

AN ENTROPIC UNCERTAINTY PRINCIPLE FOR POSITIVE OPERATOR VALUED MEASURES

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ABSTRACT. Extending a recent result by Frank and Lieb, we show an entropic uncertainty principle for mixed states in a Hilbert space, relatively to pairs of positive operator valued measures that are independent in some sense. This yields spatial-spectral uncertainty principles and log-Sobolev inequalities for invariant operators on homogeneous spaces, which are sharp in the compact case.

1. INTRODUCTION AND MAIN RESULT

A mixed state (or density matrix) in a Hilbert space H is a positive operator ρ on H with trace $\tau(\rho) = 1$. From the quantum viewpoint, one can measure ρ through positive operator valued measures (POVM) on H . We briefly recall this.

A POVM is a countably additive map P from a σ -algebra on a space X into the positive operators on H and such that $P(X) = 1_H$, see [11, 18]. Important examples are given by the following cases:

- When H is isometric to a function space $L^2(X, \mu)$, one can set $P(\Omega)$ to be the multiplication by the characteristic function χ_Ω for measurable sets $\Omega \subset X$. This is actually a *projection* valued measure.
- Other projection valued measures come from the spectral resolution of a self-adjoint operator A on H . Here $X = \mathbb{R}$ and $P(I) = \Pi_A(I)$ is the spectral projector of A associated to $I \subset \mathbb{R}$.
- Partition of unity in H , i.e., series of positive operators P_i such that $\sum_i P_i = 1_H$.

A state ρ and a POVM induce a probability measure ν_P^ρ on X by

$$\nu_P^\rho(\Omega) = \tau(\rho P(\Omega)) = \tau(\rho^{1/2} P(\Omega) \rho^{1/2}),$$

that encodes the result of the measurement of ρ through P .

The statement we want to discuss deals with the measures ν_P^ρ and ν_Q^ρ associated with a *pair* of POVM on H , say P from X and Q from Y . This pair induces a state-independent measure μ_{PQ} on $X \times Y$, called a ‘Liouville measure’ from now on, and defined by

$$\mu_{PQ}(\Omega_1 \times \Omega_2) = \tau(P(\Omega_1)Q(\Omega_2)) = \tau(P(\Omega_1)^{1/2}Q(\Omega_2)P(\Omega_1)^{1/2}).$$

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We shall see that when the Liouville measure μ_{PQ} is bounded by a σ -finite product measure $\mu_P \otimes \mu_Q$, then $\nu_P^\rho \ll \mu_P$ and $\nu_Q^\rho \ll \mu_Q$. Let then

$$\varrho_P = \frac{d\nu_P^\rho}{d\mu_P} \quad \text{and} \quad \varrho_Q = \frac{d\nu_Q^\rho}{d\mu_Q}$$

denote the corresponding density functions. Our main result is the following.

Theorem 1.1. *Using the notations above, suppose that $\mu_{PQ} \leq \mu_P \otimes \mu_Q$ for some σ -finite product measure on $X \times Y$. Then it holds*

$$(1) \quad - \int_X \varrho_P \ln \varrho_P d\mu_P - \int_Y \varrho_Q \ln \varrho_Q d\mu_Q \geq S(\rho) = -\tau(\rho \ln \rho),$$

provided the left side is well defined, i.e., we do not face $\infty - \infty$ in the integrals and their sum.

Here $S(\rho)$ is the intrinsic von Neumann entropy of the state, independent of any choice of POVM, while the integrals are minus the relative entropy from ν_P^ρ to μ_P and ν_Q^ρ to μ_Q . Hence, (1) is an entropic uncertainty principle claiming that the P and Q -measurements of ρ can't be *both* concentrated in small sets for μ_P and μ_Q , under some hypothesis on the Liouville measure. From the probabilistic viewpoint, the assumption on μ_{PQ} deals with the independence of the two POVM. The best case occurs when μ_{PQ} is a product measure itself: and we shall say that P and Q are *independent* there. We give some examples.

2. FIRST EXAMPLES

2.1. Pair of bases. Let (e_i) and (f_j) be two orthonormal bases of a Hilbert space H . This gives two projector valued measures from (finite or not) sets $X = Y \subset \mathbb{Z}$ with $P(i) = \Pi_{e_i}$ and $Q(j) = \Pi_{f_j}$. The measurements of ρ are

$$\nu_P^\rho(i) = \langle \rho e_i, e_i \rangle = p_i, \quad \nu_Q^\rho(j) = \langle \rho f_j, f_j \rangle = q_j$$

and the Liouville measure is

$$\mu_{PQ}(i, j) = \tau(P(i)Q(j)) = |\langle e_i, f_j \rangle|^2.$$

Clearly, it holds that $\mu_{PQ} \leq \mu_P \otimes \mu_Q$ with

$$\mu_P(i) = \sup_j |\langle e_i, f_j \rangle| \quad \text{and} \quad \mu_Q(j) = \sup_i |\langle e_i, f_j \rangle|.$$

Theorem 1.1 yields

$$(2) \quad - \sum_i p_i \ln p_i - \sum_j q_j \ln q_j \geq S(\rho) + K,$$

with $K = - \sum_i p_i \ln \mu_P(i) - \sum_j q_j \ln \mu_Q(j) \geq 0.$

This improves a result by Frank and Lieb in [7] where K is replaced by

$$K' = -2 \ln(\sup_{i,j} |\langle e_i, f_j \rangle|) \leq K.$$

The better estimation is obtained when P and Q are independent, i.e., when μ_{PQ} is a product measure. Taking partial i and j sums, one sees that this only happens in finite dimension d and when

$$|\langle e_i, f_j \rangle|^2 = d^{-1},$$

which is a definition of *mutually unbiased bases*, used in quantum information theory [11, 17].

We note that such results generalize to pairs of partitions of unity P_i and Q_j obtained in the following way. Given two orthonormal bases (e_i) and (f_j) of Hilbert space H' containing H , one gets two POVM on H by setting

$$P_i = \Pi_H \Pi_{e_i} \Pi_H \quad \text{and} \quad Q_j = \Pi_H \Pi_{f_j} \Pi_H.$$

One finds that the same inequality as (2) holds replacing e_i and f_j by $\Pi_H e_i$ and $\Pi_H f_j$ in formulas.

2.2. Fourier transform and uncertainty. Consider now $H = L^2(\mathbb{R}^n, dx)$ with its natural projection valued measure from $X = \mathbb{R}_x^n$ defined by $P(\Omega) = \chi_\Omega \times$. Fourier transform on H , with convention that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi\langle x, \xi \rangle} dx$$

is an isometry onto $L^2(\mathbb{R}_\xi^n, d\xi)$, and provides a second projection valued measure Q from $Y = \mathbb{R}_\xi^n$ into H by setting $Q(\Omega) = \mathcal{F}^{-1} \chi_\Omega \mathcal{F}$.

These two maps are independent in our sense. Indeed, our 'Liouville measure' is here

$$\begin{aligned} \mu_{PQ}(\Omega_1 \times \Omega_2) &= \tau(\chi_{\Omega_1} \mathcal{F}^* \chi_{\Omega_2} \mathcal{F}) = \|\chi_{\Omega_2} \mathcal{F} \chi_{\Omega_1}\|_{HS}^2 \quad (\text{Hilbert Schmidt norm}) \\ &= \int_{\Omega_2 \times \Omega_1} |e^{-2i\pi\langle x, \xi \rangle}|^2 d\xi dx \quad (\text{using kernels}) \\ &= \lambda(\Omega_1) \lambda(\Omega_2), \end{aligned}$$

which is the (genuine) Liouville measure of the phase space $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$.

We make explicit Theorem 1.1. A state may be written $\rho = \sum_i p_i \Pi_{f_i}$ for orthonormal functions f_i . Then its measurement in P reads

$$\nu_P^\rho(\Omega) = \tau(\rho \chi_\Omega) = \int_\Omega \sum_i p_i |f_i(x)|^2 dx,$$

while from Q

$$\begin{aligned} \nu_Q^\rho(\Omega) &= \tau(\rho \mathcal{F}^{-1} \chi_\Omega \mathcal{F}) = \tau(\mathcal{F} \rho \mathcal{F}^{-1} \chi_\Omega) \\ &= \int_\Omega \sum_i p_i |\mathcal{F}(f_i)(\xi)|^2 d\xi, \end{aligned}$$

since $\mathcal{F} \rho \mathcal{F}^{-1} = \sum_i p_i \Pi_{\mathcal{F}(f_i)}$. Finally, Theorem 1.1 states that

$$(3) \quad - \int_{\mathbb{R}^n} \varrho(x) \ln \varrho(x) dx - \int_{\mathbb{R}^n} \hat{\varrho}(\xi) \ln \hat{\varrho}(\xi) d\xi \geq S(\rho) = - \sum_i p_i \ln p_i$$

with

$$\varrho(x) = \frac{d\nu_P^\rho}{dx} = \sum_i p_i |f_i(x)|^2 \quad \text{and} \quad \hat{\varrho}(\xi) = \frac{d\nu_Q^\rho}{d\xi} = \sum_i p_i |\mathcal{F}(f_i)(\xi)|^2.$$

This inequality has been first proved by Frank and Lieb in [7], and is known to be sharp for states $\rho_t = e^{-t(\Delta + \|x\|^2)} / \tau(e^{-t(\Delta + \|x\|^2)})$ when $t \searrow 0$.

2.3. Frank and Lieb result. The previous example is actually a particular case of a more general theorem proved in [7].

Suppose that two σ -finite function spaces $L^2(X, \mu)$ and $L^2(Y, \nu)$ are isometric through

$$\mathcal{U} : L^2(X, \mu) \rightarrow L^2(Y, \nu).$$

Suppose moreover that \mathcal{U} is bounded from $L^1(X, \mu)$ to $L^\infty(Y, \nu)$. Let $\rho = \sum_i p_i \Pi_{f_i}$ be a (unit trace) state on $L^2(X, \mu)$ and consider $\hat{\rho} = \mathcal{U}\rho\mathcal{U}^{-1} = \sum_i p_i \Pi_{\mathcal{U}f_i}$ on $L^2(Y, \nu)$. Let

$$\varrho(x) = \sum_i p_i |f_i(x)|^2 \quad \text{and} \quad \hat{\varrho}(\xi) = \sum_i p_i |\mathcal{U}(f_i)(\xi)|^2$$

be their density as above.

Theorem 2.1. [7, Thm. 2.2] *Suppose moreover that*

$$\int_X \varrho(x) \ln^+ \varrho(x) d\mu(x) < +\infty \quad \text{and} \quad \int_Y \hat{\varrho}(y) \ln^+ \hat{\varrho}(y) d\nu(y) < +\infty$$

Then,

$$(4) \quad - \int_X \varrho(x) \ln \varrho(x) d\mu(x) - \int_Y \hat{\varrho}(y) \ln \hat{\varrho}(y) d\nu(y) \geq S(\rho) - 2 \ln \|\mathcal{U}\|_{L^1 \rightarrow L^\infty}.$$

Theorem 1.1 implies it. Indeed, as in the previous discussion on Fourier transform, one defines two projections valued measures on $H = L^2(X, \mu)$: the standard one $P(\Omega) = \chi_\Omega \times$ from X , and $Q(\Omega) = \mathcal{U}\chi_\Omega\mathcal{U}^{-1}$ from Y . Then arguing as previously, one has

$$\begin{aligned} \mu_{PQ}(\Omega_1 \times \Omega_2) &= \tau(\chi_{\Omega_1} \mathcal{U} \chi_{\Omega_2} \mathcal{U}^*) = \|\chi_{\Omega_2} \mathcal{U}^* \chi_{\Omega_1}\|_{HS}^2 \\ &= \int_{\Omega_1 \times \Omega_2} |\mathcal{U}(y, x)|^2 d\mu(x) d\nu(y) \\ &\leq \mu(\Omega_1) \nu(\Omega_2) \|\mathcal{U}\|_{L^1 \rightarrow L^\infty}^2. \end{aligned}$$

This shows that Theorem 1.1 applies and yields (4).

Comparing to Theorem 2.1, Theorem 1.1 adds some flexibility in the choice of the measures on X and Y , and applies to general positive operator valued measures, as illustrated in Section 2.1. Here the measures on X and Y are not fixed a priori but adapted to the interaction of the two maps P and Q through the intrinsic Liouville measure. We note also that Theorem 2.1 deals with L^2 function spaces, while Theorem 1.1 applies to general direct integral decompositions of Hilbert spaces, as for instance to splittings $H = \oplus_\lambda E_\lambda$ with non constant $\dim E_\lambda$ coming from spectral resolution of operators. See examples in Section 4.

Another feature of Theorem 1.1 is its invariance through general isometries of H . Indeed, if an isometry U acts on H , then two initial POVM P and Q are conjugated to $P^U = U^{-1}PU$ and $Q^U = U^{-1}QU$. The Liouville measure is preserved: $\mu_{P^U Q^U} = \tau(P^U Q^U) = \tau(PQ) = \mu_{PQ}$, as the measures associated to ρ and ρ^U : $\nu_P^\rho = \nu_{P^U}^{\rho^U}$ and $\nu_Q^\rho = \nu_{Q^U}^{\rho^U}$. In comparison, (4) is not invariant since the underlying isometry \mathcal{U}_{FL} there becomes $U^{-1}\mathcal{U}_{FL}U$, whose $L^1 \rightarrow L^\infty$ norm is a priori not controlled by $\|\mathcal{U}_{FL}\|_{1, \infty}$ alone.

3. PROOF OF THEOREM 1.1 AND RELATED COMMENTS

Our proof of Theorem 1.1 follows the lines of Frank and Lieb's argument for Theorem 2.1. It relies on two classical lemmas in quantum statistical mechanics, see e.g., [2] and [15, Thm. 8.5].

Lemma 3.1 (Gibbs variational formula). *Let A be a self-adjoint operator such that e^{-A} is trace class. Then for any unit trace state ρ , it holds*

$$\tau(A\rho) - S(\rho) \geq -\ln \tau(e^{-A}),$$

with equality iff $\rho = e^{-A}/\tau(e^{-A})$.

Lemma 3.2 (Golden–Thompson inequality). *Let A and B be self-adjoint operators with upper bounds. Then it holds that*

$$\tau(e^{A+B}) \leq \tau(e^{A/2}e^B e^{A/2}).$$

Here, following [9], $A+B$ stands for the self-adjoint operator associated with the closed upper-bounded quadratic form $q_A + q_B$ on the closure of $D(A_-^{1/2}) \cap D(B_-^{1/2})$.

We suppose that $\mu_{PQ} \leq \mu_P \otimes \mu_Q$ and first check the absolute continuity of ν_P^ρ and ν_Q^ρ with respect to μ_P and μ_Q . If $\mu_P(\Omega) = 0$, then by σ -finiteness of μ_Q one has $\mu_{PQ}(\Omega \times Y) = \tau(P(\Omega)) = 0$. Hence $P(\Omega) = 0$, and $\nu_P^\rho(\Omega) = \tau(\rho P(\Omega)) = 0$ as claimed.

Given $k \geq 0$, let $\varrho_{P,k} = \min(\varrho_P, k)$ and $\varrho_{Q,k} = \min(\varrho_Q, k)$. We consider the following bounded positive operators on H

$$A_P = \int_X \varrho_{P,k} dP \quad \text{and} \quad A_Q = \int_Y \varrho_{Q,k} dQ.$$

Using a Hilbert basis (e_i) of H and monotone convergence, one sees that

$$\tau(A_P^{1/2} A_Q A_P^{1/2}) = \sum_i \langle A_Q A_P^{1/2} e_i, A_P^{1/2} e_i \rangle = \int_Y \varrho_{Q,k} \tau(A_P^{1/2} dQ A_P^{1/2}),$$

where for any measurable set $\Omega' \subset Y$

$$\begin{aligned} \tau(A_P^{1/2} Q(\Omega') A_P^{1/2}) &= \tau(Q^{1/2}(\Omega') A_P Q^{1/2}(\Omega')) \\ &= \int_X \varrho_{P,k} \tau(Q^{1/2}(\Omega') dP Q^{1/2}(\Omega')) \\ &= \int_X \varrho_{P,k} d\mu_{P,Q}(\cdot \times \Omega'). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \tau(A_P^{1/2} A_Q A_P^{1/2}) &= \int_{X \times Y} \varrho_{P,k} \varrho_{Q,k} d\mu_{PQ} \leq \int_{X \times Y} \varrho_P \varrho_Q d\mu_{PQ} \\ &= \int_{X \times Y} \frac{d\nu_P^\rho}{d\mu_P} \frac{d\nu_Q^\rho}{d\mu_Q} d\mu_{PQ} \\ (5) \quad &\leq \int_{X \times Y} d\nu_P^\rho \otimes d\nu_Q^\rho \\ &= \nu_P^\rho(X) \nu_Q^\rho(Y) = \tau(\rho)^2 = 1. \end{aligned}$$

Then by Lemma 3.2, it holds that

$$(6) \quad \tau(e^{\ln A_P + \ln A_Q}) \leq \tau(A_P^{1/2} A_Q A_P^{1/2}) \leq 1,$$

where

$$A = \ln A_P + \ln A_Q$$

is the unique self-adjoint operator associated with the closed quadratic form $q_{\ln A_P} + q_{\ln A_Q}$ on the closure V of $D(A) = D(\ln_-^{1/2} A_P) \cap D(\ln_-^{1/2} A_Q)$; see [9] and [13, VIII.6]

We suppose now that ρ writes $\rho = \sum_i p_i \Pi_{f_i}$ with orthonormal functions f_i , and satisfies

$$\int_X \ln_- \varrho_P d\nu_P^\rho < \infty \quad \text{and} \quad \int_Y \ln_- \varrho_Q d\nu_Q^\rho < \infty.$$

Note that otherwise (1) is already satisfied. We show that $f_i \in D(A)$ for any i , and evaluate $\langle A f_i, f_i \rangle$. We can use a Jensen-type inequality for operator convex functions on POVM due to Choi, see [3, 12]. Namely, since $-\ln t$ is operator convex (see [1, Chap.V]), it holds that

$$-\ln A_P \leq - \int_X \ln \varrho_{P,k} dP,$$

hence

$$(7) \quad \begin{aligned} -\langle \ln A_P f_i, f_i \rangle &\leq - \int_X \ln \varrho_{P,k} d\langle P f_i, f_i \rangle = - \int_X \ln \varrho_{P,k} d\nu_P^{\Pi_{f_i}} \\ &\leq p_i^{-1} \int_X \ln_- \varrho_P d\nu_P^\rho < \infty \end{aligned}$$

by assumption on ρ . One gets a similar inequality for $-\langle \ln A_Q f_i, f_i \rangle$. We can then apply Lemma 3.1 to $-A$ on V with $\Pi_V \rho \Pi_V = \rho$. Summing (7) and using (6) yield

$$(8) \quad - \int_X \ln \varrho_{P,k} d\nu_P^\rho - \int_Y \ln \varrho_{Q,k} d\nu_Q^\rho \geq \tau(-A\rho) \geq S(\rho),$$

and give Theorem 1.1 by monotone convergence when $k \rightarrow +\infty$.

Equality case. We discuss here the equality case in Theorem 1.1. It holds iff (5), (7), (8) become sharp for $k \rightarrow \infty$. Equality in (5) means that the Liouville measure is a product, i.e., the POVM P and Q are independent. By [12], (7) are all equalities iff P and Q are *projection* valued measures on the support of ν_P^ρ and ν_Q^ρ . Then equality holds in (8) iff it holds:

- in Golden–Thompson inequality, which is achieved when P and Q commute,
- and in Gibbs formula, meaning that $A_P A_Q \rightarrow \rho$, i.e. ρ is a product state with respect to P, Q .

Altogether, these conditions are very restrictive. Examples are given by a space H that splits into $H_1 \otimes H_2$ with given basis $(e_i), (f_j)$. There $P(i) = \Pi_{e_i} \otimes 1$ and $Q(j) = 1 \otimes \Pi_{f_j}$, and product states $\rho = \rho_1 \otimes \rho_2$ give equalities.

Note, however, that an approximate equality in (1) may be achieved in other cases. This happens for instance for the Fourier transform discussed in Section 2.2. Here, the position-momentum maps are independent projection valued measures, hence (5) and (7) are equalities. Although P and Q don't commute here, Theorem 1.1 becomes sharp on states that spread at large scale on the (x, ξ) phase space like $\rho_t = e^{-t(\Delta + \|x\|^2)} / \tau(e^{-t(\Delta + \|x\|^2)})$ when $t \searrow 0$; see [14].

4. APPLICATIONS ON HOMOGENEOUS SPACES

4.1. Invariant operators and spatial/spectral independence. We turn back to examples and describe a setting on homogeneous spaces leading to independent spatial and spectral projections valued measures.

Let H be a closed group in a locally compact group G and let $X = G/H$ be the corresponding homogeneous space. We assume moreover that X admits a G -invariant Borel measure μ . This is equivalent to the equality of modular functions $\delta_H = \delta_G$ on H , and μ is unique up to constants; see [8, 10].

Let V be a separable Hilbert space and consider $\mathcal{H} = L^2(X, V, \mu)$. We say that an isometry U of \mathcal{H} is a gauge transform if it acts fiberwise over X , i.e., $(Uf)(x) = U_x(f(x))$ for some isometries $U_x : V \rightarrow V$.

Definition 4.1. An operator A on \mathcal{H} will be called *translation gauge-invariant*, if it is conjugated by a gauge transform U_0 to an operator $A_0 = U_0^{-1}AU_0$ whose translates $A_0^g = g^{-1}A_0g$ stay conjugated to A_0 up to gauge transforms, i.e. $A_0^g = U(g)^{-1}A_0U(g)$.

This class contains the translation-invariant operators on X , as differential operators with constant coefficients when G is a Lie group. It contains also operators conjugated to translation-invariant operators by gauge transform; for instance $A = i\frac{d}{dx} + x = e^{-ix^2/2}i\frac{d}{dx}e^{ix^2/2}$ is not translation-invariant, yet translation gauge-invariant on $L^2(\mathbb{R})$. Another classical example is given by the Hamiltonian of a constant magnetic field $B \geq 0$ in \mathbb{R}^2

$$(9) \quad H_B = \left(-i\frac{\partial}{\partial x} + \frac{By}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{Bx}{2}\right)^2 = (-i\nabla + A)^2.$$

A general gauge transform acts on H_B as $e^{-if}H_Be^{if} = (-i\nabla + A + df)^2$, and fixing $f(x, y) = \frac{B}{2}(y_0x - x_0y)$ actually translates H_B on \mathbb{R}^2 by (x_0, y_0)

Our interest for gauge-invariant self-adjoint operators A here comes from the independence of the projection valued measure on H coming from X ; namely $P(\Omega) = \chi_\Omega \times$, and the spectral resolution of A on $Y = \mathbb{R}$: $Q(I) = \Pi_A(I)$.

Proposition 4.2. *Let A be a translation gauge-invariant self-adjoint operator on $L^2(X, V, \mu)$. Then there exists a measure μ_A on \mathbb{R} such that the Liouville measure writes*

$$(10) \quad \mu_{PQ} = \mu \otimes \mu_A.$$

We shall call μ_A the *spectral measure* of A in the sequel (not to be confused with the *projection* valued measure $Q = \Pi_A$).

Proof. We first observe that conjugating any A by a gauge transform preserves μ_{PQ} . Indeed if $A^U = U^{-1}AU$ then $\Pi_{A^U}(I) = U^{-1}\Pi_A(I)U$ and

$$\begin{aligned} \mu_{PQ^U}(\Omega \times I) &= \tau(\chi_\Omega \Pi_{A^U}(I)) = \tau(U\chi_\Omega U^{-1}\Pi_A(I)) \\ &= \tau(\chi_\Omega \Pi_A(I)) = \mu_{PQ}(\Omega \times I). \end{aligned}$$

Then, if $g^{-1}Ag = A^{U(g)}$, one has

$$\begin{aligned}\mu_{PQ}(g.\Omega \times I) &= \tau(\chi_{g\Omega}\Pi_A(I)) = \tau(g\chi_{\Omega}g^{-1}\Pi_A(I)) \\ &= \tau(\chi_{\Omega}g^{-1}\Pi_A(I)g) = \tau(\chi_{\Omega}\Pi_{A^v}(I)) \\ &= \mu_{PQ}(\Omega \times I).\end{aligned}$$

Hence, given I , $\Omega \mapsto \mu_{PQ}(\Omega \times I)$ is an invariant measure on X thus proportional to μ by uniqueness, i.e. $\mu_{PQ}(\Omega \times I) = \mu(\Omega)\mu_A(I)$ as needed. \square

More concretely, $\mu_A(I)$ expresses using the Schwarz kernel $K_{\Pi_A(I)}$ of $\Pi_A(I)$. Indeed

$$\begin{aligned}\mu_{PQ}(\Omega \times I) &= \tau_{\mathcal{H}}(\chi_{\Omega}\Pi_A(I)) = \|\chi_{\Omega}\Pi_A(I)\|_{HS(\mathcal{H})}^2 \\ &= \int_{\Omega} \int_X \|K_{\Pi_A(I)}(x, y)\|_{HS(V)}^2 d\mu(y) d\mu(x),\end{aligned}$$

so that one has for μ -almost every x

$$(11) \quad \mu_A(I) = \int_X \|K_{\Pi_A(I)}(x, y)\|_{HS(V)}^2 d\mu(y).$$

This last formula, together with Plancherel formula on X , helps in computing examples.

4.2. The compact case. We first consider the case of homogeneous spaces $X = G/H$ for compact groups G . Then a scalar invariant self-adjoint operator A on X induces a spectral splitting

$$(12) \quad L^2(X) = \bigoplus_{\text{Sp } A} E_{\lambda}.$$

The spectral measure μ_A is supported on $\text{Sp } A$ and

$$\begin{aligned}\mu_{PQ}(X \times \Pi_{\lambda}) &= \tau(\Pi_{\lambda}) = \dim E_{\lambda} \\ &= \mu(X)\mu_A(\lambda)\end{aligned}$$

hence, $\mu_A(\lambda) = \dim E_{\lambda}/\mu(X)$. Therefore (1) writes here

$$(13) \quad - \int_X \varrho(x) \ln \varrho(x) d\mu(x) - \sum_{\text{Sp } A} \ln\left(\frac{\tau(\rho\Pi_{\lambda})}{\dim E_{\lambda}}\right) \tau(\rho\Pi_{\lambda}) \geq S(\rho),$$

if μ is normalized such that $\mu(X) = 1$.

A first remark here is that the spectral entropy sum, say $S_A(\rho)$, actually does not depend on the values in $\text{Sp } A$. It stays the same for any other operator $\varphi(A)$ with φ injective on $\text{Sp } A$. This sum only depends on the repartition on the state in the splitting (12).

This stays true in general for the change of self-adjoint operator A into $\varphi(A)$ on any space X . Indeed one has $\Pi_{\varphi(A)}(I) = \Pi_A(\varphi^{-1}(I))$ and the induced projection valued measure becomes $Q_{\varphi(A)} = Q_A \circ \varphi^{-1}$. One sees easily (or by Proposition 4.3 below) that this amounts in a change of variables $\lambda \mapsto \varphi^{-1}(\lambda)$ in the spectral entropy integrals over $Y = \mathbb{R}$ in (1), and thus $S_A(\rho) = S_{\varphi(A)}(\rho)$. Hence this notion does not depend on the actual values of energy levels, not even their order, but deals with the repartition of the state in the direct integral splitting of $H = \int^{\oplus} dE_{\lambda}$.

We also observe that (13) is sharp on purely spectral states such that $\rho = \varphi(A)$ with $\tau(\rho) = 1$. Indeed, these translation-invariant states have constant density $\varrho(x) = 1$ since $\mu(X) = 1 = \tau(\rho)$. Thus, the spatial entropy term vanishes in (13), while

$$S_A(\rho) = S(\rho) = - \sum_{\text{Sp } A} (\ln \varphi(\lambda)) \varphi(\lambda) \dim E_\lambda.$$

Finally, we note that $S_Q(\rho)$ decreases with the refinement of the splitting of $L^2(X)$ and is ultimately bounded from below by the finest possible one, coming from the decomposition of the unitary representation π of G in $L^2(X)$ into finite dimensional irreducible representations:

$$(14) \quad L^2(X) = \bigoplus_{\sigma \subset \pi} H_\sigma.$$

This comes from the splitting of each invariant space E_λ into such sums of H_σ (explaining the discreteness of $\text{Sp } A$), and the following general monotony property.

Proposition 4.3. *Let P be a POVM from (X, μ) to H . Suppose that P' is a POVM on (X', μ') induced from P by a measurable map $\varphi : X \rightarrow X'$, i.e. $P' = P(\varphi^{-1})$ and $\mu' = \mu(\varphi^{-1})$. Suppose moreover that X' is σ -finite. Then*

$$(15) \quad S_{P'}(\rho) = - \int_{X'} \ln\left(\frac{d\nu_{P'}^\rho}{d\mu'}\right) d\nu_{P'}^\rho \geq S_P(\rho) = - \int_X \ln\left(\frac{d\nu_P^\rho}{d\mu}\right) d\nu_P^\rho,$$

provided these integrals are defined.

Indeed, Jensen inequality applied to the conditional expectation $E\left(\frac{d\nu_P^\rho}{d\mu} \mid \varphi\right) = \frac{d\nu_{P'}^\rho}{d\mu'}$ gives

$$E\left(-\frac{d\nu_P^\rho}{d\mu} \ln\left(\frac{d\nu_P^\rho}{d\mu}\right) \mid \varphi\right) \leq -\frac{d\nu_{P'}^\rho}{d\mu'} \ln\left(\frac{d\nu_{P'}^\rho}{d\mu'}\right),$$

yielding (15) by integration.

Eventually, we illustrate this discussion on the sphere $S^{n-1} = \text{SO}(n)/\text{SO}(n-1)$ and the Laplacian Δ_S . Here $\text{Sp}(\Delta_S) = \{d(d+n-2) \mid d \geq 0\}$ and $E_d(\Delta_S)$ consists in the harmonic polynomials of degree d , with $\dim E_d = \binom{d+n-1}{n-1} - \binom{d+n-3}{n-1}$, see e.g., [8, 16]. Moreover, the representation of $\text{SO}(n)$ on E_d is irreducible, so that the spectral and Weyl decompositions (12) and (14) coincide here. That means that the measurement of ρ by Δ_S (its energy distribution), actually provides the best (lowest) spectral entropic term (among invariant operators) in (13).

4.3. Non-compact examples. We compute the spectral measure of some operators in non-compact situations. Let A be a self-adjoint differential operator with constant coefficients on $X = \mathbb{R}^n$ and σ_A its polynomial symbol. The Fourier transform of the spectral projection $\Pi_A(I)$ is the multiplication by $\chi_{\sigma_A^{-1}(I)}$ on \mathbb{R}_ξ^n . Then by (11) and Plancherel formula the spectral measure of A is

$$(16) \quad \mu_A(I) = \|\chi_{\sigma_A^{-1}(I)}\|_{L_\xi^2}^2 = \text{vol}(\sigma_A^{-1}(I)).$$

Using the coarea formula, its density with respect to Lebesgue measure on \mathbb{R} is

$$\frac{d\mu_A}{d\lambda} = \int_{\sigma_A^{-1}(\lambda)} \frac{dH_\sigma}{|\nabla\sigma|}$$

for the hypersurface measure H_σ on $\sigma^{-1}(\lambda)$. Similarly the measure of a state $\rho = \sum_i p_i \Pi_{f_i}$ in $L^2(\mathbb{R}^n)$ relatively to the spectral resolution of A reads

$$\nu_A^\rho(I) = \tau(\rho \Pi_A(I)) = \int_{\sigma_A^{-1}(I)} \widehat{\varrho}(\xi) d\xi$$

where $\widehat{\varrho}(\xi) = \sum_i p_i |\widehat{f_i}(\xi)|^2$ as in Section 2.2. Hence

$$\frac{d\nu_A^\rho}{d\mu_A}(\lambda) = \int_{\sigma_A^{-1}(\lambda)} \widehat{\varrho}(\xi) \frac{dH_\sigma}{|\nabla\sigma|} / \int_{\sigma_A^{-1}(\lambda)} \frac{dH_\sigma}{|\nabla\sigma|},$$

expliciting the spectral entropy term in (1):

$$(17) \quad S_A(\rho) = - \int_{\mathbb{R}} \ln\left(\frac{d\nu_A^\rho}{d\mu_A}\right) d\nu_A^\rho.$$

For instance in the case of the Laplacian Δ . Then $\sigma_\Delta(\xi) = \|2\pi\xi\|^2$ with our convention in Section 2.2, and the spectral measure is $\nu_\Delta([0, \lambda]) = \text{vol}(B(\sqrt{\lambda}/2\pi))$. Then, H_σ is the usual measure m on the spheres S_R , and one finds that

$$S_\Delta(\rho) = - \int_0^\infty \ln\left(\int_{S_R} \widehat{\varrho}(\xi) \frac{dm(\xi)}{m(S_R)}\right) \left(\int_{S_R} \widehat{\varrho}(\xi) dm(\xi)\right) dR$$

We note that for any invariant A , the map Π_A is actually induced (up to Fourier transform) from the momentum map $Q(\Omega) = \chi_\Omega \times$ on \mathbb{R}_ξ^n by the symbol $\sigma_A : \mathbb{R}_\xi^n \rightarrow \mathbb{R}$; i.e. $\Pi_A = Q(\sigma_A^{-1})$ and $\mu_A = \mu_Q(\sigma_A^{-1})$ by (16). Hence, Proposition 4.3 applies and gives the lower bound

$$S_A(\rho) \geq S_Q(\rho) = - \int_{\mathbb{R}^n} \widehat{\varrho}(\xi) \ln \widehat{\varrho}(\xi) d\xi,$$

arising in the Fourier uncertainty principle (3). Yet, contrarily to the Laplacian on S^{n-1} , one has in general $S_A(\rho) > S_Q(\rho)$, as due to Jensen inequality. Indeed, the knowledge of $S_A(\rho)$ requires less information on ρ than $S_Q(\rho)$, since ν_A^ρ only depends on the mean values of $\widehat{\varrho}$ on the level sets of σ_A .

We mention that the spectral measure μ_A can (in principle) be computed on other non-compact homogeneous spaces $X = G/H$, as long an explicit Plancherel formula is available. This is indeed the case for symmetric spaces G/K with G connected semi-simple Lie group with finite center and K its maximal compact subgroup; see e.g., [8]. For instance, computations of the spectral measure of the Laplacian on symmetric spaces may be found in [16].

We close this series with the example (9) of the Hamiltonian H_B in $L^2(\mathbb{R}^2)$. It is translation gauge-invariant and Proposition 4.1 applies. Its spectrum is discrete and consists in the (Landau levels) $\lambda_n = (2n+1)B$, $n \in \mathbb{N}$, with each eigenspace E_n of constant density $\mu_{H_B}(\lambda_n) = \frac{B}{2\pi}$; see e.g., [6]. Hence (1) reads here as

$$(18) \quad - \int_{\mathbb{R}^2} \varrho(x) \ln \varrho(x) dx - \sum_{n \geq 0} \tau(\rho \Pi_n) \ln \tau(\rho \Pi_n) \geq S(\rho) - \ln(B/2\pi).$$

One sees that the uncertainty constrain relaxes for a given state when B increases. Indeed, the state may concentrate on fewer Landau levels, whose density increases with B . Note also

that the spectral entropy term in the left side is always positive here and vanishes iff ρ is contained in a single level, in which case $-\int_{\mathbb{R}^2} \varrho(x) \ln \varrho(x) dx \geq S(\rho) - \ln(B/2\pi)$.

4.4. Log-Sobolev inequalities. As emphasized above, the spectral entropy associated to an operator does not depend on the actual values of energy levels, but only on the induced spectral splitting. It turns out however that one can bound it using a single estimation of the mean energy of the state $\mathcal{E}_A(\rho) = \tau(A\rho)$; yielding a log-Sobolev (entropy-energy) inequality.

Consider again a self-adjoint translation gauge-invariant operator A on an homogeneous (not necessarily compact) space $X = G/H$. For $t \geq 0$, let

$$(19) \quad L_A(t) = \int_{\mathbb{R}} e^{-t\lambda} d\mu_A(\lambda)$$

denotes the Laplace transform of the spectral measure μ_A . This is actually also the constant ratio $\tau(e^{-tA}\chi_\Omega)/\mu(\Omega)$, or heat decay of e^{-tA} . The following Gibbs inequality holds.

Proposition 4.4. *Suppose that the state ρ satisfies $\mathcal{E}_A^+(\rho) = \tau(\max(A, 0)\rho) < \infty$ and that $L_A(t) < \infty$. Then*

$$\int_{\mathbb{R}} \ln^- \left(\frac{d\nu_A^\rho}{d\mu_A} \right) d\mu_A(\rho) < \infty$$

and it holds that

$$(20) \quad S_A(\rho) \leq t\mathcal{E}_A(\rho) + \ln L_A(t).$$

Proof. From $\mathcal{E}_A(\rho) = \tau(\rho A) = \int_{\mathbb{R}} \lambda \tau(\rho d\Pi_A) = \int_{\mathbb{R}} \lambda d\nu_A^\rho$, one has

$$\begin{aligned} - \int_{\mathbb{R}} \left(\frac{d\nu_A^\rho}{e^{-t\lambda} d\mu_A} \right) \ln \left(\frac{d\nu_A^\rho}{e^{-t\lambda} d\mu_A} \right) e^{-t\lambda} d\mu_A &= S_A(\rho) - t \int_{\mathbb{R}} \lambda d\nu_A^\rho \\ &= S_A(\rho) - t\mathcal{E}_A(\rho) \\ &\leq - \left(\int_{\mathbb{R}} d\nu_A^\rho \right) \ln \left(\int_{\mathbb{R}} \frac{d\nu_A^\rho}{L_A(t)} \right) = \ln L_A(t), \end{aligned}$$

by Jensen inequality and $\int_{\mathbb{R}} d\nu_A^\rho = \nu_A^\rho(\mathbb{R}) = \tau(\rho) = 1$. Note that

$$\ln^- \left(\frac{d\nu_A^\rho}{d\mu_A} \right) - t\lambda^+ \leq (-\ln)^+ \left(\frac{d\nu_A^\rho}{e^{-t\lambda} d\mu_A} \right),$$

whose corresponding integral is finite by Jensen if $L_A(t) < \infty$. This ensures that $S_A(\rho)$ exists. \square

Remark 4.5. Note moreover that equality holds iff $d\nu_A^\rho = C e^{-t\lambda} d\mu_A = e^{-t\lambda} d\mu_A / L_A(t)$, i.e. when the spectral distribution of the state is a Gibbs measure, i.e., with exponential law with respect to energy. Such states always exist in the homogeneous case: take for instance

$$\rho_t = e^{-\frac{t}{2}A} \chi_\Omega e^{-\frac{t}{2}A} / L_A(t) \mu(\Omega).$$

As a corollary, Theorem 1.1 and (20) yield the following log-Sobolev inequality, in a parametric form.

Corollary 4.6. *Under the previous assumptions, it holds that*

$$(21) \quad - \int_X \varrho(x) \ln \varrho(x) d\mu(x) + t\mathcal{E}_A(\rho) + \ln L_A(t) \geq S(\rho).$$

When ρ is a pure state Π_f and A is the Laplacian (or e^{-tA} is Markovian) this is a well-known inequality, see [4]. It extends here for mixed states on a large class of operators: translation gauge-invariant ones on homogeneous manifolds.

One can also optimize (21) in t . Notice that $\ln L_A(t)$ is a convex function, since $L_A(t)$ is the integral of log-convex functions. We consider then the Legendre transform (Young conjugate) of $-\ln L_A(t)$, namely

$$(\ln L_A)^*(\lambda) = \inf_{t \geq 0} (t\lambda + \ln L_A(t)).$$

Then (21) reads

$$(22) \quad - \int_X \varrho(x) \ln \varrho(x) d\mu(x) + (\ln L_A)^*(\mathcal{E}_A(\rho)) \geq S(\rho).$$

Note that by Remark 4.5, (22) is also equivalent to the previous purely entropic inequality

$$- \int_X \varrho(x) \ln \varrho(x) d\mu(x) + S_A(\rho) \geq S(\rho)$$

on states at statistical equilibrium with respect to the energy A , i.e. with spectral measure $d\nu_A^\rho = C e^{-t\lambda} d\mu_A$ for some t , but is weaker on more general states. Moreover, following the discussion in Section 4.2, the inequality (22) is sharp (on mixed states) on homogeneous spaces $X = G/H$ with G compact, if e^{-tA} is trace class for some t .

Examples. In the case of the Laplacian on \mathbb{R}^n , one has $L_\Delta(t) = (4\pi t)^{-n/2}$; a well-known heat decay (that also follows from (19) with $\nu_\Delta([0, \lambda])$ given in Section 4.3). Then (22) writes

$$(23) \quad - \int_X \varrho(x) \ln \varrho(x) d\mu(x) + \frac{n}{2} \ln \left(\frac{e \mathcal{E}_\Delta(\rho)}{2\pi n} \right) \geq S(\rho).$$

This inequality is due to Dolbeault–Felmer–Loss–Paturel in [5] and asymptotically sharp on normalized harmonic oscillator heat $\rho_t = e^{-t(\Delta + \|x\|^2)} / \tau(e^{-t(\Delta + \|x\|^2)})$ for $t \searrow 0$.

We can compare this on \mathbb{R}^2 with the Hamiltonian H_B in (9). From discussion in Section 4.3, one has here

$$L_B(t) = \sum_{n \geq 0} \frac{B}{2\pi} e^{-(2n+1)Bt} = \frac{B}{4\pi \sinh(Bt)}.$$

Straightforward calculations then yield

$$(24) \quad (\ln L_B)^*(\mathcal{E}_{H_B}(\rho)) = \ln \left(\frac{\mathcal{E}_{H_B}(\rho) + B}{4\pi} \right) + n(\rho) \ln \left(1 + \frac{1}{n(\rho)} \right),$$

where

$$n(\rho) = \sum_n n \tau(\Pi_n \rho)$$

is the mean Landau level of ρ (with respect to the ground state). Note that

$$\mathcal{E}_{H_B}(\rho) = (2n(\rho) + 1)B.$$

When $B \rightarrow 0$, one has $H_B \rightarrow \Delta$ and $n(\rho) \rightarrow +\infty$ for a given state. Then

$$(\ln L_B)^*(\mathcal{E}_{H_B}(\rho)) \rightarrow \ln \left(\frac{e \mathcal{E}_\Delta(\rho)}{4\pi} \right)$$

and one recovers (23) on \mathbb{R}^2 from (22).

Further comments. The log-Sobolev inequality (22) obtained here may be compared to an other similar result proved in [14]. There, Corollary 1.6 states that

$$(25) \quad - \int_X \varrho(x) \ln \varrho(x) d\mu(x) + (\ln F_A)^c(\mathcal{E}_A(\rho)) \geq -3 - \ln \|\rho\|_{L^2 \rightarrow L^2}$$

where

$$F_A(\lambda) = \sup_{\Omega} \left(\frac{\tau(\chi_{\Omega} \Pi_A([\cdot - \infty, \lambda]))}{\mu(\Omega)} \right)$$

and $(\ln F_A)^c$ denotes the concave hull of $\ln F_A$. This statement holds on general σ -finite spaces without invariance and homogeneity assumptions.

In a homogenous situation, one has $F_A(\lambda) = \mu_A([\cdot - \infty, \lambda])$. It turns out that

$$(26) \quad (\ln L_A)^* \geq (\ln F_A)^c$$

so that the left side of (22) is larger than the one in (25). Indeed,

$$\begin{aligned} L_A(t) &= \int_{\mathbb{R}} e^{-tu} d\mu_A(u) \\ &\geq \int_{-\infty}^{\lambda} e^{-t\lambda} d\mu_A(u) = e^{-t\lambda} F_A(\lambda) \end{aligned}$$

and thus $\ln F_A(\lambda) \leq t\lambda + \ln L_A(t)$, giving (26) by concavity of $(\ln L_A)^*$. On the other hand, the right side of (25) is smaller than in (22) since on unit trace states

$$- \ln \|\rho\|_{L^2 \rightarrow L^2} \leq S(\rho) = -\tau(\rho \ln \rho),$$

with equality on uniformly distributed states, i.e., normalized projections on finite dimensional subspaces of H .

In conclusion, the two log-Sobolev inequalities we discuss here are not equivalent, even on homogeneous spaces; see also [14, §4.2] for a more precise comparison in the case of the Laplacian on \mathbb{R}^n . Yet, we have seen that the version developed here is sharp in some classical cases, including translation gauge-invariant operators on compact homogeneous spaces. Moreover it comes from the stronger entropic uncertainty principle stated in Theorem 1.1.

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