# BALANCED DISTRIBUTION-ENERGY INEQUALITIES AND RELATED ENTROPY BOUNDS

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Abstract. Let $A$ be a self-adjoint operator acting over a space $X$ endowed with a partition. We give lower bounds on the energy of a mixed state $\rho$ from its distribution in the partition and the spectral density of $A$. These bounds improve with the refinement of the partition, and generalize inequalities by Li-Yau and Lieb-Thirring for the Laplacian in $\mathbb{R}^n$. They imply an uncertainty principle, giving a lower bound on the sum of the spatial entropy of $\rho$, as measured from $X$, and some spectral entropy, with respect to its energy distribution. On $\mathbb{R}^n$, this yields lower bounds on the sum of the entropy of the densities of $\rho$ and its Fourier transform. A general log-Sobolev inequality is also shown. It holds on mixed states, without Markovian or positivity assumption on $A$.

## 1. Introduction and main results

Let $(X, \mu)$ be a $\sigma$-finite measure space, $V$ a separable Hilbert space and $A$ a self-adjoint operator acting on

$$\mathcal{H} = L^2(X, V) = L^2(X, \mu) \otimes V\,.$$

The inequalities we will consider concern *mixed states*, that is positive trace class operators on $\mathcal{H}$. From the quantum-mechanical viewpoint, they are positive linear combination of *pure states*, which are the orthogonal projections on functions in $\mathcal{H}$; see [18, §23] or [19]. More precisely, as in [15], we are looking for integral controls on the density of a state $\rho$ from its energy given by the trace

$$\mathcal{E}_A(\rho) = \tau(A\rho)\,.$$

The density function of the state, or more generally of a bounded positive operator $P$ on $\mathcal{H}$, is a notion that extends the restriction to the diagonal of $X$ of the $V$-trace of the kernel of $P$. It may be defined as follows (see e.g. [15, §1.2]): given a measurable set $\Omega \subset X$, the trace

$$\nu_P(\Omega) = \tau(\chi_\Omega P \chi_\Omega) = \tau(P^{1/2}\chi_\Omega P^{1/2}) \tag{1}$$

defines a measure on $X$. For any Hilbert basis $(e_i)$ of $\mathcal{H}$, it holds that

$$\nu_P(\Omega) = \int_\Omega D_\mu \nu_P(x) d\mu(x) \quad \text{where} \quad D_\mu \nu_P(x) = \sum_i \|(P^{1/2}e_i)(x)\|_V^2 \tag{2}$$

is called the *density function* of $P$. For instance, in the case of a pure state $P = \pi_f$ with $\|f\|_{\mathcal{H}} = 1$, one has $D_\mu\nu_P(x) = \|f(x)\|_V^2$. Also, when $V$ is finite dimensional, as for operators

*Date*: June 14, 2011.

2000 *Mathematics Subject Classification.* 58J50, 47B06, 46E35, 35P20, 94A17.

*Key words and phrases.* Lieb–Thirring inequality, entropy, uncertainty principle, log-Sobolev.

acting on scalar valued functions, it turns out that $D_\mu \nu_P$ is bounded if and only if $P$ is ultracontractive from $L^1(X)$ to $L^\infty(X)$ with

$$\|P\|_{1,\infty} \leq D_\mu(P) = \operatorname{supess} D_\mu \nu_P(x) \leq (\dim V)\|P\|_{1,\infty}\,, \tag{3}$$

see e.g. [15, Prop. 1.4].

The inequalities studied here depend on the spectral measure associated to $A$. It is defined as follows.

**Definition 1.1.** Let $A$ be a self-adjoint operator on $\mathcal{H}$ and consider the spectral projections $\Pi_\lambda = \Pi_{]-\infty,\lambda[}(A)$. Given $\lambda$, we define the *spectral measure* of a measurable set $\Omega \subset X$ by

$$F_\Omega(\lambda) = \nu_{\Pi_\lambda}(\Omega) = \tau(\Pi_\lambda \chi_\Omega \Pi_\lambda)\,, \tag{4}$$

and the *spectral density* by

$$F_x(\lambda) = D_\mu \nu_{\Pi_\lambda}(x)\,. \tag{5}$$

Note that in the case $A$ is a translation invariant operator over a group $X = \Gamma$, the spectral measure $F_\Omega(\lambda)$ is proportional to the Haar measure of $\Omega$, *i.e.* $F_\Omega(\lambda) = \mu(\Omega)F(\lambda)$. This $F(\lambda)$ is called an integrated density of states (IDS) in mathematical physics. When $\Gamma$ is discrete, $F(\lambda)$ is also known as the $\Gamma$-trace of $\Pi_\lambda$, or von Neumann's $\Gamma$-dimension of $E_\lambda = \Pi_\lambda(\mathcal{H})$.

In general, the functions $\lambda \mapsto F_x(\lambda)$ are positive increasing (in the large sense) and left continuous. In the sequel, if $\varphi : \mathbb{R} \to \mathbb{R}^+$ is an increasing function, and $y \geq 0$, we will set

$$\varphi^{-1}(y) = \sup\{x \in \mathbb{R} \mid \varphi(x) \leq y\} \in [-\infty, +\infty]\,.$$

It is a pseudo-inverse of $\varphi$, and right continuous when finite.

1.1. **Energy of a confined state and spectral bounds.** Our first purpose is to give an inequality between the trace of a state supported in a domain $\Omega$ and its energy.

**Theorem 1.2.** *Let $A$ be a self-adjoint operator acting on $\mathcal{H} = L^2(X,V)$, and let $\rho$ be a non-zero state (positive trace class operator) supported in a set $\Omega \subset X$. Suppose that*

$$\mathcal{E}_A^+(\rho) = \tau(\rho^{1/2}\max(A,0)\rho^{1/2}) \text{ is finite.}$$

*Then the integral involved below has a finite positive part and it holds that*

$$\|\rho\|_\infty \varphi_\Omega\Big(\frac{\tau(\rho)}{\|\rho\|_\infty}\Big) \leq \mathcal{E}_A(\rho)\,, \tag{6}$$

*where $\varphi_\Omega(y) = \displaystyle\int_0^y F_\Omega^{-1}(u)du$ and $\|\rho\|_\infty$ denotes the $L^2 - L^2$ norm of $\rho$.*

When applied to projections onto $N$-dimensional spaces $L$ of functions supported in $\Omega$, Theorem 1.2 gives a lower bound on the sum of the $N$-first Dirichlet eigenvalues of $\mathcal{E}_A$ in $\Omega$, namely

$$\varphi_\Omega(N) \leq \sum_{k=1}^N \lambda_k(\Omega) \leq \mathcal{E}_A(\Pi_L)\,. \tag{7}$$

Here the Dirichlet spectrum is defined using the min-max principle

$$\lambda_n(\Omega) = \inf_{L\in\mathcal{L}_n} \max_{f\in L}(\mathcal{E}_A(f)/\|f\|_2^2) \quad \text{with} \quad \mathcal{L}_n = \{\operatorname{supp} L \subset \Omega \mid \dim L = n\}\,.$$

Such lower bounds for the Dirichlet spectrum are already known in many cases. As we shall see in §2.2, they coincide for the Laplacian in $\mathbb{R}^n$ with inequalities due to Berezin and Li-Yau ([3, 12] or [13, Thm. 12.3]); and which are sharp in the semiclassical limit, i.e. when $N$ goes to $\infty$. More generally, similar results have been proved by Laptev [11] for other invariant positive pseudodifferential operators on $\mathbb{R}^n$, by Strichartz [17] for positive invariant differential operators on homogeneous manifolds, and also by Erdős-Loss-Vougalter [9] for the Laplacian in a constant magnetic field.

The Berezin-Li-Yau inequality (7) implies the following uniform controls of the whole Dirichlet spectral distribution in $\Omega$.

**Corollary 1.3.** *Let $A$ and $\Omega$ as above, and let*

$$N_\Omega(\lambda) = \sup\{\dim V \mid \operatorname{supp} V \subset \Omega \text{ and } \mathcal{E}_A(f) < \lambda \|f\|_2^2 \text{ on } V\}$$

*denotes the Dirichlet spectral distribution function of $A$ in $\Omega$. Then one has*

$$\varphi_\Omega(N_\Omega(\lambda)) \leq \lambda N_\Omega(\lambda)\,. \tag{8}$$

*If moreover $A$ is positive, then*

$$N_\Omega(\lambda) \leq 2F_\Omega(2\lambda)\,. \tag{9}$$

Hence in the positive case, the *confined* spectral distribution in $\Omega$ is controlled by twice the *free* spectral measure of $\Omega$ at twice energy level, *i.e.* by $F_\Omega(2\lambda) = \tau(\chi_\Omega \Pi_{2\lambda})$. Indeed there, $\Pi_{2\lambda}$ is the free (or unconstrained) spectral space of $A$ on the whole $X$.

One feature of the sharpness of inequalities like (6) or (8), that will be used in their proofs, lies in the fact they stay equivalent under an *energy shift* of $A$ in $A+k$. Indeed, one has then

$$F_\Omega(\lambda) \to F_\Omega(\lambda - k) \quad \text{thus} \quad F_\Omega^{-1} \to F_\Omega^{-1} + k \quad \text{and} \quad \varphi_\Omega(y) \to \varphi_\Omega(y) + ky\,.$$

Hence both sides of (6) shift by $k\tau(\rho)$, while (8) stays unchanged up to a parameter shift. This implies in particular that one can't improve (6) or (8) by a fixed multiplicative factor for any (even positive) operator and state. Indeed, suppose that for any positive operator and state it holds

$$(1+\varepsilon)\|\rho\|_\infty \varphi_\Omega(\tau(\rho)/\|\rho\|_\infty) \leq \mathcal{E}_A(\rho)\,.$$

Then one would get by a positive energy shift $A \to A+k$ that

$$0 \leq (1+\varepsilon)\|\rho\|_\infty \varphi_\Omega(\tau(\rho)/\|\rho\|_\infty) \leq \mathcal{E}_{A-k\varepsilon}(\rho) < 0$$

for $k$ large enough. Of course another stronger inequality than (6) may hold however.

In the sequel, we shall say that an inequality is *balanced* if, like (6) or (8), it stays equivalent through energy shift. None of the inequalities given in [15] is balanced; that precludes them to hold for operators of indefinite sign.

1.2. **A balanced Lieb-Thirring inequality.** We now state a version of (6), that gives lower bounds on $\mathcal{E}_A(\rho)$ knowing the distribution of the state in a partition of $X$ into $\sqcup_i \Omega_i$, i.e. given $\nu_\rho(\Omega_i) = \tau(\chi_{\Omega_i}\rho\chi_{\Omega_i})$.

**Theorem 1.4.** *Let $A$ be a self-adjoint operator on $\mathcal{H} = L^2(X, V)$, and $\rho$ a non-zero state such that $\mathcal{E}_A^+(\rho)$ is finite. Let $P = \{\Omega_i\}$ be a measurable partition of $X$.*

- *Then the sums and integral involved below have a finite positive part, and it holds that*

$$H_P(\rho) = \|\rho\|_\infty \sum_i \psi_{\Omega_i}\Big(\frac{\nu_\rho(\Omega_i)}{\|\rho\|_\infty}\Big) \leq H(\rho) = \|\rho\|_\infty \int_X \psi_x\Big(\frac{D_\mu \nu_\rho(x)}{\|\rho\|_\infty}\Big) d\mu(x) \leq \mathcal{E}_A(\rho)\,, \tag{10}$$

*where*

$$\psi_{\Omega_i}(y) = \int_0^1 \varphi_{\Omega_i,t}(y)dt \quad \text{with} \quad \varphi_{\Omega_i,t}(y) = \int_0^y F_\Omega^{-1}(t^2u)du\,, \tag{11}$$

*and similarly*

$$\psi_x(y) = \int_0^1 \varphi_{x,t}(y)dt \quad \text{with} \quad \varphi_{x,t}(y) = \int_0^y F_x^{-1}(t^2u)du\,. \tag{12}$$

- *Moreover if $P' = \{\Omega'_j\}$ is a finer partition of $X$ than $P = \{\Omega_i\}$, then $H_P(\rho) \leq H_{P'}(\rho)$.*

These balanced inequalities improve the unbalanced ones given in [15, Thm. 1.6-1.7] for positive operators. They extend an inequality due to Lieb and Thirring in the case of the Laplacian on $\mathbb{R}^n$; see [14], [13, Thm. 12.5] and §3.3.

To clarify its relation with the previous result, we first remark that since $F_\Omega^{-1}$ is increasing, one has

$$\psi_\Omega \leq \varphi_\Omega = \varphi_{\Omega,1}\,. \tag{13}$$

Hence if the state is confined in a single domain $\Omega$ of the partition, the bound (6) is stronger than $H_P(\rho) \leq \mathcal{E}_A(\rho)$ in (10). Conversely, we will see in §3.1 that if $A$ is *positive*, one has

$$\varphi_\Omega\big(\frac{x}{2}\big) \leq \psi_\Omega(x)\,, \tag{14}$$

thus (10) in the confined case actually gives $\|\rho\|_\infty \varphi_\Omega\big(\frac{\nu_\rho(\Omega)}{2\|\rho\|_\infty}\big) \leq \mathcal{E}_A(\rho)$, close to (6), but weaker.

From the quantum-mechanical viewpoint, (10) gives a lower bound on the energy that *had* a state $\rho$ *before* the measure of its distribution in the partition, given by $\nu_\rho(\Omega_i) = \tau(\chi_{\Omega_i}\rho\chi_{\Omega_i})$, is performed. Equivalently, one gets an a priori control, through $H_P(\rho)$, on the possible outcomes of a measure of the distribution of a state of known energy, before this measure is done. Indeed, in quantum physics (see e.g. [18, 19]), an actual measure of this distribution collapses $\rho$ into

$$\widetilde{\rho} = \sum_i \chi_{\Omega_i}\rho\chi_{\Omega_i}\,,$$

which is a sum of localized states $\rho_i$ in $\Omega_i$. By (6) and convexity of $\varphi_{\Omega_i}$, one has then

$$\|\widetilde{\rho}\|_\infty \sum_i \varphi_{\Omega_i}\Big(\frac{\nu_\rho(\Omega_i))}{\|\widetilde{\rho}\|_\infty}\Big) \leq \sum_i \|\rho_i\|_\infty \varphi_{\Omega_i}\Big(\frac{\nu_\rho(\Omega_i))}{\|\rho_i\|_\infty}\Big) \leq \sum_i \mathcal{E}_A(\rho_i) = \mathcal{E}_A(\widetilde{\rho})\,. \tag{15}$$

This is stronger than (10) by (13), but applies only to collapsed states as $\widetilde{\rho}$.

The monotonicity of $H_P(\rho)$ in the partition makes it behave like an *information quantity* on the state. It increases with a finer knowledge of the distribution of $\rho$, and is dominated by the continuous integral $H(\rho)$ associated to the "infinitesimal" distribution of $\rho$. Actually these inequalities imply other information-type inequalities like entropy bounds, as we see now.

1.3. **Spatial versus spectral entropy and uncertainty principle.** One interesting feature of the Lieb–Thirring inequality (10) lies in its simple behaviour under the change of $A$ into $f(A)$ for an increasing right continuous function $f$. Indeed, one has $\Pi_{f(A)}(]-\infty,\lambda[) \subset \Pi_A(]-\infty, f^{-1}(\lambda)[)$, and thus for the spectral measures

$$F_{f(A),\Omega}(\lambda) \leq F_{A,\Omega} \circ f^{-1}(\lambda)\,, \tag{16}$$

This allows to change the integrals $H(\rho)$ in (10) into many expressions, while using the corresponding energy $\mathcal{E}_{f(A)}(\rho) = \tau(f(A)\rho)$.

An attractive choice is to use $\ln \overline{F}_A(A)$, where $\overline{F}_A(\lambda)$ is the right limit of

$$F_A(\lambda) = \operatorname{supess}_x F_{A,x}(\lambda)\,.$$

For this application, it is crucial that (10) holds for non-positive operator, since $\ln \overline{F}_A(A)$ is not positive in general, even if $A$ is. This leads to entropy bounds.

**Theorem 1.5.** *Let $A$ be a self-adjoint operator and $\rho$ a state such that $\mathcal{E}^+_{\ln \overline{F}_A(A)}(\rho)$ is finite. Then the integral $S_\mu(\rho)$ below has a finite negative part and it holds that*

$$S_\lambda(\rho) + S_\mu(\rho) \geq -\tau(\rho)(3 + \ln \|\rho\|_\infty)\,, \tag{17}$$

*where*

$$S_\lambda(\rho) = \mathcal{E}_{\ln \overline{F}_A(A)}(\rho) \quad \text{and} \quad S_\mu(\rho) = -\int_X \ln(D_\mu \nu_\rho(x)) d\nu_\rho(x)\,. \tag{18}$$

The quantity $S_\mu(\rho)$ is related to the "spatial entropy" of the state $\rho$, as seen from the measure space $(X,\mu)$. Actually, when $\mu(X) = \tau(\rho) = 1$, it is minus the Kullback–Leibler divergence from $\nu_\rho$ to $\mu$, or relative entropy of $\nu_\rho$ to $\mu$. On the other hand,

$$S_\lambda(\rho) = \tau(\ln \overline{F}_A(A)\rho) = \int_{\mathbb{R}} \ln \overline{F}_A(\lambda) d\tau(\rho \Pi_\lambda)\,, \tag{19}$$

deals with the "spectral entropy" of $\rho$, as seen from its distribution within the spectrum of $A$. Indeed $\ln \overline{F}_A(\lambda)$ is an analytical *ersatz* for $\ln \dim E_\lambda$ with $E_\lambda = E_{]-\infty,\lambda]}(A)$. Namely, for invariant operators acting on groups, one has $\overline{F}_A(\lambda) = \dim_\Gamma(E_\lambda) = \tau_\Gamma(\Pi_{]-\infty,\lambda]}(A))$ with the notion of von Neumann's $\Gamma$-dimension; see e.g. [15, §1].

This spectral entropy and the inequality (17) have a striking property: they are invariant under the change of $A$ into $f(A)$ for any increasing homeomorphism $f$ of $\mathbb{R}$. Indeed the operator $\overline{F}_A(A)$ stays unchanged under such transforms, since they give equality in (16). Thus, the spectral entropy is not sensitive to the actual energy levels; it depends only on the ordered set $\{\Pi_\lambda\}$, not its parametrization.

The quantities $S_{\mu,\lambda}(\rho)$ measure the indeterminacy in position and energy of the state. They decrease respectively when $\rho$ is concentrated in a set of small measure, or in small energies. Notice that in the general case, if $X$ is not discrete and $\mu(X)$ infinite, neither $S_\mu(\rho)$ nor $S_\lambda(\rho)$ are bounded from below, even on pure states. Still, the lower bound for their sum in (17) means that a state can't be arbitrarily localized both in position and energy. This may be seen as a general statement of the uncertainty principle from the entropy viewpoint.

1.4. **Log-Sobolev inequalities.** The previous Theorem 1.5 is also related to more classical log-Sobolev inequalities, as stated for instance in [7, 13, 8] for the Laplacian. Indeed applying Jensen inequality on the spectral entropy in (17) leads to the following entropy-energy bound.

**Corollary 1.6.** *Let $A$ be a self-adjoint operator and $\rho$ a state such that $\mathcal{E}_A^+(\rho)$ is finite and $\tau(\rho) = 1$. Then it holds that*

$$S_\mu(\rho) + (\ln F_A)^c(\mathcal{E}_A(\rho)) \geq -3 - \ln \|\rho\|_\infty \,, \tag{20}$$

*where $(\ln F_A)^c$ is the concave hull of $\ln F_A$.*

This improves and extends Theorem 1.9 in [15], proved there for positive operators and with a larger energy term. The inequality (20) is balanced and even invariant under an affine rescaling of energy $A \to k_1 A + k_2$ with $k_1 > 0$. It is also equivalent to the following family of parametric log-Sobolev inequalities

$$S_\mu(\rho) + m(t)\tau(\rho) + t\mathcal{E}_A(\rho) \geq -\tau(\rho)(3 + \ln \|\rho\|_\infty) \,, \tag{21}$$

where $m(t) = \inf_{\lambda \geq 0}(\ln F(\lambda) - t\lambda)$ is minus the concave-Legendre transform of $\ln F$. Such inequalities actually hold on mixed states, without Markovian or positivity assumption on $A$.

1.5. **Fourier transform and entropy.** We now describe consequences of Theorem 1.5 on Fourier transform. Given a state $\rho$ on $X = \mathbb{R}^n$ with Lebesgue's measure $dx$, we define its Fourier transform $\widehat{\rho}$ by $\widehat{\rho}(\widehat{f}) = \widehat{\rho(f)}$. Here our convention is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx.$$

We shall see, by optimizing the choice of $A$ in (17), the following bound on the sum of the entropy of the density of $\rho$ and the entropy of the distribution of its Fourier transform.

**Theorem 1.7.** *Let $\mathrm{vol}^*$ be the (Plancherel) measure $d^*\xi = (2\pi)^{-n} d\xi$ on $\mathbb{R}^n$, and $\rho$ as above. Consider the distribution function of $\nu_{\widehat{\rho}}$ relatively to $d^*\xi$*

$$F_{\widehat{\rho}}(y) = \mathrm{vol}^*(\{\xi \in \mathbb{R}^n \mid \frac{d\nu_{\widehat{\rho}}}{d^*\xi}(\xi) \geq y\}) \,. \tag{22}$$

*Suppose that the positive part of*

$$S_F(\widehat{\rho}) = \int_0^{+\infty} \ln(F_{\widehat{\rho}}(y)) F_{\widehat{\rho}}(y) dy$$

*is finite. Then the negative part of the spatial entropy*

$$S_{dx}(\rho) = -\int_{\mathbb{R}^n} \ln\big(\frac{d\nu_\rho}{dx}\big) d\nu_\rho(x)$$

*is finite and it holds that*

$$S_{dx}(\rho) + S_F(\widehat{\rho}) \geq -\tau(\rho)(2 + \ln \|\rho\|_\infty) \,. \tag{23}$$

This gives an *operator free* version of the classical uncertainty principle stating that a function (pure state) can't be both arbitrarily localized in position and momentum. As will be seen in §4.3, the bound (23) is equivalent to the previous one (17), with $A = \Delta$, for states such that $\widehat{\rho}$ has a decaying radial density, but improves it otherwise. For instance, the repartition entropy $S_F(\widehat{\rho})$ may be much smaller than the spectral entropy $S_\lambda(\rho)$ associated to $\Delta$, if $\widehat{\rho}$ is concentrated in a set of small measure but scattered far from the origin in $\mathbb{R}^n_\xi$.

Still, the inequality (23) is not symmetric in the roles of $\rho$ and its Fourier transform $\widehat{\rho}$, because two kinds of entropies are used at the space and frequency sides. However it implies the following symmetric inequality. It extends a result proved by Hirschman [10] on pure states ($\rho = \pi_f$).

**Corollary 1.8.** *It holds that*

$$S_{dx}(\rho) + S_{d^*\xi}(\widehat{\rho}) = -\int_{\mathbb{R}^n} \ln\Big(\frac{d\nu_\rho}{dx}\Big) d\nu_\rho - \int_{\mathbb{R}^n} \ln\Big(\frac{d\nu_{\widehat{\rho}}}{d^*\xi}\Big) d\nu_{\widehat{\rho}} \geq -\tau(\rho)(\ln\tau(\rho) + \ln\|\rho\|_\infty)\,, \tag{24}$$

*provided the positive part of one of these integrals is finite.*

Besides its symmetry, this inequality has another interesting property: it is additive on tensor products of unit trace states.

As a concluding remark, we shall observe in §4.5 that at least on projections on finite dimensional spaces, the lower bounds occurring in the entropy inequalities obtained here are also related to another important entropy notion: namely to von Neumann's proper entropy, defined by $S(\rho) = -\tau(\rho\ln\rho)$. This suggests a possible improvement of Corollary 1.8 using $S(\rho)$ in the lower bound instead. This will be discussed in §4.5.

## 2. The confined states inequalities

2.1. **Proof of Theorem 1.2.** We first show Theorem 1.2 for positive operator, and use after the invariance through energy shift to extend it in the general case.

The proof in the positive case is actually an improvement of an argument given in [15, §3.1]. It is also close to the approach followed in [9]. Let $\Pi_{\geq\lambda} = \Pi_{[\lambda,+\infty[}(A) = \mathrm{Id} - \Pi_\lambda$. Using $A = \int_0^{+\infty} \Pi_{\geq\lambda} d\lambda$, we observe that

$$\mathcal{E}_A(\rho) = \tau(\rho^{1/2}A\rho^{1/2}) = \int_0^{+\infty} \tau(\rho^{1/2}\Pi_{\geq\lambda}\rho^{1/2})d\lambda\,. \tag{25}$$

Since $\operatorname{supp}\rho \subset \Omega$, one has $\rho \leq \|\rho\|_\infty\chi_\Omega$. Hence, assuming by homogeneity in $\rho$ that $\|\rho\|_\infty = 1$ in the sequel, it holds that

$$\begin{aligned}\tau(\rho^{1/2}\Pi_{\geq\lambda}\rho^{1/2}) &= \tau(\rho) - \tau(\rho^{1/2}\Pi_\lambda\rho^{1/2}) = \tau(\rho) - \tau(\Pi_\lambda\rho\Pi_\lambda)\\ &\geq \tau(\rho) - \tau(\Pi_\lambda\chi_\Omega\Pi_\lambda)\\ &= \tau(\rho) - F_\Omega(\lambda)\,.\end{aligned}$$

Using it in (25) for $\lambda < F_\Omega^{-1}(\tau(\rho)) = \sup\{u \mid F_\Omega(u) \leq \tau(\rho)\}$ yields

$$\begin{aligned}
\mathcal{E}_A(\rho) &\geq \int_0^{F_\Omega^{-1}(\tau(\rho))} \tau(\rho^{1/2}\Pi_{\geq\lambda}\rho^{1/2})d\lambda \\
&\geq \int_0^{F_\Omega^{-1}(\tau(\rho))} \big(\tau(\rho) - F_\Omega(\lambda)\big)d\lambda \\
&= \int_0^{F_\Omega^{-1}(\tau(\rho))} \Big(\int_{F_\Omega(\lambda)}^{\tau(\rho)} du\Big)d\lambda = \iint_{\{0\leq F_\Omega(\lambda)\leq u\leq\tau(\rho)\}} dud\lambda \\
&= \int_0^{\tau(\rho)} \Big(\int_0^{F_\Omega^{-1}(u)} d\lambda\Big)du \\
&= \int_0^{\tau(\rho)} F_\Omega^{-1}(u)du = \varphi_\Omega(\tau(\rho))\,,
\end{aligned} \tag{26}$$

as needed.

For a general self-adjoint operator, we consider $A_k = A\Pi_{\geq k}$. By positivity of $A_k - k$ and the behaviour of (6) in such a shift, it holds for any $k$ that

$$\varphi_{A_k,\Omega}(\tau(\rho)) \leq \mathcal{E}_{A_k}(\rho)\,. \tag{27}$$

In particular, for $k = 0$, one has $\max(F_A^{-1}, 0) \leq F_{A_0}^{-1}$ and thus

$$\int_0^{\tau(\rho)} \max(F_A^{-1}(u), 0)du \leq \varphi_{A_0,\Omega}(\tau(\rho)) \leq \mathcal{E}_{A_0}(\rho) = \mathcal{E}^+(\rho) < \infty$$

by hypothesis. Hence the integral $\varphi_{A,\Omega}(\tau(\rho)) = \int_0^{\tau(\rho)} F_{A,\Omega}^{-1}(u)du$ makes sense in $[-\infty, +\infty[$. If $\varphi_{A,\Omega}(\tau(\rho)) = -\infty$ there is nothing more to prove, and we assume henceforth that $\varphi_{A,\Omega}(\tau(\rho))$ is finite. This implies that the increasing function $F_{A,\Omega}^{-1}(u)$ is finite for $u < \tau(\rho)$. In particular, one has necessarily $F_{A,\Omega}(k)$ finite for $k \ll 0$, and thus by dominated convergence

$$F_{A,\Omega}(k) = \tau(\chi_\Omega\Pi_{]-\infty,k[}(A)\chi_\Omega) \searrow 0 \quad \text{when} \quad k \searrow -\infty\,. \tag{28}$$

Since, for $k \leq \lambda$, one has $\Pi_\lambda(A_k) = \Pi_{[k,\lambda[}(A) = \Pi_\lambda(A) - \Pi_k(A)$, it holds that

$$F_{A_k,\Omega}(\lambda) = \max(F_{A,\Omega}(\lambda) - F_{A,\Omega}(k), 0)\,. \tag{29}$$

This leads to $F_{A_k,\Omega}^{-1}(u) = F_{A,\Omega}^{-1}(u + F_{A,\Omega}(k))$, and finally

$$\varphi_{A_k,\Omega}(y) = \int_0^y F_{A_k,\Omega}^{-1}(u)du = \int_0^y F_{A,\Omega}^{-1}(u + F_{A,\Omega}(k))du\,.$$

Together with (28) and (27), this shows that $\varphi_{A_k,\Omega}(\tau(\rho)) \searrow \varphi_{A,\Omega}(\tau(\rho))$ when $k \searrow -\infty$; by dominated convergence for the positive part, and monotone convergence for the negative one. For the same reasons, one has $\mathcal{E}_{A_k}(\rho) \searrow \mathcal{E}_A(\rho)$ for $k \searrow -\infty$, giving the result by (27).

2.2. **Illustrations in $\mathbb{R}^n$.** As a first illustration of the previous result, we consider the case of the Laplacian on $X = \mathbb{R}^n$. By group invariance, the density $F_x(\lambda)$ is a constant given by the value at 0 of the kernel of $\Pi_\lambda$. To compute it, we remark that the spectral spaces $E_\Delta(\lambda)$

are functions whose Fourier transforms are supported in the ball $B_n(0, \lambda^{1/2})$. It follows easily (see e.g. [15, §4.2]) that

$$F_x(\lambda) = \widehat{\chi}_{B_n(0,\lambda^{1/2})}(0) = C_n \lambda^{n/2} \tag{30}$$
$$\text{with} \quad C_n = (2\pi)^{-n}\mathrm{vol}(B_n(0,1)) = (4\pi)^{-n/2}\Gamma\big(\frac{n}{2}+1\big)\,,$$

and thus

$$\varphi_\Omega(y) = \int_0^y F_\Omega^{-1}(u)du = \frac{n}{n+2}(C_n \mathrm{vol}(\Omega))^{-2/n} y^{1+2/n}\,. \tag{31}$$

Applying Theorem 1.2 to the orthogonal projection $\rho$ on the first $N$ Dirichlet eigenfunctions of $\Delta$ in $\Omega$, yields the following inequality, due to Berezin and Li-Yau (see [12], [13, Thm. 12.3])

$$\mathcal{E}_\Delta(\rho) = \sum_{i=1}^{N} \lambda_i(\Omega) \geq \frac{n}{n+2}(C_n \mathrm{vol}(\Omega))^{-2/n} N^{1+2/n}\,. \tag{32}$$

When $\Omega$ is a domain of finite boundary area, this bound is known to be sharp, up to lower order term in $N$, in the semiclassical limit, i.e. for $N$ goes to $\infty$; see e.g. [13, Thm. 12.11].

We observe that the previous technique also applies to other translation invariant differential operators $D$ on $\mathbb{R}^n$; the spectral density $F(\lambda)$ being still given by the volume of the level sets $\{\xi \mid \sigma(D)(\xi) \leq \lambda\}$ of the symbol $\sigma(D)$ of $D$ (see also §4.3). Note that for invariant operators, one has $F_\Omega(\lambda) = \mu(\Omega)F(\lambda)$, hence

$$\varphi_\Omega(y) = \mu(\Omega)\int_0^{y/\mu(\Omega)} F^{-1}(u)du,$$

is ruled by the function $\Sigma(x) = \int_0^x F^{-1}(u)du$, and the Berezin-Li-Yau inequality (7) writes

$$\sum_{k=1}^{N} \lambda_k(\Omega) \leq \mu(\Omega)\Sigma(N/\mu(\Omega)). \tag{33}$$

This is easily seen to coincide with the estimate proved by Strichartz in [17]. There $\Sigma$ is defined for positive operators by

$$\Sigma(F(\lambda)) = \mathrm{density}(\nu_{A\Pi_\lambda}) = \int_{[0,\lambda[} u dF(u)\,,$$

linearly interpolated between the discrete spectrum; see [17, Definition 4.1] and (34) below.

Note that a similar discussion applies more generally for invariant operators on homogeneous spaces, leading to inequalities of the same shape as (33). On symmetric spaces, where a Fourier transform is available, one can still estimate $F$ and $\Sigma$ in some classical cases, as the Laplacian, using a Plancherel formula; see examples in [17] and [15, §4.1].

Finally we mention the case of the two-dimensional Laplacian in a constant magnetic field: $H = (-i\nabla + A)^2$, where the (connection) one-form $A$ is such that $dA = B$ is constant. This example is studied in [9]. It turns out that although $H$ is not an invariant operator on $\mathbb{R}^2$, translations act on $H$ up to unitary conjugation. Hence $H$ has a constant (in space) spectral density $F$ anyway (the Landau staircase function), and (33) still holds; see [9] for details.

2.3. **Equality case and bathtub filling.** The proof of Theorem 1.2 above shows that $\mathcal{E}_A(\rho)$ gets smaller and approaches the proposed lower bound $\varphi_\Omega(\tau(\rho))$ when:

(1) $\rho$ is the largest possible, i.e close to $\chi_\Omega$, on $\Pi_\lambda$ for $\lambda < \lambda_0 = F_\Omega^{-1}(\tau(\rho))$;
(2) and $\rho$ is the smallest possible, i.e. close to 0, on $\Pi_{>\lambda_0}$.

That means that $\rho$ has to fill up, or saturate, as much as possible the lower energy levels it can, under the constraint that $\rho \subset\subset \Omega$ and until the volume $\tau(\rho)$ is reached. This kind of idea, clear from the physical viewpoint, is actually quite similar to the *bathtub principle* used in the proof of Berezin-Li-Yau inequality (32) given by Lieb and Loss in [13, Theorem 12.3], see also [9, 17].

In general, one can't have equality in (6) unless $\rho$ is pinched between $\Pi_{\lambda_0} = \Pi_{]-\infty,\lambda_0[}(A)$ and $\Pi_{\lambda_0^+} = \Pi_{]-\infty,\lambda_0]}(A)$ *and* supported in $\Omega$. Hence, if the spectral spaces of $A$ are not confined in a proper subspace $\Omega$ of the ambient space $X$, the only remaining possibility is to take $\Omega = X$ itself. This requires of course that $\dim E_{\lambda_0^-} = \tau(\Pi_{\lambda_0^-}) \leq \tau(\rho)$ be finite.

2.4. **Asymptotic sharpness and amenability.** One can go beyond the previous equality case and describe situations with $X$ infinite and where (6) is asymptotically sharp. Given $\lambda$ and $\Omega$, one considers the two states

$$\rho_\Omega = \chi_\Omega \Pi_\lambda \chi_\Omega \quad \text{and} \quad \widetilde{\rho}_\Omega = \Pi_\lambda \chi_\Omega \Pi_\lambda\,.$$

Notice that $\rho_\Omega$ is confined in $\Omega$ while $\widetilde{\rho}_\Omega$ is not. Still, one has $\tau(\rho_\Omega) = \tau(\widetilde{\rho}_\Omega) = F_\Omega(\lambda)$ and we claim that

$$\varphi_\Omega(\tau(\rho_\Omega)) = \varphi_\Omega(F_\Omega(\lambda)) = \int_{]-\infty,\lambda[} u dF_\Omega(u) = \mathcal{E}_A(\widetilde{\rho}_\Omega)\,, \tag{34}$$

if this converges. To see this we proceed as in (26), assuming first that $A$ is positive. One finds

$$\begin{aligned} \varphi_\Omega(F_\Omega(\lambda)) &= \int_0^{F_\Omega^{-1}(F_\Omega(\lambda))} (F_\Omega(\lambda) - F_\Omega(u)) du \\ &= \int_0^\lambda (F_\Omega(\lambda) - F_\Omega(u)) du\,, \end{aligned}$$

since $F_\Omega(u) = F_\Omega(\lambda)$ for $0 \leq \lambda \leq u \leq F_\Omega^{-1}(F_\Omega(\lambda))$. Thus

$$\begin{aligned} \varphi_\Omega(F_\Omega(\lambda)) &= \int_{0 \leq u \leq v < \lambda} dF_\Omega(v) du \\ &= \int_{[0,\lambda[} v dF_\Omega(v)\,. \end{aligned}$$

The general case follows by energy cut-off and shift as in §2.1.

When $\Omega$ is large, $\|\widetilde{\rho}_\Omega\|_\infty$ is close to 1, and (34) means that (6) is sharp for these states $\widetilde{\rho}_\Omega$. However they are not confined in $\Omega$. Still $\mathcal{E}_A(\rho_\Omega)$ may be compared to $\mathcal{E}_A(\widetilde{\rho}_\Omega)$ in the following situation. If $X$ is a *discrete* metric space, and $A$ is a bounded *local* operator, i.e.

$Af(x)$ depends only on the value of $f$ in the ball $B(x,r)$, then one has

$$\begin{aligned}|\mathcal{E}_A(\rho_\Omega)-\mathcal{E}_A(\widetilde{\rho}_\Omega)| &= |\tau(A\chi_\Omega\Pi_\lambda\chi_\Omega)-\tau(A\Pi_\lambda\chi_\Omega\Pi_\lambda)|\\ &= |\tau(\Pi_\lambda\chi_\Omega(A\chi_\Omega-\chi_\Omega A))|\\ &\le 2\|A\|_\infty|\partial_r\Omega|\,,\end{aligned}$$

where $|\partial_r\Omega|$ is the cardinal of $\partial_r\Omega=\{x\in X\mid d(x,\Omega)\le r \text{ and } d(x,\Omega^c)\le r\}$. This leads to the following asymptotic sharpness result for (6).

**Proposition 2.1.** *Let $X=\Gamma$ be a discrete amenable group, endowed with an invariant measure, and let $A$ be a local translation invariant symmetric operator on $X$. Suppose that $\Omega_n$ is a Fölner sequence such that $|\partial_r\Omega_n|/|\Omega_n|\to 0$ when $n\to+\infty$. Set $F=F_x$ and $\varphi=\varphi_x$ (constant in $x$). Then it holds that*

$$\lim_{n\to+\infty}\mathcal{E}_A(\rho_{\Omega_n})/|\Omega_n| = \lim_{n\to+\infty}\|\rho_{\Omega_n}\|_\infty\varphi_{\Omega_n}\big(\frac{\tau(\rho_{\Omega_n})}{\|\rho_{\Omega_n}\|_\infty}\big)/|\Omega_n| = \varphi(F(\lambda))\,. \tag{35}$$

This may be seen as the counterpart in the discrete setting to the semiclassical result recalled in §2.2; here the sharpness of (6) is achieved on large domains and fixed energy, instead of the contrary in the semiclassical limit. This statement applies for instance to the discrete Laplacians on $\ell^2$-cochains over amenable coverings of finite simplicial complex.

2.5. **Faber–Krahn inequality and the heat technique.** We can compare the lower bound on the Dirichlet spectrum, or Faber-Krahn inequality, obtained in (7):

$$\lambda_1(\Omega)\ge\varphi_\Omega(1)\,, \tag{36}$$

to the one shown in [5, Prop. II.2] using a heat kernel technique. Namely, it follows from the Nash inequality given there that if $A$ is a positive operator, one has

$$\lambda_1(\Omega)\ge\theta(\Omega)=\sup_{t>0}\frac{1}{t}\ln\Big(\frac{1}{L(t)\mu(\Omega)}\Big)\,, \tag{37}$$

where $L(t)=\|e^{-tA}\|_{1,\infty}$. This bound is actually weaker than (36), at least on scalar operators. Indeed, by (3), it holds that

$$\begin{aligned}L(t)\mu(\Omega)\ge\nu_{e^{-tA}}(\Omega) &= \tau(\chi_\Omega e^{-tA}\chi_\Omega)\\ &= \int_0^{+\infty}e^{-t\lambda}dF_\Omega(\lambda)\\ &\ge \int_{[0,F_\Omega^{-1}(1)]}e^{-t\lambda}d\widetilde{F}_\Omega(\lambda)\end{aligned}$$

with $\widetilde{F}_\Omega(\lambda)=F_\Omega(\lambda)$ for $\lambda<F_\Omega^{-1}(1)$ and $\widetilde{F}_\Omega(F_\Omega^{-1}(1))=1$. Notice that $0\le d\widetilde{F}_\Omega\le dF_\Omega$ since $F_\Omega(F_\Omega^{-1}(1))\le 1\le F_\Omega(F_\Omega^{-1}(1)^+)$ by left continuity of $F_\Omega$. Then by Jensen,

$$\begin{aligned}-\ln\big(L(t)\mu(\Omega)\big) &\le t\int_{[0,F_\Omega^{-1}(1)]}\lambda d\widetilde{F}_\Omega(\lambda)\\ &= t\int_0^1(1-F_\Omega(\lambda))du = t\varphi_\Omega(1)\,,\end{aligned}$$

by (26). This gives $\theta(\Omega)\le\varphi_\Omega(1)$ as claimed.

2.6. **Proof of Corollary 1.3.** When $A$ is a positive operator, one has for $c \in [0,1]$,

$$\begin{aligned} \varphi_\Omega(y) &= \int_0^y F_\Omega^{-1}(u)du \geq \int_{cy}^y F_\Omega^{-1}(u)du \\ &\geq (1-c)yF_\Omega^{-1}(cy)\,. \end{aligned} \tag{38}$$

Hence (8), that comes from (7), implies

$$N_\Omega(\lambda) \leq \frac{1}{c} F_\Omega\big(\frac{\lambda}{1-c}\big)\,,$$

giving (9) in the case $c = 1/2$. Unlike (8) these inequalities are not balanced.

We remark that if $F_\Omega$ is a concave function, one can sharpen (38) into $N_\Omega(\lambda) \leq 2F_\Omega(\lambda)$ by Jensen. When $F_\Omega(\lambda)/\lambda$ is increasing, for instance when $F_\Omega$ is a convex function, one sees easily that $N_\Omega(\lambda) \leq F_\Omega(2\lambda)$.

## 3. The balanced Lieb–Thirring inequality

We now consider Theorem 1.4 and begin with the continuous case. The argument is an improvement of [15, §3.2].

3.1. **Proof of $H(\rho) \leq \mathcal{E}_A(\rho)$.** Let $\rho$ be a state, $\Omega$ any measurable set in $X$, and let consider the splitting

$$\rho^{1/2}\chi_\Omega = \rho^{1/2}\Pi_\lambda\chi_\Omega + \rho^{1/2}\Pi_{\geq\lambda}\chi_\Omega\,.$$

Using Hilbert-Schmidt norm and assuming by homogeneity that $\|\rho^{1/2}\|_\infty = \|\rho\|_\infty^{1/2} = 1$ yield

$$\begin{aligned} \|\rho^{1/2}\chi_\Omega\|_{HS} &\leq \|\rho^{1/2}\Pi_\lambda\chi_\Omega\|_{HS} + \|\rho^{1/2}\Pi_{\geq\lambda}\chi_\Omega\|_{HS} \\ &\leq \|\Pi_\lambda\chi_\Omega\|_{HS} + \|\rho^{1/2}\Pi_{\geq\lambda}\chi_\Omega\|_{HS}\,. \end{aligned}$$

Since $\|P\|_{HS} = \tau(P^*P)^{1/2} = \tau(PP^*)^{1/2}$, one finds by (1) that

$$\nu_\rho(\Omega)^{1/2} \leq \nu_{\Pi_\lambda}(\Omega)^{1/2} + \nu_{\Pi_{\geq\lambda}\rho\Pi_{\geq\lambda}}(\Omega)^{1/2}\,. \tag{39}$$

This implies a similar inequality almost everywhere at the local level, i.e.

$$D_\mu\nu_\rho(x)^{1/2} \leq F_x(\lambda)^{1/2} + D_\mu\nu_{\Pi_{\geq\lambda}\rho\Pi_{\geq\lambda}}(x)^{1/2}\,. \tag{40}$$

Indeed, using (39) on the sets

$$\Omega_{a,b,c} = \{x \in X \mid D_\mu\nu_\rho(x) \geq a^2\ ,\ F_x(\lambda) \leq b^2 \text{ and } D_\mu\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x) \leq c^2\}$$

with $(a,b,c) \in D = \{a,b,c \in \mathbb{Q}^+ \mid a > b + c\}$, gives that $\mu(\Omega_{a,b,c}) = 0$. Whence

$$\{x \in X \mid\ (40) \text{ fails}\} = \bigcup_D \Omega_{a,b,c}$$

is also negligible. The author is grateful to Guy David for suggesting this level set argument.

We now suppose that $A$ is positive, and uses (25),

$$\begin{aligned}
\mathcal{E}_A(\rho) &= \int_0^{+\infty} \tau(\rho^{1/2}\Pi_{\geq\lambda}\rho^{1/2})d\lambda = \int_0^{+\infty} \tau(\Pi_{\geq\lambda}\rho\Pi_{\geq\lambda})d\lambda \\
&= \int_0^{+\infty} \nu_{\Pi_{\geq\lambda}\rho\Pi_{\geq\lambda}}(X)d\lambda \\
&= \int_{X\times\mathbb{R}^+} D_\mu\nu_{\Pi_{\geq\lambda}\rho\Pi_{\geq\lambda}}(x)d\mu(x)d\lambda \\
&\geq \int_\Omega D_\mu\nu_{\Pi_{\geq\lambda}\rho\Pi_{\geq\lambda}}(x)d\mu(x)d\lambda\,, \qquad (41)
\end{aligned}$$

where $\Omega = \{(x,\lambda) \in X \times \mathbb{R}^+ \mid F_x(\lambda) \leq D_\mu\nu_\rho(x)\}$. Then, by (40),

$$\begin{aligned}
\mathcal{E}_A(\rho) &\geq \int_\Omega \big(D_\mu\nu_\rho(x)^{1/2} - F_x(\lambda)^{1/2}\big)^2 d\mu(x)d\lambda \\
&= \int_X \psi_x(D_\mu\nu_\rho(x))d\mu(x)
\end{aligned}$$

with

$$\psi_x(y) = \int_0^{F_x^{-1}(y)} \big(y^{1/2} - F_x(\lambda)^{1/2}\big)^2 d\lambda\,. \qquad (42)$$

We shall compare this expression to $\varphi_x(y) = \int_0^y F_x^{-1}(t)dt$. First, using

$$\sqrt{y} - \sqrt{u} \geq \sqrt{\frac{y}{2} - u} \quad \text{for} \quad 0 \leq u \leq \frac{y}{2}\,,$$

and proceeding as in (26), one finds that

$$\begin{aligned}
\psi_x(y) &\geq \int_0^{F_x^{-1}(y/2)} \big(\frac{y}{2} - F_x(\lambda)\big)d\lambda \\
&= \int_0^{y/2} F_x^{-1}(t)dt = \varphi_x\big(\frac{y}{2}\big)\,,
\end{aligned}$$

This shows the comparison (14) claimed for positive operators. For the general expression (12), one uses (42)

$$\begin{aligned}
\psi_x(y) &= \int_0^{F_x^{-1}(y)} \int_{F_x(\lambda)}^{y} \big(u^{1/2} - F_x(\lambda)^{1/2}\big)\frac{du}{\sqrt{u}} d\lambda \\
&= \int_0^{F_x^{-1}(y)} \int_{F_x(\lambda)}^{y} \int_{F_x(\lambda)}^{u} \frac{dvdud\lambda}{2\sqrt{uv}} \\
&= \int_{\{0\leq F_x(\lambda)\leq v\leq u\leq y\}} \frac{dvdud\lambda}{2\sqrt{uv}} \\
&= \int_0^y \int_0^u \int_0^{F_x^{-1}(v)} d\lambda \frac{dv}{2\sqrt{v}} \frac{du}{\sqrt{u}} \\
&= \int_0^y \int_0^u F_x^{-1}(v) \frac{dv}{2\sqrt{v}} \frac{du}{\sqrt{u}} \\
&= \int_0^y \int_0^1 F_x^{-1}(t^2u) dtdu \\
&= \int_0^1 \varphi_{x,t}(y) dt\,,
\end{aligned}$$

with $\varphi_{x,t}(y) = \int_0^y F_x^{-1}(t^2u)du$ as needed. This shows that $H(\rho) \leq \mathcal{E}_A(\rho)$ for positive operators.

*Remark* 3.1. The inequality $H_P(\rho) \leq \mathcal{E}_A(\rho)$ for partitions can be proved along the same lines; just replacing (41) above by its discrete analogous

$$\mathcal{E}_A(\rho) \geq \sum_i \int_0^{F_{\Omega_i}^{-1}(\nu_\rho(\Omega_i))} \nu_{\Pi_\lambda\rho\Pi_\lambda}(\Omega_i) d\lambda\,,$$

and using (39) in place of (40). Furthermore, the previous computations on $\varphi_x$ and $\psi_x$ apply on $\varphi_{\Omega_i}$ and $\psi_{\Omega_i}$ instead.

The case of general (non-positive) operators can be handled as in §2.1; using the cut-off $A_k = \max(A,k)$ and energy shift in these balanced inequalities. From the positive case, one has

$$\int_X \int_0^{D_\mu\nu_\rho(x)} \int_0^1 \max(F_x^{-1}(t^2u), 0)\, dtdud\mu(x) \leq \mathcal{E}_A^+(\rho) < \infty\,,$$

Hence $\mathcal{E}_A^+(\rho)$ controls the positive part of the integral $H(\rho)$. Then taking $k \searrow -\infty$ yields the result: by dominated convergence for the positive part and monotone convergence for the negative one.

3.2. **Behaviour of $H_P$ under partition refinement.** We shall now prove that

$$H_P \leq H_{P'} \leq H$$

if $P' = \{\Omega'_j\}$ is a finer partition of $X$ than $P = \{\Omega_i\}$. This will actually follow by integration in $t \in ]0,1]$ of the parametric inequalities

$$H_{P,t} \leq H_{P',t} \leq H_t \tag{43}$$

where

$$H_{P,t}(\rho) = \|\rho\|_\infty \sum_i \varphi_{\Omega_i,t}\Big(\frac{\nu_\rho(\Omega_i)}{\|\rho\|_\infty}\Big) \quad \text{and} \quad H_t(\rho) = \|\rho\|_\infty \int_X \varphi_{x,t}\Big(\frac{D_\mu \nu_\rho(x)}{\|\rho\|_\infty}\Big) d\mu(x)\,.$$

*Remark* 3.2. When $t = 1$, we have seen in (15) that these expressions give energy lower bounds of collapsed states, and (43) means they also behave like an information quantity; actually finer than the averaged $H$, but restricted to such states.

We start with the discrete vs. continuous inequality, in the positive case, i.e. $F_x(0) = 0$, and assume again that $\|\rho\|_\infty = 1$. Given $t > 0$,

$$\varphi_{x,t}(y) = \int_0^y F_x^{-1}(t^2 u)du \quad \text{and} \quad \varphi_{\Omega_i,t}(y) = \int_0^y F_{\Omega_i}^{-1}(t^2 u)du$$

are convex functions whose Legendre transforms are respectively

$$\varphi^*_{x,t}(z) = \int_0^z F_x(v)\frac{dv}{t^2} \quad \text{and} \quad \varphi^*_{\Omega_i,t}(z) = \int_0^z F_{\Omega_i}(v)\frac{dv}{t^2}\,.$$

Young's inequality states that for any $y, z \geq 0$

$$yz \leq \varphi_{x,t}(y) + \varphi^*_{x,t}(z)\,.$$

Integrating it over $\Omega_i$ with $y = D_\mu \nu_\rho(x)$ yields

$$\begin{aligned} z\nu_\rho(\Omega_i) &\leq \int_{\Omega_i} \varphi_{t,x}(D_\mu \nu_\rho(x))d\mu(x) + \int_{\Omega_i}\int_0^z F_x(v)\frac{dv}{t^2}d\mu(x) \\ &= \int_{\Omega_i} \varphi_{x,t}(D_\mu \nu_\rho(x))d\mu(x) + \varphi^*_{\Omega_i,t}(z)\,, \end{aligned}$$

by Fubini and (5). Then by Legendre duality, one has

$$\varphi_{\Omega_i,t}(\nu_\rho(\Omega_i)) = \sup_{z\geq 0}\big(z\nu_\rho(\Omega_i) - \varphi^*_{\Omega_i,t}(z)\big) \leq \int_{\Omega_i} \varphi_{x,t}(D_\mu \nu_\rho(x))d\mu(x)\,. \tag{44}$$

This gives $H_{P,t}(\rho) \leq H_t(\rho)$ by summation. The discrete comparison $H_P(\rho) \leq H_{P'}(\rho)$ follows the same lines: just replacing the integration over $\Omega_i$ above by the discrete splitting of $\Omega_i$ into smaller $\Omega'_j$.

We now consider the general (non-positive) situation. From §3.1, the positive parts of $H_t(\Omega)$ and $H_{P,t}(\rho)$ are finite if $\mathcal{E}^+_A(\rho)$ is. Moreover we shall assume that the negative part of $H_{P,t}(\rho)$ is finite, or (43) is already satisfied. This implies in particular that $F_{\Omega_i}^{-1}(u) > -\infty$ for any $i$ and $u > 0$, and thus the functions $F_{\Omega_i}(\lambda) = \int_{\Omega_i} F_x(\lambda)d\mu(x) \searrow 0$ when $\lambda \searrow -\infty$. Whence, fixing an $i$, one has a.e. in $\Omega_i$ that $F_x(\lambda) \searrow 0$ when $\lambda \searrow -\infty$. We shall now apply (44) to

$$F_{k,x}(\lambda) = F_x(\lambda + k) - F_x(k) \quad \text{and} \quad F_{k,\Omega_i}(\lambda) = F_{\Omega_i}(\lambda + k) - F_{\Omega_i}(k)\,.$$

This gives

$$F_{k,x}^{-1}(u) = F_x^{-1}(u + F_x(k))) - k \quad \text{and} \quad F_{k,\Omega_i}^{-1}(u) = F_{\Omega_i}^{-1}(u + F_{\Omega_i}(k))) - k\,,$$

and

$$\int_0^{\nu_\rho(\Omega_i)} F_{\Omega_i}^{-1}(t^2 u + F_{\Omega_i}^{-1}(k))du \leq \int_{\Omega_i}\int_0^{D_\mu \nu_\rho(x)} F_x^{-1}(t^2 u + F_x(k))dud\mu(x)\,,$$

leading to the result for $k \searrow -\infty$.

3.3. **Illustration in $\mathbb{R}^n$.** We consider again the case of the Laplacian on $\mathbb{R}^n$. From (30), one has

$$F_n^{-1}(u) = C_n^{-2/n} u^{2/n} = 4\pi\Gamma(1+n/2)^{2/n} u^{2/n}\,, \tag{45}$$

giving

$$\psi_n(y) = \int_0^1 \int_0^y F_n^{-1}(t^2 u) du dt = D_n y^{1+2/n}\,,$$

with

$$D_n = \frac{4\pi}{(1+4/n)(1+2/n)} \Gamma(1+n/2)^{2/n}\,.$$

Thus, if $\rho$ is a projection onto a $N$-dimensional space of orthonormal basis $f_i$, (10) reads

$$D_n \int_{\mathbb{R}^n} \Big(\sum_{i=1}^N |f_i(x)|^2\Big)^{1+2/n} dx \leq \sum_{i=1}^N \|\nabla f_i\|_2^2\,. \tag{46}$$

Such lower bound of the kinetic energy is due to Lieb–Thirring, see [13, Thm. 12.5] or [14], and have important applications in quantum mechanics. The constant $D_n$ given here is quite sharp for large $n$. Indeed, by [13, §12.5], the (unknown) best constant has to be smaller than $B_n = (1+4/n)D_n$. This follows from the remark that

$$\varphi_n(y) = \int_0^y F_n^{-1}(u) du = B_n y^{1+2/n}\,.$$

Indeed by Jensen inequality (or (43)) and Berezin-Li-Yau inequality (32) one has both

$$\varphi_{n,\Omega}(N) = \mu(\Omega)^{-2/n} \varphi_n(N) \leq B_n \int_\Omega \Big(\sum_{i=1}^N |f_i(x)|^2\Big)^{1+2/n} dx \text{ and } \sum_{i=1}^N \|\nabla f_i\|_2^2\,,$$

for functions confined in a domain $\Omega$. As the second inequality is sharp in the semiclassical limit $N \to +\infty$, the best constant in (46) is smaller than $B_n$ as claimed.

Similar bounds can also be obtained from (10) for other examples of physical interest as $|\nabla| = \Delta^{1/2}$ or the relativistic kinetic energy $P = (\Delta + m^2)^{1/2} - m$; see [6, 13]. One replaces $F_n^{-1}$ above by respectively $F_{|\nabla|}^{-1} = (F_n^{-1})^{1/2}$ or $F_P^{-1} = (F_n^{-1} + m^2)^{1/2} - m$. By (45), one finds that $\psi_{|\nabla|} = E_n y^{1+1/n}$ with

$$E_n = (4\pi)^{1/2} \frac{\Gamma(n/2+1)^{1/n}}{(1+2/n)(1+1/n)}\,.$$

In $\mathbb{R}^3$ this gives

$$\sum_{i=1}^N (|\nabla| f_i, f_i) \geq E_3 \int_{\mathbb{R}^3} \Big(\sum_{i=1}^N |f_i(x)|^2\Big)^{4/3} dx$$

with $E_3 \simeq 1,754$, which is slightly stronger than the constant $1,63$ given in [6, eq. (3.4)]. The author is grateful to the referee for this observation.

## 4. Entropy bounds

4.1. **Proof of Theorem 1.5.** We deduce Theorem 1.5 on the entropy sum from Theorem 1.4. Consider the functions

$$F_A(\lambda) = \sup_{\Omega} \frac{F_{A,\Omega}(\lambda)}{\mu(\Omega)} = \operatorname{supess}_x F_{A,x}(\lambda) \quad \text{and} \quad \overline{F}_A(\lambda) = \lim_{\varepsilon\to 0^+} F_A(\lambda+\varepsilon)\,.$$

We observe that $F_A$ is increasing and left continuous, since the $F_{A,\Omega}$ are, while $\overline{F}_A$ is right continuous. We shall assume that $F_A(\lambda)$ is finite for $\lambda \ll 0$, in order that the hypothesis of Theorem 1.5 hold for some state. This implies in particular by dominated convergence that $F_A(\lambda) \searrow 0$ when $\lambda \searrow -\infty$. Then by (16), one has

$$F_{\ln \overline{F}_A(A)}(\lambda) \le F_A \circ (\overline{F}_A)^{-1}(e^\lambda) \le F_A \circ F_A^{-1}(e^\lambda) \le e^\lambda,$$

by left continuity of $F_A$. Hence by (12), it holds a.e. in $X$ that

$$\psi_{\ln \overline{F}_A(A),x}(y) \ge \int_0^1 \int_0^y \ln(t^2 u)\,du\,dt = y\ln y - 3y\,, \tag{47}$$

leading to Theorem 1.5.

4.2. **Illustration in $\mathbb{R}^n$.** We make explicit Theorem 1.5 in the case of the Laplacian on $\mathbb{R}^n$. Given a state $\rho$, we can express its spectral entropy $S_\lambda(\rho)$ using Fourier transform. Suppose that $\rho = \sum_i p_i \Pi_{f_i}$ for orthonormal functions $f_i$. Its Fourier transform $\widehat{\rho}$ acts on $L^2(\mathbb{R}^n_\xi)$ by $\widehat{\rho}(\widehat{f}) = \widehat{\rho(f)}$; actually $\widehat{\rho} = \sum_i p_i \Pi_{\widehat{f}_i}$ using the Plancherel measure $d^*\xi = (2\pi)^{-n} d\xi$. At the density level, this writes

$$d\nu_\rho(x) = \sum_i p_i |f_i(x)|^2 dx \quad \text{and} \quad d\nu_{\widehat{\rho}}(\xi) = \sum_i p_i |\widehat{f}_i(\xi)|^2 d^*\xi\,. \tag{48}$$

By (30), $F_n(\lambda) = C_n \lambda^{n/2}$ and the spectral entropy is

$$\begin{aligned} S_\lambda(\rho) = \tau(\ln(F_n(\Delta))\rho) &= \sum_i p_i \langle \ln(C_n \Delta^{n/2}) f_i, f_i\rangle \\ &= \sum_i p_i \int_{\mathbb{R}^n} \ln(C_n \|\xi\|^n) |\widehat{f}_i(\xi)|^2 d^*\xi \\ &= \int_{\mathbb{R}^n} \ln(\mathrm{vol}^*(B_n(0,\|\xi\|)) d\nu_{\widehat{\rho}}(\xi)\,. \end{aligned}$$

Hence the entropy bound (17) writes here

$$\int_{\mathbb{R}^n} \ln\Big(\frac{d\nu_\rho}{dx}\Big) d\nu_\rho(x) \le \int_{\mathbb{R}^n} \ln(\mathrm{vol}^*(B_n(0,\|\xi\|)) d\nu_{\widehat{\rho}}(\xi) + \tau(\rho)(3 + \ln\|\rho\|_\infty)\,. \tag{49}$$

To study the general sharpness of this bound, we first observe it implies a log-Sobolev inequality. Indeed, Jensen inequality yields

$$\begin{aligned} \int_{\mathbb{R}^n} \ln\Big(\frac{d\nu_\rho}{dx}\Big) \frac{d\nu_\rho(x)}{\tau(\rho)} &\le \ln F_n\Big(\int_{\mathbb{R}^n} \|\xi\|^2 \frac{d\nu_{\widehat{\rho}}(\xi)}{\tau(\rho)}\Big) + 3 + \ln\|\rho\|_\infty \\ &= \ln F_n\Big(\frac{\mathcal{E}_\Delta(\rho)}{\tau(\rho)}\Big) + 3 + \ln\|\rho\|_\infty\,. \end{aligned} \tag{50}$$

This in turn implies a Berezin–Li–Yau type inequality for confined states in finite measure sets $\Omega$. Namely the convexity of $y \mapsto y \ln y$ leads to

$$\tau(\rho) \leq \mu(\Omega)\|\rho\|_\infty e^3 F_n\Big(\frac{\mathcal{E}_\Delta(\rho)}{\tau(\rho)}\Big)\,. \tag{51}$$

This may be compared to (6) where $\varphi_\Omega(y) = \frac{n}{n+2} y F_n^{-1}(\frac{y}{\mu(\Omega)})$ gives

$$\tau(\rho) \leq \mu(\Omega)\|\rho\|_\infty F_n\Big(\frac{n+2}{n}\frac{\mathcal{E}_\Delta(\rho)}{\tau(\rho)}\Big)\,.$$

As recalled in §2.2 (and also §2.4 in a discrete setting) this inequality is sharp for all $n$, in the semiclassical limit of large energy. It is indeed sharper than (51), since

$$F_n\big(\frac{n+2}{n}\lambda\big) = C_n\Big(\frac{n+2}{n}\lambda\Big)^{n/2} \leq eF_n(\lambda)\,.$$

As a consequence, the inequalities (49) and (50) are sharp except possibly for the constant 3 there, which can't be taken smaller than 1 in this generality.

*Remark* 4.1. We notice that (51), with $e$ instead of $e^3$, is also an instance of the general confined states result Theorem 1.2. Indeed when applied to $\ln \overline{F}_A(A)$, one can use

$$\varphi_{\ln \overline{F}_A(A),\Omega}(y) \geq \int_0^y \ln\big(\frac{u}{\mu(\Omega)}\big)du = y\ln\big(\frac{y}{\mu(\Omega)}\big) - y\,,$$

instead of the weaker (not confined) $\psi$ version (47).

We return to the log-Sobolev inequality (50) and compare it with a similar one proved by Dolbeault, Felmer, Loss and Paturel in [8]. Namely it holds on unit trace states

$$\int_{\mathbb{R}^n} \ln\big(\frac{d\nu_\rho}{dx}\big)d\nu_\rho(x) \leq \frac{n}{2}\ln\big(\frac{e}{2\pi n}\mathcal{E}_\Delta(\rho)\big) - S(\rho)\,, \tag{52}$$

where $S(\rho) = -\tau(\rho\ln\rho)$ is von Neumann's entropy of $\rho$; see [18, 19]. From $F_n(\lambda) = C_n\lambda^{n/2}$ with

$$C_n = (2\pi)^{-n}\mathrm{vol}(B_n(0,1)) = (4\pi)^{-n/2}\Gamma(n/2+1)^{-1}\,,$$

one finds that the difference between the right sides of (50) and (52) is

$$\Delta = S(\rho) + \ln\|\rho\|_\infty + 3 + \ln\big((\frac{n}{2e})^{n/2}/\Gamma(\frac{n}{2}+1)\big)\,.$$

One has $S(\rho) \geq -\ln\|\rho\|_\infty$ in general, with equality on normalized projections on finite dimensional spaces $\rho = \Pi_V/\dim V$. Moreover by Stirling formula

$$\ln\big((\frac{n}{2e})^{n/2}/\Gamma(\frac{n}{2}+1)\big) = -\frac{1}{2}\ln(\pi n) + o(1).$$

Hence (50) is stronger than (52) on uniformly distributed states like $\rho = \Pi_V/\dim V$ and high ambient dimension $n$. On the contrary, (52) is stronger than (50) in fixed dimension on states with irregular distributions. For instance, given an integer $N \geq 2$, one can consider states $\rho_N$ on $\mathbb{R}^n$ with finite spectrum $p_0 = \frac{1}{\ln N}$ and $p_k = \frac{\ln N - 1}{N\ln N}$ for $k \in [1, N]$. One has $\tau(\rho_N) = 1$ and $-\ln\|\rho_N\|_\infty = -\ln p_0 = \ln(\ln N)$, while

$$S(\rho_N) = -\sum_0^N p_i\ln p_i \sim \ln N \gg -\ln\|\rho\|_\infty \quad \text{for large } N.$$

Another relationship between the two log-Sobolev inequalities (50) and (52) appears in the following process. Given a state $\rho$ on $\mathbb{R}^n$ with $\tau(\rho) = 1$, one can apply (50) to $\otimes^N \rho$ on $\mathbb{R}^{nN}$ for increasing $N$. One has

$$S_{dx}(\otimes^N \rho) = N S_{dx}(\rho)\,,\ \mathcal{E}_\Delta(\otimes^N \rho) = N\mathcal{E}_\Delta(\rho) \text{ and } -\ln \| \otimes^N \rho\|_\infty = -N \ln \|\rho\|_\infty\,,$$

while

$$\begin{aligned}\ln F_{nN}\big(\mathcal{E}_\Delta(\otimes^N \rho)\big) &= \frac{Nn}{2} \ln\Big((4\pi)^{-1}\Gamma\big(\frac{Nn}{2}+1\big)^{-2/Nn} N\mathcal{E}_\Delta(\rho)\Big)\\ &\sim N \times \frac{n}{2}\ln\big(\frac{e}{2\pi n}\mathcal{E}_\Delta(\rho)\big) \quad \text{for large } N\end{aligned}$$

by Stirling formula. Therefore, (50) divided by $N$ for $N \nearrow +\infty$ implies a "stabilized" inequality

$$\int_{\mathbb{R}^n} \ln\big(\frac{d\nu_\rho}{dx}\big) d\nu_\rho(x) \le \frac{n}{2}\ln\big(\frac{e}{2\pi n}\mathcal{E}_\Delta(\rho)\big) - \ln \|\rho\|_\infty. \tag{53}$$

which is closer to (52) except for the weaker term $-\ln \|\rho\|_\infty$ in place of $S(\rho)$.

4.3. **Proof of Theorem 1.7.** The right spectral term of the previous entropy bound (49) is associated to the level sets of the symbol $\sigma_\Delta(\xi) = \|\xi\|^2$ of the Laplacian; namely at some point $\xi_0$, one has $B_n(0, \|\xi_0\|)) = \{\xi \mid \sigma_\Delta(\xi) \le \sigma_\Delta(\xi_0)\}$, whose volume gives the spectral density $F_\Delta(\lambda)$ at the energy $\lambda = \|\xi_0\|^2$. Given a state $\rho$, one can consider more general translation invariant operators, associated to other fillings of $\mathbb{R}^n_\xi$, in order to minimize the spectral entropy term $S_\lambda(\rho)$. We shall proceed as follows.

Let $\sigma$ be a bounded measurable function on $\mathbb{R}^n_\xi$, and $A_\sigma$ be defined by

$$\widehat{A_\sigma(f)}(\xi) = \sigma(\xi)\widehat{f}(\xi)\,.$$

Let $\Omega^\sigma_\lambda = \{\xi \in \mathbb{R}^n \mid \sigma(\xi) \le \lambda\}$. The spectral projection $\Pi_{A_\sigma}(\lambda)$ acts through Fourier transform by multiplication by $\chi_{\Omega_\lambda}$ and, following e.g. [15, §4.1, §4.2], the spectral density of $A_\sigma$ is

$$\overline{F}_{A_\sigma}(\lambda) = \|k_{\Pi_{A_\sigma}(\lambda)}\|^2_{L^2_x} = \|\chi_{\Omega^\sigma_\lambda}\|^2_{L^2_\xi} = \mathrm{vol}^*(\Omega^\sigma_\lambda)\,.$$

This leads to the following expression for the spectral entropy of a state $\rho$, as long these integral have finite positive parts,

$$\begin{aligned} S_{A_\sigma}(\rho) = \tau(\ln \overline{F}_{A_\sigma}(A_\sigma)\rho) &= \int_{\mathbb{R}} \ln(\mathrm{vol}^*(\Omega^\sigma_\lambda)) d\tau(\Pi_{A_\sigma}(\lambda)\rho)\\ &= \int_{\mathbb{R}} \ln(\mathrm{vol}^*(\Omega^\sigma_\lambda)) d\tau(\chi_{\Omega^\sigma_\lambda}\widehat{\rho})\\ &= \int_{\mathbb{R}} \ln(\mathrm{vol}^*(\Omega^\sigma_\lambda)) d\nu_{\widehat{\rho}}(\Omega^\sigma_\lambda)\\ &= \int_{\mathbb{R}} \ln(\mathrm{vol}^*(\Omega^\sigma_\lambda)) d(\sigma_*(\nu_{\widehat{\rho}})(]-\infty,\lambda]))\,,\end{aligned}$$

using the push-forward measure $\sigma_*(\nu_{\widehat{\rho}})$. This yields

$$S_{A_\sigma}(\rho) = \int_{\mathbb{R}^n} \ln(\mathrm{vol}^*(\Omega^\sigma_{\sigma(\xi)})) d\nu_{\widehat{\rho}}(\xi)\,. \tag{54}$$

The strong functional invariance of this entropy is clear here. It stays unchanged if replacing the symbol $\sigma$ into $f(\sigma)$ for any strictly increasing function $f$ on $\sigma(\mathbb{R}^n)$, as comes from $\Omega^{f(\sigma)}_{f(\sigma)(\xi)} = \Omega^{\sigma}_{\sigma(\xi)}$. The following statement gives the minimum of these quantities and implies Theorem 1.7.

**Proposition 4.2.** *Given $\rho$, let $g = \frac{d\nu_{\widehat{\rho}}}{d^*\xi}$ and $F_{\widehat{\rho}}(y) = \mathrm{vol}^*(\{\xi \mid g(\xi) > y\})$ as in (22). Then one has*

$$S_{A_\sigma}(\rho) \geq S_F(\widehat{\rho}) - \tau(\rho) = \int_0^{+\infty} F_{\widehat{\rho}}(y) \ln F_{\widehat{\rho}}(y) dy - \tau(\rho)\,.$$

*Equality holds if $\sigma$ is a decreasing* regular filling *of the level sets of $g$ in the following sense:*

- *for each $y$, there exists $\lambda$ such that*

$$\{\xi \mid g(\xi) > y\} \subset \Omega^\sigma_\lambda = \{\xi \mid \sigma(\xi) \leq \lambda\} \subset \{\xi \mid g(\xi) \geq y\};$$

- *for all $\lambda$, $\mathrm{vol}^*(\sigma^{-1}(\lambda)) = 0$.*

Equivalently, the level sets $\Omega^\sigma_\lambda$ of a regular filling $\sigma$ are the sets $\{\xi \mid g(\xi) \geq y\}$ (up to zero measure) for regular values of $\rho$, i.e. when $\mathrm{vol}^*(g^{-1}(y)) = 0$, while on $g^{-1}(y)$ for the (discrete) non-regular values, they interpolate continuously in measure between $\{\xi \mid g(\xi) > y\}$ and $\{\xi \mid g(\xi) \geq y\}$. This can be achieved since the measure has no atom.

From Proposition 4.2, we notice that the use of the Laplacian is already optimal to minimize the spectral entropy of states such that $\widehat{\rho}$ has a decaying radial density; Theorems 1.5 and 1.7 are equivalent on such states. On anisotropic states, one advantage of (23) over (49) appears in its stronger invariance through general linear transforms $f(x) \mapsto f(Ax)$ and $\rho \mapsto \rho_A = A\rho A^{-1}$. In such cases, one checks easily that

$$\frac{d\nu_{\rho_A}}{dx}(x) = |\det A| \frac{d\nu_\rho}{dx}(Ax) \quad \text{while} \quad \frac{d\nu_{\widehat{\rho_A}}}{d^*\xi}(\xi) = |\det A|^{-1} \frac{d\nu_{\widehat{\rho}}}{d^*\xi}({}^tA^{-1}\xi)\,,$$

giving that

$$S_{dx}(\rho_A) = S_{dx}(\rho) - \tau(\rho) \ln|\det A| \quad \text{while} \quad S_F(\widehat{\rho}_A) = S_F(\widehat{\rho}) + \tau(\rho)\ln|\det A|\,,$$

which keeps the entropy sum unchanged in (49).

*Proof of Proposition 4.2.* Let $v(\xi) = \mathrm{vol}^*(\Omega^\sigma_{\sigma(\xi)})$ and $F_\sigma(z) = v_*(\nu_{\widehat{\rho}})(]0,z])$. Then by (54)

$$S_{A_\sigma}(\rho) = \int_0^{+\infty} \ln y dF_\sigma(y) = -\int_0^1 F_\sigma(y)\frac{dy}{y} + \int_1^{+\infty} (\tau(\rho) - F_\sigma(y))\frac{dy}{y}\,, \tag{55}$$

by Fubini, since $F_\sigma(+\infty) = \tau(\widehat{\rho}) = \tau(\rho)$. Hence $S_{A_{\sigma_1}}(\rho) \geq S_{A_{\sigma_2}}(\rho)$ if $F_{\sigma_1} \leq F_{\sigma_2}$, and we have to look for upper bounds for $F_\sigma$ to minimize $S_{A_\sigma}(\rho)$.

By definition, one has

$$F_\sigma(y) = \int_{D^\sigma_y} g(\xi) d^*\xi \quad \text{with} \quad D^\sigma_y = \{\xi \mid \mathrm{vol}^*(\Omega^\sigma_{\sigma(\xi)}) \leq y\}\,.$$

Clearly one has $\{\xi \mid \sigma(\xi) < \lambda\} \subset D^\sigma_y \subset \Omega^\sigma_\lambda$ where $\lambda = \sup_{D^\sigma_y} \sigma$, and thus $\mathrm{vol}^*(D^\sigma_y) \leq y$ with equality if $\mathrm{vol}^*(\sigma^{-1}(\lambda)) = 0$. Hence by the 'bathtub principle' (see [13, Theorem 12.3]) one has

$$F_\sigma(y) \leq F(y) = \int_{\{g > F^{-1}_{\widehat{\rho}}(y)\}} g(\xi) d^*\xi + F^{-1}_{\widehat{\rho}}(y)(y - F_{\widehat{\rho}}(F^{-1}_{\widehat{\rho}}(y)))\,, \tag{56}$$

with $F_{\widehat{\rho}}^{-1}(y) = \inf\{z \mid F_{\widehat{\rho}}(z) \leq y\}$. Indeed, this comes from the identity

$$F_\sigma(y) - F(y) = \int_{D_y^\sigma \cap \{g \leq F_{\widehat{\rho}}^{-1}(y)\}} (g(\xi) - F_{\widehat{\rho}}^{-1}(y)) d^*\xi \\ - \int_{(D_y^\sigma)^c \cap \{g > F_{\widehat{\rho}}^{-1}(y)\}} (g(\xi) - F_{\widehat{\rho}}^{-1}(y)) d^*\xi + F_{\widehat{\rho}}^{-1}(y)(\mathrm{vol}^*(D_y^\sigma) - y)\,.$$

Moreover this shows that equality holds in (56) if $\mathrm{vol}^*(D_y^\sigma) = y$ and, up to zero measure sets, $\{g > F_{\widehat{\rho}}^{-1}(y)\} \subset D_y^\sigma \subset \{g \geq F_{\widehat{\rho}}^{-1}(y)\}$; which is fulfilled for regular fillings by the discussion above.

We rewrite the function $F(y)$ in a more convenient form. Since $\tau(\rho) = \tau(\widehat{\rho}) = \int_{\mathbb{R}^n} g(\xi) d^*\xi$, one has

$$\begin{aligned} \int_{\{g > F_{\widehat{\rho}}^{-1}(y)\}} g(\xi) d^*\xi &= \tau(\rho) - \int_{\{g \leq F_{\widehat{\rho}}^{-1}(y)\}} g(\xi) d^*\xi \\ &= \tau(\rho) - \int_{\{0 \leq u < g(\xi) \leq F_{\widehat{\rho}}^{-1}(y)\}} du d^*\xi \\ &= \tau(\rho) - \int_0^{F_{\widehat{\rho}}^{-1}(y)} (F_{\widehat{\rho}}(u) - F_{\widehat{\rho}}(F_{\widehat{\rho}}^{-1}(y))) du \\ &= \tau(\rho) - \int_0^{F_{\widehat{\rho}}^{-1}(y)} F_{\widehat{\rho}}(u) du + F_{\widehat{\rho}}^{-1}(y) F_{\widehat{\rho}}(F_{\widehat{\rho}}^{-1}(y))\,. \end{aligned}$$

Then by (56),

$$\begin{aligned} F(y) &= \tau(\rho) - \int_0^{F_{\widehat{\rho}}^{-1}(y)} (F_{\widehat{\rho}}(u) - y) du \\ &= \tau(\rho) - \int_{\{y < v < F_{\widehat{\rho}}(u)\}} dv du\,, \end{aligned}$$

since by right continuity of $F_{\widehat{\rho}}$, one has $u < F_{\widehat{\rho}}^{-1}(y)$ iff $F_{\widehat{\rho}}(u) > y$. Therefore

$$F(y) = \tau(\rho) - \int_y^{+\infty} F_{\widehat{\rho}}^{-1}(v) dv = \int_0^y F_{\widehat{\rho}}^{-1}(v) dv\,, \tag{57}$$

since

$$\begin{aligned} \tau(\rho) &= \int_{\mathbb{R}^n} g(\xi) d^*\xi = \int_{\{0 \leq u < g(\xi)\}} du d^*\xi = \int_0^{+\infty} F_{\widehat{\rho}}(u) du \\ &= \int_{\{0 \leq v < F_{\widehat{\rho}}(u)\}} dv du = \int_0^{+\infty} F_{\widehat{\rho}}^{-1}(v) dv\,. \end{aligned}$$

Finally, (55) and (57) lead to

$$\begin{aligned}
S_{A_\sigma}(\rho) &\geq \int_0^{+\infty} \ln y F_{\widehat{\rho}}^{-1}(y)dy = \int_{\{0<z<F_{\widehat{\rho}}^{-1}(y)\}} \ln y dz dy \\
&= \int_{\{0<y<F_{\widehat{\rho}}(z)\}} \ln y dy dz \\
&= \int_0^{+\infty} F_{\widehat{\rho}}(z)(\ln F_{\widehat{\rho}}(z) - 1)dz = S_F(\widehat{\rho}) - \tau(\rho)\,,
\end{aligned}$$

as claimed in Proposition 4.2. □

4.4. **Proof of Corollary 1.8.** We deduce Corollary 1.8 from Theorem 1.7. This relies on the following entropy comparison:

$$S_F(\widehat{\rho}) \leq S_{d^*\xi}(\widehat{\rho}) + \tau(\rho)(1 + \ln \tau(\rho))\,. \tag{58}$$

Indeed, one has

$$\begin{aligned}
-S_{d^*\xi}(\widehat{\rho}) - \tau(\rho) &= \int_{\mathbb{R}^n} \ln\Big(\frac{d\nu_{\widehat{\rho}}}{d^*\xi}\Big) d\nu_{\widehat{\rho}} - \tau(\widehat{\rho}) \\
&= \int_{\{0<y<\frac{d\nu_{\widehat{\rho}}}{d^*\xi}\}} \ln y dy d^*\xi \\
&= \int_0^{+\infty} F_{\widehat{\rho}}(y) \ln y dy\,,
\end{aligned}$$

thus

$$\begin{aligned}
S_F(\widehat{\rho}) - S_{d^*\xi}(\widehat{\rho}) - \tau(\rho) &= \int_0^{+\infty} \ln(yF_{\widehat{\rho}}(y))F_{\widehat{\rho}}(y)dy \\
&\leq \int_0^{+\infty} \ln(\tau(\rho))F_{\widehat{\rho}}(y)dy = \tau(\rho)\ln\tau(\rho)\,,
\end{aligned}$$

since $yF_{\widehat{\rho}}(y) = y\mathrm{vol}^*\{\xi \mid \frac{d\nu_{\widehat{\rho}}}{d^*\xi}(\xi) > y\} \leq \int_{\mathbb{R}^n} \frac{d\nu_{\widehat{\rho}}}{d^*\xi} d^*\xi = \tau(\rho)$. Then, (23) and (58) give

$$S_{dx}(\rho) + S_{d^*\xi}(\widehat{\rho}) \geq -\tau(\rho)(3 + \ln\tau(\rho) + \ln\|\rho\|_\infty)\,.$$

Then we observe that, except for the term $-3\tau(\rho)$, this expression is additive in taking tensor product of unit trace states. Therefore, applying it to $\otimes^N \rho$ on $\mathbb{R}^{nN}$ as in §4.2, and dividing by $N$ for $N \nearrow +\infty$, gives (24) on unit trace states, and the general statement by homogeneity.

4.5. **The role of von Neumann's entropy.** We finally discuss a possible improvement of the last result Corollary 1.8 using von Neumann's entropy $S(\rho)$ in the lower bound for $S_{dx}(\rho) + S_{d^*\xi(\rho)}$. Following e.g. [18, 19], this intrinsic entropy is defined for unit trace states by

$$S(\rho) = -\tau(\rho \ln \rho)\,.$$

For such states, one has $S(\rho) \geq -\ln\|\rho\|_\infty$, with equality on normalized projections on finite dimensional spaces $\rho = \Pi_V/\dim V$. Hence on these projections (24) reads

$$S_{dx}(\rho) + S_{d^*\xi}(\widehat{\rho}) \geq S(\rho)\,(= \ln\dim V). \tag{59}$$

We don't know whether this holds for general unit trace states. An interesting family of examples here is given by the heat of the harmonic oscillator, which is the semigroup

$$\rho_t = \exp(-t(\Delta + \|x\|^2))\,,$$

acting on $L^2(\mathbb{R}^n_x)$. This state is the $n$-th tensor product of the one-dimensional case. Furthermore it is self-dual in Fourier transform, i.e. $\widehat{\rho}_t = \rho_t$. The kernel of $\rho_t$ is given by Mehler's formula (see [4, Chap. 4.2]):

$$\rho_t(x,y) = (2\pi \sinh 2t)^{-n/2} \exp\big(-\frac{1}{2}(\coth 2t)(\|x\|^2 + \|y\|^2) + (\sinh 2t)^{-1}\langle x,y\rangle\big)\,,$$

so that its density and trace are

$$\frac{d\nu_{\rho_t}}{dx} = \rho_t(x,x) = (2\pi \sinh 2t)^{-n/2} \exp(-(\tanh t)\|x\|^2) \quad \text{and} \quad \tau(\rho_t) = (2\sinh t)^{-n}\,.$$

This leads easily to the entropies of the normalized states $\lambda_t = \rho_t/\tau(\rho_t)$, namely

$$S_{dx}(\lambda_t) = \frac{n}{2} - \frac{n}{2}\ln(\frac{\tanh t}{\pi}) \quad \text{while} \quad S_{d^*\xi}(\widehat{\lambda}_t) = \frac{n}{2} - \frac{n}{2}\ln(\frac{\tanh t}{\pi}) - n\ln(2\pi)\,,$$

hence

$$S_{dx}(\lambda_t) + S_{d^*\xi}(\widehat{\lambda}_t) = n - n\ln 2 - n\ln(\tanh t)\,.$$

To compute von Neumann's entropy of $\lambda_t$, we recall that on $\mathbb{R}$, the spectrum of $\rho_t$ is given by $p_k = e^{-(2k+1)t}$, $k \in \mathbb{N}$. One finds that

$$\begin{aligned} S(\lambda_t) = nS(\lambda_t^{\mathbb{R}}) &= -n\sum_{k\geq 0}(2\sinh t)p_k \ln((2\sinh t)p_k) \\ &= -n\ln(2\sinh t) - 2nt\sinh t\sum_{k\geq 0}(-2k-1)e^{-(2k+1)t} \\ &= -n\ln(2\sinh t) - 2nt\sinh t\big(\frac{1}{2\sinh t}\big)' \\ &= nt\coth t - n\ln 2 - n\ln(\sinh t)\,. \end{aligned}$$

Therefore we obtain

$$S_{dx}(\lambda_t) + S_{d^*\xi}(\widehat{\lambda}_t) - S(\lambda_t) = n(1 + \ln(\cosh t) - t\coth t)\,,$$

which is easily seen to be increasing in $t$ and positive. Hence these states also satisfy the entropy bound (59), even sharply when $t$ goes to 0. It is not sharp in the opposite limit $t$ goes to $+\infty$. Indeed there $\lambda_t$ converges to the pure ground state of the harmonic oscillator. One has $S(\lambda_t) \to 0$ while $S_{dx}(\lambda_t) + S_{d^*\xi}(\widehat{\lambda}_t) \to n(1 - \ln 2)$, which, following [1, 2, 20], is actually the best lower bound for this entropy sum on pure states.

Another clue in favour of (59) on general (unit trace) states is that it implies the log-Sobolev inequality (52) proved in [8]:

$$S_{dx}(\rho) + \frac{n}{2}\ln\Big(\frac{e}{2\pi n}\mathcal{E}_\Delta(\rho)\Big) \geq S(\rho)\,.$$

Indeed by a classical inequality ([16, 10, 20]), a probability $\mu$ in $\mathbb{R}^n_\xi$ always satisfies

$$S_{d\xi}(\mu) = -\int_{\mathbb{R}^d} \ln\big(\frac{d\mu}{d\xi}\big)d\mu \leq \frac{n}{2}\ln\big(\frac{2\pi e}{n}\sigma^2(\mu)\big)\,,$$

where $\sigma^2(\mu) = \int_{\mathbb{R}^n} \|\xi - E_\mu(\xi)\|^2 d\mu$ is the variance of $\mu$, and equality achieved on Gaussian measures. Hence for $\mu = \nu_{\widehat{\rho}}$, one has

$$\begin{aligned} S_{d^*\xi}(\widehat{\rho}) &= S_{d\xi}(\widehat{\rho}) - n\ln(2\pi) \\ &\leq \frac{n}{2}\ln\big(\frac{e}{2\pi n}\sigma^2(\nu_{\widehat{\rho}})\big) \end{aligned}$$

with

$$\sigma^2(\nu_{\widehat{\rho}}) \leq \int_{\mathbb{R}^n} \|\xi\|^2 d\nu_{\widehat{\rho}} = \tau(\|\xi\|^2\widehat{\rho}) = \tau(\Delta\rho) = \mathcal{E}_\Delta(\rho)\,.$$

*Acknowledgements.* The author is grateful to the referees for their valuable comments and references.

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