SPECTRAL DENSITY AND SOBOLEV INEQUALITIES FOR PURE AND MIXED STATES

MICHEL RUMIN

Abstract. We prove some general Sobolev–type and related inequalities for positive operators A of given ultracontractive spectral decay $F(\lambda) = \|\chi_A(]0,\lambda])\|_{1,\infty}$, without assuming e^{-tA} is sub-Markovian. These inequalities hold on functions, or pure states, as usual, but also on mixed states, or density operators in the quantum-mechanical sense. As an illustration, one can relate the Novikov–Shubin numbers of coverings of finite simplical complexes to the vanishing of the torsion of the $\ell^{p,2}$-cohomology for some $p \geq 2$.

1. Introduction and main results

Let A be a strictly positive self-adjoint operator on a σ-finite measure space (X, μ). Suppose moreover that the semigroup e^{-tA} is equicontinuous on $L^1(X)$ (sub-Markovian for instance). Then, according to Varopoulos [32], a polynomial heat decay

$$\|e^{-tA}\|_{1,\infty} \leq Ct^{-\alpha/2} \quad \text{with} \quad \alpha > 2\,,$$

is equivalent to the Sobolev inequality

$$\|f\|_p \leq C'\|A^{1/2}f\|_2 \quad \text{for} \quad 1/p = 1/2 - 1/\alpha. \tag{1}$$

This result applies in particular in the case A is the Laplacian acting *on scalar functions* of a complete manifold, either in the smooth or discrete graph setting.

1.1. **General Sobolev–Orlicz inequalities.** The first purpose of this paper is to present short proofs of general Sobolev–Orlicz inequalities that hold for positive self-adjoint operators, without equicontinuity or polynomial decay assumption, knowing either their heat decay, as above, or the "ultracontractive spectral decay" $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$ of their spectral projectors $\Pi_\lambda = \chi_A(]0,\lambda])$ on E_λ. As will be seen, the interest for this former $F(\lambda)$ mostly comes from geometric considerations. For instance if A is a scalar invariant operator over an unimodular group Γ, then $F(\lambda)$ coincides with von Neumann's Γ-dimension of E_λ, and thus F represents the non-zero spectral density function of A, see Proposition 1.4. In the general setting the spectral decay F remains a right continuous increasing function as follows from the identity

$$\|P^*P\|_{1,\infty} = \|P\|_{1,2}^2 = \sup_{\|f\|_1,\|g\|_1\leq 1} |\langle Pf, Pg\rangle|. \tag{2}$$

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We first state the Sobolev–Orlicz inequality we shall prove on a single function, or "pure state", as usual. The technique is inspired by the proof of the $L^2 - L^p$ classical Sobolev inequality in $\mathbb{R}^n$ given by Chemin and Xu in [7]. In the sequel, if φ is a monotonic function, φ^{-1} will denote its right continuous inverse. All measure spaces (X, μ) will be assumed σ-finite.

Theorem 1.1. *Let A be a positive self-adjoint operator on (X, μ) with ultracontractive spectral projections $\Pi_\lambda = \chi_A(]0, \lambda])$, i.e. $F(\lambda) = \|\Pi_\lambda\|_{1,\infty} < +\infty$.*

Suppose moreover that the Stieljes integral $G(\lambda) = \displaystyle\int_0^\lambda \frac{dF(u)}{u}$ converges. Then any non zero $f \in L^2(X) \cap (\ker A)^\perp$ of energy $\mathcal{E}(f) = \langle Af, f\rangle_2$ satisfies

$$\int_X H\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big) d\mu \leq 1\,, \tag{3}$$

where $H(y) = y\, G^{-1}(y)$.

The heat version of this result has a similar statement (and proof).

Theorem 1.2. *Let A be a positive self-adjoint operator on (X, μ) such that $L(t) = \|e^{-tA}\Pi_V\|_{1,\infty}$ is finite, with $V = L^2(X) \cap (\ker A)^\perp$.*

Suppose moreover that $M(t) = \displaystyle\int_t^{+\infty} L(u) du < +\infty$. Then any non zero $f \in V$ of finite energy satisfies

$$\int_X N\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big) d\mu \leq \ln 2\,, \tag{4}$$

where $N(y) = y/M^{-1}(y)$

Both results give (effective) Sobolev inequalities (1) in the polynomial decay case for F or L. At first, we will see in (26) that the transform from F to G is increasing in general, while G to H is decreasing. Therefore, if $F(\lambda) \leq C\lambda^\alpha$ for $\alpha > 1$, then $G(\lambda) \leq C_1 \lambda^{\alpha-1}$ with $C_1 = \frac{C\alpha}{\alpha-1}$, and $H(y) \geq C_1^{\frac{1}{1-\alpha}} y^{\frac{\alpha}{\alpha-1}}$. Hence (3) reads $\|f\|_{2\alpha/(\alpha-1)} \leq 2C_1^{\frac{1}{2\alpha}} \|A^{1/2} f\|_2$.

Other related inequalities: generalised Moser, Nash and Faber–Krahn inequalities are stated in Theorem 2.2. Also an application of Theorem 1.1 is given below to the study of ℓ^2-cohomology of coverings of simplical complexes, but we will first consider another issue.

1.2. **From pure to mixed states.** Namely we note that, from the quantum-mechanical viewpoint, Theorems 1.1 and 1.2 are inequalities dealing with the density $|f(x)|^2$ of a single particle, or pure state. Since we start from the knowledge of a strong "collective data", related to the vector space E_λ, it is tentative to look for a collective version of (3); that would handle many functions simultaneously. A classical approach in statistical quantum mechanics consists in replacing the orthogonal projection Π_f on f, by a mixed state, that is a positive linear combination of such projections, or more generally by a positive operator ρ of finite trace, see [33, Chap. IV] or short presentations in [19, §23] or [35].

When dealing with a pure state, $|f(x)|^2$ interprets as the diagonal value $K_\rho(x, x)$ of the kernel of $\rho = \Pi_f$. We need then to extend this notion to general density operators ρ. Moreover it is also useful in geometry to consider operators acting on vector valued, or even Hilbert valued functions. For instance, one may work on differential forms of higher degrees. One can also consider Γ-coverings $\widetilde{M}$ of compact manifolds M; in which case one may set $X = \Gamma$ and use $L^2(\widetilde{M}) = L^2(\Gamma, H)$ with $H = L^2(\mathcal{F})$ for a fundamental domain $\mathcal{F}$. To handle such cases, we will rely on the following approach.

Definition 1.3. Let (X, μ) be a σ-finite measure space, H a separable Hilbert space and P a bounded positive operator acting on $\mathcal{H} = L^2(X, H) = L^2(X, \mu) \otimes H$. Then, given a measurable $\Omega \subset X$, the trace

$$\nu_P(\Omega) = \tau(\chi_\Omega P \chi_\Omega) = \tau(P^{1/2}\chi_\Omega P^{1/2}) \tag{5}$$

defines an absolutely continuous measure on X with respect to μ. Its Radon–Nikodym derivative $D\nu_P = \dfrac{d\nu_P}{d\mu}$ will be called the *density function* of P, and

$$D(P) = \operatorname{supess} D\nu_P = \sup_\Omega \frac{\nu_P(\Omega)}{\mu(\Omega)}$$

the *density* of P.

The following properties summarize the relationships between this density, ultracontractivity and von Neumann Γ-trace. We refer for instance to [27, §2] for an introduction on this last subject.

Proposition 1.4. • *With P and $\mathcal{H}$ as above, one has for any Hilbert basis (e_i) of $\mathcal{H}$*

$$\nu_P(\Omega) = \int_\Omega \sum_i \|(P^{1/2}e_i)(x)\|_H^2 d\mu(x) \quad \text{hence} \quad D\nu_P(x) = \sum_i \|(P^{1/2}e_i)(x)\|_H^2 \text{ a.e.} \tag{6}$$

without finiteness assumption.

• *If H is finite dimensional, a bounded positive P is ultracontractive if and only if it has a bounded density and*

$$\|P\|_{1,\infty} \leq D(P) \leq (\dim H)\|P\|_{1,\infty}\,. \tag{7}$$

• *When X is a locally compact group Γ with its Haar measure, and P a translation invariant operator, then the density function of P is a constant number, so that $\nu_P(\Omega) = D(P)\mu(\Omega)$. Moreover, it coincides with von Neumann Γ-trace of P when Γ is discrete. Namely if K_P denotes the kernel of P, one has in this case*

$$D(P) = \tau_H(K_P(e,e)) = \tau_\Gamma(P)\,. \tag{8}$$

In this setting, the mixed state version of Theorem 1.1 is the following.

Theorem 1.5. *Let (X, μ) be a measure space, H an Hilbert space, and A a positive self-adjoint operator on $\mathcal{H} = L^2(X, H)$.*

Suppose that the spectral projections $\Pi_\lambda = \chi_A(]0,\lambda])$ have finite density $F(\lambda) = D(\Pi_\lambda)$, and that $G(\lambda) = \int_0^\lambda \frac{dF(u)}{u}$ converges. Let ρ be a positive operator such that $\rho = 0$ on $\ker A$. Then

$$\int_X G^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}}\Big) d\nu_\rho \leq 4\mathcal{E}(\rho)\,, \tag{9}$$

where

$$\mathcal{E}(\rho) = \tau(\rho^{1/2}A\rho^{1/2}) = \tau(A^{1/2}\rho A^{1/2}) \ (= \tau(A\rho) \text{ if finite})\,,$$

and $\|\rho^{1/2}A\rho^{1/2}\|_{2,2}$ *is the* $L^2 - L^2$ *norm of* $\rho^{1/2}A\rho^{1/2}$.

The heat version of the Sobolev inequality Theorem 1.2 has also a mixed state or ρ-version, replacing F above by $L(t) = D(e^{-tA})$, and G by $\widetilde{M}$ where $\widetilde{M}(1/t) = \int_t^{+\infty} L(s)ds$.

To illustrate Theorem 1.5, suppose again that F has a polynomial growth

$$F(\lambda) \leq C\lambda^\alpha \quad \text{for some} \quad \alpha > 1\,. \tag{10}$$

As examples of mixed states, we take ρ to be the projection onto a N-dimensional linear space V of functions such that $\mathcal{E}(f) = \langle Af, f\rangle_2 \leq \lambda \|f\|_2^2$ for all $f \in V$. Then (6) yields

$$D\nu_\rho(x) = \sum_{i=1}^N \|f_i(x)\|_H^2\,,$$

for any orthonormal basis (f_i) of V. Since as previously $G(\lambda) \leq C_1 \lambda^{\alpha-1}$ with $C_1 = \frac{C\alpha}{\alpha-1}$, the ρ-Sobolev inequality (9) provides

$$\int_X \big(\sum_{i=1}^N \|f_i(x)\|_H^2\big)^{\frac{\alpha}{\alpha-1}} d\mu \leq C_2 \lambda^{\frac{1}{\alpha-1}} \sum_{i=1}^N \mathcal{E}(f_i) \leq C_2 N \lambda^{\frac{\alpha}{\alpha-1}}\,, \tag{11}$$

with $C_2 = 4^{\frac{\alpha}{\alpha-1}} C_1^{\frac{1}{\alpha-1}}$. In comparison, the sum of Sobolev inequalities only gives

$$\int_X \big(\sum_{i=1}^N \|f_i(x)\|_H^2\big)^{\frac{\alpha}{\alpha-1}} d\mu \leq C_3 \Big(\sum_{i=1}^N \mathcal{E}(f_i)\Big)^{\frac{\alpha}{\alpha-1}} \leq C_3 N^{\frac{\alpha}{\alpha-1}} \lambda^{\frac{\alpha}{\alpha-1}}\,,$$

but holds without the assumptions done in (11) that the functions (f_i) are orthonormal and that $\mathcal{E}(f) \leq \lambda \|f\|_2^2$ on their span V.

Another feature of the mixed state inequalities (11) is that they give back some control of the spectral distribution of the Dirichlet spectrum of $\mathcal{E}$ on any domain Ω of finite measure. Namely, if the states f_i are supported in Ω, then by Jensen, one finds that

$$(N/\mu(\Omega))^{\alpha/\alpha-1} = \Big(\int_\Omega \sum_{i=1}^N \|f_i(x)\|_H^2 d\mu/\mu(\Omega)\Big)^{\alpha/\alpha-1} \leq C_2 (N/\mu(\Omega)) \lambda^{\frac{\alpha}{\alpha-1}}\,,$$

yielding

$$\frac{\dim V}{\mu(\Omega)} \leq \frac{4^\alpha \alpha}{\alpha-1} C\lambda^\alpha = C_3 \lambda^\alpha, \tag{12}$$

for any V supported in Ω and such that $\mathcal{E}(f) \leq \lambda \|f\|_2^2$ on V.

In the case $\dim V = 1$, i.e. the pure state case, this interprets as a Faber–Krahn inequality, on the lower bound for the Dirichlet spectrum of $\mathcal{E}$ on Ω. While using mixed states, one gets actually the control of the *whole* geometrical spectral repartition function

$$F_\Omega^{\dim}(\lambda) = \sup_{V \subset\subset \Omega} \{\dim V \mid \mathcal{E} \leq \lambda \text{ on } V\} \leq \mu(\Omega) C_3 \lambda^\alpha\,. \tag{13}$$

Note that if X itself has a finite measure, this control is coherent with the starting hypothesis (10) on the spectral density F of E_λ, since in finite measure

$$F_X^{\dim} = \dim E_\lambda \leq \mu(X) F\,, \tag{14}$$

as follows from Definition 1.3 and

$$\dim V = \tau(\Pi_V) = \nu_{\Pi_V}(X) \leq \mu(X) D(\Pi_V)\,. \tag{15}$$

We note also that for invariant operators and spaces on finite groups, (14) and (15) are equalities by Proposition 1.4. Hence the "ρ-Sobolev" inequalities (4) capture back the bound on the spectral density F, at least in this simple setting.

In the case of the Laplacian in $\mathbb{R}^n$, or more general Schrödinger operators, spectral bounds like (13), have been proved independently by Cwikel, Lieb and Rosenbłjum, see [20] and [21, §12]. It will be called a CLR inequality in the sequel, like its generalization we will show.

1.3. **Moser-like inequalities for mixed states and CLR inequalities.** As done in the polynomial case above, one can show that the previous ρ-Sobolev inequalities also imply Faber–Krahn and CLR inequalities for other spectral densities, see Proposition 4.4. However this approach assumes some thinness of the spectrum, as required by the convergence of G or M. As Faber–Krahn inequalities make sense for thick spectrum, we now present another viewpoint.

Our starting point is an inequality for mixed state, that extends the Moser–Wang one stated in Theorem 2.2 on functions. We will give an integral version, as above, but also a discrete one, associated to a partition of X into $\bigsqcup_i \Omega_i$. We state the result under two close sets of hypothesis; depending whether one remove the kernel of A from the spectral density and the states, as needed in the previous approach, or not. The integral version is the following.

Theorem 1.6. *Let A be a positive self-adjoint operator on $L^2(X,H)$ and ρ be a non-zero positive operator. Suppose either*

- *$\rho = 0$ on $\ker A$ and $F_x(\lambda) = D\nu_{\Pi_{]0,\lambda]}}(x)$ denotes the density at x of $\Pi_{]0,\lambda]} = \chi_A(]0,\lambda])$,*
- *or ρ is any and $F_x(\lambda) = D\nu_{\Pi_{[0,\lambda]}}(x)$ is the density of $\Pi_{[0,\lambda]} = \chi_A([0,\lambda])$.*

Then the following inequality holds for the density operator ρ

$$\int_X F_x^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}\Big) d\nu_\rho \leq 4\mathcal{E}(\rho)\,. \tag{16}$$

As an illustration, suppose again that $F(\lambda) \leq C\lambda^\alpha$, and ρ is a projection on a N-dimensional linear space V span by orthonormal functions (f_i). Then (16) gives

$$\int_X \Big(\sum_{i=1}^N \|f_i(x)\|_H^2\Big)^{\frac{\alpha+1}{\alpha}} d\mu(x) \leq 4(4C)^{\frac{1}{\alpha}} \sum_{i=1}^N \mathcal{E}(f_i)\,.$$

For the Laplacian and Schrödinger operators on $\mathbb{R}^n$, such an inequality has been proved, and used in quantum mechanics, by Lieb–Thirring; see [22, Thm 4] and [21, §12].

The "discrete" version to Theorem 1.6 for partitions is written:

Theorem 1.7. *Let A as above and $X = \bigsqcup_I \Omega_i$ be a discrete measurable partition of X.*

Consider the Π_λ-measures of Ω_i

$$F_{\Omega_i}(\lambda) = \tau(\chi_{\Omega_i}\Pi_\lambda\chi_{\Omega_i}) = \nu_{\Pi_\lambda}(\Omega_i)\,,$$

where Π_λ denotes either $\Pi_{]0,\lambda]}$ or $\Pi_{[0,\lambda]}$. Let ρ be a non-zero positive operator, with the additional assumption that $\rho = 0$ on $\ker A$ if using $\Pi_{]0,\lambda]}$.

In this discrete setting the ρ-Moser, or Lieb-Thirring inequality, can be written as

$$\sum_i F_{\Omega_i}^{-1}\Big(\frac{\nu_\rho(\Omega_i)}{4\|\rho\|_{2,2}}\Big)\nu_\rho(\Omega_i) \leq 4\mathcal{E}(\rho)\,. \tag{17}$$

This yields the following CLR spectral bounds on set of finite measure.

Corollary 1.8. *With the same notations as above, suppose moreover that ρ is supported in a domain Ω of finite measure and has finite energy $\mathcal{E}(\rho)$, then*

$$\frac{\tau(\rho)}{4\|\rho\|_{2,2}} \leq F_\Omega\big(4\langle A\rangle_\rho\big) \leq \mu(\Omega)F\big(4\langle A\rangle_\rho\big)\,. \tag{18}$$

where $\langle A\rangle_\rho = \dfrac{\mathcal{E}(\rho)}{\tau(\rho)} = \dfrac{\tau(A\rho)}{\tau(\rho)}$ is the expectation value of A with respect to ρ.

- *In particular the whole Dirichlet spectrum of $\mathcal{E}$ on Ω is controlled for all λ by*

$$F_\Omega^{\dim}(\lambda) \leq 4\mu(\Omega)F(4\lambda)\,, \tag{19}$$

where as in (13)*, we define $F_\Omega^{\dim}(\lambda) = \sup\{\dim V \mid \mathrm{supp}(V) \subset \Omega \text{ and } \mathcal{E} \leq \lambda \text{ on } V\}$.*

Thus (19) extends uniformly, whatever F, the CLR inequality (13) obtained previously from the ρ-Sobolev inequalities. In some sense it means that the spectral density of a confined system is controlled by the spectral density of the free system, up to universal multiplicative constants in volume and energy.

We note that except for these constants 4, the formula (19) looks quite sharp in general. Indeed, as recalled above, one has $F_\Gamma^{\dim} = \mu(\Gamma)F$ when A is an invariant operator on a finite group $\Gamma = X$. Hence in general

$$F_\Omega^{\dim}(\lambda) \leq \mu(\Omega)F(\lambda) \tag{20}$$

is certainly an ideal bound for inequalities like (19).

1.4. **Log-Sobolev and entropy inequalities.** As a last illustration of the ρ-Moser inequality, we note that they lead easily to another family of Sobolev-like inequalities: namely parametric Log-Sobolev and entropy-energy inequalities, see [10, 11, 21]. They hold here without sub-Markovian assumption on the heat semigroup, and in the mixed state setting.

Theorem 1.9. • *With the assumptions of Theorem 1.6, let*

$$m(t) = \sup_{\lambda\geq 0}(\ln F(\lambda) - t\lambda)\,.$$

Then the following parametric log-Sobolev inequality holds for any $t > 0$

$$\int_X \ln\Big(\frac{D\nu_\rho}{4\|\rho\|_{2,2}}\Big)d\nu_\rho \leq m(t)\tau(\rho) + 4t\mathcal{E}(\rho)\,. \tag{21}$$

• *Let* $(\ln F)^c$ *denote the concave hull of* $\ln F$*, i.e. the smallest concave function dominating* $\ln F$*. Then the entropy-energy inequality holds*

$$\int_X \ln\Big(\frac{D\nu_\rho}{4\|\rho\|_{2,2}}\Big)\frac{d\nu_\rho}{\tau(\rho)} \le (\ln F)^c(4\langle A\rangle_\rho)\,, \tag{22}$$

for the mean energy $\langle A\rangle_\rho = \tau(\rho A)/\tau(\rho)$*.*

Of course (22) is more interesting if $\ln F$ is a concave function itself (for instance if F has polynomial growth), or more generally if $(\ln F)^c(\lambda) \le \ln F(k\lambda)$ for some constant k.

For its meaning, we note that both sides of (22) have some entropic flavour: the left side is related to the "spatial entropy" of the state ρ, as seen from X, while the right side deals with its "spectral entropy" relative to its energy. Indeed heuristically, at least on groups, one has $\ln F(\lambda) = \ln(\tau_\Gamma(E_\lambda)) = \ln \dim_\Gamma(E_\lambda)$ using the Γ-dimension of E_λ at the energy level λ.

1.5. **Sobolev inequalities and ℓ^2-cohomology.** We conclude with an application to geometric analysis of the pure state case of Sobolev inequalities in Theorem 1.1 or 1.2. As they are not restricted to Markovian operators, these results apply in the following setting. Let K be a finite simplicial complex and $X \to K = X/\Gamma$ some covering. One considers on X the complex of ℓ^2 k-cochains with the discrete coboundary

$$d_k : \ell^2 X^k \to \ell^2 X^{k+1}$$

dual to the usual boundary ∂ of simplexes, see e.g. [27, §3].

Its ℓ^2-cohomology $H_2^{k+1} = \ker d_{k+1}/\operatorname{Im} d_k$ splits in two components :

- the reduced part $\overline{H}_2^{k+1} = \ker d_{k+1}/\overline{\operatorname{Im} d_k}$, isomorphic to ℓ^2-harmonic cochains $\mathcal{H}_2^{k+1} = \ker d_{k+1} \cap \ker d_k^*$,
- and the torsion $T_2^{k+1} = \overline{\operatorname{Im} d_k}/\operatorname{Im} d_k$.

Although this torsion is not a normed space, one can study it by "measuring" the unboundedness of d_k^{-1} on $\operatorname{Im} d_k$. We will consider here two different means.

- The first one is inspired by $\ell^{p,q}$-cohomology. One enlarges the space $\ell^2 X^k$ to $\ell^p X^k$ for $p \ge 2$, and asks whether, for p large enough, one has

$$\overline{d_k(\ell^2 X^k)}^{\ell^2} \subset d_k(\ell^p X^k)\,, \tag{23}$$

This is satisfied in case the following Sobolev identity holds

$$\exists C \quad \text{such that} \quad \|\alpha\|_p \le C\|d_k\alpha\|_2 \quad \text{for all } \alpha \in (\ker d_k)^\perp \subset \ell^2\,. \tag{24}$$

The geometric interest of the rougher formulation (23) lies in its stability under the change of X into other bounded homotopy equivalent spaces, as stated in Proposition 5.2. Moreover if $\overline{H}_2^{k+1}(X)$ vanishes, then (23) is equivalent to the vanishing of the torsion of the $\ell^{p,2}$-cohomology of X, as will be seen in Section 5.

- The second approach is spectral and relies on the von Neumann Γ-trace of $\Pi_{]0,\lambda]}$, i.e to the spectral density by Proposition 1.4. Consider the Γ-invariant self-adjoint $A = d_k^* d_k$ acting on $(\ker d_k)^\perp$ and the spectral density $F_{\Gamma,k}(\lambda) = \tau_\Gamma(\Pi_{]0,\lambda]})$. This function vanishes near zero if and only if zero is isolated in the spectrum of A, which is equivalent to the vanishing of the torsion T_2^{k+1}. The asymptotic behaviour of $F_{\Gamma,k}(\lambda)$ when $\lambda \searrow 0$ has a geometric interest in general since, given Γ, it is an homotopy invariant of the quotient space K, as shown by Efremov, Gromov and Shubin in [14, 18, 17].

One can compare these two notions in the spirit of Varopoulos' result (1) on functions. In the case of polynomial decay one obtains.

Theorem 1.10. *Let K be a finite simplicial space and $X \to K = X/\Gamma$ a covering. Let $F_{\Gamma,k}(\lambda) = \dim_\Gamma E_\lambda$ denotes the spectral density function of $A = d_k^* d_k$ on $(\ker d_k)^\perp$.*

If $F_{\Gamma,k}(\lambda) \leq C\lambda^{\alpha/2}$ for some $\alpha > 2$, then the Sobolev inequality (24)*, and the inclusion* (23)*, hold for $1/p \leq 1/2 - 1/\alpha$.*

If moreover the reduced ℓ^2-cohomology $\overline{H}_2^{k+1}(X)$ vanishes, this implies the vanishing of the $\ell^{p,2}$-torsion of X, as stated in Corollary 5.4.

Other spectral decays than polynomial can be handled with Theorem 1.1, leading then to a bounded inverse of d_k from $\operatorname{Im} d_k \cap \ell^2$ into a more general Orlicz space given by H.

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2. Proof of the pure state inequalities

The first step towards Theorems 1.1 to 1.2 is to consider the ultracontractivity of the auxiliary operators $A^{-1}\Pi_\lambda$ and $A^{-1}e^{-tA}\Pi_V$.

Proposition 2.1. • *Let A, F and G be given as in Theorem 1.1. Then $A^{-1}\Pi_\lambda$ is ultracontractive with*

$$\|A^{-1}\Pi_\lambda\|_{1,\infty} \leq G(\lambda) = \int_0^\lambda \frac{dF(u)}{u}\,. \tag{25}$$

• *Let A, L and M be given as in Theorem 1.2. Then $A^{-1}e^{-tA}\Pi_V$ is ultracontractive with*

$$\|A^{-1}e^{-tA}\Pi_V\|_{1,\infty} \leq M(t) = \int_t^{+\infty} L(s)ds\,. \tag{26}$$

Proof. • The spectral calculus gives

$$A^{-1}(\Pi_\lambda - \Pi_\varepsilon) = \int_{]\varepsilon,\lambda]} u^{-1} d\Pi_u = \lambda^{-1}\Pi_\lambda - \varepsilon^{-1}\Pi_\varepsilon + \int_{]\varepsilon,\lambda]} u^{-2}\Pi_u du\,, \tag{27}$$

thus taking norms, one obtains

$$\begin{aligned}\|A^{-1}(\Pi_\lambda - \Pi_\varepsilon)\|_{1,\infty} &\leq \lambda^{-1}F(\lambda) + \varepsilon^{-1}F(\varepsilon) + \int_{]\varepsilon,\lambda]} u^{-2}F(u)du \\ &= G(\lambda) - G(\varepsilon) + 2\varepsilon^{-1}F(\varepsilon)\,.\end{aligned}$$

Now by finiteness of G, one has $\|\Pi_\varepsilon/\varepsilon\|_{1,\infty} = F(\varepsilon)/\varepsilon \leq G(\varepsilon) \to 0$ when $\varepsilon \searrow 0$, hence by (2)

$$\begin{aligned}\|A^{-1}\Pi_\lambda\|_{1,\infty} &= \|\Pi_\lambda A^{-1/2}\Pi_\lambda\|_{1,2}^2 \\ &= \lim_{\varepsilon\to 0} \|(\Pi_\lambda - \Pi_\varepsilon)A^{-1/2}\Pi_\lambda\|_{1,2}^2 \quad \text{by Beppo-Levi,} \\ &= \lim_{\varepsilon\to 0} \|A^{-1}(\Pi_\lambda - \Pi_\varepsilon)\|_{1,\infty} \leq G(\lambda)\,.\end{aligned}$$

We note that we also have

$$G(\lambda) = \lambda^{-1}F(\lambda) + \int_0^\lambda u^{-2}F(u)du\,, \tag{28}$$

which shows the useful monotonicity of the transform from F to G and H.

- The heat case (26) is clear since $A^{-1}e^{-tA}\Pi_V = \int_t^{+\infty} e^{-sA}\Pi_V ds$ by the spectral calculus.

□

The sequel of the proofs of Theorems 1.1 and 1.2 relies on a classical technique from real interpolation theory, as used for instance in the elementary proof of the $L^2 - L^p$ Sobolev inequality in $\mathbb{R}^n$ given by Chemin and Xu in [7]. This consists here in estimating each level set $\{x, |f(x)| > y\}$ by using an appropriate spectral splitting of $f \in V$ into

$$f = \chi_A(]0,\lambda])f + \chi_A(]\lambda,+\infty[)f = \Pi_\lambda f + \Pi_{>\lambda} f\,. \tag{29}$$

2.1. **Proof of Theorem 1.1.** By (2) and (25) one has $\|A^{-1/2}\Pi_\lambda\|_{2,\infty}^2 \leq G(\lambda)$, hence

$$\|\Pi_\lambda f\|_\infty^2 \leq G(\lambda)\|A^{1/2}f\|_2^2 = G(\lambda)\mathcal{E}(f)\,. \tag{30}$$

Then suppose that $|f(x)| \geq y$, with $y^2 = 4G(\lambda)\mathcal{E}(f)$. As $|\Pi_\lambda f(x)| \leq y/2$ by (30), one has necessarily by (29) that $|\Pi_{>\lambda} f(x)| \geq y/2 \geq |\Pi_\lambda f(x)|$ and finally

$$|f(x)|^2 \leq 4|\Pi_{>\lambda} f(x)|^2 \quad \text{on} \quad \big\{x \in X \mid |f(x)|^2 \geq 4G(\lambda)\mathcal{E}(f)\big\}\,. \tag{31}$$

Hence a first integration in x gives,

$$\int_{\{x\,,\,|f(x)|^2 \geq 4\mathcal{E}(f)G(\lambda)\}} |f(x)|^2 d\mu \leq 4\|\Pi_{>\lambda} f\|_2^2\,,$$

and a second integration in λ,

$$\int_X \frac{|f(x)|^2}{4\mathcal{E}(f)} G^{-1}\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big) d\mu(x) \leq \int_0^{+\infty} \frac{\|\Pi_{>\lambda} f\|_2^2}{\mathcal{E}(f)} d\lambda\,,$$

where $G^{-1}(y) = \sup\{\lambda \mid G(\lambda) \leq y\}$. At last the spectral calculus provides

$$\begin{aligned}\int_0^{+\infty} \|\Pi_{>\lambda} f\|_2^2\, d\lambda &= \int_0^{+\infty}\int_\lambda^{+\infty} \langle d\Pi_\mu f, f\rangle \\ &= \int_0^{+\infty} \mu\, \langle d\Pi_\mu f, f\rangle = \langle Af, f\rangle = \mathcal{E}(f)\,,\end{aligned}$$

proving Theorem 1.1.

2.2. **Proof of Theorem 1.2.** We follow the same lines as above. First by (2) and (26) one has for $f \in V$

$$\|e^{-tA/2}f\|_\infty^2 \leq M(t)\mathcal{E}(f)\,,$$

leading to

$$|f(x)|^2 \leq 4|(1-e^{-tA/2})f(x)|^2 \quad \text{on} \quad \big\{x \in X \mid |f(x)|^2 \geq 4M(t)\mathcal{E}(f)\big\}\,. \tag{32}$$

Then integrations in x and dt/t^2 give

$$\int_X \frac{|f(x)|^2}{4\mathcal{E}(f)} / M^{-1}\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big)\, d\mu(x) \leq \frac{1}{\mathcal{E}(f)}\int_0^{+\infty} \|(1-e^{-tA/2})f\|_2^2\, \frac{dt}{t^2}\,,$$

where now $M^{-1}(y) = \inf\{t \mid M(t) \leq y\}$ for the decreasing M. The right integral is computed by spectral calculus

$$\begin{aligned}\int_0^{+\infty} \|(1-e^{-tA/2})f\|_2^2 \,\frac{dt}{t^2} &= \int_0^{+\infty}\int_0^{+\infty} (1-e^{-t\lambda/2})^2 \langle d\Pi_\lambda f, f\rangle \,\frac{dt}{t^2} \\ &= \int_0^{+\infty} \Big(\int_0^{+\infty} \frac{(1-e^{-u})^2}{2u^2} du\Big) \lambda \langle d\Pi_\lambda f, f\rangle \\ &= I\mathcal{E}(f)\,,\end{aligned}$$

where $2I = \displaystyle\int_0^{+\infty} \frac{(1-e^{-u})^2}{u^2} du = 2\ln 2$ as seen developing $I_\varepsilon = \displaystyle\int_\varepsilon^{+\infty} \frac{(1-e^{-u})^2}{u^2} du$ when $\varepsilon \searrow 0$.

2.3. **Related inequalities.** Using the same technique as above one can also show some generalised Moser, Nash and Faber–Krahn inequalities for functions. From the heat decay to Nash and Faber–Krahn, a general approach has already been obtained by Coulhon, without markovianity assumption on the semigroup, see [8, 9] and also [16] for the Laplacian. We will state here inequalities using the spectral density F instead, and compare them to the heat kernel result after. The starting point is a Moser-like inequality that resembles to the "F-Sobolev" inequality introduced by Wang in [34] for some Schrödinger operators.

Theorem 2.2. *Let A be a positive self-adjoint operator on (X,μ). Suppose either*

- *f is a non-zero function in $V = L^2(X) \cap (\ker A)^\perp$ and $F(\lambda)$ denotes $\|\Pi_{]0,\lambda]}\|_{1,\infty}$ as above,*
- *or f is any non-zero function in $L^2(X)$, and $F(\lambda) = \|\Pi_{[0,\lambda]}\|_{1,\infty}$.*

- *Then the following generalised L^2-Moser inequality holds*

$$\int_X |f(x)|^2 F^{-1}\Big(\frac{|f(x)|^2}{4\|f\|_2^2}\Big) d\mu \leq 4\mathcal{E}(f)\,, \tag{33}$$

and also

$$\int_X |f(x)|^2 F^{-1}\Big(\frac{|f(x)|}{2\|f\|_1}\Big) d\mu \leq 4\mathcal{E}(f)\,. \tag{34}$$

- *Both inequalities imply a Nash–type inequality (with weaker constants starting from* (33)*)*

$$\|f\|_2^2 F^{-1}\Big(\frac{\|f\|_2^2}{4\|f\|_1^2}\Big) \leq 8\mathcal{E}(f)\,. \tag{35}$$

- *In particular if f is supported in a domain Ω of finite measure and has finite energy, the following Faber–Krahn inequality, or "uncertainty principle", is satisfied*

$$4\mu(\Omega) F\Big(\frac{8\mathcal{E}(f)}{\|f\|_2^2}\Big) \geq 1\,. \tag{36}$$

Proof. Here one compares levels of f to $\|f\|_2$ or $\|f\|_1$ instead of $\mathcal{E}(f)$. This does not rely on Proposition 2.1, and one can work either with $f \in (\ker A)^\perp$ and $F(\lambda) = \|\Pi_{]0,\lambda]}\|_{1,\infty}$, as before, or with a general $f \in L^2(X)$ and $F(\lambda) = \|\Pi_{[0,\lambda]}\|_{1,\infty}$. In any case, starting from (29) one gets

$$|f(x)|^2 \leq 4|\Pi_{>\lambda} f(x)|^2 \quad \text{on} \tag{37}$$
$$\left\{x \in X \mid |f(x)|^2 \geq 4F(\lambda)\|f\|_2^2\right\} \quad \text{or} \quad \left\{x \in X \mid |f(x)| \geq 2F(\lambda)\|f\|_1\right\}.$$

This yields the generalised Moser inequalities (33) and (34) by integrations as in Theorem 1.1.

Note that in the case where one works without restriction on f and $F(\lambda) = \|\Pi_{[0,\lambda]}\|_{1,+\infty}$, one has to complete the definition of F^{-1} by setting

$$F^{-1}(y) = \begin{cases} 0 & \text{if} \quad y < F(0)\,, \\ \sup\{\lambda \mid F(\lambda) \leq y\} & \text{elsewhere}\,. \end{cases} \tag{38}$$

This means that the inequalities (33) and (34) cut off small values of f in that case.

To deduce the Nash–type inequality (35), we argue as in [10, p. 97]. Observe that for all non-negative s and t one has

$$st \leq sF(s) + tF^{-1}(t)\,. \tag{39}$$

Applying to $t = \frac{|f(x)|}{2\|f\|_1}$ gives

$$s\frac{|f(x)|}{2\|f\|_1} - sF(s) \leq \frac{|f(x)|}{2\|f\|_1} F^{-1}\Big(\frac{|f(x)|}{2\|f\|_1}\Big)\,.$$

By integration against the measure $|f(x)|d\mu$ and using (34), this yields

$$s\frac{\|f\|_2^2}{2\|f\|_1} - sF(s)\|f\|_1 \leq \int_X \frac{|f(x)|^2}{2\|f\|_1} F^{-1}\Big(\frac{|f(x)|}{2\|f\|_1}\Big) d\mu \leq \frac{2\mathcal{E}(f)}{\|f\|_1}\,.$$

This provides (35) using

$$s \nearrow F^{-1}\Big(\frac{\|f\|_2^2}{4\|f\|_1^2}\Big) = \sup\Big\{s \mid F(s) \leq \frac{\|f\|_2^2}{4\|f\|_1^2}\Big\}\,.$$

One can proceed similarly starting from the L^2 Nash–type inequality (33) instead of (34). One replaces (39) by

$$st \leq s\sqrt{F(s)} + tF^{-1}(t^2)$$

with $t = \frac{|f(x)|}{2\|f\|_2}$. Integrating against $|f(x)|d\mu$ and using $s \nearrow F^{-1}(\frac{\varepsilon^2\|f\|_2^2}{\|f\|_1^2})$ yields

$$(\frac{1}{2} - \varepsilon)\|f\|_2^2 F^{-1}\Big(\frac{\varepsilon^2\|f\|_2^2}{\|f\|_1^2}\Big) \leq 2\mathcal{E}(f)\,,$$

which is similar to (35), but with weaker constants.

When is f is supported in a domain Ω of finite measure, one has $\|f\|_1^2 \leq \mu(\Omega)\|f\|_2^2$, and thus (35) implies that

$$F^{-1}\Big(\frac{1}{4\mu(\Omega)}\Big) \leq \frac{8\mathcal{E}(f)}{\|f\|_2^2}\,.$$

If $\mathcal{E}(f)$ is finite, this leads to the Faber–Krahn inequality (36) since by right continuity of F and (38), one has $F(F^{-1}(\lambda)) \geq \lambda$ when $F^{-1}(\lambda)$ is finite.

□

In comparison to the previous result, from the heat decay to Nash inequality, the following statement is proved in [8, Prop. II.2]:

Theorem 2.3. *Let A be a positive operator on (X,μ) with ultracontractive heat decay $L(t) = \|e^{-tA}\|_{1,\infty}$. Then for any $f \in D(A) \cap L^1(X,\mu)$ one has*

$$\|f\|_2^2 \theta\Big(\frac{\|f\|_2^2}{\|f\|_1^2}\Big) \leq \mathcal{E}(f)\,, \tag{40}$$

where $\theta(x) = \sup_{t>0} \frac{1}{t} \log\big(\frac{x}{L(t)}\big)$.

For invariant operators on groups Γ, this result, starting from the heat decay, follows from (35), up to multiplicative constants inside and outside θ. Indeed we will see in Proposition 4.3 that

$$\int_0^{+\infty} e^{-t\lambda} dF(\lambda) \leq nL(t)$$

for invariant operators on $L^2(\Gamma, V)$ with $\dim V = n$. In that case one has

$$\theta(x) \leq F^{-1}(nx)\,. \tag{41}$$

Namely, for all $t, y > 0$, it holds that

$$nL(t) \geq \int_0^y e^{-t\lambda} dF(\lambda) \geq e^{-ty} F(y)\,,$$

yielding (41) with $y = F^{-1}(nx)$.

We note also that, from the geometrical viewpoint, the shape of the Faber–Krahn inequality (36) looks rather natural and optimal in general, except for the multiplicative constants (that may be tightened a little bit under a convexity assumption on $yF^{-1}(y)$, see also §3.3 below). Indeed, (36) shows that, on any space X, if there exists a non-zero f of energy $\mathcal{E}(f) \leq \lambda \|f\|_2^2$ supported in a domain Ω of finite measure, then

$$4F(8\lambda) \geq 1/\mu(\Omega)\,. \tag{42}$$

Now it may happen that some spaces X of finite measure are tiled by $N = \mu(X)/\mu(\Omega)$ copies of such domains, leading by the min-max principle and (14) to

$$F(\lambda) \geq \dim E_\lambda/\mu(X) \geq N/\mu(X) = 1/\mu(\Omega)\,.$$

This gives the geometric meaning of a control like (42) (also called an L^2-isoperimetric profile) in such cases. Strikingly, its shape in general is the same as in the tiled situation.

2.4. **Remarks on the reverse problems.** In the previous proofs, it appears clearly that the proposed controls of ultracontractive norms of spectral or heat decay are much stronger than the Sobolev or Nash-Moser inequalities deduced. Indeed these inequalities are twice integrated versions, in space and frequency, of the "local" inequalities (31), (32) and (37), that come directly from the ultracontractive controls. Therefore it seems hopeless to get the converse statements in general, even on groups, although we don't have a counterexample.

However, we recall that one can get back from Sobolev or Nash inequalities to the heat decay, in the case the heat is *equicontinuous* on L^1 or L^∞, see [31, 5]. This stems from ideas of Nash [25]; see also [8, 9, 16, 32] for further references and details. This strong equicontinuity hypothesis holds for the Laplacian on scalar functions, as comes for instance from the maximum principle, but unfortunately only in a positive curvature setting for Hodge-de Rham Laplacians acting on forms of higher degrees.

As a last comment here, the problem of finding converse statements is quite different starting from the mixed state inequalities. Indeed, Proposition 4.4 and Corollary 1.8 state that the ρ-Sobolev, or better, ρ-Moser Lieb–Thirring inequalities, imply CLR bounds

$$F_\Omega^{\dim}(\lambda) \le 4\mu(\Omega)F_{\text{Moser}}(4\lambda) \tag{43}$$

for the Dirichlet spectral distribution on domains Ω. Now, according to Dodziuk and Mathai [13] it can be shown that, on amenable discrete groups $X = \Gamma$ and for invariant discrete local operators (as the discrete Laplacian on cochains), one has

$$F_{\Omega_n}^{\dim}(\lambda)/\mu(\Omega_n) \to F_\Gamma(\lambda)$$

for a Fölner exhaustion Ω_n of Γ, and where $F_\Gamma(\lambda) = \tau_\Gamma(\Pi_\lambda) = F(\lambda)$ by (8). Together with (43), this gives $F(\lambda) \le 4F_{\text{Moser}}(4\lambda)$, which means that the function ruling the mixed stated ρ-Moser inequality, or the CLR bounds, controls back the spectral density F in these cases.

3. Proof of the mixed state inequalities

3.1. **Proof of ρ-Sobolev.** The proof of the mixed state version of Sobolev inequality follows the same lines as the pure state one.

The first step is adapted to use the density of operators, as given in Definition 1.3, instead of their ultracontractive norm.

Proposition 3.1. • *Let A, F and G be given as in Theorem 1.5. Then $A^{-1}\Pi_\lambda$ has finite density and*

$$D(A^{-1}\Pi_\lambda) \le G(\lambda) = \int_0^\lambda \frac{dF(u)}{u}\,. \tag{44}$$

• *Suppose $L(t) = D(e^{-tA})$ and $M(t) = \int_s^{+\infty} L(s)ds$ are finite. Then $A^{-1}e^{-tA}\Pi_V$ has finite density with*

$$D(A^{-1}e^{-tA}) \le M(t) = \int_t^{+\infty} L(s)ds\,. \tag{45}$$

Proof. We follow the proof of Proposition 2.1. First by (27) and linearity of trace one has

$$\begin{aligned}
\nu_{A^{-1}\Pi_{]\varepsilon,\lambda]}}(\Omega) &= \tau(\chi_\Omega A^{-1}\Pi_{]\varepsilon,\lambda]}\chi_\Omega) \\
&= \lambda^{-1}\nu_{\Pi_\lambda}(\Omega) - \varepsilon^{-1}\nu_{\Pi_\varepsilon}(\Omega) + \int_{]\varepsilon,\lambda]} u^{-2}\nu_{\Pi_\varepsilon}(\Omega)du \\
&\le \mu(\Omega)\big(\lambda^{-1}F(\lambda) + \int_{]\varepsilon,\lambda]} u^{-2}F(u)du\big) \\
&= \mu(\Omega)(G(\lambda) + F(\varepsilon)/\varepsilon) \longrightarrow \mu(\Omega)G(\lambda)\,,
\end{aligned}$$

when $\varepsilon \searrow 0$ since $F(\varepsilon)/\varepsilon \leq G(\varepsilon) \to 0$. Now using an Hilbert basis f_n of $\mathcal{H}$, one sees by Beppo-Levi and the spectral theorem that

$$\begin{aligned}\tau(\chi_\Omega A^{-1}\Pi_{]\varepsilon,\lambda]}\chi_\Omega) &= \sum_i \|A^{-1/2}\Pi_{]\varepsilon,\lambda]}\chi_\Omega f_i\|_2^2 \\ &\longrightarrow \sum_i \|A^{-1/2}\Pi_\lambda \chi_\Omega f_i\|_2^2 \quad \text{when} \quad \varepsilon \searrow 0\,, \\ &= \tau(\chi_\Omega A^{-1}\Pi_\lambda\chi_\Omega) = \nu_{A^{-1}\Pi_\lambda}(\Omega)\,.\end{aligned}$$

Therefore we obtain that $\nu_{A^{-1}\Pi_\lambda}(\Omega) \leq \mu(\Omega) G(\lambda)$ yielding $D(A^{-1}\Pi_\lambda) \leq G(\lambda)$ as claimed.

The proof at the heat level also follows Proposition 2.1 and starts from

$$\nu_{A^{-1}e^{-tA}}(\Omega) = \int_s^{+\infty} \nu_{e^{-sA}}(\Omega) ds\,.$$

□

Remark 3.2. In these proofs, we note that for invariant operators on groups Γ, endowed with their Haar measure, the previous inequalities (44) and (45) become equalities, as due to $\nu_P(\Omega) = D(P)\mu(\Omega)$ in such cases by Proposition 1.4.

The sequel of the proof also follows the pure state case. Let $\rho^{1/2}$ be the positive square root of ρ and consider for a measurable $\Omega \subset X$ the following splitting

$$\rho^{1/2}\chi_\Omega = \rho^{1/2}A^{1/2}A^{-1/2}\Pi_\lambda\chi_\Omega + \rho^{1/2}\Pi_{>\lambda}\chi_\Omega\,.$$

Taking a Hilbert–Schmidt norm gives

$$\|\rho^{1/2}\chi_\Omega\|_{HS} \leq \|\rho^{1/2}A^{1/2}\|_{2,2}\|A^{-1/2}\Pi_\lambda\chi_\Omega\|_{HS} + \|\rho^{1/2}\Pi_{>\lambda}\chi_\Omega\|_{HS}\,.$$

Then using the following classical properties, see e.g. [28, Chap. VI],

$$\tau(P^*P) = \|P\|_{HS}^2 = \|P^*\|_{HS}^2 \quad \text{and} \quad \|P\|_{2,2}^2 = \|P^*\|_{2,2}^2 = \|P^*P\|_{2,2}\,, \tag{46}$$

one finds that for any $\Omega \subset X$

$$\nu_\rho(\Omega) = \|\rho^{1/2}\chi_\Omega\|_{HS}^2 \leq 2\|\rho^{1/2}A\rho^{1/2}\|_{2,2}\nu_{A^{-1}\Pi_\lambda}(\Omega) + 2\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(\Omega)\,.$$

Therefore taking densities gives almost everywhere

$$\begin{aligned}D\nu_\rho(x) &\leq 2\|\rho^{1/2}A\rho^{1/2}\|_{2,2}D\nu_{A^{-1}\Pi_\lambda}(x) + 2D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x) \\ &\leq 2\|\rho^{1/2}A\rho^{1/2}\|_{2,2}G(\lambda) + 2D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x)\,,\end{aligned}$$

and finally

$$D\nu_\rho(x) \leq 4D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x) \quad \text{if} \quad D\nu_\rho(x) \geq 4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}G(\lambda)\,. \tag{47}$$

By integration on $\Omega = \{(x,\lambda) \mid D\nu_\rho(x) \geq 4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}G(\lambda)\}$ this leads to

$$\begin{aligned}
\int_X G^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}}\Big)d\nu_\rho &= \int_\Omega D\nu_\rho(x)d\mu(x)d\lambda \quad \text{by Fubini,}\\
&\leq 4\int_\Omega D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x)d\mu(x)d\lambda \quad \text{by} \quad (47),\\
&\leq 4\int_{X\times\mathbb{R}^+} D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x)d\mu(x)d\lambda\\
&= 4\int_{\mathbb{R}^+} \tau(\Pi_{>\lambda}\rho\Pi_{>\lambda})d\lambda\\
&= 4\int_{\mathbb{R}^+} \tau(\rho^{1/2}\Pi_{>\lambda}\rho^{1/2})d\lambda \quad \text{by} \quad (46)\,,\\
&= 4\tau\Big(\rho^{1/2}\big(\int_{\mathbb{R}^+}\Pi_{>\lambda}d\lambda\big)\rho^{1/2}\Big)\\
&= 4\tau(\rho^{1/2}A\rho^{1/2}) = 4\mathcal{E}(\rho)\,.
\end{aligned}$$

3.2. **Proof of the mixed state Moser inequalities.** The proofs of Theorem 1.6 and 1.7 follow the same technique. Here one compares the level sets of $D\nu_\rho(x)$ to $\|\rho\|_{2,2}$ instead of $\|\rho^{1/2}A\rho^{1/2}\|_{2,2}$ in (47). This does not rely on Proposition 3.1 and one can work either with $\rho = 0$ on $\ker A$ and $\Pi_{]0,\lambda]}$, as before, but also with a general positive ρ and $\Pi_{[0,\lambda]}$.

In any case, given $\Omega \subset X$, one considers the splitting

$$\rho^{1/2}\chi_\Omega = \rho^{1/2}\Pi_\lambda\chi_\Omega + \rho^{1/2}\Pi_{>\lambda}\chi_\Omega\,.$$

Taking Hilbert–Schmidt norms yields

$$\|\rho^{1/2}\chi_\Omega\|_{HS} \leq \|\rho^{1/2}\|_{2,2}\|\Pi_\lambda\chi_\Omega\|_{HS} + \|\rho^{1/2}\Pi_{>\lambda}\chi_\Omega\|_{HS}\,,$$

and using (46) as above

$$\nu_\rho(\Omega) = \|\rho^{1/2}\chi_\Omega\|_{HS}^2 \leq 2\|\rho\|_{2,2}\nu_{\Pi_\lambda}(\Omega) + 2\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(\Omega)\,, \tag{48}$$

whence

$$\begin{aligned}
D\nu_\rho(x) &\leq 2\|\rho\|_{2,2}D\nu_{\Pi_\lambda}(x) + 2D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x)\\
&= 2\|\rho\|_{2,2}F_x(\lambda) + 2D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x)\,.
\end{aligned}$$

Thus in place of (47) one finds that

$$D\nu_\rho(x) \leq 4D\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(x) \quad \text{if} \quad D\nu_\rho(x) \geq 4\|\rho\|_{2,2}F_x(\lambda)\,. \tag{49}$$

This leads to (16) by integration on $B = \{(x,\lambda) \mid D\nu_\rho(x) \geq 4\|\rho\|_{2,2}F_x(\lambda)\}$ as done above. Note that when one works on general ρ and $\Pi_{[0,\lambda]}$ one has to complete the definition of F_x^{-1} by setting

$$F_x^{-1}(u) = \begin{cases} 0 & \text{if} \quad u < F_x(0) = D\nu_{\Pi_{\ker A}}(x)\,,\\ \sup\{\lambda \mid F_x(\lambda) \leq u\} & \text{elsewhere}\,. \end{cases} \tag{50}$$

This means that the inequality (16) cuts off small values of $D\nu_\rho$ in this setting.

• For the discrete Moser inequality, (48) provides

$$\nu_\rho(\Omega) \leq 4\nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(\Omega) \quad \text{if} \quad \nu_\rho(\Omega) \geq 4\|\rho\|_{2,2} F_\Omega(\lambda)\,, \tag{51}$$

in place of the local (49). Given a partition $X = \bigsqcup \Omega_i$, and summing (51) on

$$C = \{(i,\lambda) \mid \nu_\rho(\Omega_i) \geq 4\|\rho\|_{2,2} F_{\Omega_i}(\lambda)\} \subset I \times \mathbb{R}^+$$

leads now to (17), where again one defines in general

$$F_\Omega^{-1}(u) = \begin{cases} 0 & \text{if} \quad u < F_\Omega(0) = \nu_{\ker A}(\Omega)\,, \\ \sup\{\lambda \mid F_\Omega(\lambda) \leq u\} & \text{elsewhere}\,. \end{cases} \tag{52}$$

• When the density ρ is supported in Ω, one has $\nu_\rho(\Omega) = \nu_\rho(X) = \tau(\rho)$ and the Moser inequality (17) can be written as

$$F_\Omega^{-1}\Big(\frac{\tau(\rho)}{4\|\rho\|_{2,2}}\Big) \leq 4\frac{\mathcal{E}(\rho)}{\tau(\rho)} = 4\langle A\rangle_\rho. \tag{53}$$

This yields the Faber–Krahn inequality (18) when $\langle A\rangle_\rho$ is finite since by right continuity of F_Ω and (52), one has $F_\Omega(F_\Omega^{-1}(\lambda)) \geq \lambda$ when $F_\Omega^{-1}(\lambda)$ is finite.

3.3. **Note on constants.** We observe that one can balance differently the multiplicative constants in the proofs of inequalities. Given $\varepsilon \in]0,1[$, one can replace for instance (51) by

$$\varepsilon^2\nu_\rho(\Omega) \leq \nu_{\Pi_{>\lambda}\rho\Pi_{>\lambda}}(\Omega) \quad \text{if} \quad (1-\varepsilon)^2\nu_\rho(\Omega) \geq \|\rho\|_{2,2} F_\Omega(\lambda)\,,$$

yielding

$$\sum_i F_{\Omega_i}^{-1}\Big(\frac{(1-\varepsilon)^2\nu_\rho(\Omega_i)}{\|\rho\|_{2,2}}\Big)\nu_\rho(\Omega_i) \leq \varepsilon^{-2}\mathcal{E}(\rho)$$

instead of (17), and in particular to the Faber–Krahn inequality

$$(1-\varepsilon)^2 F_\Omega^{\dim}(\lambda) \leq F_\Omega(\varepsilon^{-2}\lambda) \leq \mu(\Omega)F(\varepsilon^{-2}\lambda) \tag{54}$$

in place of (19). With respect to the ideal bound (20), one sees in (54) that tightening the energy multiplicative constant to 1 blows up the volume one, and reversely.

3.4. **Proof of the log-Sobolev inequality.** We show how to deduce Theorem 1.9 from the ρ-Moser Lieb–Thirring inequality. We follow a general argument in [10, §6].

Given s and $t > 0$, one has $s \leq F(F^{-1}(s))$ if $\lambda = F^{-1}(s)$ is finite, hence it holds that

$$\ln s - tF^{-1}(s) \leq \ln F(\lambda) - t\lambda \leq m(t)\,.$$

Setting $s = D\nu_\rho(x)/4\|\rho\|_{2,2}$ and integrating on X with $d\nu_\rho$ gives

$$\begin{aligned}\int_X \ln\Big(\frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}\Big)d\nu_\rho(x) &\leq \tau(\rho)m(t) + t\int_X F^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}\Big)d\nu_\rho(x)\\ &\leq \tau(\rho)m(t) + 4t\mathcal{E}(\rho)\,,\end{aligned}$$

by ρ-Moser, proving the log-Sobolev inequality (21).

For the entropy inequality (22), we first observe that since $m(t) = \sup_{\lambda\geq 0}(\ln F(\lambda) - t\lambda)$ one has

$$(\ln F)^c(\lambda) = \inf_{t\geq 0}(m(t) + t\lambda)\,,$$

by a classical property of Legendre transform; indeed $-m(t)$ is the concave-Legendre transform of $\ln F$. Then the optimization in t of the log-Sobolev inequality (21) leads to (22).

4. Relationships between inequalities

4.1. **Ultracontractivity, density and von Neumann trace.** We start the study of the relationships existing between the different inequalities we considered. We first discuss Proposition 1.4, that relates the three measurements of positive operators used here: through ultracontractivity, density function, or Γ-trace.

At first, the existence and expression for the density function $D\nu_P(x)$ given in (6), directly comes from the definition (5) written in the form

$$\nu_P(\Omega) = \tau(P^{1/2}\chi_\Omega P^{1/2}) = \|\chi_\Omega P^{1/2}\|_{HS}^2 = \int_\Omega \sum_i \|(P^{1/2}e_i)(x)\|_H^2 d\mu(x)\,.$$

Note that this holds without finiteness assumption. This expression of the density is useful when P is a positive compact operator. Namely if $P = \sum_i \lambda_i \Pi_{e_i}$ is the spectral decomposition of P, it reads

$$D\nu_P(x) = \sum_i \lambda_i \|e_i(x)\|_H^2 \quad \text{almost everywhere.}$$

Also, (6) clearly implies that $\|P\|_{1,\infty} = \|P^{1/2}\|_{2,\infty}^2 \le D(P) = \operatorname{supess} D\nu_P(x)$ holds in general. For the opposite inequality, we suppose H is finite d-dimensional. Then given a basis (e_i) of $\mathcal{H}$ and (h_j) of H, one has by (6)

$$D\nu_P(x) = \sum_i \|P^{1/2}e_i(x)\|_H^2 = \sum_{j=1}^d \sum_i \langle (P^{1/2}e_i)(x), h_j\rangle^2 \quad \text{a.e.}$$

where, for each j, by Cauchy–Schwarz

$$\begin{aligned}
\sum_i \langle (P^{1/2}e_i)(x), h_j\rangle^2 &= \sup_{\sum c_i^2 \le 1} \big(\sum_i c_i \langle (P^{1/2}e_i)(x), h_j\rangle\big)^2 \\
&= \sup_{\sum c_i^2 \le 1} \Big(\langle \big(P^{1/2}(\sum_i c_i e_i)\big)(x), h_j\rangle\Big)^2 \\
&= \sup_{\|f\|_2 \le 1} \big(\langle (P^{1/2}f)(x), h_j\rangle\big)^2 \\
&\le \|P^{1/2}\|_{2,\infty}^2 = \|P\|_{1,\infty}\,.
\end{aligned}$$

This gives $D\nu_P(x) \le d\|P\|_{1,\infty}$ a.e as needed.

We would like to illustrate here the relevance of the spectral density $D(\Pi_\lambda)$ when dealing with general mixed state inequalities, while the ultracontractive norm $\|\Pi_\lambda\|_{1,\infty}$ is adapted to functions. Indeed, suppose that A is a scalar positive operator with finite $F(\lambda) = \|\Pi_\lambda\|_{1,\infty} = D(\Pi_\lambda)$ and consider n-copies A_n of A acting diagonally on $\mathcal{H}_n = L^2(X, \mathbb{C}^n)$. Then one finds easily that

$$\|\Pi_\lambda(A_n)\|_{1,\infty} = \|\Pi_\lambda(A)\|_{1,\infty} (= \|\Pi_\lambda(A)\|_{2,\infty}^2) \quad \text{while} \quad D(\Pi_\lambda(A_n)) = nD(\Pi_\lambda(A))\,,$$

if one sets $\|f\|_\infty = \sup_X \|f(x)\|_2$ and $\|f\|_1 = \int_X \|f(x)\|_2 d\mu$ with the hermitian norm of $\mathbb{C}^n$. Hence the Sobolev inequality (3) is independent of the phase space dimension n, as the Faber–Krahn inequality (36) for the *first* eigenvalue $\lambda_{1,n}$ of A_n on a domain Ω, that can be written as

$$4\mu(\Omega)F(8\lambda_{1,n}(\Omega)) \geq 1 .$$

In comparison, the CLR inequality (19) only yields

$$4n\mu(\Omega)F(4\lambda_{1,n}(\Omega)) \geq 1$$

for the first eigenvalue, but also gives the linear bound in n of the whole spectral distribution, namely: $F^{\dim}_{\Omega,A_n}(\lambda) \leq 4n\mu(\Omega)F(4\lambda)$.

We come back to the study of the density of invariant operators on locally compact groups Γ with their Haar measure. In such a case, the measure $\nu_P(\Omega) = \tau(\chi_\Omega P \chi_\Omega)$ is clearly invariant too, thus its density function is constant, as claimed in Proposition 1.3. Moreover, when Γ is discrete, this density coincides with von Neumann trace since

$$D(P) = D\nu_P(e) = \tau_{\mathcal{H}}(\chi_e P \chi_e) = \tau_H(K_P(e,e)) \stackrel{\text{def}}{=} \operatorname{Tr}_\Gamma(P) ,$$

where K_P is the kernel of P. More generally, one can characterize finite density operators on not necessarily discrete groups as follows.

Proposition 4.1. *Let Γ be a locally compact group with its Haar measure and Q be a bounded Γ-invariant operator on $\mathcal{H} = L^2(\Gamma,\mu) \otimes H$. Let E be the space of Hilbert–Schmidt operators on H endowed with the Hilbert–Schmidt norm.*

- *Then $P = Q^*Q$ has a finite density $D(P)$ iff Q has a kernel $K_Q(x,y) = k_Q(y^{-1}x)$ with $k_Q \in L^2(\Gamma, E)$ and $D(P) = \int_\Gamma \|k_Q(x)\|^2_{HS} d\mu$.*
- *If moreover Γ is* unimodular, *one has $D(Q^*Q) = D(QQ^*)$ and the density actually defines a faithful trace in that case.*

Remark 4.2. We recall that this last trace property allows one to get a meaningful notion of dimension for closed Γ-invariant subspaces $L \subset \mathcal{H} = L^2(\Gamma) \otimes H$. Indeed, one sets then $\dim_\Gamma L = D(\Pi_L)$. This satisfies the key property $\dim_\Gamma f(L) = \dim_\Gamma L$ for any closed densely defined invariant injective operator $f : L \to H$, see e.g. [27, §2] or [30, §3.2, §6].

Proof. We recall that an operator is Hilbert–Schmidt if and only if it possesses an L^2 kernel, see e.g. [28, Chap VI]. Then by definition

$$\begin{aligned}
\nu_P(\Omega) &= \tau_{\mathcal{H}}(\chi_\Omega Q^* Q \chi_\Omega) = \|Q\chi_\Omega\|^2_{\mathcal{H}S} \\
&= \int_{\Gamma\times\Omega} \|K_Q(x,y)\|^2_{HS} d\mu(x) d\mu(y) \\
&= \mu(\Omega) \int_\Gamma \|k_Q(x)\|^2_{HS} d\mu(x) \quad \text{by invariance} .
\end{aligned}$$

Hence $D(P) = \int_\Gamma \|k_Q(x)\|^2_{HS} d\mu(x)$. Moreover, one has

$$\|k_Q(x)\|_{HS} = \|(k_Q(x))^*\|_{HS} = \|k_{Q^*}(x^{-1})\|_{HS}$$

giving

$$D(Q^*Q) = \int_\Gamma \|k_Q(x)\|_{HS}^2 d\mu(x) = \int_\Gamma \|k_{Q^*}(y)\|_{HS}^2 d\mu(y) = D(QQ^*)$$

on unimodular groups since there $\mu(\Omega^{-1}) = \mu(\Omega)$. □

One can finally express the density using Fourier analysis on some family of groups. Namely, following Dixmier [12, §18.8], if the group Γ is *locally compact, unimodular and postliminaire*, there exists a Plancherel measure μ^* on its unitary dual $\widehat{\Gamma}$, together with a Plancherel formula on $L^2(\Gamma)$. In particular on positive operators $P = Q^*Q$, one has using Proposition 4.1

$$D(P) = \|k_Q\|_2^2 = \int_{\widehat{\Gamma}} \|\widehat{k_Q}(\xi)\|_{HS}^2 d\mu^*(\xi)\,. \tag{55}$$

This allows one to estimate the spectral density $F(\lambda)$ in some simple cases as in $\mathbb{R}^n$.

4.2. **Illustration and comparison with known inequalities on $\mathbb{R}^n$.** For instance, in the case of the Laplacian Δ on $\mathbb{R}^n$, the spectral space $E_\lambda(\Delta)$ is the Fourier transform of functions supported in the ball $B(0,\sqrt{\lambda})$ in $(\widehat{\mathbb{R}^n}, d\mu^*) \simeq (\mathbb{R}^n, (2\pi)^{-n}dx)$, hence $\widehat{k_{\Pi_\lambda}} = \chi_{B(0,\sqrt{\lambda})}$ and (55) provides

$$F(\lambda) = \mu^*(B(0,\sqrt{\lambda})) = C_n\lambda^{n/2}, \tag{56}$$

with $C_n = (2\pi)^{-n}\mathrm{vol}(B_n)$. This leads to

$$G(\lambda) = \frac{nC_n}{n-2}\lambda^{n/2-1} \quad \text{and} \quad H(x) = xG^{-1}(x) = \Big(\frac{n-2}{nC_n}\Big)^{\frac{2}{n-2}} x^{\frac{n}{n-2}}\,,$$

so that finally (3) gives the classical Sobolev inequality in $\mathbb{R}^n$

$$\|f\|_{2n/(n-2)} \le \frac{1}{\pi}\Big(\frac{n\,\mathrm{vol}(B_n)}{n-2}\Big)^{\frac{1}{n}} \|df\|_2 = D_n\|df\|_2\,.$$

One finds that the constant D_n has the correct rate of decay in n, namely $D_n \sim_{+\infty} \sqrt{\frac{2e}{n\pi}}$. Indeed according to [2] or [21, Thm. 8.3], the best constant here is

$$D_n^* = \frac{2}{\sqrt{n(n-2)}}\mathrm{area}(S_n)^{-1/n} \sim_{+\infty} \sqrt{\frac{2}{n\pi e}}\,.$$

We now consider the Moser inequality (33). On functions this gives the classical L^2-Moser inequality with constants with the right decay in n on $\mathbb{R}^n$. Indeed from (56), one finds that

$$\|f\|_{2+4/n}^{2+4/n} \le 4^{1+2/n}C_n^{2/n}\|f\|_2^{4/n}\|df\|_2^2 = E_n\|f\|_2^{4/n}\|df\|_2^2\,,$$

with $E_n \sim_{+\infty} \frac{2e}{n\pi}$ while, following Beckner, see [3] or [10, Appendix], the best constants in the L^2-Moser inequality are asymptotic to $\frac{2}{n\pi e}$.

Still on $\mathbb{R}^n$, one can get some general algebraic expression of $F(\lambda)$ for the positive invariant differential operator $A = \sum_I a_I\partial_{x_I}$. Let $\sigma(A)(\xi) = \sum_I a_I(i\xi)^I$ be its polynomial symbol. Then again the spectral space $E_\lambda(A)$ consists in functions whose Fourier transform is supported in

$$D_\lambda = \{\xi \in \mathbb{R}^n \mid \sigma(A)(\xi) \le \lambda\}\,,$$

and as above

$$F(\lambda) = (2\pi)^{-n}\mathrm{vol}(D_\lambda).$$

The asymptotic behaviour of $F(\lambda)$ when $\lambda \searrow 0$ can be obtained from the resolution of the singularity of the polynomial $\sigma(A)$ at 0. Indeed, there exists $\alpha \in \mathbb{Q}^+$ and $k \in [0, n-1] \cap \mathbb{N}$ such that

$$F(\lambda) \underset{\lambda \to 0^+}{\sim} C\lambda^\alpha |\ln \lambda|^k ,$$

see e.g. Theorem 7 in [1, §21.6]. Moreover, under a non-degeneracy hypothesis on $\sigma(A)$, the exponents α and k can be read from its Newton polyhedra. Then if $\alpha > 1$, Proposition 4.5 yields that $G(\lambda) \underset{0}{\asymp} \lambda^{\alpha-1}|\ln \lambda|^k$. Therefore $G^{-1}(u) \underset{0}{\asymp} u^{1/(\alpha-1)}|\ln u|^{-k/(\alpha-1)}$ and finally the H-Sobolev inequality (3) is governed in small energy by the function

$$H(u) \asymp u^{\frac{\alpha}{\alpha-1}}|\ln(u)|^{-\frac{k}{\alpha-1}} \quad \text{for} \quad u \ll 1 .$$

4.3. **Spectral versus heat decay.** We now compare the two Theorems 1.1 and 1.2. They both state Sobolev inequalities for functions starting either from the heat or spectral decay. One can compare F and G to L and M through Laplace transform of associated measures.

Proposition 4.3. • *In any case it holds that*

$$L(t) \leq \mathcal{L}(dF)(t) = \int_0^{+\infty} e^{-\lambda t} dF(\lambda) \tag{57}$$

$$M(t) \leq \mathcal{L}(dG)(t) = \int_0^{+\infty} e^{-\lambda t} dG(\lambda) . \tag{58}$$

• *If A is an invariant operator acting on $\mathcal{H} = L^2(\Gamma, H)$ over a locally compact group Γ, then reverse inequalities hold up to the multiplicative factor $n = \dim H$, i.e.*

$$\mathcal{L}(dF) \leq nL \quad \text{and} \quad \mathcal{L}(dG) \leq nM .$$

Moreover $G(y) \leq neM(y^{-1})$ and H-Sobolev inequality (3) implies N-Sobolev (9), up to multiplicative constants.

• *Reversely, for any operator, if G satisfies the exponential growing condition :*

$$\exists C \text{ such that } \forall u, y > 0 ,\ G(uy) \leq e^{Cu} G(y) , \tag{59}$$

then $M(y^{-1}) \leq 3G(2Cy)$. Hence H and N-Sobolev are equivalent on groups in that case.

Proof. • By spectral calculus $e^{-tA}\Pi_V = \int_0^{+\infty} e^{-t\lambda} d\Pi_\lambda = t\int_0^{+\infty} e^{-t\lambda}\Pi_\lambda d\lambda$, hence

$$L(t) = \|e^{-tA}\Pi_V\|_{1,\infty} \leq t\int_0^{+\infty} e^{-t\lambda}\|\Pi_\lambda\|_{1,\infty} d\lambda = \mathcal{L}(dF)(t) ,$$

and thus

$$M(t) = \int_t^{+\infty} L(s)ds \leq \int_t^{+\infty}\int_0^{+\infty} e^{-\lambda s} dF(\lambda) ds = \int_0^{+\infty} \frac{e^{-\lambda t}}{\lambda} dF(\lambda) = \mathcal{L}(dG)(t) .$$

• For positive invariant operators P on groups, the ultracontractive norm $\|P\|_{1,\infty}$ is pinched between the density $D(P)$ and $nD(P)$ by (7). This gives the reverse inequalities by positive linearity of $D(P)$ on such operators. In particular one gets

$$\begin{aligned} nM(y^{-1}) &\geq \int_0^{+\infty} e^{-\lambda/y} dG(\lambda) = y^{-1} \int_0^{+\infty} e^{-\lambda/y} G(\lambda) d\lambda \\ &\geq y^{-1} \int_y^{+\infty} e^{-\lambda/y} G(y) d\lambda = e^{-1} G(y). \end{aligned}$$

Therefore $N(y) = y/M^{-1}(y) \leq yG^{-1}(ney) = (ne)^{-1}H(ney)$ and H-Sobolev implies

$$\int_X N\Big(\frac{|f(x)|^2}{4ne\mathcal{E}(f)}\Big) d\mu \leq (ne)^{-1} .$$

- If G satisfies the growing condition (59), one has by (58)

$$\begin{aligned} M(1/y) &\leq \int_0^{+\infty} e^{-\lambda/y} dG(\lambda) = \int_0^{+\infty} e^{-u} G(uy) du \\ &\leq \int_0^{2C} e^{-u} G(2Cy) du + \int_{2C}^{+\infty} e^{-u/2} G(2Cy) du \\ &\leq 3G(2Cy) . \end{aligned}$$

□

We note that it may happen that $N \ll H$ for very thin near–zero spectrum. In an extreme case there may be a gap in the spectrum, i.e. $A \geq \lambda_0 > 0$, hence $F = G = 0$ on $[0, \lambda_0[$ and $H(y) \geq \lambda_0 y$, while $L(t) \asymp Ce^{-ct}$, $M(t) \asymp C'e^{-ct}$ and $N(y) \asymp C''y/\ln(y/C')$.

We mention also that a similar statement holds at the mixed state level between the two ρ-Sobolev inequalities obtained from the spectral density $F(\lambda) = D(\Pi_\lambda)$ or the heat density $L(t) = D(e^{-tA})$. Namely the F version is stronger in general to the L one on groups, although of the same strength under the same growing assertion (59) on G, that excludes extremely thin spectrum near zero.

4.4. **From ρ-Sobolev to CLR inequalities.** We have seen in the introduction how the mixed state Sobolev inequality (9) implies a CLR inequality (13) in the case of a polynomial spectral density. We extend this to other profiles and compare the result to Corollary 1.8, obtained from the mixed state Moser (Lieb–Thirring) approach.

Proposition 4.4. *Suppose that a ρ-Sobolev inequality* (9) *holds for some increasing right continuous function G. Then for any mixed state ρ supported in a domain Ω of finite volume, one has*

$$\tau(\rho) \leq 8\mu(\Omega)\|\rho^{1/2}A\rho^{1/2}\|_{2,2} G(8\langle A\rangle_\rho) . \tag{60}$$

- *In particular the following CLR inequality holds*

$$F_\Omega^{\dim}(\lambda) \leq 8\mu(\Omega)\lambda G(8\lambda) . \tag{61}$$

Hence the ρ-Sobolev inequality obtained in Theorem 1.5 implies a CLR inequality which is a priori weaker that (19), since $F(\lambda) \leq \lambda G(\lambda)$ there.

Proof. When the function $tG^{-1}(t)$ is convex, Jensen inequality easily gives the result, with better coefficients. Without convexity assumption, we can argue again as in §2.3. Observe that for all non-negative s and t one has

$$st \leq sG(s) + tG^{-1}(t)\,.$$

Applying to $t = \frac{D\nu_\rho(x)}{4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}}$ and integrating over Ω yields

$$\begin{aligned}\frac{s\tau(\rho)}{4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}} - sG(s)\mu(\Omega) &\leq \int_X \frac{D\nu_\rho(x)}{4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}} G^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho^{1/2}A\rho^{1/2}\|_{2,2}}\Big) d\mu \\ &\leq \frac{\tau(A\rho)}{\|\rho^{1/2}A\rho^{1/2}\|_{2,2}} \quad \text{by} \quad (9)\,.\end{aligned}$$

Using

$$s \nearrow G^{-1}\Big(\frac{\tau(\rho)}{8\mu(\Omega)\|\rho^{1/2}A\rho^{1/2}\|_{2,2}}\Big) = \sup\Big\{ s \mid G(s) \leq \frac{\tau(\rho)}{8\mu(\Omega)\|\rho^{1/2}A\rho^{1/2}\|_{2,2}} \Big\}\,,$$

gives

$$G^{-1}\Big(\frac{\tau(\rho)}{8\mu(\Omega)\|\rho^{1/2}A\rho^{1/2}\|_{2,2}}\Big) \leq 8\langle A\rangle_\rho\,,$$

and (60) since $G(G^{-1}(s)) \geq s$ when $G^{-1}(s)$ is finite by right continuity of G. □

Observe that one may have $F(\lambda) \ll \lambda G(\lambda)$ for very thick near-zero spectrum, even when G converges. For instance if $F(\lambda) = \lambda/\ln^2\lambda$ then $\lambda G(\lambda) = (-\ln\lambda + 1)F(\lambda)$. Except this "low dimensional" phenomenon, one has $\lambda G(\lambda) \underset{0}{\asymp} F(\lambda)$ in the other cases, and thus the two CLR inequalities (19) and (61) obtained through ρ-Sobolev or Moser inequalities have the same strength. For instance this holds if $F(\lambda) \underset{0}{\sim} \lambda^{1+\varepsilon}\varphi(\lambda)$ for some $\varepsilon > 0$ and an increasing $\varphi > 0$. This comes from the following remark.

Proposition 4.5. *Suppose there exists $\varepsilon > 0$ such that, for small λ, F satisfies the growing condition $F(2\lambda) \geq 2(1+\varepsilon)F(\lambda)$, then $(2+\varepsilon^{-1})F(\lambda) \geq \lambda G(\lambda) \geq F(\lambda)$.*

Proof. By (28), one has

$$\begin{aligned}G(\lambda) &= \int_0^\lambda \frac{dF(u)}{u} = \frac{F(\lambda)}{\lambda} + \int_0^\lambda \frac{F(u)}{u^2} du \\ &= \frac{F(\lambda)}{\lambda} + \Big(\int_0^{\lambda/2} + \int_{\lambda/2}^\lambda\Big)\frac{F(u)}{u^2} du \\ &\leq \frac{2F(\lambda)}{\lambda} + \int_0^{\lambda/2} \frac{F(2u)}{2(1+\varepsilon)u^2} du \quad \text{by hypothesis on } F\,, \\ &\leq \frac{2F(\lambda)}{\lambda} + \frac{1}{1+\varepsilon}\Big(G(\lambda) - \frac{F(\lambda)}{\lambda}\Big)\,,\end{aligned}$$

leading to $\lambda G(\lambda) \leq (2+\varepsilon^{-1})F(\lambda)$. □

As a curiosity, we note that under the growing hypothesis on F above, the spectral density of states F and the spatial repartition function H have symmetric expressions with respect to G and G^{-1}. Indeed, one has simply there

$$F(\lambda) \asymp \lambda G(\lambda) \quad \text{while} \quad H(x) = xG^{-1}(x)\,. \tag{62}$$

4.5. **Discrete and integral ρ-Moser inequalities.** We now study the relationships between the two versions of the ρ-Moser inequalities given in Theorem 1.6 and 1.7.

A first remark is that the discrete case may be seen as an instance of the integral version. Indeed if $X = \bigsqcup_I \Omega_i$, then one may split $\mathcal{H} = L^2(X,\mu) \otimes H = \bigoplus_I L^2(\Omega_i,\mu) \otimes H$ and, given a positive P on $\mathcal{H}$, define a measure ν'_P on I by

$$\nu'_P(J) = \sum_{i \in J} \tau(\chi_{\Omega_i} P \chi_{\Omega_i}) = \nu_P(\cup_J \Omega_i)\,.$$

Then the density function of ν'_P can be written as

$$D\nu'_P(i) = \tau(\chi_{\Omega_i} P \chi_{\Omega_i}) = \nu_P(\Omega_i)\,.$$

In this setting the integral formula (16) yields the discrete one (17).

Another feature of these formulae is that, up to multiplicative constants, the integral formula dominates all the discrete ones, whatever the partition of X.

Proposition 4.6. *Given a measurable Ω in X, it holds that*

$$F_\Omega^{-1}\Big(\frac{\nu_\rho(\Omega)}{8\|\rho\|_{2,2}}\Big)\nu_\rho(\Omega) \le 2\int_\Omega F_x^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}\Big)d\nu_\rho\,. \tag{63}$$

In particular, for any partition $X = \bigsqcup_I \Omega_i$ one has

$$\sum_i F_{\Omega_i}^{-1}\Big(\frac{\nu_\rho(\Omega_i)}{8\|\rho\|_{2,2}}\Big)\nu_\rho(\Omega_i) \le 2\int_X F_x^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}\Big)d\nu_\rho\,.$$

Hence Theorem 1.6 implies Theorem 1.7 and Corollary 1.8, up to weaker constants 8 instead of 4.

Proof. Given $x \in X$, one has for all non negative s and t

$$st \le sF_x(s) + tF_x^{-1}(t)\,,$$

where $F_x(\lambda) = D\nu_{\Pi_\lambda}(x)$. Applying to $t = \frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}$ and integrating on Ω yields

$$s\frac{\nu_\rho(\Omega)}{4\|\rho\|_{2,2}} - sF_\Omega(s) \le \int_\Omega F_x^{-1}\Big(\frac{D\nu_\rho(x)}{4\|\rho\|_{2,2}}\Big)\frac{d\nu_\rho}{4\|\rho\|_{2,2}}\,.$$

This gives (63) using

$$s \nearrow F_\Omega^{-1}\Big(\frac{\nu_\rho(\Omega)}{8\|\rho\|_{2,2}}\Big) = \sup\Big\{s \mid F_\Omega(s) \le \frac{\nu_\rho(\Omega)}{8\|\rho\|_{2,2}}\Big\}\,.$$

$\square$

5. Spectral density and cohomology

We conclude with an application of the Sobolev inequalities in the pure state setting. Let K be a finite simplicial complex and consider a covering $\Gamma \to X \to K$. Let d_k be the coboundary operator on k-cochains X^k of X. As a purely combinatorial and local operator, it acts boundedly on all ℓ^p-spaces of cochains $\ell^p X^k$, see e.g. [4, 27].

Let $F_{\Gamma,k}(\lambda)$ denotes the Γ-trace of the spectral projector $\Pi_\lambda = \chi_A(]0,\lambda])$ of $A = d_k^* d_k$. By Proposition 1.4 this function coincides with the density of Π_λ relatively to Γ and is also equivalent, up to multiplicative constants, to the ultracontractive spectral decay $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$. Thus Theorem 1.10 is a direct application of Theorem 1.1 or 1.5 in the polynomial case. This statement compares two measurements of the torsion of ℓ^2-cohomology $T_2^{k+1} = \overline{d_k(\ell^2)}^{\ell^2} / d_k(\ell^2)$ that share some geometric invariance. We describe this more precisely.

We first recall the main invariance property of $F_{\Gamma,k}(\lambda)$. We say that two increasing functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are equivalent if there exists $C \geq 1$ such that $f(\lambda/C) \leq g(\lambda) \leq f(C\lambda)$ for λ small enough. According to [14, 18, 17] we have:

Theorem 5.1. *Let K be a finite simplicial complex and $\Gamma \to X \to K$ a covering. Then the equivalence class of $F_{\Gamma,k}$ only depends on Γ and the homotopy class of the $(k+1)$-skeleton of K.*

One tool in the proof is the observation that an homotopy of finite simplicial complexes X and Y induces bounded Γ-invariant homotopies between the Hilbert complexes $(\ell^2 X^k, d_k)$ and $(\ell^2 Y^k, d'_k)$. That means there exist Γ-invariant bounded maps

$$f_k : \ell^2 X^k \to \ell^2 Y^k \quad \text{and} \quad g_k : \ell^2 Y^k \to \ell^2 X^k$$

such that

$$f_{k+1} d_k = d'_k f_k \quad \text{and} \quad g_{k+1} d'_k = d_k g_k$$

and

$$g_k f_k = \mathrm{Id} + d_{k-1} h_k + h_{k+1} d_k \quad \text{and} \quad f_k g_k = \mathrm{Id} + d'_{k-1} h'_k + h'_{k+1} d'_k$$

for some bounded maps

$$h_k : \ell^2 X^k \to \ell^2 X^{k-1} \quad \text{and} \quad h'_k : \ell^2 Y^k \to \ell^2 Y^{k-1}.$$

Actually all these maps are purely combinatorial and local, see e.g. [4, §1], and thus extend on all ℓ^p spaces of cochains.

One can show a similar invariance property of the inclusion (23) we recall below, but that holds more generally on uniformly *locally finite simplicial complexes*, without requiring a group invariance. These are simplicial complexes such that each point lies in a bounded number $N(k)$ of k-simplexes.

Proposition 5.2. *Let X and Y be uniformly locally finite simplicial complexes. Suppose that they are boundedly homotopic in ℓ^2 and ℓ^p norms for some $p \geq 2$. Then one has*

$$\overline{d_k(\ell^2 X^k)}^{\ell^2} \subset d_k(\ell^p X^k)\,, \tag{64}$$

if and only if a similar inclusion holds on Y.

Proof. Suppose that $\overline{d_k(\ell^2X^k)}^{\ell^2} \subset d_k(\ell^pX^k)$ and consider a sequence $\alpha_n = d'_k(\beta_n) \in d'_k(\ell^2Y^k)$ that converges to $\alpha \in \overline{d_k(\ell^2Y^k)}^{\ell^2}$ in ℓ^2.

Then $g_{k+1}\alpha_n = d_k(g_k\beta_n) \to g_{k+1}\alpha \in \overline{d_k(\ell^2X^k)}^{\ell^2}$. Therefore there exists $\beta \in \ell^pX^k$ such that $g_{k+1}\alpha = d_k\beta$. Then taking ℓ^2-limit in the sequence

$$f_{k+1}g_{k+1}\alpha_n = \alpha_n + d'_kh'_{k+1}\alpha_n + h'_{k+2}d'_{k+1}\alpha_n = \alpha_n + d'_kh'_{k+1}\alpha_n$$

gives

$$d'_k(f_k\beta) = f_{k+1}d_k\beta = \alpha + d'_kh'_{k+1}\alpha\,,$$

and finally $\alpha \in d'_k(\ell^pY^k)$ since $\ell^2Y^k \subset \ell^pY^k$ for $p \geq 2$. □

The inclusion (64) we consider here is related to problems studied in $\ell^{p,q}$ cohomology. We briefly recall this notion and refer for instance to [15] for more details. If X is a simplicial complex as above, one considers the spaces

$$Z_q^k(X) = \ker d_k \cap \ell^qX^k \quad \text{and} \quad B_{p,q}^k(X) = d_{k-1}(\ell^pX^k) \cap \ell^qX^k\,.$$

Then the $\ell^{p,q}$-cohomology of X is defined by

$$H_{p,q}^k(X) = Z_q^k(X)/B_{p,q}^k(X)\,.$$

Its reduced part is the Banach space

$$\overline{H}_{p,q}^k(X) = Z_q^k(X)/\overline{B}_{p,q}^k(X)\,,$$

while its torsion part

$$T_{p,q}^k(X) = \overline{B}_{p,q}^k(X)/B_{p,q}^k(X)$$

is not a Banach space. These spaces fit into the exact sequence

$$0 \to T_{p,q}^k(X) \to H_{p,q}^k(X) \to \overline{H}_{p,q}^k(X) \to 0\,.$$

It is straightforward to check as above that, for $p \geq q$, these spaces satisfy the same homotopical invariance property as in Proposition 5.2.

Proposition 5.3. *Let X and Y be uniformly locally finite simplicial complexes. Suppose that they are boundedly homotopic in ℓ^p and ℓ^q norms for $p \geq q$. Then the maps $f_k : \ell^*X^k \to \ell^*Y^k$ and $g_k : \ell^*Y^k \to \ell^*X^k$ induce reciprocal isomorphisms between the $\ell^{p,q}$ cohomologies of X and Y, as well as their reduced and torsion components.*

In this setting, the vanishing of the $\ell^{p,2}$-torsion $T_{p,2}^{k+1}(X)$ is equivalent to the closeness of $B_{p,2}^{k+1}(X) = d_k(\ell^pX^k) \cap \ell^2X^{k+1}$ in ℓ^2X^{k+1}, i.e. to the inclusion

$$\overline{d_k(\ell^pX^k) \cap \ell^2X^{k+1}}^{\ell^2} \subset d_k(\ell^pX^k) \cap \ell^2X^{k+1}\,.$$

This implies the weaker inclusion (64), but is stronger in general unless the following holds

$$d_k(\ell^pX^k) \cap \ell^2X^{k+1} \subset \overline{d_k(\ell^2X^k)}^{\ell^2}\,. \tag{65}$$

Now by Hodge decomposition in ℓ^2X^{k+1}, one has always

$$d_k(\ell^pX^k) \cap \ell^2X^{k+1} \subset \ker d_{k+1} \cap \ell^2X^{k+1} = \overline{H}_2^{k+1}(X) \oplus^\perp \overline{d_k(\ell^2X^k)}^{\ell^2}\,.$$

Hence (65) holds if the reduced ℓ^2-cohomology $\overline{H}_2^{k+1}(X)$ vanishes, proving in that case the equivalence of (64) to the vanishing of the $\ell^{p,2}$-torsion, and even to the identity

$$B_{p,2}^{k+1} := d_k(\ell^p X^k) \cap \ell^2 X^{k+1} = \overline{d_k(\ell^2 X^k)}^{\ell^2}, \tag{66}$$

which is clearly closed in ℓ^2.

Corollary 5.4. *Let K be a finite simplicial space and $\Gamma \to X \to K$ a covering. Suppose that the spectral distribution $F_{\Gamma,k}$ of $A = d_k^* d_k$ on $(\ker d_k)^\perp$ satisfies $F_{\Gamma,k}(\lambda) \leq C\lambda^{\alpha/2}$ for some $\alpha > 2$. Suppose moreover that the reduced ℓ^2-cohomology $\overline{H}_2^{k+1}(X)$ vanishes.*

Then (66) and the vanishing of the $\ell^{p,2}$-torsion $T_{p,2}^{k+1}(X)$ hold for $1/p \leq 1/2 - 1/\alpha$.

A comprehensive introduction to L^2-invariants may be found in [24]. For instance by [6], the reduced ℓ^2-cohomology vanishes in all degrees for infinite amenable groups; but actually this vanishing also occurs in other classical situations.

Namely, if Γ is a co-compact lattice in a non-compact Riemannian symmetric space $M = G/K$, then the reduced ℓ^2-cohomology $\overline{H}_2^k(\Gamma)$ is non-zero if and only if $\operatorname{rank} G = \operatorname{rank} K$ and $k = \dim M/2$; see e.g. [24, Chap. 5]. Moreover, by [23, 26], all the Novikov-Shubin exponents α_k are known. With our notation, one has $F_{\Gamma,k}(\lambda) \neq 0$ near 0 if and only if

$$r = \operatorname{rank} G - \operatorname{rank} K > 0 \quad \text{and} \quad k \in [(\dim M - r)/2, (\dim M + r)/2[\,,$$

in which case

$$F_{\Gamma,k}(\lambda) \asymp \lambda^{r/2}\,.$$

In comparison to such examples, not so much is known on general groups on the geometric meaning and estimations of these exponents, except for the first one on functions. Indeed, following [32], α_0 is finite only for virtually nilpotent groups, and coincides with the growth rate of Γ. We mention also that some estimations and computations of higher exponents can be done on lattices in Carnot-Carathéodory groups G; see [29]. For instance, the exponent α_1 in degree one is pinched between $\alpha_0/(w_{\max} - 1)$ and $\alpha_0/(w_{\min} - 1)$, where α_0 is the growth rate of G and $w_{\min}$, $w_{\max}$ denote the minimal and maximal weights of the relations required to define G.

In some sense Theorem 1.10 and Corollary 5.4 relate a spectral invariant to cruder integration properties on cocycles at large scale. Such a relation was a key in the geometric understanding of α_0 recalled above, see e.g. [9, 32], and hopefully may be useful in higher degree too.

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Laboratoire de Mathématiques d'Orsay, UMR 8628, CNRS et Université Paris-Sud, 91405 Orsay Cedex, France

E-mail address: michel.rumin@math.u-psud.fr