity after the coordinate change $r = r' + \log f$. Therefore the asymptotic behaviour of the metric g_0 depends only on the CR structure. We note moreover that

$$h_0(r) = \mathrm{e}^r\,(\mathrm{e}^r\,\theta^2 + \gamma) = \varepsilon^{-1} h_\varepsilon,$$

where h_ε is the metric introduced in equation (1), with $\varepsilon = \mathrm{e}^{-r}$.

We can now also extend J, initially defined on M, to an almost complex structure J_0 on the whole N. Indeed, Catlin [18] showed that a strictly pseudoconvex CR structure always determines the infinite jet of a complex structure of N along M. One can perform this explicitly as follows: first define

$$J_0 \partial_r = \mathrm{e}^{-r}\,T,$$

where T is the Reeb field associated to θ. A (formal) integrable complex structure on N can then be deduced as a series in powers of e^{-r} step by step by requiring the Nijenhuis tensor to vanish at all orders. More precisely, one finds an infinite series $J(r) = J_0 + J_1\,\mathrm{e}^{-r} + J_2\,\mathrm{e}^{-2r} + \cdots$ on N, whose coefficients are given in terms of the covariant derivatives of Tanaka-Webster curvature R and torsion τ of the pseudohermitian manifold (M, θ). The first terms are

$$J(r) = J_0 - 2\,\mathrm{e}^{-r}\,\tau + \mathrm{e}^{-2r}(2|\tau|^2 - J_0 \nabla_T \tau) + \cdots .$$

This construction is in fact independent of the choice of the pseudohermitian structure: following classical ideas in complex analysis, it is easily shown that conformally changing the contact form on the boundary leads to an equivalent complex structure in the inside which is related to the original one by a diffeomorphism continuously extending as the identity on the boundary, see [11, proposition 3.5] for details.

This data can be complemented by that of a (formal) *Kähler-Einstein metric* on N whose dominant term is given by the metric g_0 described above: as explained in [11, Section 2], g_0 is an asymptotically Kähler metric, and its curvature is asymptotic when r goes to $+\infty$ to the curvature of the complex hyperbolic plane with holomorphic sectional curvature -1.

It is then *asymptotically Einstein*, in the sense that

$$\mathrm{Ric}(g_0) + \frac{3}{2}\, g_0 = O(\mathrm{e}^{-r}).$$

One can now add higher order corrections (in powers of e^{-r}) to g_0 to get a *uniquely defined jet of a Kähler-Einstein metric* g_{KE} up to order e^{-2r} (relatively to g_0), when r tends to infinity. This development is again expressed with the covariant derivatives of Tanaka-Webster curvature R and torsion τ of the pseudohermitian manifold (M, θ), and has been calculated in [11, Theorem 3.3 and Corollary 3.4]: given some choice of coframe $\theta^1 \in \Omega^{1,0}H$, the expression of its Kähler form ω is

$$\begin{aligned} \omega &= \mathrm{e}^r(dr \wedge \theta + d\theta) - \frac{R}{2} d\theta \\ &+ \frac{4}{3}\big(\frac{i}{8} R_{,\bar 1}\vartheta^0 \wedge \theta^{\bar 1} - \frac{i}{8} R_{,1}\vartheta^{\bar 0} \wedge \theta^1 - \frac{1}{2}\tau^1_{\bar 1,1}\vartheta^0 \wedge \theta^{\bar 1} - \frac{1}{2}\tau^{\bar 1}_{1,\bar 1}\vartheta^{\bar 0} \wedge \theta^1\big) \\ &- \frac{\Delta_H R}{2}\,\mathrm{e}^{-r}\, d\theta - \frac{2}{3}\Big(\frac{R^2}{8} - |\tau|^2 - \frac{\Delta_H R}{6} + \frac{2i}{3}(\tau^1_{\bar 1,11} - \tau^{\bar 1}_{1,\bar 1\bar 1})\Big)\,\mathrm{e}^{-r}\, dr \wedge \theta \\ &+ \frac{2}{3}\Big(\frac{R^2}{8} - |\tau|^2 - \frac{\Delta_H R}{12} - \frac{i}{3}(\tau^1_{\bar 1,11} - \tau^{\bar 1}_{1,\bar 1\bar 1})\Big)\,\mathrm{e}^{-r}\, d\theta + o(\mathrm{e}^{-2r}), \end{aligned} \tag{9}$$

where

$$\begin{cases} \vartheta^0 &= \mathrm{e}^{-r}\, dr + i\theta, \\ \vartheta^1 &= \theta^1 + i\,\mathrm{e}^{-r}\,\tau^1 + \cdots \end{cases}$$

is a coframe of $\Omega^{1,0}N$ associated to $J(r)$.

We will denote by g_{KE} the metric on N given by this second order jet of Kähler metric: $g_{KE} = \omega(\cdot, J(r)\cdot)$. Its development may be pursued further (at least up to order 3), but it is explained in [11] why terms in ω (resp. g_{KE}) of order strictly higher than 2 are irrelevant in all that concerns the ν-invariant to be defined below. Roughly speaking, the ν-invariant will be defined by taking a limit of Riemannian invariants associated with g_{KE} when r goes to infinity, and it is shown in [11] that any extra term in g_{KE} that is $o(\mathrm{e}^{-2r})$ contributes to 0 in the limit.

We now observe that g_{KE} has an *universal* polynomial expression in the powers of e^r, with coefficients that are tensorial in the covariant derivatives of R and τ. By construction the leading term of g_{KE} is g_0 as given in (8), and the family of metrics $h(r)$ induced on

$$M_r = \{r\} \times M \simeq M$$

is asymptotic to $h_0(r)$ in (8). Finally, an important point here is that, although we have chosen a contact form to write down the formulas for g_{KE}, actually it does depend only on the CR structure, not on the pseudohermitian structure. This is because the filling complex structure on N depends only on J, as does the zeroth order term of g_0, and the finite development of the Kähler-Einstein metric that we need is uniquely determined by requiring it to be Kähler-Einstein, see [11, Theorem 3.6].

Definition of the ν-invariant. According to [11], the ν-invariant is obtained by taking the limit as r goes to infinity (*i.e.* by taking the *diabatic limit*) of the boundary contribution on M_r of the Atiyah-Patodi-Singer formula for the characteristic number $\chi - 3\tau$ of $[r_0, r] \times M \subset N$, with respect to the metric g_{KE}. More precisely, recall that the Euler characteristic and signature formulas for a closed Riemannian manifold with boundary (Y, h) are given by

$$\chi(Y) = \int_Y P_\chi(h) + \int_{\partial Y} B_\chi(h), \quad \tau(Y) = \int_Y P_\tau(h) + \int_{\partial Y} B_\tau(h) + \eta(h_{\partial Y}),$$

where $P_\chi(h)$, $P_\tau(h)$ $B_\chi(h)$, $B_\tau(h)$ are universal polynomial expressions in the curvature of h (and second fundamental form of the boundary for B_χ and B_τ), and $\eta(h_{\partial Y})$ is the η-invariant of the boundary operator for the signature $S = (-1)^p(*d - d*)$ on $\Omega^{2p}\partial Y$ with the metric h restricted to ∂Y (see [2]). The explicit expressions of the universal polynomials involved in the above formulas, taken from [11, Section 7, p. 86–88], are given as follows: we denote R, W, Ric_0, and Scal, the Riemann, Weyl, tracefree Ricci, and scalar curvatures of h, and $\mathbb{I} = \nabla^h \mathrm{n}$ the second fundamental form of ∂Y (n being its outer unit normal). For 1-forms α and α', a 2-form β, and a 3-tensor F in $\otimes^3 T^*M$, we let

$$\begin{aligned} \alpha \wedge \alpha' (X, Y) &= \alpha(X)\, \alpha'(Y) - \alpha'(X)\, \alpha(Y) \ , \\ \alpha \wedge \beta\, (X, Y, Z) &= \alpha(X)\, \beta(Y, Z) + \alpha(Y)\, \beta(Z, X) + \alpha(Z)\, \beta(X, Y) \ , \\ \mathfrak{S}(F)(X, Y, Z) &= F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) \ ; \end{aligned}$$

if μ and ν are forms with values in bundles E and F, $\mu \wedge \nu$ is the obviously defined form with values in $E \otimes F$; last, if ρ is a 3-form and X_1, X_2, X_3 are three vectors, we define (a definition that immediately extends to any 3-form with values in $\otimes^3 TM$):

$$\mathcal{T}\left(\rho \otimes X_1 \otimes X_2 \otimes X_3\right) = d\mathrm{vol}_{\partial Y}\left(X_1, X_2, X_3\right)\, \rho.$$

The desired formulas for the Euler characteristic and signature then read:

$$\begin{aligned} P_\chi(h) &= \frac{1}{8\pi^2}\left(|W|^2 - \frac{1}{2}|\mathrm{Ric}_0|^2 + \frac{1}{24}\,\mathrm{Scal}^2\right) \\ B_\chi(h) &= \frac{1}{12\pi^2}\,\left(\mathcal{T}(\mathbb{I} \wedge \mathbb{I} \wedge \mathbb{I}) + 3\,\mathcal{T}(\mathbb{I} \wedge R)\right) \\ P_\tau(h) &= \frac{1}{12\pi^2}\,\left(|W^+|^2 - |W^-|^2\right) \\ B_\tau(h) &= \frac{1}{12\pi^2}\,\left(\mathfrak{S}(\mathbb{I}(., R(., .)\mathrm{n})\right) \end{aligned}$$

where the second fundamental form and the curvature must be seen as a 1- or 2-form with values in vectors or 2-vectors in the second line, and the second form as a quadratic form and the curvature as a 2-form with values in endomorphisms in the last line. For simplicity's sake, we shall now denote, in the setting adapted to 3-dimensional strictly pseudoconvex CR manifolds described above,

$$B(g_{KE}, M_r) = \int_{M_r} B_\chi(h(r)) \ - 3 \int_{M_r} B_\tau(h(r)) \ .$$

Note however that we will not need the precise form of $B(g_{KE}, M_r)$ in this paper, and will only use the fact that it is tensorially constructed from the curvature of g_{KE} and the second fundamental form of M_r in N.

2.1. **Definition.** The ν-invariant of M is

$$\nu(M) = \lim_{r\to+\infty} \nu(r) = \lim_{r\to+\infty} B(g_{KE}, M_r) - 3\,\eta(h(r)) \ .$$

It is shown in [11, Section 7] that this limit always exists for any 3-dimensional strictly pseudoconvex compact CR manifold M, and actually gives rise to a CR invariant of M (independent of the choice of the contact form θ).

3. The renormalized η-invariant

From its very definition, the invariant ν is a renormalisation of η-invariants of a jet $h(r)$ of the very natural Kähler metric g_{KE} restricted to slices of large radii M_r. However, these metrics are quite intricate (as formula (9) obviously shows), and ν itself is given by a limit of some complicated expressions built from these metrics. For these reasons we would like to describe how ν is related to the η-invariants of the much simpler contact-rescaling family of metrics of formula (1):

$$h_\varepsilon = \varepsilon^{-1}\theta^2 + \gamma.$$

This can be done by relying onto the following simple observation: although η is *a priori* not locally computable from the metric, its variation is. Indeed from the Atiyah-Patodi-Singer formula [2] and Chern-Simons' theory [22] one has

$$\eta(h_{\varepsilon_1}) - \eta(h_{\varepsilon_0}) = \frac{1}{3}\int_M Tp_1(\nabla_{\varepsilon_1}, \nabla_{\varepsilon_0}), \tag{10}$$

where $Tp_1(\nabla_{\varepsilon_1}, \nabla_{\varepsilon_0})$ is the Chern-Simons' transgression form of the first Pontrjagin class relative to the Levi-Civita connections of the *product* metrics

$$\widetilde{g}_\varepsilon = dr^2 + h_\varepsilon \quad \text{on} \quad N = \mathbb{R}\times M.$$

If $\nabla_{\varepsilon_1} = \nabla_{\varepsilon_0} + \alpha$ and Ω_t is the curvature 2-form of $\nabla_{\varepsilon_0} + t\alpha$, then

$$Tp_1(\nabla_{\varepsilon_1}, \nabla_{\varepsilon_0}) \ = \ 2\int_0^1 P_1(\alpha, \Omega_t)dt \ = \ -\frac{1}{4\pi^2}\int_0^1 \mathrm{Tr}(\alpha\wedge\Omega_t)dt. \tag{11}$$

Calculating the integral gives the usual formula

$$\eta(h_{\varepsilon_1}) - \eta(h_{\varepsilon_0}) \ = \ -\frac{1}{24\pi^2}\int_0^1 \mathrm{Tr}\left(2\Omega_0\wedge\alpha + d^{\nabla_{\varepsilon_0}}\alpha\wedge\alpha + \frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right). \tag{12}$$

This leads quickly to the following lemma.

3.1. **Lemma.** *Let (M^3, J, θ) be a strictly pseudoconvex pseudohermitian manifold, with metric $\gamma = d\theta(\cdot, J\cdot)$ on the contact distribution. Then the η-invariants of the family of metrics $h_\varepsilon = \varepsilon^{-1}\theta^2 + \gamma$ have a decomposition in homogeneous terms:*

$$\eta(h_\varepsilon) = \sum_{i=-2}^{2} \eta_i(M,\theta)\varepsilon^i. \tag{13}$$

The terms η_i for $i \neq 0$ are integral of local pseudohermitian invariants of (M, θ), and the η_i for $i > 0$ vanish when the torsion vanishes.

Proof. Denote by ∇ the Tanaka-Webster connection, with τ being the torsion seen as a trace-free symmetric endomorphism of $H = \ker\theta$, τ^1 (resp. $\tau^{\bar 1}$) being its expression as a $(0,1)$-form (resp $(1,0)$-form) relative to a choice of complex coframe θ^1. One computes easily the difference $a = \nabla_\varepsilon - \nabla$ (see the formulas in [46, page 316]), and the result is a decomposition into homogeneous terms of degrees -1, 0 and 1:

$$\nabla_\varepsilon - \nabla = a = \sum_{-1}^{1} a^{(i)}\varepsilon^i, \tag{14}$$

where each $a^{(i)}$ is locally defined by the pseudohermitian structure: $a^{(0)}$ and $a^{(-1)}$ are horizontal, but $a^{(1)}$ is vertical, more precisely, for horizontal $X, Y \in H$ one has

$$\begin{aligned} a^{(1)}_X Y &= -\gamma(\tau(X), Y)T, \\ a^{(0)}_X T &= \tau(X), \\ a^{(-1)}_T Y &= \frac{1}{2} JY. \end{aligned}$$

The output is the following decomposition for the curvature

$$\Omega(\nabla_\varepsilon) = \Omega(\nabla) + d^\nabla a + a \wedge a \tag{15}$$

$$= \sum_{-1}^{1} \Omega^{(i)}\varepsilon^i. \tag{16}$$

Indeed, the terms $\Omega^{(\pm 2)} = a^{(\pm 1)} \wedge a^{(\pm 1)}$ clearly vanish. Moreover,

$$\Omega^{(1)} = da^{(1)} + a^{(1)} \wedge a^{(0)} + a^{(0)} \wedge a^{(1)}$$

vanishes when the torsion vanishes. The existence of a decomposition in terms of powers of ε now follows from (12). From equation (11) one has

$$\varepsilon \frac{d}{d\varepsilon}\eta(h_\varepsilon) \;=\; -\frac{1}{12\pi^2}\int_M \mathrm{Tr}(\Omega \wedge \varepsilon\frac{da}{d\varepsilon}) \;=\; \sum_{\substack{-2\leqslant i\leqslant 2\\ i\neq 0}} i\,\eta_i\,\varepsilon^i$$

where the η_i $(i \neq 0)$ are local pseudohermitian invariants. When the torsion vanishes, $a^{(1)}$ and $\Omega^{(1)}$ vanish, so that η_i vanishes for each $i > 0$. $\square$

From the conformal invariance of the η-invariant, one deduces moreover immediately that, for a real number $\lambda > 0$,

$$\eta_i(M, \lambda\theta) = \lambda^{-i}\, \eta_i(M, \theta). \tag{17}$$

so that $\eta_0(M, \theta)$ is scale (but not conformally) invariant.

3.2. **Definition.** Let (M^3, θ) be a compact strictly pseudoconvex pseudohermitian 3-dimensional manifold. The *renormalized* η*-invariant of* (M, θ) is the constant term $\eta_0(M, \theta)$ in the expansion (13) for the η-invariants of the family of metrics $h_\varepsilon = \varepsilon^{-1}\theta^2 + d\theta(\cdot, J\cdot)$.

In the case where the torsion of (M, θ) vanishes, the terms $\eta_i(M, \theta)$ in (13) for $i > 0$ vanish, so that, when ε goes to infinity instead of 0, one has

$$\eta_0(M, \theta) = \lim_{\varepsilon \to \infty} \eta(h_\varepsilon) := \eta_{\mathrm{ad}}. \tag{18}$$

This corresponds to the geometric situation when the Reeb flow preserves the metric. Then, when $\varepsilon \to \infty$, the family of metrics h_ε collapses with bounded connection and curvature. This is the well-known *adiabatic limit*, and $\eta_0(M, \theta)$ is then the adiabatic limit η_{ad} of the η-invariant. It has been much studied, in particular in the geometrically meaningful situation when the Riemannian flow comes from some fibration in circles over a surface [12, 23]. The renormalized η-invariant depends on the choice of θ (as well, of course, on that of J).

However, we are more interested in this paper in the opposite direction: the *diabatic limit*, or equivalently the case where ε goes to 0. Although we will not need its precise expression, making the calculations in the proof of lemma 3.1 explicit shows the term $\eta_{-2}(M, \theta)$ never vanishes on contact manifolds, and is a non-zero multiple of $\int_M \theta \wedge d\theta$. Therefore $\eta(h_\varepsilon)$ always diverges at speed ε^{-2} in the diabatic limit, but the constant term $\eta_0(M, \theta)$ is still well-defined. We called it the *renormalized* η-invariant, as it is reminiscent of other similar contexts where renormalized invariants have been defined [26, 28, 42, 49].

4. The relation between ν and η_0

Our goal now is to prove Theorem 1.1, *i.e.* to show that on any CR manifold the ν-invariant is related to η_0 in a simple way.

4.1. **Lemma.** *There exist two constants C_1 and C_2 such that for any CR strictly pseudoconvex pseudohermitian manifold (M^3, J, θ), one has*

$$\nu(M) + 3\,\eta_0(M, \theta) = C_1 \int_M R^2\, \theta \wedge d\theta \;+\; C_2 \int_M |\tau|^2\, \theta \wedge d\theta, \tag{19}$$

where $\eta_0(M,\theta)$ is the renormalized η-invariant of (M,θ), and R, τ are the Tanaka-Webster curvature and torsion of M.

One can therefore look at $-\nu(M)/3$ as a local CR-conformal invariant correction of $\eta_0(M,\theta)$ (recall that $\eta_0(M,\theta)$ is *a priori* only invariant under the rescaling $\theta \to \lambda\theta$ for λ *constant*).

Proof. The metrics g_{KE} and $h(r) = g_{KE}|_{\{r\}\times M}$ issued from (9) are quite complicated, but are corrections of the model metrics g_0 and $h_0(r)$ defined in (8). More precisely, their expressions are universal polynomials in e^r and pseudohermitian invariant of (M,θ), and they do not actually depend on the choice of framing (except θ) and the constants in front of each such term are universal, *i.e.* independent of the manifold. Therefore, using a transgression formula as in (10) and (11), but between $h(r)$ and $h_0(r)$, we see that $\eta(h(r)) - \eta(h_0(r))$ has to be an invariant universal expression of type

$$\sum_{k=-n}^{n} \mathrm{e}^{kr} \int_M P_k(R,\tau,\nabla R,\nabla\tau,\ldots). \tag{20}$$

From Lemma 3.1, and the fact that the metric $h_0(r)$ is $\varepsilon^{-1}h_\varepsilon$ with $\varepsilon = \mathrm{e}^{-r}$, the same holds true for $\eta(h(r)) - \eta_0(M,\theta)$.

Moreover, the boundary contribution $B(g_{KE}, M_r)$ arising in definition 2.1 of ν is the integral of a secondary class built from the curvature of g_{KE} and has therefore a development of the same type as (20). The expression

$$\nu(r) + 3\,\eta_0(M,\theta) = B(g_{KE}, M_r) - 3\left(\eta(h(r)) - \eta_0(M,\theta)\right)$$

has then a development of the same kind. Note that this expression is void of terms in e^{kr} for $k > 0$ since we already know from definition 2.1 and [11] that it converges when r goes to infinity. As a result, the local boundary contribution necessarily cancels *all* divergent terms, and adds (still local) convergent terms. Identifying the constant terms we get eventually:

$$\nu(M) + 3\eta_0(M,\theta) = \int_M P_\theta(R,\tau,\nabla R,\nabla\tau,\ldots)\,\theta\wedge d\theta$$

where P_θ is some pseudohermitian local tensorial invariant. The invariance under the rescaling $\theta \to \lambda^2\theta$ shows that the polynomial P_θ must satisfy

$$P_{\lambda^2\theta} = \lambda^{-4}P_\theta.$$

The list of all possible expressions is easily established. Indeed, elementary invariant theory yields that such $U(1)$-invariant polynomials have to be sums of full contractions. Curvature R and torsion τ (here we see the torsion τ as a tensor of type $\tau = A_{11}\theta^1 \otimes \theta^1$ using some coframe θ^1 of $T^{1,0}H$) are homogeneous of weight -2 with respect to the previous rescaling, while a covariant differentiation along T decreases the weight by 2, and an horizontal one by 1. Following Proposition 5.13 in [51], we find that P_θ is a combination of

$$\begin{gathered} R^2\ ,\ |\tau|^2=|A_{11}|^2\ ,\ R_{,0}=dR(T)\ ,\ \Delta_H R\ , \\ (\nabla_{0,1})^2\,\tau=A_{11,\bar{1}\bar{1}}\ ,\ (\nabla_{1,0})^2\,\bar{\tau}=A_{\bar{1}\bar{1},11} \end{gathered} \tag{21}$$

where ∇ is the Tanaka-Webster connection and $\nabla_{1,0}$, resp. $\nabla_{0,1}$, is its restriction to horizontal vectors of type $(1,0)$, resp. $(0,1)$. Full divergences do not contribute after integration over M, so that one may forget the last four expressions, and the proof of Lemma 4.1 is over. $\square$

Computation of the constants. We are left with the determination of C_1 and C_2 in Lemma 4.1. This shall come from an explicit study of left-invariant CR structures on the 3-dimensional sphere.

Choose a basis $(\alpha_1,\alpha_2,\alpha_3)$ of left-invariant 1-forms on the sphere $\mathbb{S}^3$, such that $d\alpha_1=\alpha_2\wedge\alpha_3$, etc. The η-invariant of the left-invariant metric $\lambda_1^2\alpha_1^2+\lambda_2^2\alpha_2^2+\lambda_3^2\alpha_3^2$ has been computed[1] by Hitchin [29, formula (10)]:

$$\eta(\lambda_1^2\alpha_1^2+\lambda_2^2\alpha_2^2+\lambda_3^2\alpha_3^2)=\frac{2}{3}\left(\frac{s_1^3-4s_1s_2}{s_3}+9\right) \tag{22}$$

where the s_i are the symmetric polynomials in the λ_i^2. As a result, we get

$$\begin{aligned} &\eta(\alpha_1^2+\lambda_2^2\alpha_2^2+\lambda_3^2\alpha_3^2) \\ &\qquad=\frac{2}{3\lambda_2^2\lambda_3^2}\big(\lambda_3^6-(1+\lambda_2^2)\lambda_3^4-(\lambda_2^4-3\lambda_2^2+1)\lambda_3^2+(\lambda_2^6-\lambda_2^4-\lambda_2^2+1)\big) \end{aligned}$$

and taking the constant term in the diabatic limit $\lambda_3\to\infty$ (*i.e.* taking $\theta=\alpha_3$) leads to

$$\eta_0(\mathbb{S}^3,\alpha_1^2+\lambda^2\alpha_2^2)=\frac{2}{3\lambda^2}(-\lambda^4+3\lambda^2-1). \tag{23}$$

On the other hand, the ν-invariant can be estimated from the μ-invariant introduced by Burns and Epstein for embeddable CR structures, or more generally CR manifolds with trivial holomorphic part of the contact bundle [14]: for the contact form $\theta=\alpha_3$ and a metric $\gamma=\lambda^{-1}(\alpha_1)^2+\lambda(\alpha_2)^2$, μ is calculated in [14, 4.1.A]. Since

$$R=\frac{1+\lambda^2}{2\lambda},\quad |\tau|=\frac{1-\lambda^2}{2\lambda}, \tag{24}$$

one has

$$\mu(\lambda^{-1}\alpha_1^2+\lambda\alpha_2^2)=-\frac{1}{16\pi^2}\int_{S^3}(4|\tau|^2-R^2)\theta\wedge d\theta=-1+\frac{3(1-\lambda^2)^2}{4\lambda^2}.$$

[1]There is a slight mistake in [29] by a factor 2, as can be seen by comparing the results in [29] for the standard sphere to those of Theorem 5.2 below: one must find $\eta_0(\mathbb{S}^3,std)=\frac{2}{3}$ as computed by the equation (23), rather than $\frac{4}{3}$ as computed by [29].

It is proved in [11] that, for a deformation of the standard CR 3-sphere, one has $\nu = 3\mu+2$, and therefore

$$\nu(\lambda^{-1}\alpha_1^2 + \lambda\alpha_2^2) \; = \; -1 \; + \; \frac{9(1-\lambda^2)^2}{4\lambda^2} \; . \tag{25}$$

From equations (23), (24), and (25), we deduce

$$(\nu + 3\eta_0)(\lambda^{-1}\alpha_1^2 + \lambda\alpha_2^2) \; = \; \frac{(1+\lambda^2)^2}{4\lambda^2} \; = \; \frac{1}{16\pi^2} \int_{S^3} R^2 \, \theta \wedge d\theta \; .$$

This yields $16\pi^2 C_1 = 1$ and $C_2 = 0$ and the proof of Theorem 1.1 is done. □

4.2. *Remark.* From Theorem 1.1, we see that

$$-3\,\eta_0 + \frac{1}{16\pi^2} \int_M R^2 \theta \wedge d\theta$$

is a CR invariant. This fact can be proved directly: standard calculations in pseudohermitian geometry lead easily to the conclusion that it is invariant under conformal transformations $\theta \to f\theta$.

This remark provides an alternative (and independent) definition of the ν-invariant. The latest is clearly simpler than the one explained in Section 2: this is useful for computations and theoretical aspects, in particular the relation with the η-invariant of the pseudohermitian operator $D*$ on vertical 2-forms, as we shall see in the following sections. On the other hand, very important for the applications is the fact that ν arises as a boundary term in the integral of characteristic classes (see for example Corollary 1.9), and this can be obtained only through the first definition and the work done in [11].

One may also think that this remark could serve as a basis for defining a version of ν in higher dimensions, by looking for local corrections of η_0 that would lead to a CR invariant. However, this seems a very difficult task, as the range of possible terms of the right weight is in general much larger than in (21), even in the next relevant dimension 7.

5. Computation of the ν-invariant on Seifert manifolds

This section is devoted to explicit computations of the ν-invariant on $\mathbb{S}^1$-invariant CR manifolds of dimension 3. Although certainly a digression from our main route towards Theorems 1.4 and 1.5, this appears as a nice direct application of the results obtained in the previous section. We have thus chosen to interrupt the pace of our proofs, and to offer this section as a refreshing *intermezzo* before the analytical technicalities that will follow.

We first describe our family of spherical 3-dimensional compact strictly pseudoconvex CR manifolds in greater detail.

5.1. **Definition.** A CR Seifert manifold is a 3-dimensional compact manifold endowed with both a pseudoconvex CR structure (H, J) and a Seifert structure, that are compatible in the following sense: the circle action $\varphi : \mathbb{S}^1 \to \mathrm{Diff}(M)$ preserves the CR structure and is generated by a Reeb field T.

Any $\mathbb{S}^1$-invariant CR structure admits a $\mathbb{S}^1$-invariant contact form θ if the manifold is orientable (this is proved in [31]). Moreover it is easily proved that existence of a Reeb field T (defined by $\theta(T) = 1$ and $\iota_T d\theta = 0$) satisfying $\varphi_*(\frac{d}{dt}) = T$ and $\mathcal{L}_T\theta = 0$, $\mathcal{L}_T J = 0$, is equivalent to the existence of a locally free action of $\mathbb{S}^1$ whose (never vanishing) infinitesimal generator preserves H and J and is transverse everywhere to H. Hence, our CR Seifert manifolds could also be called *transverse* $\mathbb{S}^1$*-invariant CR manifolds*; note moreover that there exists a much larger class of $\mathbb{S}^1$-invariant CR manifolds, with the infinitesimal generator being sometimes tangent to the contact distribution [31, 39].

As we do not assume the action to be free but only locally free, the quotient space $\Sigma = M/\mathbb{S}^1$ is a surface with possibly conical singularities. Each CR Seifert manifold is then an orbifold bundle over the compact Riemannian orbifold surface Σ. If Σ is such a surface, endowed with a complex structure, orbifold $\mathbb{S}^1$-bundles are classified by their (rational) degrees d. Singularities of the bundle are located above the singularities of Σ in such a way that the resulting 3-manifold is smooth: if the local fundamental group is $\mathbb{Z}/\alpha\mathbb{Z}$ ($\alpha \in \mathbb{N}^*$), a generator acts on a local chart around p of the basis manifold as $\mathrm{e}^{i\frac{2\rho\pi}{\alpha}}$ and on the fiber as $\mathrm{e}^{i\frac{2\pi\beta}{\alpha}}$ with ρ and β prime to α (the extra parameter ρ may seem pointless as it is always possible to reduce oneself to two parameters by taking $\rho' = 1$ and $\beta' = \beta\rho^{-1}$ mod. α, but this extended description will prove useful when specializing our computations to the case of lens spaces in Section 10). Any choice of equivariant connection 1-form θ on M endows it with an invariant CR structure, H being chosen as the horizontal space for the connection and J being pulled back from the base. It is strictly pseudoconvex if $d < 0$. The interested reader is referred to [40] for a very readable account on orbifold bundles over orbifold surfaces. Note moreover that, taking the length of the fiber to be 2π, one has

$$\int_M \theta \wedge d\theta = -4\pi^2 d,$$

and that the metric $\gamma = d\theta(\cdot, J\cdot)$ projects downwards to a metric on Σ of volume

$$\int_\Sigma d\theta = -2\pi d,$$

(see [40] again for integration of forms over orbifolds). Its curvature R equals the Tanaka-Webster curvature of (M, θ) and Gauss-Bonnet reads

$$\int_\Sigma R\, d\theta = 2\pi\, \chi,$$

where χ is the (rational) Euler characteristic of Σ.

Computations in constant curvature. In the first half of this section, we moreover assume that γ has constant curvature R. In this case, the CR structure is spherical, that is M is locally isomorphic to the standard 3-sphere. Conversely, it is known that spherical CR Seifert manifolds are exactly those of constant Tanaka-Webster curvature R, except if the base is a sphere, see for instance [7].

The computations now rely on the explicit derivation of the η-invariant of (orbifold) circle bundles over (orbifold) Riemannian surfaces with constant curvature that have been done by Komuro [33] and more generally by Ouyang [41]. In our conventions and notations, their results read:

5.2. **Theorem** (Ouyang). *The η-invariant of the metric $t^2\,\theta^2+\gamma$ on M is equal to*

$$\frac{1}{3}\left(d+3+2d\left(\frac{\pi t^2}{V}\chi-\frac{\pi^2t^4}{V^2}d^2\right)\right)+4\sum_{j=1}^{p}s(\alpha_j,\rho_j,\gamma_j),$$

where $s(\alpha,\rho,\gamma)=\frac{1}{4\alpha}\sum_{k=1}^{\alpha-1}\cot(\frac{k\rho\pi}{\alpha})\cot(\frac{k\beta\pi}{\alpha})$ is the classical Rademacher-Dedekind sum.

We can now proceed to the computation of ν in the constant curvature case. We have to show Corollary 1.3, which we restate here:

5.3. **Corollary.** *Let M be a compact $\mathbb{S}^1$-orbifold bundle of rational degree $d<0$ over a compact orbifold surface Σ of constant curvature and rational Euler characteristic χ. Then,*

$$\nu(M)=-d-3-\frac{\chi^2}{4d}-12\sum_{j=1}^{p}s(\alpha_j,\rho_j,\beta_j). \tag{26}$$

Let us remark that the ν-invariant depends only on the topology for this class of CR manifolds, and not, for instance, on the complex structure of Σ. This is *a priori* known, since the gradient of ν is the Cartan curvature [11, Theorem 8.1], which vanishes for spherical CR manifolds.

Proof. According to Theorem 1.1, the ν-invariant is given by adding a local term to the renormalized η-invariant. On $\mathbb{S}^1$-invariant CR manifolds with constant curvature, the renormalized invariant is easily read from Ouyang's theorem 5.2 above:

$$\eta_0 \;=\; 1+\frac{d}{3}+4\sum_{j=1}^{p}s(\alpha_j,\rho_j,\beta_j)\;. \tag{27}$$

Moreover, the integral term is just

$$\frac{1}{16\pi^2}\int_M R^2\theta\wedge d\theta \;=\; \frac{-4\pi^2 d\left(-\frac{\chi}{d}\right)^2}{16\pi^2} \;=\; -\frac{\chi^2}{4d},$$

which shows also Theorem 1.3 in the constant curvature case. ☐

5.4. *Remark.* Corollary 1.3 can also be obtained by direct calculation from the original definition of ν and Ouyang's formula. Indeed the asymptotically Kähler-Einstein metric g_{KE} on $[r_0, +\infty[\times M$ can be handled with bare hands in this simple situation, and the boundary contribution counterbalancing the divergence of the sequence of η-invariants can be explicitly derived. Putting together Ouyang's theorem 5.2 and these local computations yield the value of ν, see [28] for similar computations. This is of course a painful method, but it is still a reasonably simple case where the cancellation of divergences by local terms can be observed in detail.

Extension to cases of non-constant curvature. We now extend the computations of ν to an (almost) complete proof of Theorem 1.2. It is shown in [31, 39] that there always exist a unique (up to equivalence) transverse $\mathbb{S}^1$-contact form on an orientable Seifert manifold (careful: this might be wrong for a non-transverse action). Given the natural contact form that fixes the length of the regular fibers to 2π, the choice of a CR structure is then equivalent to the choice of a downwards orbifold Riemannian metric γ of fixed volume $d\theta$, and this metric might or might not be of constant curvature.

In case the base is smooth (no orbifold singularities), it is known that the adiabatic limit $\eta_{\rm ad}$ does not depend on the underlying metric on Σ, see *e.g.* [55]. As one can always find a constant curvature metric of volume $d\theta$ (easy consequence of Moser's lemma on volume forms), the previous formula (27) for $\eta_0 = \eta_{\rm ad}$ applies. Then Theorem 1.1 enables to conclude that

$$\nu(M) = -d - 3 - 12 \sum_{j=1}^{p} s(\alpha_j, \rho_j, \beta_j) + \frac{1}{8\pi} \int_\Sigma R^2 \, d\theta. \tag{28}$$

If orbifolds singularities are present, it is known that every orbifold surface has a constant curvature metric, except some exceptional cases on the sphere described in [8]. As the set of compatible complex structures with a given contact structure is contractible, this means that, except on the exceptional cases we have just alluded to, it suffices to check the following:

5.5. **Lemma.** *Without any assumption on the quotient structure of M by the Reeb flow, the variations of η_0 with respect to the complex structure vanish when the torsion is zero.*

Proof. From Theorem 1.1, η_0 has the same variation as

$$-\frac{\nu}{3} + \frac{1}{48\pi^2} \int_M R^2 \, \theta \wedge d\theta.$$

The variation of ν with respect to J has been computed in [11, Theorem 8.1], namely

$$\frac{d\nu}{dJ} = \frac{-3}{8\pi^2} \int_M \langle Q_J, \dot{J} \rangle \, \theta \wedge d\theta \,, \tag{29}$$

where $Q_J = iQ_1{}^{\bar 1}\theta^1 \otimes Z_{\bar 1} - iQ_{\bar 1}{}^1\theta^{\bar 1} \otimes Z_1 \in \mathrm{End}(H)$ is Cartan's tensor. Its expression in term of derivatives of Tanaka-Webster curvature and torsion is given by

$$Q_1{}^{\bar 1} = \frac{1}{6}R_{,1}{}^{\bar 1} + \frac{i}{2}RA_1{}^{\bar 1} - A_1{}^{\bar 1}{}_{,0} - \frac{2i}{3}A_1{}^{\bar 1}{}_{,\bar 1}{}^{\bar 1}\,. \tag{30}$$

On the other hand the variation of the Tanaka-Webster curvature is computed e.g in [19, (2.20)], and is given by

$$\dot R = i(E_1{}^{\bar 1}{}_{,\bar 1}{}^1 - E_{\bar 1}{}^1{}_{,1}{}^{\bar 1}) - (A_1{}^{\bar 1}E_{\bar 1}{}^1 + A_{\bar 1}{}^1E_1{}^{\bar 1})\,, \tag{31}$$

where

$$\dot J = 2E_1{}^{\bar 1}\theta^1 \otimes Z_{\bar 1} + 2E_{\bar 1}{}^1\theta^{\bar 1} \otimes Z_{\bar 1}. \tag{32}$$

Putting everything together and integrating by parts shows that, in vanishing torsion, η_0 does not depend on the complex structure as needed. □

5.6. *Remark.* This computation of variations may be seen as an alternative mean to determine the constant $C_1 = \frac{1}{16\pi^2}$ in Lemma 4.1, independently of the computations of examples done in Section 4. Moreover, we stress that η_0 is independent of J (whenever the torsion vanishes) *without any assumption on the quotient structure of M by the Reeb flow* (smooth or not). This last fact will be used in Section 9.

In the remaining exceptional cases over $\mathbb{S}^2$ described in [8], the results stay the same but the proof above does not apply anymore and one has to rely on a different technique: this will be done below in Section 8.

6. The contact complex and the diabatic limit.

Theorem 1.1 gives a simple formula relating the ν-invariant and the renormalized η-invariant η_0 of the contact-rescaling. According to (18), η_0 coincides with the adiabatic limit of η in the case the CR manifold has vanishing torsion, and this enables computations, for explicit expressions of the adiabatic limit are known in a number of cases. But a deeper question is to relate directly the ν-invariant to the geometry and spectral theory of the CR or pseudohermitian manifold.

In the sequel we shall consider a natural η-invariant arising in pseudohermitian geometry. One actually knows by [47] a candidate for this, coming from the contact-de Rham complex. We shall briefly recall its construction in dimension 3 and its relation with the diabatic limit.

Let M be a 3-dimensional contact manifold and H its contact distribution. We denote by Ω^*H the space of horizontal forms, *i.e.* the space of sections of the alternating algebra over the dual of the bundle H. Let also Ω^*V be the subspace of vertical forms on M, by which we mean "true" forms in Ω^*M vanishing on H. Equivalently, one has

$$\Omega^*V = \{\theta \wedge \alpha \ , \ \alpha \in \Omega^*M\} = \theta \wedge \Omega^*H$$

for any local choice of contact form θ. The contact-de Rham complex is then the following:

$$C^\infty(M) \xrightarrow{d_H} \Omega^1 H \xrightarrow{D} \Omega^2 V \xrightarrow{d_H} \Omega^3 M, \tag{33}$$

where for $f \in C^\infty(M)$, $d_H f \in \Omega^1 H$ stands for the restriction of df to H, while

$$d_H : \Omega^2 V \to \Omega^3 M$$

is just de Rham's differential restricted to $\Omega^2 V$ in $\Omega^2 M$, and D is defined as follows: since d induces an isomorphism

$$d_0 : \Omega^1 V \to \Omega^2 H \ \text{ with } \ d_0(f\theta) = f\, d\theta_{|\Lambda^2 H},$$

then any α in $\Omega^1 H$ admits a unique extension $\ell(\alpha)$ in $\Omega^1 M$ such that $d\ell(\alpha)$ belongs to $\Omega^2 V$; namely, given any initial extension $\overline{\alpha}$ of α, one has

$$\ell(\alpha) = \overline{\alpha} - d_0^{-1}(d\overline{\alpha})_{|\Lambda^2 H}. \tag{34}$$

We then define

$$D\alpha = d\ell(\alpha). \tag{35}$$

This differential D is a second order operator, since the lifting $\ell : \Omega^1 H \to \Omega^1 M$ is a first order one. Moreover one sees easily that ℓ induces an homotopy equivalence between the contact and de Rham complexes, together with the natural restrictions, and the retraction $\ell' : \Omega^2 M \longrightarrow \Omega^2 V$ defined by

$$\ell'(\alpha) = \alpha - d d_0^{-1} \alpha_{|\Lambda^2 H}.$$

From now on we will suppose moreover that the contact manifold M is endowed with a strictly pseudoconvex CR structure J, together with some choice of contact form θ. We consider the contact-rescaling sequence of metrics of (8)

$$h_0(r) = \mathrm{e}^{2r}\, \theta^2 + \mathrm{e}^r\, d\theta(\cdot, J\cdot).$$

Let $\varepsilon = \mathrm{e}^{-r}$, as before, and define

$$g_\varepsilon = \varepsilon^{-2}\theta^2 + \varepsilon^{-1} d\theta(\cdot, J\cdot) = h_0(r). \tag{36}$$

This metric induces an orthogonal splitting $TM = H \oplus \mathbb{R}T$ where T is the Reeb field of θ, and one can identifies $\Omega^1 H$ with "true" 1-forms on M vanishing on T. Observing that Hodge $*$-operator exchanges $\Omega^1 H$ and $\Omega^2 V$ and one can consider $D*$ acting on closed vertical 2-forms $\Omega^2_D V = \Omega^2 V \cap \operatorname{im} D$.

Following [2, Theorem 4.14], we define the boundary operator for the signature attached to the Riemannian metric g_ε as

$$S_\varepsilon = (-1)^p(*_\varepsilon d - d*_\varepsilon),$$

acting on $\Omega^{2p}M = C^\infty M \oplus \Omega^2 M$. As observed in [2, Prop. 4.20], one may remove some spectral symmetry, and its η-function

$$\eta(S_\varepsilon)(s) = \mathrm{Tr}^*(S_\varepsilon|S_\varepsilon|^{-(s+1)}) = \sum_{\lambda_i\in\mathrm{spec}(S_\varepsilon)\setminus\{0\}} \frac{\lambda_i}{|\lambda_i|^{s+1}} \tag{37}$$

actually coincides with that of $d*_\varepsilon$ when restricted to $\Omega^2_d M = \Omega^2 M \cap \operatorname{im} d$. Note that we have used Tr^* to denote a trace taken outside the 0-eigenspace. In the same vein, the notation spec^* used below will denote a spectrum *where the* 0*-eigenvalue has been removed.*

From [4, p. 74] or [25, Chap. 1.10], the series (37) is absolutely convergent for $\operatorname{Re} s > 3$ and has a meromorphic extension to $\mathbb{C}$, with possibly simple poles at $s = 3 - n$, $n \in \mathbb{N}$. By Atiyah-Patodi-Singer's theorem [2], $\eta(S_\varepsilon)(s)$ is actually regular at $s = 0$ and its value there is called the η-invariant of (M, g_ε). Similarly, an η-function and its value at 0 can be defined for the operator $D*$ in dimension 3. This mainly follows by applying the same ideas, but with the adequate symbolic calculus for hypoelliptic operators, see Section 9.

In order to compare them, let us now compute $d*_\varepsilon$ and $D*_\varepsilon$ using the decomposition of $\Omega^2 M$ into vertical and horizontal 2-forms:

$$\alpha = \theta \wedge \alpha_T + \alpha_H,$$

with $\alpha_T \in \Omega^1 H$, $\alpha_H \in \Omega^2 H$. From (36) one sees that

$$*_\varepsilon \alpha = \theta \wedge *_H \alpha_H + \varepsilon *_H \alpha_T$$

where $*_H$ denotes the induced Hodge duality on H. In matrix form, one gets

$$d*_\varepsilon = \begin{pmatrix} \varepsilon\,\mathcal{L}_T *_H & -d_H *_H \\ \varepsilon\, d_H *_H & 1 \end{pmatrix} \tag{38}$$

on $\Omega^2 M$, where $\mathcal{L}_T$ is the Lie derivative along T.

We now consider $D*_\varepsilon$. Using (34) and (35) one finds that

$$\ell(\beta) = \beta - (*_H d_H\beta)\theta$$

on $\Omega^1 H$, so that

$$D\beta = \theta \wedge (\mathcal{L}_T + d_H *_H d_H)\beta\ ,$$

and hence

$$D *_\varepsilon (\theta \wedge \alpha_T) = \varepsilon\, \theta \wedge (\mathcal{L}_T + d_H *_H d_H) *_H \alpha_T \tag{39}$$

on $\Omega^2 V = \theta \wedge \Omega^1 H$.

The whole spectrum of $D*_\varepsilon = \varepsilon D*_1$ then collapses at speed ε in the diabatic limit $\varepsilon \to 0$, whereas part of the spectrum of $d*_\varepsilon$ is not collapsing: for instance $(d*_\varepsilon)(d\theta) = d\theta$. Hence the diabatic behaviour of the *whole* spectrum of $d*_\varepsilon$ cannot be related to $D*_\varepsilon$ alone, and indeed only the collapsing spectra are related. This shows up in the following formulae, which are direct consequences of (38) and (39), or even more directly from the definitions (34) and (35) of ℓ and D. If $P_\varepsilon = \varepsilon^{-1} d*_\varepsilon$,

$$\begin{aligned} P_\varepsilon = \varepsilon^{-1} d*_\varepsilon &= \begin{pmatrix} D*_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -(d_H *_H)^2 & -\varepsilon^{-1} d_H *_H \\ d_H *_H & \varepsilon^{-1} \end{pmatrix} \\ &= \Pi_{\Omega^2 V}(D*_1)\Pi_{\Omega^2 V} + \varepsilon P_\varepsilon \Pi_{\Omega^2 H} P_\varepsilon. \end{aligned} \tag{40}$$

It follows that in the diabatic limit $\varepsilon \to 0$ all the eventually bounded spectrum of $P_\varepsilon = \varepsilon^{-1} d*_\varepsilon$ converges, at least weakly, towards the spectrum of $D*_1$. Actually its turns out that this spectral convergence is uniform over bounded intervals, as a consequence of the uniform convergence in the diabatic limit of the resolvents $(\lambda - P_\varepsilon)^{-1}$ on $\ker d$ towards $(\lambda - D*_1)^{-1}$, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ [47, Theorem 3.6].

Such a spectral convergence is unfortunately only a first step in the study of a global spectral invariant like η. To illustrate this, recall that by [13] an equivalent expression of the Riemannian η-invariant is given by

$$\eta(P_\varepsilon)(0) = \pi^{-1/2} \int_0^\infty \mathrm{Tr}\big(P_\varepsilon\, \mathrm{e}^{-tP_\varepsilon^2}\big) \frac{dt}{\sqrt{t}}. \tag{41}$$

Now by [47, Theorem 7.1] the following global trace convergence holds

$$\mathrm{Tr}(P_\varepsilon\, \mathrm{e}^{-tP_\varepsilon^2}) \longrightarrow \mathrm{Tr}(D * \mathrm{e}^{-tDD^*}),$$

when ε goes to 0, but uniformly on t *only for* $t \geqslant t_0 > 0$. It cannot be true for small t since the η-invariants and the integrals (41) diverge in the diabatic limit (although one knows by transgression formulas that these divergences of $\eta(P_\varepsilon)(0)$ are given by local expressions). From the analytic viewpoint, these divergences are rooted in the transition from elliptic towards hypoelliptic operators, that cannot be uniform in all (t, ε) regimes. For instance, the asymptotic spectral densities (Weyl's laws), or the powers of t occurring in the asymptotic expansions of the heat kernels for $t \to 0$ are not the same for the elliptic P_ε and the hypoelliptic $D*$. However it is possible, as is usual in such asymptotic spectral problems, that the divergences occurring in the $(d*_\varepsilon, D*)$ transition when ε and t go to 0, are ruled again by local expressions in the curvature, see also Remark 8.6. This would provide directly a relation like (7) between the finite part η_0 of $\eta(P_\varepsilon)$ in the diabatic limit and the pseudohermitian η-invariant $\eta(D*)$. Unfortunately, the techniques used in [47] cannot handle these problems in the general case. The analysis can however be done in the particular case of CR Seifert manifolds, and we will now restrict ourselves to this case.

7. Spectral analysis on Seifert manifolds.

As explained above, we will now deal with CR-manifolds endowed with both a Seifert and a CR structure compatible in the sense that the circle action $\varphi : \mathbb{S}^1 \to \mathrm{Diff}(M)$ preserves the CR structure (H, J) and is generated by a Reeb field T. An invariant contact form θ has then been chosen, and we note that in this section, in opposite to Section 5, we will never assume the Webster curvature to be constant.

The circle action allows to perform a Fourier decomposition of functions or forms on M without referring to the quotient structure. For instance, given $n \in \mathbb{Z}$ and $f \in C^0(M)$, its n-th component is the function on M defined by

$$\pi_n f = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-int} (f \circ \varphi_t) dt.$$

It satisfies $(\pi_n f) \circ \varphi_t = \mathrm{e}^{int}(\pi_n f)$, so that $\mathcal{L}_T(\pi_n f) = in\pi_n f$ on $C^1(M)$. The projections π_n preserve and are clearly bounded on all $C^p(M)$, $L^p(M)$ or Sobolev spaces. Moreover, the Hilbert sum of all π_n for n in $\mathbb{Z}$ is the identity on $L^2(M)$. Last, this circle action preserves all structures and operators related to the above choice of contact form, so that we will be able to split their spectra into Fourier components.

We can now study the spectral aspects of the contact rescaling g_ε in (36) on a CR Seifert manifold M. Of course the adiabatic limit exists in this situation, and has already been much studied, see *e.g.* [12, 23], but we will need a different approach here, focusing on the diabatic behaviour of $d{*_\varepsilon}$ and $\eta(d{*_\varepsilon})$, and their relations with the spectrum of $D*$ and its η-invariant.

One computes easily the Laplacian on $\Omega^2 M$, relatively to the splitting

$$\Omega^2 M = \theta \wedge \Omega^1 H \oplus \Omega^2 H,$$

namely

$$\Delta_\varepsilon = \begin{pmatrix} \varepsilon\Delta_H - \varepsilon^2 T^2 & -d_H *_H \\ \varepsilon d_H *_H & 1 + \varepsilon\Delta_H - \varepsilon^2 T^2 \end{pmatrix}, \tag{42}$$

where $\Delta_H = d_H\delta_H + \delta_H d_H$ is the horizontal Laplacian (not to be confused with the contact Laplacian introduced in [46, 47]), T denotes here the Lie derivative along T, and we have used that $T^* = -T$ and $[T, \delta_H] = 0$ since T is a Killing Reeb field on the CR Seifert manifold. We observe from (38) that the non-diagonal part of Δ_ε is the same as that of $d{*_\varepsilon}$, so that

$$\Delta_\varepsilon = d *_\varepsilon \; + \varepsilon \begin{pmatrix} \Delta_H - T *_H & 0 \\ 0 & \Delta_H \end{pmatrix} \; - \; \varepsilon^2 T^2.$$

When studying spectral asymmetry, we shall now restrict ourselves to the subspace $\Omega^2_d M = \operatorname{im} d$ of $\Omega^2 M$, on which $\Delta_\varepsilon = (d{*_\varepsilon})^2$. We get therefore the following expression relating pairwise commuting operators:

$$(d*_\varepsilon)^2 = d*_\varepsilon + \varepsilon K - \varepsilon^2 T^2\,, \tag{43}$$

with

$$K = \begin{pmatrix} \Delta_H - T*_H & 0 \\ 0 & \Delta_H \end{pmatrix}.$$

Therefore if $\alpha \in \Omega^2_d M \setminus \{0\}$ satisfies

$$(d*_\varepsilon)\alpha = \lambda_\varepsilon \alpha\ , \quad K\alpha = k\alpha \quad \text{and} \quad T^2\alpha = -n^2\alpha, \tag{44}$$

for λ_ε a non-zero eigenvalue of $d*_\varepsilon$, then (43) gives

$$\lambda_\varepsilon^2 = \lambda_\varepsilon + \varepsilon k + \varepsilon^2 n^2 \neq 0\ , \tag{45}$$

and, necessarily,

$$\lambda_\varepsilon = \lambda_\varepsilon^+ \text{ or } \lambda_\varepsilon^- \quad \text{with} \quad \lambda_\varepsilon^\pm = \frac{1 \pm \sqrt{1 + 4\varepsilon(k + \varepsilon n^2)}}{2}. \tag{46}$$

Hence the spectrum of $d*_\varepsilon$ splits in two families which behave differently in the diabatic limit $\varepsilon \to 0$. Eigenvalues of type λ_ε^- all collapse, while those of type λ_ε^+ all converge to 1. According to the general results of [47] discussed in Section 6, only eigenvalues of type λ_ε^- are related to $D*$, after rescaling by ε^{-1}.

The previous eigenvalue equation (46) is only a necessary condition and we have to determine which of the possible $\lambda_\varepsilon^\pm$ are effectively present in spec$(d*_\varepsilon)$ and to compute their multiplicities. To do this, we use the splitting induced by the choice of the Reeb field: suppose $\alpha = \theta \wedge \alpha_T + \alpha_H$ is a 2-form in the image of d. By (38), the system $(d*_\varepsilon)\alpha = \lambda_\varepsilon \alpha$ is

$$(\lambda_\varepsilon - \varepsilon T*_H)\,\alpha_T = -d_H *_H \alpha_H \tag{47}$$

$$(\lambda_\varepsilon - 1)\,\alpha_H = \varepsilon\, d_H *_H \alpha_T. \tag{48}$$

Suppose now that

$$(d*_\varepsilon)\alpha = \lambda_\varepsilon \alpha\ , \quad K\alpha = k\alpha \quad \text{and} \quad T^2\alpha = -n^2\alpha. \tag{49}$$

Then we observe that $*_H = -J$ on $\Omega^1 H$ and $(T*_H)^2 = -T^2 = n^2$. Therefore (47) gives

$$(\lambda_\varepsilon^2 - \varepsilon^2 n^2)\alpha_T = -(\lambda_\varepsilon + \varepsilon T*_H)\ d_H *_H \alpha_H, \tag{50}$$

so that α_H determines uniquely α_T when $\lambda_\varepsilon^2 \neq \varepsilon^2 n^2$. A first (quite large) part of the non-zero spectrum is then handled as follows.

7.1. **Proposition.** • *Forms $\alpha = \theta \wedge \alpha_T + \alpha_H$ in $\Omega^2_d M$ satisfying*

$$(d*_\varepsilon)\alpha = \lambda_\varepsilon^+ \alpha\ , \quad K\alpha = k\alpha \quad \textit{and} \quad T^2\alpha = -n^2\alpha \tag{51}$$

such that $(\lambda_\varepsilon^+)^2 \neq \varepsilon^2 n^2$ *are in one-to-one linear correspondence with forms* α_H *in* $\Omega^2 H$ *satisfying*

$$\Delta_H \alpha_H = k\alpha_H \quad \textit{and} \quad T^2\alpha_H = -n^2\alpha_H. \tag{52}$$

- *Forms* $\alpha = \theta \wedge \alpha_T + \alpha_H$ *in* $\Omega^2_d M$ *satisfying*

$$(d*_\varepsilon)\alpha = \lambda_\varepsilon^- \alpha\ , \quad K\alpha = k\alpha \quad \textit{and} \quad T^2\alpha = -n^2\alpha \tag{53}$$

such that $(\lambda_\varepsilon^-)^2 \neq \varepsilon^2 n^2$ *are in one-to-one linear correspondence with forms* α_H *in* $\Omega^2 H$ *satisfying*

$$\Delta_H \alpha_H = k\alpha_H \quad \textit{and} \quad T^2\alpha_H = -n^2\alpha_H \tag{54}$$

with $k \neq |n|$*.*

Proof. In the case $\lambda_\varepsilon^2 \neq \varepsilon^2 n^2$, α_H determines α by (50). Hence one always has $\alpha_H \neq 0$ here, and, by $\Delta_H\alpha_H = k\alpha_H$, k is necessarily non-negative. Moreover by (45) one has $\lambda_\varepsilon^2 \neq \varepsilon^2 n^2$ if and only if $\lambda_\varepsilon \neq -\varepsilon k$. This is always satisfied in our case $k \geqslant 0$ when $\lambda_\varepsilon = \lambda_\varepsilon^+ > 0$, and only for $k \neq |n|$ when $\lambda_\varepsilon = \lambda_\varepsilon^-$.

Conversely, suppose now given α_H, n, k, λ_ε as needed. From (50), one defines

$$\alpha_T = -(\lambda_\varepsilon^2 - \varepsilon^2 n^2)^{-1}(\lambda_\varepsilon + \varepsilon T *_H)\ d_H *_H \alpha_H,$$

which satisfies (47). To check (48), recall that

$$\delta_H = - *_H\ d_H\ *_H \quad \text{and} \quad d_H^2 = -LT = -TL,$$

where $L(f) = f\,d\theta$ (the last equation being a consequence of $d^2 = 0$ see e.g. [47, p. 415]). One finds

$$\begin{aligned}(\lambda_\varepsilon^2 - \varepsilon^2 n^2)\, d_H *_H \alpha_T &= (\lambda_\varepsilon d_H \delta_H \alpha_H + \varepsilon d_H^2 T *_H \alpha_H)\\ &= (\lambda_\varepsilon \Delta_H - \varepsilon T^2)\alpha_H\\ &= (\lambda_\varepsilon k + \varepsilon n^2)\alpha_H.\end{aligned}$$

The eigenvalue equation (45) then easily leads to (48). □

For later use, note that the choice $(k, n) = (0, 0)$ in the positive case leads to $\alpha_H = C\,d\theta$ and $\lambda_\varepsilon = 1$, hence $\alpha_T = 0$ by (47), and this is the only case where this might happen by (48).

Proposition 7.1 and (46) show that a large part of $\mathrm{spec}^*(d*_\varepsilon)$ is symmetric with respect to $\frac{1}{2}$ and is parametrised by the spectrum $\{k + \varepsilon n^2\}$ of the non-negative elliptic Laplacian $L_{\varepsilon,H} = \Delta_H - \varepsilon T^2$ acting on $\Omega^2 H$, or, equivalently *via* Hodge duality, by the spectrum of

$$\Delta_\varepsilon = \Delta_H - \varepsilon T^2 \tag{55}$$

acting on functions. However from (54) there are "holes" in this symmetry corresponding to the eigenvalues $\lambda_\varepsilon^- = -\varepsilon k$ when $k = |n|$. This means that in the case $\lambda_\varepsilon = \lambda_\varepsilon^-$, we have to remove from the parameter space the horizontal forms α_H in

$$\mathcal{H}^0 = \ker(\Delta_H^2 + T^2). \tag{56}$$

This space has a simple description using the complex structure J and the associated splitting $\Omega^1 H \otimes \mathbb{C} = \Omega^{1,0}H \oplus \Omega^{0,1}H$. We recall that the component $d_H^{0,1}$ of d_H from functions to $\Omega^{0,1}H$ is called the $\overline{\partial}_b$ operator, and its kernel is the space of CR functions.

7.2. **Proposition.** *On a CR Seifert manifold, the space $*_H\mathcal{H}^0$ is the space of pluri-CR functions,* i.e. *real parts of CR functions.*

Proof. Consider the Kohn Laplacians $\overline{\Box}_b = \overline{\partial}_b^*\overline{\partial}_b$ and $\Box_b = \partial_b^*\partial_b$ acting on functions. Following, say, [35, Theorem 2.3], one has in dimension 3

$$\Delta_H = \overline{\Box}_b + \Box_b \quad \text{and} \quad iT = \overline{\Box}_b - \Box_b. \tag{57}$$

Since T commutes with everything here one gets

$$\Delta_H^2 + T^2 = 4\overline{\Box}_b\Box_b = 4\Box_b\overline{\Box}_b.$$

If f is a real function in $\mathcal{H}^0$ then $g = \Box_b f$ is CR since its image by $\overline{\Box}_b$ is zero, and is in the image of Δ_H since its integral vanishes. Hence

$$\Delta_H f = \overline{\Box}_b f + \Box_b f = \bar{g} + g = 2\,\mathrm{Re}\, g,$$

and $f = 2\,\mathrm{Re}\, h$ with $h = \Delta_H^{-1} g$ is a CR function as needed. □

7.3. *Remark.* CR Seifert manifolds arise as boundaries of Seifert disk bundles, hence the $\overline{\partial}_b$ operator has closed range and infinite dimensional kernel. We shall see this very explicitly in Section 8.

We now study the missing case $\lambda_\varepsilon^2 = \varepsilon^2 n^2$. We first recall that complex vertical forms $\Omega^* V \otimes \mathbb{C} \simeq \theta \wedge \Omega^* H \otimes \mathbb{C}$ also have a natural bigrading inherited from J on H, independently from θ. Of particular interest here is the

7.4. **Definition.** The bundle $K_M \simeq \theta \wedge \Omega^{1,0} H$ of 2-forms vanishing on $H^{0,1}$ is called the canonical CR bundle. We denote by $\mathcal{H}^{2,0}$ its subspace of closed sections, also called holomorphic $(2,0)$-forms, and $\mathcal{H}_+^2$ the real part of $\mathcal{H}^{2,0}$.

When the CR manifold M can be locally embedded in a 4-dimensional complex manifold N, K_M is the restriction to M of the canonical bundle $K_N = \Omega^{2,0} N$ of N, and holomorphic forms are local restrictions of holomorphic $(2,0)$-forms in N, see [35] for instance. This explains the notation in the previous definition, as $\mathcal{H}^{2,0}$ (resp. $\mathcal{H}_+^2$) is related to the space of holomorphic $(2,0)$-forms in the usual sense on N (resp. to the space of self-dual 2-forms, orthogonal to the Kähler form). Note that this is indeed the case for our CR Seifert manifolds for one can take $N = M \times \mathbb{R}$ with the extension of J considered in Section 2.

We now show that the remaining spectrum of $d*_\varepsilon$ is entirely given by holomorphic forms.

7.5. **Proposition.** *A 2-form $\alpha \in \Omega^2_d M$ satisfies*

$$(d*_\varepsilon)\alpha = \lambda_\varepsilon \alpha \ , \quad K\alpha = k\alpha \quad \text{and} \quad T^2\alpha = -n^2\alpha \tag{58}$$

with $\lambda_\varepsilon^2 = \varepsilon^2 n^2$ if and only if $\alpha_H = 0$ and $\alpha = \theta \wedge \alpha_T$ belongs to $\mathcal{H}^2_+$.

Proof. Let $\alpha = \theta\wedge\alpha_T + \alpha_H$ in $\Omega^2_d M$ be a eigenfunction of $d*_\varepsilon$ satisfying (58) with $\lambda_\varepsilon^2 = \varepsilon^2 n^2$. By (45) one has also $\lambda_\varepsilon = -\varepsilon k$. We first show that $\alpha_H = 0$.

Since $(T*_H)^2 = -T^2 = n^2 = k^2$ on $\Omega^1 H$, one can split

$$\alpha_T = \alpha_T^+ + \alpha_T^- \quad \text{with} \quad (T*_H)\alpha_T^\pm = \pm k \alpha_T^\pm .$$

Then (47) and (48) are equivalent to

$$2\varepsilon\, k\, \alpha_T^+ = d_H *_H \alpha_H \tag{59}$$

$$-(\varepsilon k + 1)\alpha_H = \varepsilon\, d_H *_H \alpha_T \tag{60}$$

Moreover $K\alpha = k\alpha$ gives $(\Delta_H - T*_H)\alpha_T = k\alpha_T$, which implies $\Delta_H \alpha_T^- = 0$ since $[\Delta_H, T*_H] = 0$ on $\Omega^1 H$. Therefore α_T^- lives in $\ker \delta_H$, so that (60) becomes

$$-(\varepsilon k + 1)\alpha_H = \varepsilon\, d_H *_H \alpha_T^+ \, ,$$

leading by (59) to

$$\Delta_H \alpha_H = -2\varepsilon k\, d_H *_H \alpha_T^+ = 2k(\varepsilon k + 1)\alpha_H \, . \tag{61}$$

But $K\alpha = k\alpha$ gives $\Delta_H \alpha_H = k\alpha_H$. This together with (61) implies that $\alpha_H = 0$ since otherwise one would have both $k > 0$ and $\varepsilon k = -1/2$. Hence $\alpha = \theta \wedge \alpha_T$ is a vertical form as claimed.

Now (60) reads $\delta_H \alpha_T = 0$, or equivalently

$$d_H(J\alpha_T) = 0.$$

Moreover α belongs to $\Omega^2_d M$, hence is closed. The $(1,0)$-part of α_T is then in $\ker d_H$ and $\alpha = \theta \wedge \alpha_T$ lives in $\mathcal{H}^2_+$ as needed.

Conversely, $\mathcal{H}^2_+$ is preserved by J and T. Thus it can be split in eigenspaces of $T *_H = -JT = k$, on which $d*_\varepsilon = \varepsilon k$ by definition, see (38). ☐

We now summarize our spectral study of $d*_\varepsilon$ in relation to the diabatic limit $\varepsilon \to 0$.

7.6. **Corollary.** *The spectrum of $d*_\varepsilon$ splits into the following families:*

(i) *A converging part Λ_ε^+, converging to 1 and parametrised by the whole spectrum of $\Delta_\varepsilon = \Delta_H - \varepsilon T^2$ (acting on functions) by the formula*

$$\Lambda_\varepsilon^+ = \operatorname{spec}\left(\frac{1 + \sqrt{1 + 4\varepsilon\Delta_\varepsilon}}{2}\right).$$

(ii) *A collapsing part, converging to 0, itself divided into two families:*

(a) *the first one* Λ_ε^-, *nearly symmetric to* Λ_ε^+*:*

$$\Lambda_\varepsilon^- = \operatorname{spec}\left(\frac{1-\sqrt{1+4\varepsilon\Delta_\varepsilon}}{2}\right),$$

but Δ_ε *has here to be restricted to the orthogonal of the space of pluri-CR functions* $\mathcal{H}^0$.

(b) *the spectrum* Λ_ε^0 *of* $\varepsilon T *_H = -\varepsilon JT$ *acting on* $\mathcal{H}_+^2$, *the real parts of holomorphic forms in the canonical CR bundle.*

The signs of the eigenvalues in the first two families are clear. About the third one, we can notice:

7.7. **Proposition.** *Up to some finite dimensional space,* $d*_\varepsilon$ *is* positive *on* $\mathcal{H}_+^2$.

Proof. Recall that $d*_\varepsilon = -JT$ on $\mathcal{H}_+^2$. Consider then the splitting of the Tanaka-Webster connection $\nabla_H = \nabla_{1,0} + \nabla_{0,1}$ on $H \otimes \mathbb{C}$. Then on $K_M = \theta \wedge \Omega^{1,0}H$ one has in dimension 3,

$$R = \nabla_{0,1}^*\nabla_{0,1} - \nabla_{1,0}^*\nabla_{1,0} - i\nabla_T.$$

On holomorphic forms $\mathcal{H}^{2,0}$ in K_M, the Lie derivative in T equals ∇_T and the previous equation reduces to

$$-iT = R + \nabla_{1,0}^*\nabla_{1,0},$$

which implies that $-(iT+R)$ is a non-negative operator. As the spectrum of $d*_\varepsilon$ (on closed forms) is discrete and without accumulation points, there is only a finite dimensional space of eigenvectors with non-positive eigenvalues. □

In order to get more symmetry in the spectral decomposition of $d*_\varepsilon$, one can fill in the holes in Λ_ε^- by adding Δ_ε on $\mathcal{H}_0$. As already discussed, this corresponds to adding the cases $k = |n|$ and $\lambda_\varepsilon = -\varepsilon k \neq 0$. Given k, the multiplicity of each added *virtual* eigenvalue $-\varepsilon k$ is equal to $2h_0(k)$ by Proposition 7.2, where we have denoted

$$h_0(k) = \dim_{\mathbb{C}}\big\{\text{CR functions } f \text{ such that } iTf = -kf\big\}.$$

Observe that by (57), $h_0(k) = 0$ if $k < 0$. In the same spirit, the holomorphic part Λ_ε^0 above consists in $\{\varepsilon k \mid k \in \mathbb{Z}^*\}$, with multiplicity $2h_2(k)$ given by

$$h_2(k) = \dim_{\mathbb{C}}\big\{\text{holomorphic } (2,0)\text{-forms } \alpha \in \mathcal{H}^{2,0} \text{ such that } iT\alpha = -k\alpha\big\}.$$

Considering the positive operators

$$Q_\varepsilon^\pm = \frac{\pm 1 + \sqrt{1+4\varepsilon\Delta_\varepsilon}}{2\varepsilon},$$

leads to the more suggestive decomposition:

$$\operatorname{spec}^*\left(\frac{d*_\varepsilon}{\varepsilon}\right) = \pm\operatorname{spec}^*\left(Q_\varepsilon^\pm\right) \,\cup\, 2\times\operatorname{spec}^*\left(-iT_{|\mathcal{H}^{2,0}}\right) \,\setminus\, 2\times\operatorname{spec}^*\left(iT_{|\ker\overline{\partial}_b}\right). \tag{62}$$

This formula shows that the virtual spectrum of $d*_\varepsilon$ consists in two completely different parts: a (nearly) symmetric part to 1/2, that varies with ε, and a *constant* holomorphic part. We will see in Lemma 8.5 that the symmetric part always contributes to 1 in the renormalized η-invariant η_0 when torsion vanishes. Hence the computation of η_0 finally reduces to counting holomorphic objects, as will be done in Section 8. This phenomenon has already been observed on a smooth base in [55] and over orbifolds, in the adiabatic context and constant curvature, in [40].

8. The spectrum of $D*$ and comparison of the η-invariants

Our goal is now to relate our description of the spectrum of $P_\varepsilon = \varepsilon^{-1} d*_\varepsilon$ to the spectrum of the middle operator of the contact complex $D*$. We already know (see the discussion at the end of Section 6) that the bounded spectrum of P_ε converges towards that of $D*$ in the diabatic limit [47]. Therefore from Corollary 7.6 the non-zero spectrum of $D*$ has to split as follows

$$\operatorname{spec}^*(D*) = \operatorname{spec}^*(-\Delta_{H|(\mathcal{H}^0)^\perp}) \ \cup \ \operatorname{spec}^*(-JT_{|\mathcal{H}^2_+}) \tag{63}$$

(note the lack of uniformity already noted in the introduction in the convergence of Λ_ε^- when $\varepsilon \to 0$, as each eigenvalue μ in the spectrum of Δ_H is approached at a speed approximately $\varepsilon\mu$). This is enough to compare the needed η-invariant to η_0 and conclude (see (67) below and the discussion following it), but we would like first to spend a few lines to reinterpret this more precisely in the CR Seifert context.

The spectrum of $D*$ from the CR viewpoint. First of all, the second spectral family of eigenvalues in (63) is clearly embedded in $\operatorname{spec}^*(D*)$, as (39) shows that $D* = -TJ$ on $\mathcal{H}^2_+$. To understand where the first one comes from, we consider the closure in L^2 of the following operator

$$Q = d_H J : \ker d_H \subset \Omega^2 V \longrightarrow \Omega^3 M.$$

By definition $\mathcal{H}^2_+ = \ker Q$. We also remark that

$$(Q^*)*_M = (\Pi_{\ker d_H} J\delta_H)*_M = - *_M (\Pi_{\ker \delta_H} J d_H)$$

so that $\ker Q^* = *_M \mathcal{H}^0$ and $\overline{\operatorname{im} Q} = *_M (\mathcal{H}^0)^\perp$. To complete the landscape, we of course define $\mathcal{H}^2_- = \overline{\operatorname{im} Q^*}$, so that

$$\ker d_H \cap \Omega^2 V = \ker Q \oplus \overline{\operatorname{im} Q^*} = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-. \tag{64}$$

Then in vanishing Webster torsion, one has by (39) that

$$\begin{aligned} Q(D*) &= d_H J(-TJ - (d_H *_H)^2) = T d_H + (d_H *_H)^3 \\ &= -\Delta_H Q, \end{aligned} \tag{65}$$

on $\ker d_H \subset \Omega^2 V$, where $\Delta_H = d_H \delta_H$ is the contact Laplacian on $\Omega^3 M$, conjugate to Δ_H on functions through $*_M$. This shows that $D*$ is conjugate to $-\Delta_H$ on $*_M (\mathcal{H}^0)^\perp$ by Q, and that $D*$ preserves the splitting (64). We therefore recover the decomposition of spec$(D*)$ in two families (63), but now entirely seen within $\Omega^2 V$:

$$\text{spec}^*(D*) = \text{spec}^*(D*_{|\mathcal{H}^2_- = \overline{\operatorname{im} Q^*}}) \,\cup\, \text{spec}^*(D*_{|\mathcal{H}^2_+ = \ker Q}). \tag{66}$$

The space $\mathcal{H}^2_-$ is actually a CR invariant, as is $\mathcal{H}^2_+$. Indeed Δ_H is surjective on $\Omega^3 M$ up to "constant" 3-forms $C\theta \wedge d\theta$; as Q^* is zero on these,

$$\begin{aligned}
\mathcal{H}^2_- &= \overline{\operatorname{im} Q^*} = \overline{\operatorname{im} Q^* \Delta_H} \\
&= \overline{\operatorname{im} D * J\delta_H} \ , \quad \text{by} \quad (65), \\
&= \overline{\operatorname{im} DJd_H}.
\end{aligned}$$

We now have two splittings of $\Omega^2 V \cap \operatorname{im} D$: the spectral splitting

$$\operatorname{im} D = E^+ \oplus E^-$$

in the positive and negative eigenspaces of $D*$, and the CR invariant splitting given by

$$\operatorname{im} D = (\mathcal{H}^2_+ \cap \operatorname{im} D) \oplus \mathcal{H}^2_- \ .$$

It follows from prop. 7.7, (63) and (64) that, on a CR Seifert manifold, the pair (E^+, E^-) is in Fredholm position with respect to $(\mathcal{H}^2_+, \mathcal{H}^2_-)$. More precisely,

$$\mathcal{H}^2_+ = E^+ \oplus V \oplus H^2(M, \mathbb{R}) \quad \text{and} \quad E^- = \mathcal{H}^2_- \oplus V$$

with the finite dimensional space $V = \mathcal{H}^2_+ \cap E^-$. This enlightens the CR meaning of the spectral asymmetry of $D*$ we are studying here.

Observe however that if the formal definitions of $\mathcal{H}^2_\pm$ make sense on any 3-dim CR manifold, their use is highly problematic in general. For instance, if M does not bound a Stein manifold, the spaces E^+ and E^- still exist and keep their nice analytic features by hypoellipticity of $D*$ on $\operatorname{im} D$, while $\mathcal{H}^2_+$ may be null as the range of $\overline{\partial}_b$ may not be closed. The previous Fredholm picture then definitely breaks down. Anyway, from the pseudodifferential viewpoint, the projection on E^+ is a natural quantization of the real part of the Szegö projector on holomorphic $(2,0)$-forms, as seen at the Heisenberg symbolic level, see e.g [5, Chap. 4] for more details on this notion.

We now come back to the comparison between the Riemannian and contact spectra. In (63), we can proceed as in (62) by "filling the holes" in the spectrum of $-\Delta_H$ on $\mathcal{H}^0$. From (57) we still have $\Delta_H = -iT$ on CR functions, and this leads to the following decomposition:

$$\text{spec}^*(D*) \ = \ \text{spec}^*(-\Delta_H) \,\cup\, 2 \times \text{spec}^*(-iT_{|\mathcal{H}^{2,0}}) \ \setminus \ 2 \times \text{spec}^*(iT_{|\ker \overline{\partial}_b}). \tag{67}$$

8.1. *Remark.* In a slightly more tricky way, one can add $\mathrm{spec}^*(\Delta_H)$ to both sides of (67): the operator Δ_H on functions is conjugate to $\Delta_H = d_H\delta_H$ on $\Omega^3 M$ and, wedging by θ, to $\delta_H d_H$ on $\Omega^2 V$. The spectrum of the contact Laplacian

$$\Delta_2 = D * + \delta_H d_H \quad \text{on } \Omega^2 V$$

(see Section 9 for more on this one) appears then in a very symmetric manner, namely

$$\begin{aligned} \mathrm{spec}^*(\Delta_2) &= \mathrm{spec}^*(D*) \;\cup\; \mathrm{spec}^*(\Delta_H) \\ &= \mathrm{spec}^*(\Delta_H) \;\cup\; \mathrm{spec}^*(-\Delta_H) \\ &\quad \bigcup\; 2\;\times \mathrm{spec}^*(-iT_{|\mathcal{H}^{2,0}}) \setminus 2\;\times \mathrm{spec}^*(iT_{|\ker \overline{\partial}_b})\,. \end{aligned} \tag{68}$$

This spectral symmetry can also be seen directly. Equation (39) yields

$$\Delta_2 = T *_H - d_H\delta_H + \delta_H d_H = T *_H + P$$

on $\Omega^2 V = \theta \wedge \Omega^1 H$. As $[*_H, T*_H] = 0$ while $*_H P = -P *_H$, $\Delta_2(*_H P) = -(*_H P)\Delta_2$ and $\mathrm{spec}(\Delta_2)$ is symmetric except maybe on $\ker P$, where $\Delta_2 = T*_H = -TJ$. It is then easily seen that the kernel splits into

$$(\ker P)^{2,0} = \mathcal{H}^{2,0} \oplus \overline{\partial}_b^{\,-1}(*_M \ker \overline{\partial}_b),$$

yielding (68).

8.2. *Remark.* Let us mention that this decomposition and the spectral symmetry of Δ_2 also hold on contact manifolds of any dimension, in vanishing Tanaka-Webster torsion, see [46, Prop. 8]. This leads to the same kind of formulae as (68), with a “residual spectrum” given by sum of η-functions counting holomorphic objects.

Comparison of pseudohermitian and Riemannian η-invariants. Comparing the spectrum of P_ε given by (62) with that of $D*$ in (67) yields an immediate relation between their η-functions, up to combinations of ζ-functions of positive operators:

8.3. **Proposition.** *On a CR Seifert manifold,*

$$\eta(P_\varepsilon) - \eta(D*) = \zeta(\Delta_H) + \zeta(Q_\varepsilon^+) - \zeta(Q_\varepsilon^-), \tag{69}$$

where $Q_\varepsilon^\pm = \dfrac{1}{2\varepsilon}(\pm 1 + \sqrt{1 + 4\varepsilon\Delta_\varepsilon})$*, and* $\Delta_\varepsilon = \Delta_H - \varepsilon T^2$ *on functions.*

8.4. *Remark.* Note that here and in the sequel we include the eigenvalue 0 in the definition of zeta functions of non-negative operators P. This means we take

$$\zeta(P)(s) = \sum_{\lambda_i > 0} \lambda_i^{-s} + \dim\ker P = \zeta^*(P)(s) + \dim\ker P\,.$$

The reason for this convention is that $\zeta(P)(0)$, defined in this way, is local for large classes of non-negative operators, as seen by considering $P + h$ with $h \to 0^+$, see also [4]. In particular in (69), one has

$$\zeta^*(\Delta_H) + \zeta^*(Q_\varepsilon^+) - \zeta^*(Q_\varepsilon^-) = \zeta(\Delta_H) + \zeta(Q_\varepsilon^+) - \zeta(Q_\varepsilon^-)\,,$$

since $Q_\varepsilon^+ > 0$ and $\dim\ker\Delta_H = \dim\ker Q_\varepsilon^- = 1$, being achieved by constant functions.

Following Definition 3.2, the renormalized η-invariant $\eta_0(M,\theta)$ is the constant term in the development of $\eta(P_\varepsilon)(0) = \eta(M, g_\varepsilon)$ in powers of ε. It is then immediately extracted from (69) as follows:

$$\eta_0(M,\theta) = \eta(D*)(0) + \zeta(\Delta_H)(0) + \zeta_0(Q), \tag{70}$$

where $\zeta_0(Q)$ is the constant term in the development in powers of ε

$$\zeta(Q_\varepsilon^+)(0) - \zeta(Q_\varepsilon^-)(0) = \sum_{i=-2}^{2} \zeta_i(Q)\,\varepsilon^i, \tag{71}$$

which we already know to exist by (13) and (69), since it is the same as that of $\eta(P_\varepsilon)$ except for the constant term. Moreover, it turns out that $\zeta_0(Q)$ can be evaluated without too much harm on *arbitrary* CR manifolds of dimension 3.

8.5. **Lemma.** *On any* 3*-dimensional CR manifold,*

$$\zeta(Q_\varepsilon^+)(0) = -\zeta(Q_\varepsilon^-)(0),$$

and

$$\zeta_0(Q) = \frac{1}{24\pi^2}\int_M |\tau|^2\theta\wedge d\theta.$$

where $\tau = -\frac{1}{2}J\mathcal{L}_T J$ is the Tanaka-Webster torsion.

Proof. In view of

$$2\varepsilon\, Q_\varepsilon^\pm = \pm 1 + \sqrt{1+4\varepsilon\Delta_\varepsilon},$$

we consider for $\lambda \geqslant -1$ the family of positive operators

$$Q(\lambda) = \lambda + \sqrt{1+4\varepsilon\Delta_\varepsilon},$$

where actually

$$\varepsilon\Delta_\varepsilon = \varepsilon\Delta_H - \varepsilon^2 T^2 = \Delta_{g_\varepsilon}$$

is the standard Laplacian on functions for the rescaled metric $g_\varepsilon = \varepsilon^{-2}\theta^2 + \varepsilon^{-1}\gamma_H$ we use here.

Seeley's classical results [48] imply that $Q(\lambda)$ is a smooth family of positive elliptic pseudo-differential operators of order 1, and that their ζ-functions

$$P(\lambda)(s) := \zeta(\lambda + \sqrt{1+4\Delta_{g_\varepsilon}})(s)$$

are meromorphic with possibly simple poles at $s = 1$, 2 and 3. According to [4, Prop. 2.9] or [25, Lemma 1.10.2] one can differentiate $P(\lambda)(s)$ with respect to λ to get

$$\frac{d}{d\lambda}P(\lambda)(s) = -sP(\lambda)(s+1).$$

Therefore $\dfrac{d^4}{d\lambda^4}P(\lambda)(0) = 0$ since $P(\lambda)$ is regular at $s = 4$, and $P(\lambda)(0)$ is a polynomial of degree 3 in λ:

$$P(\lambda) = R_0 - \lambda R_1 + \lambda^2\frac{R_2}{2} - \lambda^3\frac{R_3}{3}, \tag{72}$$

where $R_0 = \zeta(\sqrt{1+4\Delta_{g_\varepsilon}})(0)$ and R_n for $n > 0$ stands for the residue at $s = n$ of

$$\zeta(\sqrt{1+4\Delta_{g_\varepsilon}})(s) = \zeta(1+4\Delta_{g_\varepsilon})(s/2).$$

Actually these residues are related to the development of the heat kernel of Δ_{g_ε} on functions in a simple way. Let

$$\operatorname{Tr}(\mathrm{e}^{-t\Delta_{g_\varepsilon}}) \overset{t\to 0^+}{\sim} \frac{a_0(g_\varepsilon)}{t^{3/2}} + \frac{a_2(g_\varepsilon)}{t^{1/2}} + \cdots .$$

According to [25, Theorem 4.8.18d], the constants are computed in terms of the volume and the Riemannian scalar curvature of g_ε as:

$$a_0(g_\varepsilon) = \frac{\operatorname{Vol}(M, g_\varepsilon)}{(4\pi)^{3/2}} \quad \text{and} \quad a_2(g_\varepsilon) = \frac{1}{6(4\pi)^{3/2}}\int_M \operatorname{Scal}(g_\varepsilon)d\operatorname{vol}_{g_\varepsilon} . \tag{73}$$

This yields

$$\operatorname{Tr}(\mathrm{e}^{-t(1+4\Delta_{g_\varepsilon})}) = e^{-t}\operatorname{Tr}(\mathrm{e}^{-4t\Delta_{g_\varepsilon}}) \sim \frac{a_0(g_\varepsilon)}{8t^{3/2}} + \frac{4a_2(g_\varepsilon) - a_0(g_\varepsilon)}{8t^{1/2}} + \cdots ,$$

and by Mellin's transform [25, Lemma 1.10.1],

$$\Gamma(s/2)\,\zeta(1+\Delta_{g_\varepsilon})(s/2) \;=\; \frac{a_0(g_\varepsilon)}{4(s-3)} \;+\; \frac{4a_2(g_\varepsilon) - a_0(g_\varepsilon)}{4(s-1)} \;+\; h(s),$$

with h holomorphic for $\operatorname{Re} s > -1$. Hence

$$R_0 = \zeta((1+4\Delta_{g_\varepsilon})^{1/2})(0) = 0$$

as this is the only way to cancel the simple pole of the Γ-function at $s = 0$, and

$$R_2 = 0,$$

(because the Γ-function does not vanish at $s = 2$ and the r.h.s. has no pole at this point) so that $P(\lambda)$ is an odd polynomial. This gives $P(1) = -P(-1)$ or, equivalently,

$$\zeta(Q_\varepsilon^+)(0) = -\zeta(Q_\varepsilon^-)(0)$$

as announced. Moreover one has

$$R_1 = \frac{4\,a_2(g_\varepsilon) - a_0(g_\varepsilon)}{4\sqrt{\pi}} \quad \text{and} \quad R_3 = \frac{a_0(g_\varepsilon)}{2\sqrt{\pi}},$$

and thus by (72) and (73)

$$\begin{aligned}\zeta(Q_\varepsilon^+)(0) &= -R_1 - R_3/3 \\ &= \frac{1}{\sqrt{\pi}}\Big(\frac{a_0(g_\varepsilon)}{12} - a_2(g_\varepsilon)\Big) \qquad (74) \\ &= \frac{1}{48\pi^2\varepsilon^2}\Big(\frac{1}{2}\int_M \theta\wedge d\theta - \int_M \mathrm{Scal}(g_\varepsilon)\,\theta\wedge d\theta\Big).\end{aligned}$$

The Riemannian curvature of g_ε can be developed in powers of ε using the links between Tanaka-Webster and Levi-Civita connections underlined in (14). According to *e.g.* [46, p 318], one finds in dimension 3 that

$$\mathrm{Scal}(g_\varepsilon) = -\frac{1}{2} + 2\varepsilon R - \varepsilon^2\,|\tau|^2,$$

where R and τ are Tanaka-Webster curvature and torsion. The constant term in the full development of $\zeta(Q_\varepsilon^+)$ is then necessarily equal to the integral of $\frac{1}{48\pi^2}|\tau|^2$ on M. ☐

8.6. *Remark.* According to (62), Q_ε^+ describes the non-collapsing spectrum of $d*_\varepsilon$, on Seifert CR manifolds. We have seen that this spectrum only contributes by a local expression $\zeta(Q_\varepsilon^+)(0)$ to $\eta(d*_\varepsilon)$. We expect this to hold in the general case. Indeed on any CR manifold, the non-collapsing spectrum is always strictly positive, since it converges to 1 and $d*_\varepsilon$ has no spectral flow. It therefore always contributes through a *zeta* function, whose value at 0 is local for a wide class of operators.

A computation of η_0. The previous Lemma 8.5, together with the spectral decomposition (62), leads to a general computation of the renormalized η-invariant on all CR Seifert manifolds, including the still missing exceptional cases of Section 5. Indeed, one has using our convention for zeta in Remark 8.4

$$\zeta^*(Q_\varepsilon^+) - \zeta^*(Q_\varepsilon^-) = \zeta(Q_\varepsilon^+) - \zeta(Q_\varepsilon^-) + 1,$$

since 0 belongs to $\mathrm{spec}(Q_\varepsilon^-)$ with multiplicity 1 (corresponding to the constant functions). It follows then from (62) that

$$\eta_0(d*) = \eta_{\mathrm{ad}}(d*) = 1 + 2\left(\eta(-iT_{|\mathcal{H}^{2,0}})(0) - \eta(iT_{|\ker\overline{\partial}_b})(0)\right). \qquad (75)$$

These holomorphic counting functions can be nicely expressed as dimensions of spaces of sections on adequate orbifold line bundles over the basis orbifold Riemann surface, which in turn are easily computed with the help of Riemann-Roch-Kawasaki's theorem [32]. Note that this has already been observed in the adiabatic setting and constant curvature by L. Nicolaescu in [40, Sec. 1]. We give below only a short description of the computation, and refer to [40] for more details.

Following Section 5, the CR Seifert manifold M may be seen as the unit circle bundle of some orbifold line bundle L over Σ, with singular data $(\alpha_i, \rho_i, \beta_i)$ at points $m_i \in \Sigma$. Let

$K_\Sigma = \Lambda^{1,0}T^*\Sigma$ denotes the orbifold canonical bundle of Σ. Now, given a Fourier component $iT = n \in \mathbb{Z}$, the space of CR functions f such that $f \circ \varphi_t = e^{-int} f$ are interpreted as the space of holomorphic sections of L^n, and we denote by $h_0(L^n)$ its dimension. Moreover the space of holomorphic forms σ in the canonical CR bundle $K_M \simeq \theta \wedge K_\Sigma \otimes L$ such that $-iT\sigma = n\sigma$ may be seen as the space of holomorphic sections of $K_\Sigma \otimes L^n$, *i.e.* $(1,0)$-holomorphic forms in L^n. Let $h_1(L^n)$ denotes its dimension. Hence we get

$$\tag{76}\begin{aligned}\eta(-iT_{|\mathcal{H}^{2,0}})(s) - \eta(iT_{|\ker \overline{\partial}_b})(s) &= -\sum_{n\in\mathbb{Z}^*} \operatorname{sgn}(n)\frac{h_0(L^n) - h_1(L^n)}{|n|^s}\\ &= \sum_{n\in\mathbb{Z}^*} \operatorname{sgn}(n)\frac{\chi_{\overline{\partial}}(L^{-n})}{|n|^s}.\end{aligned}$$

Following the method in [40, Sec. 1], this sum can be computed explicitly using Riemann-Roch-Kawasaki theorem (extension of the classical Riemann-Roch to the orbifold case) [32]. Using the (rational) orbifold Euler characteristic χ of the base Σ and the (rational) degree d of L, it reads

$$\tag{77} \chi_{\overline{\partial}}(L^{-n}) = \frac{\chi}{2} - nd + \sum_i \frac{1}{2}\left(1 - \frac{1}{\alpha_i}\right) - \left\{\frac{-n\beta_i\rho_i'}{\alpha_i}\right\},$$

where $\{x\} = x - [x]$ denotes the fractional part of x, and ρ_i' is the inverse of ρ_i mod. α_i. This purely topological formula holds true, irrespective of the curvature value. The result should then be the same in the constant and non-constant curvature cases, so that Ouyang's formula (27) for η_0 holds true on any CR Seifert manifold.

To get explicitly the formula, one can argue as follows: the constant terms in (77) do not contribute to the sum (76), whereas

$$\sum_{n\in\mathbb{Z}^*} -d|n|^{-s+1} = -2d\,\zeta(s-1)$$

has value $\frac{d}{6}$ at $s = 0$. The Dedekind-Rademacher sums $s(\alpha_i, 1, \beta_i\rho_i') = s(\alpha_i, \rho_i, \beta_i)$ appear from the periodic orbifold contribution in (77), as in Nicolaescu's work using [40, Proposition 1.4]. Inserting in (75) leads to the desired expression (27) for η_0.

8.7. *Remark.* This last computation shows that Theorem 1.4 could have been proved in a quicker way on constant curvature CR Seifert manifolds: applying the previous formulae and using the computation of $\zeta(\Delta_H)(0)$ given below leads to an expression for $\eta(D*)$ that can be compared directly to Ouyang's formula for η_0. We have however omitted this proof since the links between $\eta(D*)$ and η_0 proved in this way would have appeared as the result of a possibly completely fortuitous or miraculous equality between explicitly known numerical expressions. On the contrary, our proof stresses the fact that the relation between $D*$ and $d*$ is deeply rooted in the nature of CR geometry and the diabatic limit. Moreover, it applies to *the whole family* of CR Seifert manifolds, irrespective of their curvature, and especially the exceptional cases that do not admit constant curvature contact forms.

We now complete the comparison between the renormalized η-invariant $\eta_0(M,\theta)$ and the pseudohermitian η-invariant $\eta(D*)$.

8.8. **Theorem.** *Let M be a CR Seifert manifold. Then,*

$$\eta_0(M,\theta) = \eta(D*)(0) + \zeta(\Delta_H)(0) \tag{78}$$

with

$$\zeta(\Delta_H)(0) = \frac{1}{512}\int_M R^2\,\theta\wedge d\theta\ . \tag{79}$$

Proof. From Proposition 8.3 and Lemma 8.5 it remains to compute $\zeta(\Delta_H)(0)$. The development of the heat kernel $\mathrm{e}^{-t\Delta_H}$ of the Kohn Laplacian Δ_H has been studied by Beals, Greiner and Stanton in [6, Theorem 7.30]. On any CR manifold of dimension 3,

$$\mathrm{Tr}(\mathrm{e}^{-t\Delta_H}) \ \sim\ \sum_{n=0}^{\infty} t^{n-2} b_n(M,\theta) \quad \text{as} \quad t\to 0^+,$$

where $b_n(M,\theta)$ are integrals on M of polynomials of covariant derivatives of Tanaka-Webster curvature and torsion. Mellin's transform yields again

$$\Gamma(s)\,\zeta(\Delta_H)(s) \ =\ \sum_{n\leqslant N}\frac{b_n(M,\theta)}{s-2+n} \ +\ h_N(s)$$

with h_N holomorphic for $\mathrm{Re}\, s > N-2$, and hence

$$\zeta(\Delta_H)(0) = b_2(M,\theta).$$

As $\zeta(\Delta_H)(0)$ stays unchanged when θ becomes $k\theta$, one must have $b_2(M,k\theta) = b_2(M,\theta)$, and the same argument as in Lemma 4.1 gives that

$$b_2(M,\theta) \ =\ C_1\int_M R^2\,\theta\wedge d\theta \ +\ C_2\int_M |\tau|^2\,\theta\wedge d\theta,$$

for some constants C_1, C_2.

Thanks to N. Stanton's work [51] it is possible to determine C_1 on the sphere $\mathbb{S}^3$. Indeed, let $L = 4\Delta_H + R$ be the CR-conformal Laplacian on $\mathbb{S}^3$. Stanton states in [51, Theorem 4.34] that for the contact form $\theta = i\overline{\partial}r = \frac{i}{2}(z^1 d\bar z^1 + z^2 d\bar z^2)$,

$$\mathrm{Tr}(\mathrm{e}^{-tL}) = \frac{\pi^2}{256t^2} + O\big(\frac{1}{t^2}\,\mathrm{e}^{-\pi^2/4t}\big) \quad \text{as} \quad t\to 0^+.$$

Now Tanaka-Webster curvature R equals 4 here, so that the heat development of Δ_H is

$$\mathrm{Tr}(\mathrm{e}^{-t\Delta_H}) \ =\ \mathrm{e}^t\,\mathrm{Tr}(\mathrm{e}^{-tL/4}) \ =\ \mathrm{e}^t\,\frac{\pi^2}{16t^2} + O\big(\frac{1}{t^2}\,\mathrm{e}^{-\pi^2/4t}\big),$$

and the constant term $b_2(M,\theta)$ has value $\frac{\pi^2}{32}$. Hence

$$\zeta(\Delta_H)(0) \;=\; \frac{\pi^2}{32} \;=\; C_1 \int_{\mathbb{S}^3} R^2\,\theta\wedge d\theta \;=\; 16\pi^2\, C_1$$

yields $C_1 = \frac{1}{32\times 16}$ on the sphere, hence on any CR Seifert manifold. □

Putting together this last result and Theorem 1.1 leads to Corollary 1.5.

9. The pseudohermitian η-invariant and its CR invariant correction

We first begin by showing existence of the pseudohermitian η-invariant in dimension 3. It follows mostly the classical method of Chapter 1 of [25], using pseudo-differential calculi developed on contact manifolds. As a consequence, we shall put below the emphasis mainly on the steps where the hypoelliptic context introduces differences with the well-known elliptic theory.

9.1. **Theorem.** *Let (M,H,J) be a compact 3-dimensional strictly pseudoconvex CR manifold endowed with a compatible contact form θ and the associated horizontal metric $g_1 = \theta^2 + d\theta(\cdot, J\cdot)$. Then the series*

$$\eta(D*)(s) = \mathrm{Tr}^*(D*|D*|^{-(s+1)}) = \sum_{\lambda_i\in\mathrm{spec}(D*)\setminus\{0\}} \frac{\lambda_i}{|\lambda_i|^{s+1}}$$

converges absolutely for $\mathrm{Re}\, s > 2$, *and has an meromorphic extension with possible simple poles at $s = 2 - n/2$ for $n\in\mathbb{N}$. Moreover $\eta(D*)(s)$ is regular at $s=0$; its value $\eta(D*)(0)$ is the* pseudohermitian η-invariant.

Proof. From [46] the two Laplacians

$$\Delta_2 \;=\; D* + \delta_H d_H \text{ on } \Omega^2 V \quad\text{and}\quad \Delta_3 = d_H\delta_H \text{ on } \Omega^3 M,$$

are maximally hypoelliptic (be careful: Δ_3 is non-negative, but Δ_2 is not, despite the notation). This means that they control two horizontal derivatives in L^2 norms (and one vertical derivative). By the associated Sobolev embeddings, their resolvents are compact and their spectra are discrete. By orthogonality and conjugation, the non-zero spectrum of Δ_2 splits into

$$\mathrm{spec}^*(\Delta_2) = \mathrm{spec}^*(D*) \cup \mathrm{spec}^*(\Delta_3)\ , \tag{80}$$

and $D*$ has discrete pure point spectrum with finite multiplicities on $\mathrm{im}\, D$. Sobolev embeddings also yields that $(i+\Delta_2)^{-n}$, $(i+\Delta_3)^{-n}$ are trace class for n large enough, hence the same for $(D*)^{-n}$. The series $\eta(D*)(s)$ is then well defined and holomorphic for $\mathrm{Re}\, s$ large.

Getting more information on η relies in the Riemannian (elliptic) case on the use of the classical pseudo-differential calculus for elliptic operators. Such a symbolic calculus has also been developed on contact manifold by Beals, Greiner and Stanton in [5, 6] or Taylor in [53], a concise account may also be found in [24]. The symbols of the hypoelliptic operators Δ_2 and Δ_3 are invertible in this calculus: this follows from [30, Lemmas 5.18, 5.19], or else by observing that in dimension 3 their principal symbols are sums of invertible Folland-Stein ones.

The parameter calculus adapted to the Heisenberg setting developed in Propositions 5.20 to 5.26 of [30] yields pseudo-differential approximations $R(\lambda)$ of the resolvents $((\Delta_2)^2-\lambda)^{-1}$, when $\lambda \notin \mathbb{R}^+$. This uses the classical iteration process described in [25, p. 51] or [50, Sec. 9.1] for instance, where the standard pseudo-differential symbolic product has to be replaced by the Heisenberg one, see [6, 24]. The symbol of these $R(\lambda)$ are universal expressions involving the symbol of $(\Delta_2)^2 - \lambda$, its inverse, and tensorial expressions of the Webster-Tanaka curvature and its derivatives.

Then, as explained in [25, Sec. 1.7], $R(\lambda)$ can be used in place of $((\Delta_2)^2 - \lambda)^{-1}$ in the contour integral

$$\Delta_2 \, \mathrm{e}^{-t(\Delta_2)^2} \;=\; \frac{1}{2i\pi} \int_\gamma \mathrm{e}^{-t\lambda} \, \Delta_2(\Delta_2^2 - \lambda)^{-1} \, d\lambda,$$

with $\gamma \subset \mathbb{C} \setminus \mathbb{R}^+$ the correctly oriented boundary of the cone $\{\operatorname{Im}\lambda \leqslant \operatorname{Re}\lambda + 1\}$, in order to get good approximations of $\Delta_2 \, \mathrm{e}^{-t(\Delta_2)^2}$ when t goes to 0. Following Lemma 1.7.7 of [25], homogeneity arguments then easily lead to the asymptotic development of $\operatorname{Tr}(\Delta_2 \, \mathrm{e}^{-t\Delta_2^2})$ when $t \to 0^+$. Namely,

$$\operatorname{Tr}(\Delta_2 \, \mathrm{e}^{-t\Delta_2^2}) \;\sim\; \sum_{n=0}^{\infty} t^{(n-6)/4} R_n(M,\theta), \tag{81}$$

where $R_n(M,\theta)$ are integrals over M of universal polynomials in Tanaka-Webster curvature and covariant derivatives (with respect to the classical elliptic development given in [25, Lemma 1.7.7], the only changes here concern the powers of t: this is due to the fact that, in the Heisenberg calculus, horizontal directions have weight 1, while T is of weight 2. For instance, this implies that the "Heisenberg-dimension" of M is 4 instead of 3).

9.2. *Remark.* Another more direct track, if steeper, also leads to such kernel developments. One can follow Beals-Greiner-Stanton's approach to heat kernels asymptotics in the contact setting. In [6] they have extended their symbolic calculus on $M \times \mathbb{R}$ to include the heat operator $\partial_t + P$ for some positive sub-Laplacians P. They show that in the case P is a positive Folland-Stein type operator, one can inverse the symbol of $\partial_t + P$ inside this calculus, which gives rather directly developments like (81) for $\operatorname{Tr}(Q \, \mathrm{e}^{-tP})$ from the symbol of $Q(\partial_t + P)^{-1}$, see also [24, Sec 4]. By R. Ponge's recent work [43, 44], this approach leads to a relatively simple proof of the index theorem, and also applies to more general positive hypoelliptic P as $(\Delta_2)^2$.

Let us now complete the proof of Theorem 9.1. Mellin transform and the functional calculus relate the asymptotic development in small time of the heat kernel to η and ζ

functions [25, Section 1.10]. In particular, [25, p 81] and (81) yield:

$$\eta(\Delta_2, s)\,\Gamma((s+1)/2) \;=\; \sum_{n=0}^{N} \frac{4}{2s+n-4} R_n(M,\theta) + h_N(s)$$

where h_N is an holomorphic function for $s > 2-N/2$. Hence we get the required meromorphic extension of $\eta(\Delta_2)(s)$. The same technique applies to Δ_3 on $\Omega^3 M$, but this is a positive operator whose heat kernel development has been extensively treated in [6, Theorem 7.30]: the η-function is here a ζ-function which is regular at $s = 0$.

Using the spectral decomposition (80), we get that $\eta(D*)(s)$ is meromorphic with $s = 0$ being possibly a simple pole. It remains to show that this function is regular at $s = 0$. We first note that the value of the residue of $\eta(D*)$ at $s = 0$ is $2R_4(M,\theta)$. It is easily seen in (39) that $D*$ becomes $k\,D*$ in the contact rescaling $\theta \to k\theta$. Therefore, $\eta(D*_{k\theta})(s) = k^s \eta(D*_\theta)(s)$ and

$$R_4(M, k\theta) = R_4(M,\theta).$$

Following the proof of Lemma 4.1, this implies that, *in dimension* 3,

$$R_4(M,\theta) \;=\; C_1 \int_M R^2\,\theta\wedge d\theta \;+\; C_2 \int_M |\tau|^2\,\theta\wedge d\theta \tag{82}$$

where R and τ are Tanaka-Webster curvature and torsion and C_1, C_2 are universal constants.

The residue is moreover invariant under smooth deformation of the pseudohermitian and CR structures (*i.e.* both θ and J): as underlined in [25, Lemma 1.10.2] this general feature stems from the existence of a local variation formula for η-functions, namely in the absence of spectral flow here:

$$\dot\eta(\Delta_2)(s) \;=\; -s\ \mathrm{Tr}(\dot\Delta_2 |\Delta_2|^{-(s+1)}).$$

The point here is that the trace on the right has a meromorphic extension coming from the development of $\mathrm{Tr}(\dot\Delta_2\,\mathrm{e}^{-t(\Delta_2)^2})$, but the possible simple pole at $s = 0$ is actually cancelled out by the s in front of the whole expression.

The conclusion is that the integrals in (82) have to be independent of variations of θ and J, and this implies $C_1 = C_2 = 0$: indeed, the variations of R^2 and $|\tau|^2$ when $\theta \to \theta_f = \mathrm{e}^{2f}\,\theta$ have been computed in [35, Sec. 5]. One finds that

$$\frac{d}{df}(R^2\,\theta\wedge d\theta) = 8R\,(\Delta_H f)\,\theta\wedge d\theta \tag{83}$$

while (if $\tau = A_{11}\theta^1 \otimes \theta^1$)

$$\frac{d}{df}(|\tau|^2\,\theta\wedge d\theta) = 2i(A_{\bar1\bar1} f_{,11} - A_{11} f_{,\bar1\bar1})\,\theta\wedge d\theta. \tag{84}$$

After integration by parts, this yields

$$\frac{d}{df} R_4(M,\theta) \;=\; 8C_1 \int_M f\Delta_H R\,\theta\wedge d\theta \;+\; 2iC_2 \int_M f(A_{\bar1\bar1,11} - A_{11,\bar1\bar1})\,\theta\wedge d\theta. \tag{85}$$

Testing on a circle bundle (with vanishing torsion) over a Riemann surface of non-constant curvature cancels out C_1. General expression for torsion of hypersurfaces in [54, Sec. 4] shows that $A_{\bar{1}\bar{1},11} - A_{11,\bar{1}\bar{1}}$ does not vanish identically: actually, following [36] the only Bianchi identity of order 2 between R and τ in dimension 3 is $R_{,0} = A_{11,\bar{1}\bar{1}} + A_{\bar{1}\bar{1},11}$, which does not occur in (85) so that $C_2 = 0$. □

9.3. *Remark.* The contact-de Rham complex exists on contact manifolds of any dimension, and the pseudohermitian signature operator $D*$ is still self-adjoint in dimension $4n-1$. Therefore the properties of $\eta(D*)(s)$ stated in Theorem 9.1 make sense on contact manifolds of any dimension. Most of the previous discussion, and its conclusions, still applies, but the last argument about the regularity at $s = 0$ of $\eta(D*)$. The residue is still both a *contact* invariant, independent of the choices of θ and J, and an integral of some universal pseudohermitian polynomial of the right weight. But many possibilities are now left, which cannot be so easily analysed (even in the next relevant dimension 7, the algebra becomes quite complicated). At the present time, one still ignores whether this residue always vanishes or not.

The CR invariant correction of $\eta(D*)$. Having now a well-defined object at hand, we can proceed to the construction of a *modified pseudohermitian η-invariant.*

9.4. **Theorem.** *There exists a unique choice of universal constants C_1 and C_2 such that, for any compact strictly pseudoconvex CR 3-manifold M, the following pseudohermitian invariant*

$$\overline{\eta}(D*) \;=\; \eta(D*) \,+\, C_1 \int_M R^2\theta\wedge d\theta \,+\, C_2 \int_M |\tau|^2\,\theta\wedge d\theta \;, \tag{86}$$

formed from a contact form θ, its Tanaka-Webster curvature R and torsion τ, is in fact a CR invariant *of M, which we shall call the* modified pseudohermitian η-invariant.

The key point for the proof of Theorem 9.4 is the following: on an oriented CR 3-manifold M, the space of adapted contact forms for the contact structure (let us denote it by Θ) is contractible and non empty. Then, for a pseudohermitian invariant, being CR invariant simply means being independent of the choice of the contact form, *i.e.* having a vanishing derivative in the direction of any variation in θ.

Using the analysis above, we get that $\eta(D*)$, seen as a function on the space Θ of contact forms adapted to a given CR structure, has the following features :

(i) $\eta(D*_{k\theta}) = \eta(D*_\theta)$ for any positive k;
(ii) its derivative is local: if $\theta_t = (1+tf)\theta$ is a small variation of contact forms,

$$\frac{d}{dt}\eta(D*_{\theta_t})_{t=0} = \int_M f\,\mathcal{E}_\theta\,\theta\wedge d\theta \;,$$

where $\mathcal{E}_\theta$ is a local pseudohermitian invariant of θ built algebraically and *universally* from a finite jet of θ and its Tanaka-Webster curvature R and torsion τ.

One then deduces from (i) and (ii) that, necessarily,

$$\mathcal{E}_{k\theta} = k^{-4}\mathcal{E}_\theta \ , \tag{87}$$

and moreover

$$\int_M \mathcal{E}_\theta \, \theta \wedge d\theta \ = \ 0 \ . \tag{88}$$

Said otherwise, $\mathcal{E}_\theta$ is of weight -4 and vanishing integral. One can then remark a basic fact:

9.5. **Lemma.** *Let α be a smooth* closed, *and* real 1-*form on Θ where $T_\theta\Theta$ is identified to the space of functions on M through $f \to \frac{d}{dt}(1+tf)\theta$. If α is of the type*

$$\alpha_\theta : f \in C^\infty(M) \longmapsto \alpha_\theta(f) = \int_M f \, \mathcal{A}_\theta \, \theta \wedge d\theta \tag{89}$$

where $\mathcal{A}_\theta$ is an universal local pseudohermitian invariant of a finite jet of θ. If $\mathcal{A}_\theta$ is of weight -4 and of vanishing integral, then α is a linear combination of the derivatives in θ of

$$\int_M R^2 \theta \wedge d\theta \quad \textit{and} \quad \int_M |\tau|^2 \theta \wedge d\theta \ .$$

Proof. We argue as in Section 4, classifying local pseudohermitian invariants that are real and of weight 4. We have seen that the sole possibilities are the linear combinations:

$$\mathcal{A}_\theta = c_1 R^2 + c_2|\tau|^2 + c_3 R_{,0} + c_4 \Delta_H R + c_5 i(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11})$$

for some real constants $c_1, \dots, c_5$. Now

$$\int_M \mathcal{A}_\theta \, \theta \wedge d\theta = \int_M (c_1 R^2 + c_2 |\tau|^2) \, \theta \wedge d\theta,$$

and this integral does not vanish in general, so that $c_1 = c_2 = 0$.

From (83) and (84), the expression

$$\int_M f \left(c_4 \Delta_H R + c_5 i(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11}) \right) \theta \wedge d\theta$$

is the variation of

$$\int_M \left(c_4 \frac{R^2}{8} - c_5 \frac{|\tau|^2}{2} \right) \theta \wedge d\theta.$$

In particular it induces a closed 1-form on on Θ.

The lemma will now follow from the vanishing of c_3. It suffices to check that the term $R_{,0} = A_{11,\bar{1}\bar{1}} + A_{\bar{1}\bar{1},11}$ does not yield a closed form on Θ. According to [35, Sec. 5], a change of contact form $\theta \to \theta_f = e^f\theta$ induces the following changes

$$R_f = e^{-f}(R + 2\Delta_H f - 2|f_{,\bar{1}}|^2) \quad \text{and} \quad T_f = e^{-f}(T + i f_1 Z_{\bar{1}} - i f_{\bar{1}} Z_1),$$

and therefore

$$\frac{d}{df}(R_{,0}\theta\wedge d\theta) = \big(-f_{,0}R + if_{,1}R_{,\bar 1} - if_{,\bar 1}R_{,1} + 2(\Delta_H f)_{,0}\big)\,\theta\wedge d\theta\,.$$

When restricted on the sphere $\mathbb{S}^3$ with its constant curvature pseudohermitian structure this gives

$$\begin{aligned}\int_M (g\frac{d}{df} - f\frac{d}{dg})(R_{,0}\,\theta\wedge d\theta) &= 2\int_M ((\Delta_H f)_{,0}g - (\Delta_H g)_{,0}f)\theta\wedge d\theta\\ &= -4\int_M (\Delta_H f)(T.g)\,\theta\wedge d\theta.\end{aligned}$$

This expression does not vanish identically: for instance when taking any non T-invariant function g and f such that $\Delta_H f = T.g$. This completes the proof. □

This shows Theorem 9.4, exhibiting a new CR invariant

$$\overline{\eta}(D*) \;=\; \eta(D*) \;+\; C_1\int_M R^2\,\theta\wedge d\theta + C_2\int_M |\tau|^2\,\theta\wedge d\theta\;. \tag{90}$$

Uniqueness in the choice of the constants is obtained because no linear combination in the integrals of R^2 and $|\tau|^2$ can be a CR invariant. □

9.6. *Remark.* An analogous line of reasoning yields: there exists a universal constant C' such that, for any compact strictly pseudoconvex Cauchy-Riemann 3-manifold M,

$$\overline{\eta}(D*) \;-\; C'\,\nu(M) \tag{91}$$

is a *contact* invariant, *i.e.* is independent of the choice of the complex structure. The proof (left to the reader) consists in proving that the only tensorial choice for the differential of the CR invariant $\overline{\eta}$ is (up to some multiplicative constant) the Cartan curvature like in (29) and (30).

The best one can get is the following: it has already been remarked earlier that the value of the renormalized η-invariant η_0 is purely topological on CR Seifert manifolds. Keeping the contact form fixed, this means that it has to be independent of the complex structure. As $\eta(D*) = \eta_0 - \frac{1}{512}\int R^2\theta\wedge d\theta$ and

$$\overline{\eta} - C'\nu = (1+3C')\eta_0 + (C_1 - \frac{1}{512} - \frac{C'}{16\pi^2})\int R^2\theta\wedge d\theta$$

must be a contact invariant, this implies that

$$C_1 - \frac{1}{512} - \frac{C'}{16\pi^2} = 0,$$

since the integral of R^2 has non-zero variations with respect to the complex structures.

Guessing the values of C in Conjecture 1.6 and C_2 in Theorem 9.4 seems much harder. Having a precise value for them would (for instance) involve a precise computation of the spectrum of $\eta(D*)$ in a case where the torsion does not vanish. This seems difficult to achieve either with our methods, which rely on Fourier decomposition under the circle action, or with classical tools of representation theory, which require a high degree of homogeneity.

Of course, one knows that the derivative of $\eta(D*)$ is given by algebraic expressions of the jet of the hypoelliptic symbols of the involved operators. However these expressions are so intricate that the constants are only computable this way "in theory", and not in practice.

9.7. *Remark.* The same arguments also apply to the renormalized η-invariant η_0 introduced in Section 3, instead of $\eta(D*)$. This explains *a priori* the existence of some local correction of η_0 leading to a CR invariant, itself related (up to some contact invariant) to a multiple of ν; this might be compared with Lemma 4.1.

10. Applications

Corollaries 1.7 and 1.9 rely on the formula discovered by the first and second authors [11, Theorem 1.2]: for any Einstein asymptotically hyperbolic manifold (N^4, g),

$$\frac{1}{8\pi^2}\int_N \left(3|W^-|^2 - |W^+|^2 + \frac{1}{24}\,\mathrm{Scal}^2\right) - \chi(N) + 3\,\tau(N) = \nu(M). \tag{92}$$

For complex hyperbolic surfaces, the integral term is zero. If $\bar{N}$ is smooth, with M as the only end, then the topological contributions always are integers. Corollary 1.7 is then proved.

It is instructive to check the results for a holomorphic disk bundle over a hyperbolic Riemann surface Σ, with M as its boundary. Clearly one has $\chi(N) = \chi(\Sigma) = \chi$ and $\tau(N) = -1$. If N carries a complex hyperbolic metric with M as its boundary at infinity, then Corollary 1.7 gives the equation

$$\chi \;-\; 3\,\tau \;=\; -\,\nu(M) \;=\; d + 3 + \frac{\chi^2}{4d}$$

and the only solution is $d = \frac{\chi}{2}$. We then recover the well-known fact that the only disk bundles carrying a complex hyperbolic metric are the square roots of the (complex) tangent bundle.

Corollary 1.9 is again a direct consequence of (92), since for a Kähler-Einstein metric, the integral term is non-negative. For an Einstein metric, the story is more complicated, but positivity is achieved if solutions of the Seiberg-Witten equations exist, and it is proved in [45, Corollary 31] that it is a consequence of the nonvanishing of the Kronheimer-Mrowka invariants [34].

From [17, Theorem 5.12], one knows that pseudoconvex complex hyperbolic surfaces N have vanishing third homology group $H_3(N, \mathbb{Z})$. Hence no multiple ends can occur, but one expects orbifold singularities or cusps to appear in the interior of a complex hyperbolic filling. The complex hyperbolic cusps can be compactified to yield a complex orbifold surface that we note again N, by adding at the infinity of each cusp a quotient Σ_i of a 2-torus. The corollaries 1.7 and 1.9 remain true in this case, with the Euler characteristic and the signature of N being replaced by their orbifold versions: In case ℓ cusps are present, there is an additional contribution in the signature coming from the self-intersection of each 2-torus at infinity. Namely, one has to consider the modified signature [9, Proposition 3.4]

$$\tau_{\mathrm{cusp}}(N) = \tau(N) - \frac{1}{3}\sum_{1}^{\ell}[\Sigma_i]\cdot[\Sigma_i].$$

Of course, Corollary 1.8 is no more true, since the characteristic numbers are now rational; the denominator of ν only gives an hint on the order of the singularities needed to fill M.

Explicitation for lens spaces. We now specialize the formula obtained in Corollary 1.3 to the lens space $L(p,q)$ obtained as a quotient of the 3-sphere $\mathbb{S}^3$ in $\mathbb{C}^2$ by $\mathbb{Z}/p\mathbb{Z}$, with its generator acting on $\mathbb{C}^2$ by $(\mathrm{e}^{\frac{2i\pi}{p}},\mathrm{e}^{\frac{2iq\pi}{p}})$, where q is prime with p. They are interesting in connection with filling by Einstein metrics, since some of them appear as boundary at infinity of selfdual Einstein metrics [16]. On the other hand, it has been shown that large families of them admit symplectic fillings [37], so that Corollary 1.9 may be applied to these.

10.1. **Proposition.** *One has:* $\nu(L(p,q)) = -\frac{1}{p} + 12\, s(p,q,1)$.

For sake of comparison, we recall to the interested reader the value of the classical η-invariant on lens spaces with the standard round metric, as computed by Atiyah-Patodi-Singer [3, Proposition 2.12]:

$$\eta(L(p,q)) = -4\, s(p,q,1). \tag{93}$$

Proof. For simplicity, we shall assume that $(q-1)$ is prime with p (as a matter of fact this implies that we take $q \neq 1$), and we leave the general case to the reader. Let us see the 3-sphere as the bundle $\mathcal{O}(-1)$ over the projective line $\mathbb{C}P^1$. The induced action on $\mathbb{C}P^1$ has two fixed points: the two antipodal points, with action of $\mathbb{Z}/p\mathbb{Z}$ generated by $\mathrm{e}^{\pm i2\pi\frac{q-1}{p}}$, and action in the fiber by $\mathrm{e}^{i\frac{2\pi}{p}}$ and $\mathrm{e}^{i2\pi\frac{q}{p}}$ respectively. Therefore $L(p,q)$ is a $\mathbb{S}^1$-orbifold bundle over an orbifold projective line with two orbifold points with angle $\frac{2\pi}{p}$. The Euler characteristic is $\chi = \frac{2}{p}$ and the degree (first Chern number) is $d = -\frac{1}{p}$. Now Corollary 1.3 and Ouyang's theorem 5.2 give the formulae

$$\nu(L(p,q)) = -3 + \frac{2}{p} - 12\big(s(p,q-1,1) + s(p,1-q,q)\big)\ ,$$
$$\eta(L(p,q)) = 1 - \frac{1}{p} + 4\big(s(p,q-1,1) + s(p,1-q,q)\big)\ ,$$

(note that the extra parameter ρ in Theorem 5.2 appears naturally on lens spaces), so that $\nu(L(p,q)) = -\frac{1}{p} - 3\eta(L(p,q))$. The proposition then follows from (93). $\square$

Comparison with the Burns-Epstein invariant. Another interesting point is to compare these results with those obtained by use of the Burns-Epstein μ-invariant [14, 15] (it is already suggested at the end of [15] that obstructions follow from computations of μ). The μ-invariant is only defined on strictly pseudoconvex CR 3-manifolds whose tangent holomorphic bundle is a torsion class in homology. Roughly speaking, it comes from Chern-Simons-type constructions (integration of a local formula), whereas the ν-invariant is extracted from the Atiyah-Patodi-Singer η-invariant. The relation between μ and ν is similar to that between the η and the Chern-Simons invariants: more precisely, when μ is defined, then for a CR structure J one has

$$\nu(J) \;=\; 3\,\mu(J) \;+\; \text{constant},$$

with the constant depending only of the underlying contact structure [11, Theorem 1.3]. Burns-Epstein's version of Miyaoka-Yau [15] then reads, if M is the boundary at infinity of a Kähler-Einstein N:

$$\chi(N) - \frac{1}{3}\,\bar{c}_1(N)^2 \geqslant -\mu(M), \tag{94}$$

with equality if the metric is complex hyperbolic; here $\bar{c}_1$ is a lift in $H^2(N,M)$ of $c_1(N)$.

A first important difference here is that our obstruction in Corollary 1.9 (filling by an ACH Einstein metric) is purely topological, whereas (94) involves a complex structure and a Kähler-Einstein metric.

Another important fact to be noticed, at least in the case when the quotient has no orbifold singularities, is that the obstructions obtained by both methods are different: if M is a $\mathbb{S}^1$-bundle over the Riemann surface Σ, then the μ-invariant, being defined by a local formula, is multiplicative on finite coverings [14, 15]. Hence the values are

$$\mu = \frac{\chi^2}{4d} \quad \text{whereas} \quad \nu = -\frac{\chi^2}{4d} - d - 3. \tag{95}$$

Equation (94) implies that 3μ must be an integer, *i.e.* $\frac{3\chi^2}{4d}$ must belong to $\mathbb{Z}$, a condition that is weaker than Corollary 1.8, by a factor 3.

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