An introduction to spectral and differential geometry in Carnot-Carathéodory spaces

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1 Introduction

Starting with a compact manifold M, or even a finite simplicial complex, one can consider the large time behaviour of heat on forms (or cochains) on its fundamental cover $\widetilde{M}$. The large time heat decay exponents, called Novikov-Shubin numbers, are known to be homotopical invariants of M. On functions this exponent is related to the growth of $\pi_1(M)$. Yet, in higher degrees very few is known about their geometric signification.

We will consider the case of $\widetilde{M}$ being a graded nilpotent group G (or Carnot group), that is a nilpotent group with a dilation. We will show how the study on one forms, or even on discrete one cochains, leads to introduce differential operators of high orders, that fit into complexes. These constructions extend to Carnot-Carathéodory spaces, that is manifolds with a bracket generating distribution given in the their tangent bundle. The ideas and results will be precised on examples.

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I have tried to follow an elementary and self-contained presentation, in order to keep it the most accessible. Related developments around the topics presented here, but relying on more analytic techniques, may be found in [39, 40].

2 Some motivation from spectral geometry

2.1 Discrete groups, Cayley graphs and cochain complex

Discrete groups seen as graphs. The spectral geometric problem we want to discuss actually makes sense not only on fundamental cover $\widetilde{M}$ of smooth compact manifolds M, but on any finitely presented discrete group Γ.

One says that a discrete group Γ has a finite presentation if

- Γ is generated by a finite set $S = \{s_1, s_2, \cdots, s_n\}$, meaning that any element of Γ can be written as a word (product) of $s_i^{\pm 1} \in S \cup S^{-1}$,
- any relation in Γ, that is any word w in the $s_i^{\pm 1}$ such that $w = e$ neutral element of Γ, can be factorized as a product of elements of the form $\gamma^{-1} r_i^{\pm 1} \gamma$, where the r_i runs within a finite set $R = \{r_1, r_2, \cdots, r_k\}$ of "*elementary*" relations.

In other words, that means that we can identify Γ with the quotient of $\mathrm{Free}(S)$, the free group on S, by the normal subgroup of relations generated by R. The basic examples of such groups are fundamental groups $\pi_1(M)$ of compact manifolds M.

Associated to any choice of generating set S of Γ is a graph $\mathcal{C}$, called *Cayley graph*, and defined as follows

- vertices of $\mathcal{C}$ are elements of Γ
- two elements γ, γ' of Γ are related by an (oriented) edge in $\mathcal{C}$ if $\gamma' = \gamma s_i^{\pm 1}$ for some $s_i \in S$.

Figure 1 shows two Cayley graphs associated to $\mathbb{Z}^2$.
In the left one we see $\mathbb{Z}^2$ as generated by two elements, while three (!) are used on the right one. (This academic example is just here to stress on the fact that

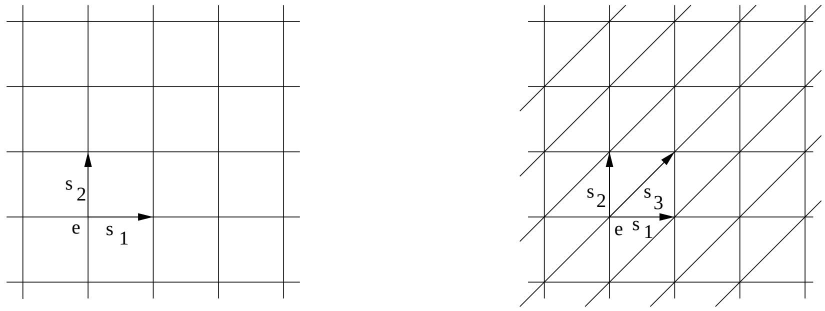


Figure 1: Dupond and Dupont.

they are many Cayley graphs associated to a single group, and that in general there is no preferred presentation.)

We observe that, given S, relations of Γ appears as closed loops in $\mathcal{C}$. So one can naturally complete $\mathcal{C}$ by adding two dimensional disks at any vertex γ, along the chosen elementary relations $r_i \in R$. One obtains then a two dimensional polyhedra $\mathcal{P}$ called *Cayley polyhedra.* For instance, in the left part of fig 1, one adds squares corresponding to the relation $r_1 = s_1 s_2 s_1^{-1} s_2^{-1}$, while one the right one we have to choose two basic relations, for example $r_1 = s_1 s_2 s_3^{-1}$ and $r_2 = s_2 s_1 s_3^{-1}$, and glue corresponding triangles. For a general Γ, the polyhedra $\mathcal{P}$ is simply-connected as comes from the fact that R generates all the relations of Γ. In fact $\mathcal{P}$ can be seen as the fundamental cover of a *finite* two dimensional polyhedra $\mathcal{P}/\Gamma$ where Γ acts on $\mathcal{P}$ by left translations.

The presentation complex. We now describe two natural operators associated to $\mathcal{P}$. Let $\ell^2(\Gamma)$, $\ell^2(S)$ and $\ell^2(R)$ denote respectively square integrable functions on Γ, $\Gamma \times S$ and $\Gamma \times R$. These interpret as the ℓ^2 functions spaces on the vertices, edges and two cells of $\mathcal{P}$ (or basic loops of $\mathcal{C}$).

The first operator d_0 goes from $\ell^2(\Gamma)$ to $\ell^2(S)$ and is defined by

$$(d_0 f)(\gamma, s) = f(\gamma s) - f(\gamma), \tag{1}$$

this is the *difference operator* of the function f along the edges of $\mathcal{P}$ (or $\mathcal{C}$). One can also define a *circulation operator* $d_1 : \ell^2(S) \to \ell^2(R)$ by

$$(d_1 \alpha)(\gamma, r) = \oint_{(\gamma, \gamma r)} \alpha, \tag{2}$$

to be understood as the finite sum of values of α encountered in the oriented closed loop in $\mathcal{P}$ (or $\mathcal{C}$) starting at γ and travelling along the elementary relation r.

One sees that $d_1 \circ d_0 = 0$. Moreover one has, at least locally, $\ker d_1 = \operatorname{Im} d_0$, as seen using the fact that any relation in Γ (closed loop in $\mathcal{C}$) can be solved by elementary relations (filled by two cells in $\mathcal{P}$). The piece of complex we described

$$\ell^2(\Gamma) \xrightarrow{d_0} \ell^2(S) \xrightarrow{d_1} \ell^2(R)\,, \tag{3}$$

will be called the *presentation complex* in the sequel. It is the beginning of the (ℓ^2) simplicial cochain complex one can introduce on the two dimensional Cayley polyhedra $\mathcal{P}$. The maps d_0 and d_1 are actually dual to the two *boundary* operators ∂_1 and ∂_2 available here. These linear ∂_1, ∂_2 are defined respectively on (ℓ^2) sums of oriented edges and two cells of $\mathcal{P}$ by

$$\partial_1(\text{edge } e) = \text{end } e - \text{origin } e \ ,$$

and

$$\partial_2(\text{two cell } r) = \sum \text{edges bounding } r\,.$$

Then the formula (2) for the circulation operator d_1 reads

$$\langle d_1\alpha, \text{two cell } r\rangle = \langle \alpha, \partial_2 r\rangle\,.$$

The simplicial cochain complex (3) may be extended if (locally finitely many) higher dimensional cells are available. This happens for instance on coverings of triangulations of smooth compact manifolds. Again the coboundary maps are dual to the boundary operators on the extra simplexes.

2.2 Measuring the heat decay and the spectrum

Heat operators and spectra. Let $\delta_0 : \ell^2(S) \to \ell^2(\Gamma)$ and $\delta_1 : \ell^2(R) \to \ell^2(S)$ denote the adjoints of the (bounded) simplicial differentials d_0 and d_1. We have two positive self-adjoint Laplacians

$$\Delta_0 = \delta_0 d_0 \quad \text{acting on} \quad \ell^2(\Gamma)\,,$$

and

$$\Delta_1 = d_0\delta_0 + \delta_1 d_1 \quad \text{acting on} \quad \ell^2(S).$$

We can consider the associated heat operators. By spectral resolution (see [37]) we have,

$$e^{-t\Delta} = \int_0^\infty e^{-t\lambda} dE_\Delta(\lambda),$$

where $E_\Delta(\lambda)$ denotes the spectral projection associated to $[0,\lambda]$ by $\Delta = \Delta_0$ or Δ_1. We see that for $t \to +\infty$, the heat operators $e^{-t\Delta_0}$ and $e^{-t\Delta_1}$ strongly converge towards orthogonal projections onto $\ker\Delta_0$ and $\ker\Delta_1$. (Recall that $P_n \to P$ strongly if $P_n f \to f$ in norm, for any *fixed* f.) As there is no harmonic ℓ^2 functions, if Γ is infinite, the first space actually vanishes. This is not necessarily the case for the space of Δ_1-harmonic ℓ^2 one cochains, which is isomorphic to

$$\ell^2 H^1_{\text{red}}(\Gamma) = (\ker d_1 \cap \ell^2)/\overline{\operatorname{Im} d_0}\,,$$

called the first reduced ℓ^2-cohomology of Γ (see e.g. [34] for an introduction).

According to [8], this cohomology group vanishes on amenable groups and in particular in the case of nilpotent groups we will study. In general anyway, one can split the asymptotic analysis of $e^{-t\Delta_1}$ more precisely. Using Hodge decomposition

$$\ell^2(S) = \ker\Delta_1 \oplus \overline{\operatorname{Im} d_0} \oplus \overline{\operatorname{Im}\delta_1},$$

we see that $e^{-t\Delta_1}$ may be written

$$e^{-t\Delta_1} = \Pi_{\ker \Delta_1} + e^{-td_0\delta_0}\Pi_{\overline{\operatorname{Im} d_0}} + e^{-t\delta_1 d_1}\Pi_{\overline{\operatorname{Im}\delta_1}}\,. \tag{4}$$

Therefore the study of large time behaviour of $e^{-t\Delta_1}$ divides in two cases,

1. study of $e^{-td_0\delta_0}$ on $\overline{\operatorname{Im} d_0} = (\ker \delta_0)^\perp$, where in fact this heat is conjugated by δ_0 to $e^{-t\Delta_0}$ on functions,
2. study of $e^{-t\delta_1 d_1}$ on $\overline{\operatorname{Im}\delta_1} = (\ker \delta_1)^\perp$, which a priori contains the new spectral information with respect to functions.

Γ-trace and the spectral density function. We now describe a way to measure the speed of the strong convergences we faced. All the operators P we met act on $\ell^2(\Gamma)\otimes V$, for some finite dimensional space V, and are Γ-invariant under left translations. So their (End(V) valued) Schwartz kernels $k(\gamma_1,\gamma_2)$ are actually of the form $k(\gamma_2^{-1}\gamma_1, e)$. In particular they are determined by their value on δ_e, the characteristic function of the neutral element $e \in \Gamma$, through the formula

$$P\delta_e = \sum_{\gamma\in\Gamma} k(\gamma, e)\delta_\gamma\,.$$

Furthermore the single $k(e,e)$ controls all the $k(\gamma, e)$ since, by positivity and symmetry of P, one sees that for all u, $v \in V$,

$$|(k(\gamma,e)u,v)|^2 \leq (k(e,e)u,u)(k(\gamma,\gamma)v,v) = (k(e,e)u,u)(k(e,e)v,v)\,,$$

and in particular using the trace on End(V),

$$\operatorname{Tr}_V(k^*(\gamma,e)k(\gamma,e)) \leq \big(\operatorname{Tr}_V(k(e,e))\big)^2.$$

Definition 2.1. Let P be a Γ-invariant bounded operator acting on some $\ell^2(\Gamma)\otimes V$, the number

$$\tau(P) = \operatorname{Tr}_V(k(e,e)) = \operatorname{Tr}_V\big((P\delta_e)(e)\big)$$

is called the Γ-trace of P.

We have seen that, for P positive symmetric, one has

- $\tau(P) \geq 0$, and also $\tau(P) = 0$ iff $P = 0$.

We can use this τ to define the *spectral density function* of P by

$$F_P(\lambda) = \tau(E_P(\lambda)) \tag{5}$$

where $E_P(\lambda)$ is the spectral projection associated to $[0,\lambda]$ by P. When Γ is trivial, F_P reduces to Weyl's repartition function, counting the eigenvalues lower than λ.

By the spectral theorem this increasing function is the building block to express the trace of $f(P)$ as the Stieljes integral

$$\tau(f(P)) = \int f(\lambda)dF_P(\lambda).$$

Applying this to $f(\lambda) = e^{-t\lambda}$ we have

$$\tau(e^{-tP}) = \int_0^{+\infty} e^{-t\lambda} dF_P(\lambda), \tag{6}$$

meaning that the function $\tau(e^{-tP})$ is actually the Laplace transform of the spectral density function of P. In particular by dominated convergence, we see that when $t \to +\infty$,

$$\tau(e^{-tP}) \to F_P(0) = \tau(\Pi_{\ker P}) \,.$$

More precisely it is a classical result (see appendix of [23]), that the asymptotics of the two functions $\tau(e^{-tP})$ when $t \to +\infty$ and $F_P(\lambda)$ when $\lambda \to 0^+$ are related in the following way. There exists $\alpha \in [0, +\infty]$ such that,

$$\tau(e^{-tP}) - F_P(0) \asymp t^{-\alpha} \quad \text{when} \quad t \to +\infty$$

iff

$$F_P(\lambda) - F_P(0) \asymp \lambda^{\alpha} \quad \text{when} \quad \lambda \to 0^+ \,,$$

where $f \asymp g$ means $\exists c, c' > 0$ such that $cf \leq g \leq c'f$. Observe that $\alpha = +\infty$ in the case $e^{-tP} - \Pi_{\ker P}$ has a super polynomial decay, for instance an exponential one, which happens when P has a spectral gap around zero.

A more general definition of this exponent α is

$$\alpha(P) = \liminf_{\lambda \to 0} \Big(\frac{\ln\big(F_P(\lambda) - F_P(0)\big)}{\ln \lambda} \Big) \tag{7}$$

which always exists in $[0, +\infty]$.

Remark 2.2. We could have considered a lim sup instead. All the results we will mention here will be independent of this choice. Also, as far as we know, there is no geometric example where these limits are distinct (for natural operators acting on Galois coverings of finite simplicial complex). Very few of these exponents have been computed so far anyway!

Novikov-Shubin numbers. Going back to our problem, we can now define two analytic exponents of the Cayley polyhedra $\mathcal{P}$ we have described in 2.1.

Definition 2.3. The Novikov-Shubin numbers of $\mathcal{P}$ are

$$\alpha_0(\mathcal{P}) = 2\alpha(\Delta_0) \quad \text{and} \quad \alpha_1(\mathcal{P}) = 2\alpha(\delta_1 d_1),$$

describing respectively large-time heat decay on functions and one cochains in $\overline{\operatorname{Im} \delta_1} = (\ker d_1)^{\perp}$.

Remark 2.4. According to (4) we could have also defined a third exponent $\alpha(d_0\delta_0)$, describing heat decay of $e^{-t\Delta_1}$ on one cochains in $\overline{\operatorname{Im} d_0} = (\ker \delta_0)^{\perp}$. Yet, one always has $\alpha(d_0\delta_0) = \alpha(\Delta_0)$, as will be seen in §3.2.

As mentioned in §2.1 the presentation complex (3) can be continued if higher dimensional cells exist. Namely, let K be a $(p+1)$-dimensional finite simplicial complex K and $\widetilde{K}$ denotes its universal cover, or even any Galois cover $\Gamma \to \widetilde{K} \to K$ with Γ a quotient group of $\pi_1(K)$.

Then for $k \leq p$ the kth Novikov-Shubin exponent $\alpha_k(\widetilde{K})$ is defined by

$$\alpha_k(\widetilde{K}) = 2\alpha(\delta_k d_k)\,, \tag{8}$$

where d_k is dual to the boundary map between $k+1$ and k-simplexes.

Remark 2.5. About the 2 factor in these formulas. This is a convenient normalisation as we will see for α_0. Also it disappears if one uses instead in the definition $|d_k| = (\delta_k d_k)^{1/2}$, the symmetric part of the polar decomposition of d_k. Namely, one has

$$\alpha(|d_k|) = 2\alpha(\delta_k d_k)$$

as follows from $E_{P^2}(\lambda^2) = E_P(\lambda)$ and $F_{P^2}(\lambda^2) = F_P(\lambda)$ for positive P, λ.

So far we have only considered analytic exponents associated to the discrete simplicial cochain complex. One can do a similar work for the de Rham complex on a smooth manifold. There is a notion of *de Rham Novikov-Shubin numbers* that can be defined as follows.

Let M be a smooth compact manifold. De Rham differential d acts between smooth p and $p+1$-forms on the universal cover $\widetilde{M}$. One is interested again in the bottom of the spectrum (or the speed of heat decay) of the essentially self-adjoint Laplacian $\Delta = d\delta + \delta d$, or more precisely of δd as acting on $(\ker d)^{\perp}$.

In order to get numerical invariants, one has to extend the function τ introduced in the discrete setting in Definition 2.1. One uses again the Schwartz kernels, but now doing some average of the values taken on the diagonal of $\widetilde{M} \times \widetilde{M}$. Precisely it is well known, as a consequence of ellipticity, that the heat operators $e^{-t\Delta}$ and the spectral projections $E_\Delta(\lambda)$ are smoothing operators. Therefore their Schwartz kernels k are smooth on $\widetilde{M} \times \widetilde{M}$.

In general, let $\mathcal{F} \subset \widetilde{M}$ be some fundamental domain of the $\Gamma = \pi_1(M)$ action and P be a smoothing Γ-invariant operator P acting on a Γ-invariant bundle V over $\widetilde{M}$. Then following Atiyah [1] we define the Γ-trace of P as

$$\tau(P) \text{ or } \operatorname{Tr}_\Gamma(P) = \int_{\mathcal{F}} \operatorname{Tr}(k(x,x))dx\,,$$

where Tr represents the point wise trace on $\operatorname{End}(V_x)$. Finally the pth de Rham Novikov-Shubin number of $\widetilde{M}$ is defined (nearly) as in (7) and (8) by,

$$\alpha_p^{\mathrm{dR}}(\widetilde{M}) = 2\liminf_{\lambda\to 0}\Big(\frac{\ln F(\lambda)}{\ln \lambda}\Big) \tag{9}$$

where

$$F(\lambda) = \tau\big(E_{\delta d}(]0,\lambda])\big)$$

is the Γ-trace of the spectral projection associated to $]0,\lambda]$ by δd acting on p-forms (which is also the projection $E_{\Delta_p}(\lambda) \circ \Pi_{\overline{\operatorname{Im}\delta}}$).

2.3 Topological invariance and around

Basic results. It turns out that these various analytic exponents, as defined on de Rham and discrete cochain complexes, tend to coincide for a given degree. Even more they are known to be topological invariants.

Theorem 2.6. *(Gromov-Shubin [23, 24], Efremov [17], Lott [29])*

- *Let K be a finite simplicial complex and $\Gamma \to \widetilde{K} \to K$ some Galois covering. Then $\alpha_p(\widetilde{K})$ only depends on the choice of Γ and the* homotopy class *of the $(p+1)$-skeleton of K (cells of K of dim $\leq p+1$).*

- *Let K be a triangulation of a compact smooth manifold M, and Γ some covering group. Then the pth simplicial Novikov-Shubin number $\alpha_p(\widetilde{K})$ coincides with de Rham's one $\alpha_p^{\mathrm{dR}}(\widetilde{M})$. In particular, given Γ, these numbers are* homotopical invariants *of M.*

An interesting particular case here is the following, corresponding to $K = \mathcal{P}/\Gamma$, where $\mathcal{P}$ is a Cayley polyhedra of Γ (see §2.1).

Corollary 2.7.
- *Let $\mathcal{P}$ be a Cayley polyhedra associated to a presentation of a discrete group Γ. Then the two Novikov-Shubin numbers $\alpha_0(\mathcal{P})$ and $\alpha_1(\mathcal{P})$ actually do not depend on the choice of the presentation (and $\mathcal{P}$) but only on Γ.*

- *If M is any smooth compact manifold with $\pi_1(M) = \Gamma$, then*

$$\alpha_0(M) = \alpha_0(\Gamma) \quad \text{and} \quad \alpha_1(M) = \alpha_1(\Gamma).$$

We will describe the basic ideas and ℓ^2 tools involved in the proof of these results in §2.

For the moment, we recall that the geometric signification of the first of these analytic exponents, $\alpha_0(\Gamma)$, describing large-time heat decay on functions, is known.

Theorem 2.8. *(Varopoulos [43], Gromov)*

- *$\alpha_0(\Gamma) < +\infty$ iff Γ is a group of polynomial growth,*

- *that is iff Γ is virtually nilpotent (has a finite nilpotent cover), and then*

$$\alpha_0(\Gamma) = \mathrm{growth}\,(\Gamma) =_{\mathrm{def}} \lim_{N \to +\infty} \frac{\ln\big(\mathrm{card}(B_\Gamma^S(N))\big)}{\ln N}$$
$$= \sum_n n\,\mathrm{rank}_{\mathbb{Z}}\,(\Gamma_n/\Gamma_{n+1}) \in \mathbb{N}\,,$$

where $B_\Gamma^S(N)$ stands for the set of elements of Γ that can be written as a product of at most N terms from a generating set S of Γ,

and Γ_n is the lower central series : defined by $\Gamma_1 = \Gamma$ and $\Gamma_n = (\Gamma, \Gamma_{n-1})$. This is the normal subgroup of Γ generated by iterated commutators

$$(\gamma_1, (\cdots(\gamma_{n-1}, \gamma_n))).$$

From the geometric side, it appears clearly that the growth, and therefore $\alpha_0(\Gamma)$, is a large scale invariant of Γ. We can also easily see that the growth does not depend on the choice of generators.

Indeed, suppose given two generating sets $S = \{s_1, \cdots, s_p\}$ and $S' = \{s'_1, \cdots, s'_n\}$. One can write each s_i in a finite product of elements of S' and reversely. Let c and c' the maximum lengths needed. This gives bounds to translate words in S in S', and reversely, so that one gets inclusions

$$B_\Gamma^S(N) \subset B_\Gamma^{S'}(cN) \quad \text{and} \quad B_\Gamma^{S'}(N) \subset B_\Gamma^{S'}(c'N),$$

from which follows the claim.

Analytic and geometric aspects. Hidden in the statement of the previous theorem, but crucial in Varopoulos proof, is that α_0 is also related to other important analytic and geometric properties of Γ.

One of them is that α_0 rules the Sobolev injections. Namely one has in ℓ^p norm for $p < \alpha_0$,

$$\|f\|_q \leq C\|d_0 f\|_p \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{\alpha_0}\,, \tag{10}$$

for functions with finite support on Γ. This provides a link to the geometrical interpretation of α_0. Indeed, using (10) with $p = 1$ and $f = 1\!\!1_\Omega$ gives the following isoperimetric inequality, between the size of sets $\Omega \subset \Gamma$ and their boundaries $\partial\Omega$,

$$\operatorname{card}(\Omega) \leq C\big(\operatorname{card}(\partial\Omega)\big)^{\frac{\alpha_0}{\alpha_0 - 1}}. \tag{11}$$

Applying this to the balls $\Omega = B_\Gamma^S(N)$ and summing (see [11]), one obtains then

$$\operatorname{card}\big(B_\Gamma^S(N)\big) \leq C' N^{\alpha_0}\,,$$

which gives the relation with the growth upper bound.

The relationships between large-time heat decay, Sobolev inequalities, isoperimetry and volume bounds has been much clarified and extended by many authors since Varopoulos work, see for instance the survey [10]. In all these approaches the use of ℓ^p spaces and analysis is required to translate the basic ℓ^2 spectral invariant α_0 into a geometric information, and reversely. Unfortunately (or luckily for geometers), most of the interpolation techniques needed deeply rely on the fact we are dealing with *functions* here, at least through the maximum principle that heat operator $e^{-t\Delta_0}$ decays sup (or ℓ^1) norms. (As was patiently explained to me by Thierry Coulhon.) They do not extend automatically when working on *forms*.

Let us mention anyway that a more 'elementary' approach (with respect to analysis) exists in the particular case we will restrict of graded nilpotent (Carnot) groups, that is nilpotent groups with dilations. Namely, we will see there that a direct link of α_0 with the volume growth can be obtained from an *homogeneity* argument. The trick, from the geometric side, is to use a more convenient (homogeneous) differential on functions, instead of the standard one, to compute α_0. This is suggested by the underlying idea, in Theorem 2.6 and

Corollary 2.7, that α_0 is a very 'stable' number that can be computed using many geometric approaches.

We will play a similar game, based on homogeneity of modified differentials, to estimate the next Novikov-Shubin exponent $\alpha_1(\Gamma)$ on such groups. This will allow us to relate it to the depth of the relations necessary to present Γ from the free group.

Yet, there are many examples where such an elementary approach only gives a geometric *pinching* of $\alpha_1(\Gamma)$. Going ahead, even in the case of Carnot groups, should rely on ℓ^p techniques, or more powerful analytic tools, as was the case for α_0.

Some relevant analysis, based on hypoellipticity notion has been presented in [38, 40]. Nevertheless, we would like to stress on the fact that the picture of possible results, both from the geometric and analytic viewpoints, is still very unclear. In particular, a large part of the ℓ^p machinery used on functions is not available here, due as already observed, to the lack of basic tools like the maximum principle on forms (or discrete cochains) in non-positive curvature.

Discrete time. We conclude with an alternative "discrete in time" presentation of the exponent α_1, more attractive from the numerical viewpoint.

Recall that α_1 has been introduced as being twice the (continuous) large-time heat decay exponent of $e^{-t\delta_1 d_1}$ on the one-cochains in $H = (\ker d_1)^\perp \subset \ell^2(S)$. This abstract Hilbert space H is not so convenient to use numerically. A first fact is that, on nilpotent groups, one can use instead the heat decay of the full discrete Laplacian

$$\Delta_1 = d_0\delta_0 + \delta_1 d_1$$

acting on $\ell^2(S)$. This is because, as we will see (see also [31]), the heat decay associated to $d_0\delta_0$ (conjugated to $\Delta_0 = \delta_0 d_0$ on functions) is always quicker that the one induced on H by $\delta_1 d_1$. From the numerical viewpoint, Δ_1 is a Γ-invariant linear operator acting on functions on the discrete space $\Gamma \times S$ (space of edges of the Cayley graph of Γ associated to the generating set S). Moreover, Δ_1 is a *local* operator in the sense that the value of $\Delta_1\alpha$ at $(\gamma, s) \in \Gamma \times S$ is a linear combination of $\alpha(\gamma', s')$ with $\gamma'\gamma^{-1}$ in a fixed finite neighbourhood of the neutral element.

Now we describe the discrete time approach, starting with the case of functions. Recall that for functions on Γ, the (continuous in time) heat decay can also be obtained from the asymptotic return probability of random walk on Γ (see [10]). This is due to the relation

$$\Delta_0 = \mathrm{Id} - P_s$$

where P_s is the Markov operator of the standard random walk on the Cayley graph $\mathcal{C}$ (a particle on a vertex of $\mathcal{C}$ jumps to any of its neighbours with equal probability). Using instead the more convenient random walk associated to

$$P = \frac{\mathrm{Id} + P_s}{2} = \mathrm{Id} - \frac{\Delta_0}{2} \tag{12}$$

(here the particle doesn't move with probability 1/2), we see that the return to origin probability after n steps is given by

$$P^n(\delta_e)(e) = \tau(P^n),$$

with notation of Definition 2.1. By the spectral theorem, and $\|P_s\| \leq 1$, we have

$$\tau(P^n) = \int_0^2 \big(1 - \frac{\lambda}{2}\big)^n dF_{\Delta_0}(\lambda), \tag{13}$$

where, following (5), $F_{\Delta_0}(\lambda) = \tau(E_{\Delta_0}(\lambda))$ is the spectral density function of Δ_0. This gives that the rate decays of $\tau(P^n)$ when $n \to +\infty$ and $F_{\Delta_0}(\lambda)$ when $\lambda \to 0^+$ are the same, more precisely

$$\liminf_{n\to+\infty}\Big(\frac{\ln\tau(P^n)}{-\ln n}\Big) = \liminf_{\lambda\to 0^+}\Big(\frac{F_{\Delta_0}(\lambda)}{\ln\lambda}\Big) = 2\alpha_0(\Gamma), \tag{14}$$

(and the same for lim sup).

Proof. Cutting the integral (13) for $\lambda \leq \lambda_n = 2(1 - 2^{-1/n})$, gives

$$\tau(P^n) \geq \frac{1}{2} F_{\Delta_0}(\lambda_n)$$

and a first inequality in (14), using $\ln\lambda_n \sim -\ln n$ when $n \to +\infty$.

In the opposite direction, if $F_{\Delta_0}(\lambda) \leq C\lambda^\alpha$, integration by parts gives

$$\begin{aligned}
\tau(P^n) &= \int_0^2 \frac{n}{2}\big(1 - \frac{\lambda}{2}\big)^{n-1} F_{\Delta_0}(\lambda)d\lambda \\
&\leq \frac{C}{2}\int_0^2 n\big(1 - \frac{\lambda}{2}\big)^{n-1}\lambda^\alpha d\lambda \\
&= C'\int_0^{2n} \big(1 - \frac{t}{2n}\big)^{n-1}\big(\frac{t}{n}\big)^\alpha dt \\
&\sim C' n^{-\alpha}\int_0^{+\infty} e^{-t/2} t^\alpha dt \quad \text{for} \quad n \to +\infty.
\end{aligned}$$

□

This approach does not rely on probability techniques (except for its intuitive meaning) but only on the spectral theorem, and therefore applies also for other combinations of operators. In particular, on discrete one cochains, one can use instead of (12),

$$P = \mathrm{Id} - k\Delta_1,$$

for any k such that $k \leq \|\Delta_1\|^{-1}_{(2,2)}$. Then using the trace at e,

$$\tau(P^n) = \mathrm{Tr}\big((P^n\delta_e)(e)\big)$$

as defined in Definition 2.1, one obtains similarly that

$$\liminf_{n\to+\infty}\Big(\frac{\ln\tau(P^n)}{-\ln n}\Big) = \liminf_{\lambda\to 0^+}\Big(\frac{\ln(F_{\Delta_1}(\lambda))}{\ln\lambda}\Big) = 2\alpha_1(\Gamma), \tag{15}$$

which is also the large-time heat decay, as mentioned in §2.2.

From the numerical viewpoint, the computation of the iteration $P^n\delta_e$ on a nilpotent group Γ uses a memory space polynomial in n (of degree $\alpha_0(\Gamma) = \mathrm{growth}(\Gamma)$, since the kernel support of P^n spreads this way), and a polynomial time. In practice however the convergence in (15) may be slow since this is a $\ln / \ln$ limit.

3 Quick review of the basic ℓ^2 tools

3.1 Homotopy of Hilbert complexes

Homotopies from the analytic viewpoint. We now present the basic ideas leading to the homotopical invariance of the Novikov-Shubin numbers, as stated in Theorem 2.6 and Corollary 2.7.

At first sight, it seems unlikely that exponents built from the spectrum may possess a strong topological invariance. Indeed, spectrum of Laplacians depends on the metric and thus should only be isometry invariants of the manifold. This is (more or less) the case for the *full* spectrum, but we are only concern here in the *near-zero* spectrum, more precisely in the asymptotic behaviour of the spectral density function at zero. Then, the topological invariance of this behaviour may be understood as an extension of the well known fact that the kernel, or zero spectrum, of Laplacians has actually a topological sense, since it represents cohomology.

The tools needed to grasp this idea have been introduced by Gromov and Shubin in [23, 24]. The general setting of the problem is on *Hilbert complexes.* Indeed we have met several sequences of Hilbert spaces H_k

$$(H,d) \qquad\qquad 0 \to H_0 \xrightarrow{d_0} H_1 \xrightarrow{d_1} H_2 \cdots$$

with closed densely defined operators d_k such that $d_{k+1} \circ d_k = 0$ on the domain $D(d_k)$ of d_k. A relevant notion of homotopy here is the following.

Definition 3.1. Two Hilbert complexes (H,d) and (H',d') are said *homotopy equivalent* up to degree p, if there exists *bounded* maps

$$f_k : H_k \to H'_k \quad \text{and} \quad g_k : H'_k \to H_k$$

for $k \leq p+1$, such that for $k \leq p$,

$$f_{k+1}d_k = d'_{k+1}f_k \text{ on } D(d_k) \quad \text{and} \quad g_{k+1}d'_k = d_{k+1}g_k \text{ on } D(d'_k),$$

with

$$g_kf_k = \mathrm{Id}_{H_k} + d_{k-1}h_k + h_{k+1}d_k \quad \text{on} \quad D(d_k)$$

and

$$f_kg_k = \mathrm{Id}_{H'_k} + d'_{k-1}h'_k + h'_{k+1}d'_k \quad \text{on} \quad D(d'_k),$$

for some bounded maps h_k and h'_k. The corresponding diagram is

$$\begin{array}{ccccc}
H_{k-1} & \underset{h_k}{\overset{d_{k-1}}{\rightleftarrows}} & H_k & \underset{h_{k+1}}{\overset{d_k}{\rightleftarrows}} & H_{k+1} \\
{\scriptstyle g_{k-1}}\uparrow\downarrow{\scriptstyle f_{k-1}} & & {\scriptstyle g_k}\uparrow\downarrow{\scriptstyle f_k} & & {\scriptstyle g_{k+1}}\uparrow\downarrow{\scriptstyle f_{k+1}} \\
H'_{k-1} & \underset{h'_k}{\overset{d'_{k-1}}{\rightleftarrows}} & H'_k & \underset{h'_{k+1}}{\overset{d'_k}{\rightleftarrows}} & H'_{k+1}
\end{array}$$

Remark 3.2. If moreover some discrete group Γ is acting both on H and H', one asks that all involved operators commute with this action.

We give some examples useful to our study.

The case of the presentation complex. Consider first two Cayley polyhedras $\mathcal{P}$ and $\mathcal{P}'$ associated to two presentations of a finitely presented discrete group Γ, as in §2.1. We have two presentation complexes as described in (3)

$$\ell^2(\Gamma) \xrightarrow{d_0} \ell^2(S) \xrightarrow{d_1} \ell^2(R) \quad \text{and} \quad \ell^2(\Gamma) \xrightarrow{d'_0} \ell^2(S') \xrightarrow{d'_1} \ell^2(R').$$

As a first step to Corollary 2.7, let us show:

Proposition 3.3. *These Hilbert complexes are homotopy equivalent up to degree one. In particular, their homotopy class only depends on* Γ.

Proof. The required maps are maybe easier to see on the dual chain complexes

$$\ell^2(\Gamma) \xleftarrow{\partial_1} \ell^2(S) \xleftarrow{\partial_2} \ell^2(R) \quad \text{and} \quad \ell^2(\Gamma) \xleftarrow{\partial'_1} \ell^2(S') \xleftarrow{\partial'_2} \ell^2(R'),$$

so that we focus on them.

Let $S = \{s_1, \cdots, s_n\}$, $S' = \{s'_1, \cdots, s'_m\}$ be the two generating sets of Γ. Each $s \in S$ can be written as a word $f_1(s)$ in elements of $S' \cup S'^{-1}$, and reversely, each $s' \in S'$ is a word $g_1(s')$ in $S \cup S^{-1}$. These $f_1(s)$ and $g_1(s')$ correspond to paths in the Cayley polyhedras $\mathcal{P}'$ and $\mathcal{P}$ (see §2.1). To these paths are also associated one chains which are the sums of the edges encountered. The boundaries of these chains satisfy

$$\partial'_1(f_1(s)) = s - e = \partial_1(s) \quad \text{and} \quad \partial_1(g_1(s')) = s' - e = \partial'_1(s'),$$

so that

$$\partial'_1 f_1 = \partial_1 \quad \text{and} \quad \partial_1 g_1 = \partial'_1. \tag{16}$$

We still denote by

$$f_1 : \ell^2(S) \to \ell^2(S') \quad \text{and} \quad g_1 : \ell^2(S') \to \ell^2(S)$$

the linear Γ left-invariant extensions of the previous maps.

For each $s \in S$, (16) gives that $g_1(f_1(s))$ has same endpoints (boundary) than s . Therefore, by simple connectivity of $\mathcal{P}$, one can fill the cycle $g_1(f_1(s)) - s$ with a (finite) two chain $h_1(s) \in \mathrm{Vect}(R) \subset \ell^2(R)$, that is

$$g_1(f_1(s)) - s = \partial_2 h_1(s),$$

which extends as before on $\ell^2(S)$ in

$$g_1 f_1 = \mathrm{Id} + \partial_2 h_1.$$

Lastly, for each two cell $r \in R$ attached at e, one has by (16)

$$\partial_1'(f_1 \partial_2 r) = \partial_1 \partial_2 r = 0.$$

Again, one can choose some two chain $f_2(r) \in \mathrm{Vect}(R') \subset \ell^2(R')$ such that

$$f_1 \partial_2 r = \partial_2'(f_2(r)).$$

Therefore by extension we have a map $f_2 : \ell^2(R) \to \ell^2(R')$ satisfying

$$f_1 \partial_2 = \partial_2' f_2,$$

and a similar one $g_2 : \ell^2(R') \to \ell^2(R)$, completing the picture

$$\begin{array}{ccccc}
\ell^2(\Gamma) & \underset{0}{\overset{\partial_1}{\rightleftarrows}} & \ell^2(S) & \underset{h_1}{\overset{\partial_2}{\rightleftarrows}} & \ell^2(R) \\
\mathrm{Id} \Big\| & & g_1 \uparrow\downarrow f_1 & & g_2 \uparrow\downarrow f_2 \\
\ell^2(\Gamma) & \underset{0}{\overset{\partial_1'}{\rightleftarrows}} & \ell^2(S') & \underset{h_1'}{\overset{\partial_2'}{\rightleftarrows}} & \ell^2(R')
\end{array}$$

All these Γ-invariant maps are bounded and even local. □

The previous proof is purely combinatorial and is indeed a special instance of general topological constructions on simplicial complexes [42]. Here, the *analytic* (partial) homotopies of these Hilbert complexes are induced from the *geometric* ones, between the two finite simplicial complexes $K = \mathcal{P}/\Gamma$ and $K' = \mathcal{P}'/\Gamma$. The existence of the latter is due to $\pi_1(K) = \pi_1(K') = \Gamma$.

More generally, if K and K' are two finite simplicial complexes which are homotopy equivalent up to degree d, then there exists a simplicial map $f : K_{d+1} \to K'_{d+1}$ which formally induces an homotopy equivalence, up to degree d, between the ℓ^2 cochain complexes of given regular covers of K and K'.

From de Rham to simplicial complexes. The previous principle, that homotopy in the topological sense implies homotopy of relevant Hilbert complexes, also applies between L^2-de Rham complexes on covers of smooth compact manifolds. This was proved by Gromov and Shubin in [23, 22].

Theorem 3.4. *Let M and N be homotopic smooth compact manifolds and Γ some covering group. Then the L^2-de Rham complexes on the Γ-coverings $\widetilde{M}$ and $\widetilde{N}$ are homotopy equivalent.*

Given a triangulation K of a smooth manifold M, it remains to compare the L^2 de Rham complex of $\widetilde{M}$ with the ℓ^2 simplicial cochain complex of $\widetilde{K}$. They are also homotopy equivalent. Some natural but delicate proof, working in L^p, is given by Gol'dshtein, Kuz'minov and Shvedov in [19]. As in earlier work of

Dodziuk [15], it relies on the use of (regularized) de Rham and Whitney maps, between de Rham and cochain complexes.

Another approach, unifying these results, and much more elementary analytically, has been proposed by Pansu in [33] and [34, Chapter 4]. It consists in adapting classical principles from sheaf theory (see eg [18] or [20, Chapter 0.3]) to complexes of Hilbert sheaves (or more generally Banach sheaves).

Loosely speaking (see [33, 34] for more precise and general statements), one obtains that *any* Γ-invariant Hilbert complex of sheaves on a cover $\widetilde{X}$ of a metric space X, which is uniformly acyclic relatively to some open covering $\mathcal{U}$ of X, is homotopy equivalent to the ℓ^2 Čhech complex of the covering $\widetilde{\mathcal{U}}$ of $\widetilde{X}$, also the ℓ^2-simplicial cochain complex of the nerve of the covering.

The uniformity assumption is easily checked on Alexander-Spanier cochain complex of small size, but also on the de Rham complex, where it reduces to the following local integration lemma.

Lemma 3.5. *(L^2 Poincaré Lemma on the unit ball, [33, 34])*

Let B^n be the unit ball in $\mathbb{R}^n$. There exists a constant C such that any closed L^2 form α on B^n can be written $d\beta$ for some β with $\|\beta\|_2 \leq C\|\alpha\|_2$.

This is proved by averaging over the ball Poincaré's integration formula.

3.2 Near-cohomology and Γ-trace

Quadratic forms versus spectra. Now we return to the presentation of the tools leading to the homotopy invariance of the Novikov-Shubin numbers. Let (H, d) and (H', d') be homotopy equivalent Hilbert complexes.

By Definition 3.1, the maps $f : H \to H'$ and $g : H' \to H$, induce inverse topological isomorphisms between the cohomology spaces $\mathcal{H} = \ker d/\operatorname{Im} d$ of H and H', and also for the reduced cohomology $\mathcal{H}_r = \ker d/\overline{\operatorname{Im} d}$, but not at all between the spectrum of say, the induced Laplacians Δ and Δ'.

In comparison to these quite delicate analytic data, the (Dirichlet) quadratic forms, defined on $D(d)$ by

$$Q_d(\alpha) = \|d\alpha\|^2,$$

better behave since one has obviously

$$Q_{d'}(f\alpha) \leq C\, Q_d(\alpha) \quad \text{and} \quad Q_d(g\alpha) \leq C'\, Q_{d'}(\alpha),$$

for fixed constants $C = \|f\|^2$ and $C' = \|g\|^2$.

Taking account of this fact, Gromov and Shubin considered in [23, 24] the family of closed cones for $\varepsilon > 0$,

$$C_d(\varepsilon) = \{\alpha \in \widehat{H} = H/\ker d \mid Q_d(\alpha) \leq \varepsilon^2\|\alpha\|^2\}. \tag{17}$$

These cones are shrinking to $\{0\}$ when $\varepsilon \to 0^+$, and contains forms which are 'nearly' closed. One defines an equivalence relation called *near-cohomology* on such families of cones.

Definition 3.6. Two Hilbert complexes (H, d') and (H, d) have same near-cohomology if for ε small enough there exists a constant $k > 0$ and bounded injective maps $f : C_d(\varepsilon) \to C_{d'}(k\varepsilon)$ and $g : C_{d'}(\varepsilon) \to C_d(k\varepsilon)$.

This notion is compatible with homotopy equivalence as defined in 3.1.

Theorem 3.7. *([23, 24]) Homotopy equivalent Hilbert complexes, up to degree p, have the same near-cohomology, up to degree p.*

Proof. We give the proof for completeness. With the notations of Definition 3.1, we want to show that $f : H \to H'$ induces an injective map from $C_d(\varepsilon)$ into $C_{d'}(k\varepsilon)$, for ε small enough and some $k > 0$. Let

$$\widehat{f} : \widehat{H} = H/\ker d \simeq (\ker d)^{\perp} \longrightarrow \widehat{H}' = H'/\ker d'$$

be the quotient (or projection) map induced by f, and let $\alpha \in C_d(\varepsilon) \subset \widehat{H}$, then

$$\begin{aligned} \|d'(\widehat{f}\alpha)\| &= \|d'(f\alpha)\| = \|fd\alpha\| \\ &\leq \|f\|\|d\alpha\| \leq \varepsilon\|f\|\|\alpha\|. \end{aligned} \tag{18}$$

We need to control α by $\widehat{f}\alpha$. One has $\widehat{f}\alpha = f\alpha + \beta$ with $\beta \in \ker d'$, so that using the homotopy formula for $g \circ f$ (valid on H_k for $k \leq p$),

$$g(\widehat{f}\alpha) = g(f\alpha) + g\beta = \alpha + dh\alpha + hd\alpha + g\beta.$$

In this decomposition, $\alpha \in (\ker d)^{\perp}$ is orthogonal to $dh\alpha + g\beta \in \ker d$, and therefore,

$$\begin{aligned} \|\alpha\| &\leq \|\alpha + dh\alpha + g\beta\| = \|g(\widehat{f}\alpha) - hd\alpha\| \\ &\leq \|g(\widehat{f}\alpha)\| + \|h\|\varepsilon\|\alpha\|. \end{aligned}$$

Hence, for $\varepsilon < \|h\|^{-1}$,

$$\|\alpha\| \leq \frac{\|g\|}{1 - \varepsilon\|h\|}\|\widehat{f}\alpha\|.$$

This proves the injectivity of $\widehat{f}$ acting on $C_d(\varepsilon)$ and, together with (18), that $\widehat{f}(C_d(\varepsilon)) \subset C_{d'}(k\varepsilon')$ for $\varepsilon = (2\|h\|)^{-1}$ and $k = 2\|f\|\|g\|$. □

Basic properties of τ. We still have to relate this abstract notion to numerical information as contained in the Novikov-Shubin invariants. This will rely on properties of the Γ-trace τ introduced in §2.2. We briefly review them for the reader's convenience. More details can be found in Atiyah's original work [1], or the survey [34, Chapter 2].

Recall that we are working on Hilbert spaces H of the type $\ell^2(\Gamma) \otimes V$. Here V is either finite dimensional, in the case of the ℓ^2 cochain complex, or for the de Rham complex, an Hilbert space $L^2(\mathcal{F}, \Lambda^* M)$ of L^2 sections of the exterior bundle $\Lambda^* M$ over a fundamental domain $\mathcal{F}$ of the Γ action. In any case, one can define the trace Tr_V on positive operators acting on V by

$$\mathrm{Tr}_V(P) = \sum_j (Pv_j, v_j) \in [0, +\infty], \tag{19}$$

for any Hilbert basis v_j of V. Let $i_e : V \to H$ be the injection defined by $i_e(v) = \delta_e \otimes v$, and $\pi_e = i_e^* : H \to V$ be the evaluation map at e. Then (19) extends on positive Γ-invariant operators P acting on $H = \ell^2(\Gamma) \otimes V$, by

$$\tau(P) = \mathrm{Tr}_V(\pi_e P i_e) = \sum_j (P(\delta_e \otimes v_j), \delta_e \otimes v_j). \tag{20}$$

Here the important facts about τ are the following.

Proposition 3.8. *τ is a positive faithful trace, meaning that for Γ-invariant bounded operators P*

- *$\tau(P^*P) \geq 0$ and $\tau(P) = 0$ iff $P = 0$,*
- *$\tau(P^*P) = \tau(PP^*)$.*

Proof. The first property has already been seen in §2.2. For the second one, by (20),

$$\begin{aligned}
\tau(P^*P) &= \sum_j \|P(\delta_e \otimes v_j)\|^2 = \sum_{j,\gamma} |(P(\delta_e \otimes v_j), \delta_\gamma \otimes v_j)|^2 \\
&= \sum_{j,\gamma} |(P^*(\delta_\gamma \otimes v_j), \delta_e \otimes v_j)|^2 \\
&= \sum_{j,\gamma} |(P^*(\delta_e \otimes v_j), \delta_{\gamma^{-1}} \otimes v_j)|^2, \quad \text{by } \Gamma \text{ invariance}, \\
&= \sum_j \|P^*(\delta_e \otimes v_j)\|^2 = \tau(PP^*).
\end{aligned}$$

□

Using τ, we can 'measure' a Γ-invariant subspace $L \subset H$. We define its Γ-dimension by

$$\dim_\Gamma L = \tau(\Pi_L), \tag{21}$$

where Π_L is the orthogonal projection on the closure $\overline{L}$ of L. A striking property of $\dim_\Gamma$ is the following invariance.

Proposition 3.9. *Let L be a closed Γ-invariant subspace of H, and $f : L \to H$ be an* injective *closed densely defined Γ-invariant operator, then*

$$\dim_\Gamma(f(L)) = \dim_\Gamma(L).$$

Proof. Let $f\Pi_L = US$ be the polar decomposition of $f\Pi_L$ (see [37, Section VIII.9]). Recall that $S = |f\Pi_L|$ is positive and self-adjoint, while U is a partial isometry from $(\ker(f\Pi_L))^\perp = L$ (by injectivity of f on L here) to $\overline{f(L)}$. Hence

$$U^*U = \Pi_L \quad \text{and} \quad UU^* = \Pi_{f(L)},$$

and Proposition 3.8 gives

$$\dim_\Gamma(L) = \tau(U^*U) = \tau(UU^*) = \dim_\Gamma(f(L)).$$

□

Example 3.10. As a first use, let us apply this to the problem in remark 2.4. We want to show that the heat decay of $e^{-t\Delta_0}$ on functions is the same as for $e^{-td\delta}$ on one forms $\overline{\operatorname{Im} d}$. By Laplace transform (6) we need to compare the trace of the two spectral projectors $E_{\Delta_0}(]0,\lambda])$ and $E_{\delta d}(]0,\lambda])$. In fact by Proposition 3.9 they are even equal, since d maps injectively $\operatorname{Im} E_{\Delta_0}(]0,\lambda])$ into $\operatorname{Im} E_{\delta d}(]0,\lambda])$ and reversely for δ.

Measuring near-cohomology. We conclude by relating the near-cohomology to the Novikov-Shubin exponents. Recall they were defined in §2.2 as the polynomial decay when $\lambda \to 0$ of the spectral density function

$$F_{\delta d}(\lambda^2) = \tau(\Pi_\lambda),$$

where $\Pi_\lambda = E_{\delta d}(]0,\lambda^2])$ is the spectral projection associated to $]0,\lambda^2]$ by δd.

By the spectral theorem, the space $L_\lambda = \operatorname{Im} \Pi_\lambda$ is a closed Γ-invariant linear *subspace* of the (near-cohomology) cone

$$C_d(\lambda) = \{\alpha \in \widehat{H} = (\ker d)^\perp \mid \|d\alpha\|^2 = (\delta d\alpha, \alpha) \leq \lambda^2 \|\alpha\|^2\}.$$

It is in some sense the largest one. Namely, if $L' \subset C_d(\lambda)$ is another such space, then it projects *injectively* on L_λ by Π_λ, since the spectral theorem gives,

$$\|d\alpha\|^2 = (\delta d\alpha, \alpha) > \lambda^2 \|\alpha\|^2$$

for any non zero $\alpha \in \ker \Pi_\lambda \cap \widehat{H} = \operatorname{Im} E_{\delta d}(]\lambda^2, +\infty[)$. Using the Γ-dimension this translates numerically into the following variational principle, due to Shubin.

Lemma 3.11. *[23] Let $\mathcal{L}_\lambda$ be the set of all Γ-invariant linear subspace in $C_\lambda(d)$. Then*

$$F_{\delta d}(\lambda^2) = \sup_{L \in \mathcal{L}_\lambda} \dim_\Gamma L.$$

Proof. If $L \in \mathcal{L}_\lambda$ then by the previous argument and Proposition 3.9,

$$\dim_\Gamma L = \dim_\Gamma(\Pi_\lambda(L)) \leq \dim_\Gamma(\operatorname{Im} \Pi_\lambda) = F_{\delta d}(\lambda^2),$$

since $\dim_\Gamma$ is an increasing function by positivity of τ. □

Connecting this with Theorem 3.7, and using again Proposition 3.9, we can now compare spectral functions of homotopy equivalent Hilbert complexes.

Theorem 3.12. *[23, 24] Let (H,d) and (H',d') be homotopy equivalent Hilbert complexes (of type $\ell^2(\Gamma) \otimes V$). Then there exists $C, C' > 0$ such that*

$$F_{\delta d}(C\lambda) \leq F_{\delta' d'}(\lambda) \leq F_{\delta d}(C'\lambda),$$

and in particular (H,d) and (H',d') have the same Novikov-Shubin numbers (like any other dilatationally invariant limit built from $F_{\delta d}(\lambda)$).

This together with the results of §3.1, linking homotopy of Hilbert complexes to homotopy of metric spaces, implies the topological invariance of these numbers as stated in Theorem 2.6 and Corollary 2.7.

These general techniques will also be very useful in the particular case we will restrict now of Carnot groups.

4 The presentation complex seen from far

4.1 Carnot groups

Why? We would like to describe here some formal asymptotic rescaling of the presentation complex (3) that can be done for discrete groups embedded in nilpotent Lie groups with dilations.

Nilpotent groups provide an interesting class with respect to the study of Novikov-Shubin numbers. Recall that by Theorem 2.8 they are, up to finite coverings, the only one with finite first exponent α_0 on functions. Moreover, on forms or cochains of higher degree, one can show that for such groups, zero is never isolated in the spectrum of the Laplacian [30, Prop. 20] : a first necessary condition for finiteness of the next exponents α_p.

Lastly by a theorem of Mal'cev [36, Chap. 2], any finitely generated torsion-free nilpotent discrete group Γ can be cocompactly embedded into a nilpotent *Lie group* G. This Lie group G, called Mal'cev completion of Γ, is such that $\ln \Gamma$ spans $\mathfrak{g}$, even more, $\ln \Gamma$ is a finite index subgroup of a lattice (additive subgroup) of $\mathfrak{g}$. Reversely, Mal'cev has shown that a nilpotent Lie group G admits a cocompact discrete group Γ iff its Lie algebra $\mathfrak{g}$ has a rational structure $\mathfrak{g}_{\mathbb{Q}}$, *i.e.* admits a basis with brackets given by rational coefficients.

By Corollary 2.7 these contractible Lie groups G provide us natural smooth models with the same Novikov-Shubin numbers as their discrete cocompact Γ. Thus differential geometry, and even Lie group techniques, are available to investigate the problem on nilpotent groups (a pity for the pure topologist but a chance for us!).

We note that the irruption of smooth structures is not so artificial in this study. After all these exponents are large scale invariants (stable under finite coverings), and one knows for instance on $\mathbb{Z}^d$, that at large time the random walk

$$P_s^n = (\mathrm{Id} - \Delta_0^{\mathrm{simpl}})^n$$

(see §2.2) do converges under appropriate rescaling to the kernel of the 'smooth heat' on $\mathbb{R}^d$ (the Gaussian law),

$$n^d P_s^{[tn^2]}([nx],[ny]) \xrightarrow[n\to+\infty]{} e^{-t\Delta_0^{\mathrm{smooth}}}(x,y).$$

Such a rescaling (central limit) result actually holds on nilpotent groups with dilations [13].

Definition 4.1. A connected nilpotent Lie group G is called a Carnot group if its Lie algebra $\mathfrak{g}$ splits in

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r \quad \text{with} \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}.$$

The one-parameter family of Lie group automorphisms induced by

$$h_\varepsilon = \varepsilon^k \,\mathrm{Id} \quad \text{on} \quad \mathfrak{g}_k$$

are called dilations of G.

Remark 4.2. 'Carnot group' seems to be a relatively recent terminology. In other places, like in subelliptic theory, such groups have been called *filtered* nilpotent Lie groups. This is a particular case of *graded* groups, where one only asks

$$[\mathfrak{g}_1, \mathfrak{g}_k] \subset \mathfrak{g}_{k+1},$$

and which also possess dilations.

Shrinking d_0. Suppose given now a rational Carnot group G together with a discrete cocompact group Γ. We assume moreover that Γ is generated by elements $\gamma_i = \exp X_i$ with $X_i \in \mathfrak{g}_1$. Choose some 'elementary' relations $R = \{r_j\}$ associated to this generating set $S = \{\gamma_i\}$ of Γ (see §2.1). We would like to look at the presentation complex (see (3))

$$\ell^2(\Gamma) \xrightarrow{d_0} \ell^2(S) \xrightarrow{d_1} \ell^2(R)$$

at large scale. Equivalently we can consider the presentation complexes of the shrunk groups $\Gamma_\varepsilon = h_\varepsilon \Gamma$ at fixed scale. We don't deal with analytical problems here since we just want some formal hint of what's happening when $\varepsilon \to 0$.

Recall that the first map d_0 is the difference operator (1) so that for smooth functions f restricted to Γ_ε

$$\begin{aligned} d_0^{\Gamma_\varepsilon} f(\gamma, h_\varepsilon \gamma_i) &= f(\gamma h_\varepsilon(\gamma_i)) - f(\gamma) \\ &= f(\gamma \exp(\varepsilon X_i)) - f(\gamma) \\ &\sim \varepsilon (X_i.f)(\gamma) = \varepsilon df(\gamma)(X_i) \quad \text{when} \quad \varepsilon \to 0. \end{aligned}$$

Hence $\varepsilon^{-1} d_0^{\Gamma_\varepsilon} f$ converges to $d_H f$: the differential of f along the *horizontal* bundle $H = \mathfrak{g}_1$ ($= \mathrm{span}_{\mathbb{R}}(\ln S)$ also here).

Our next issue will be to describe the asymptotic of the differential d_1

$$d_1^{\Gamma_\varepsilon} \alpha(r_\varepsilon) = \oint_{r_\varepsilon} \alpha \tag{22}$$

on a shrinking relation $r_\varepsilon = h_\varepsilon r$ or Γ_ε.

4.2 Discrete and infinitesimal relations

For free. In the treatment of shrinking relations, as in (22), it is useful to introduce an analogous notion of *infinitesimal* relations for Lie algebras. These are defined with respect to *free Lie algebras*. We briefly describe this framework.

The free associative algebra $\mathcal{A}(H)$ over the vector space $H = \mathfrak{g}_1$ is the direct sum of all tensor products $\otimes^p H$. Given a basis $\{X_1, \cdots, X_n\}$ of H, $\mathcal{A}(H)$ identifies with the space of non-commutative polynomials in X_i. The bracket

$$[P, Q]_A = PQ - QP$$

defines a Lie algebra structure on $\mathcal{A}(H)$, and by (one possible) definition, the *Free Lie algebra* generated by H is the Lie sub-algebra $\mathcal{F}(H) \subset \mathcal{A}(H)$ generated by $H \subset \mathcal{A}(H)$. In other words

$$\mathcal{F}(H) = \bigoplus_{p \geq 1} F_p$$

with

$$\begin{cases} F_1 = H \\ F_{p+1} = [H, F_p]_A = \mathrm{span}\{XP - PX \mid (X, P) \in H \times F_p\}. \end{cases}$$

Now since our Carnot Lie algebra $\mathfrak{g}$ is bracket generated by $H = \mathfrak{g}_1$, it can be naturally identifies with the quotient

$$\mathfrak{g} = \mathcal{F}(H)/\mathcal{R}(\mathfrak{g}) \tag{23}$$

where the ideal $\mathcal{R}(\mathfrak{g})$ stands for the *infinitesimal relations* of G. This is the Lie version of the presentation of a discrete group by generators and relations as described in §2.1, namely

$$\Gamma = \mathrm{Free}(S)/R(\Gamma),$$

where S is a generating set, and $R(\Gamma)$ is the normal subgroup generated by the chosen 'elementary' relations of Γ. We can take profit of the two viewpoints here.

From discrete to infinitesimal relations. From the discrete side, a relation r of Γ is a finite product of the generators $\gamma_i \in S$, equals to e in Γ. Since $S \subset \exp \mathfrak{g}_1 = \exp H$ here, r can be lifted as an element $\widetilde{r} \in \mathcal{F}(H)$ using Baker-Campbell-Hausdorff formula. Namely this formula expresses

$$\begin{aligned} X * Y &= \ln(\exp X \exp Y) \\ &= X + Y + (XY - YX)/2 + \cdots \\ &= X + Y + \frac{1}{2}[X, Y]_A + \cdots \end{aligned} \tag{24}$$

as a formal polynomial series in brackets of X, Y, and provides a product on $\mathcal{F}(H)$ compatible with the one on $G = \exp \mathfrak{g}$. Since $r = e$ in $\Gamma \subset G$, one has necessarily $\widetilde{r} \in \mathcal{R}(\mathfrak{g})$.

Actually this lifting map from $R(\Gamma)$ into $\mathcal{R}(\mathfrak{g})$ induces an isomorphism between the vector spaces

$$R_c(\Gamma) = R(\Gamma)/(F, R(\Gamma)) \otimes \mathbb{R} \quad \text{and} \quad \mathcal{R}_c(\mathfrak{g}) = \mathcal{R}(\mathfrak{g})/[\mathcal{F}(H), \mathcal{R}(\mathfrak{g})].$$

This comes from two classical facts (see e.g [4, Thm. 5.3] and [36, Prop. 7:18])

- Hopf's formula identifying $R_c(\Gamma)$ with $H_2(\Gamma, \mathbb{R})$ and $\mathcal{R}_c(\mathfrak{g})$ with $H_2(\mathfrak{g}, \mathbb{R})$.
- The isomorphism $H_2(\Gamma, \mathbb{R}) \simeq H_2(\mathfrak{g}, \mathbb{R})$ coming from the cocompact embedding of Γ in G.

Therefore in our situation, the lifts $\widetilde{r}$ of the elementary relations r of Γ also generate the infinitesimal relations ideal $\mathcal{R}(\mathfrak{g})$.

Lastly we observe that the series $\widetilde{r}$ may be decomposed into its homogeneous components

$$\widetilde{r} = \sum_{p \geq d(r)} \widetilde{r}_p \tag{25}$$

with $\widetilde{r}_p \in F_p$, and $\widetilde{r}_{d(r)} \neq 0$ or $d(r) = +\infty$.

Definition 4.3. We will call $d(r)$ the *order* of r and $D(r) = \widetilde{r}_{d(r)}$ its *direction*.

Since G is a graded Lie group, its relation ideal $\mathcal{R}(\mathfrak{g})$ is too. In particular the directions of elementary relations of Γ still belong to $\mathcal{R}(\mathfrak{g})$ and generate it.

A few examples. We give simple examples to clarify previous things.

• Consider $\Gamma = \mathbb{Z}^n \subset G = \mathbb{R}^n$. Since $\mathfrak{g} = \mathfrak{g}_1 = F_1$, the relation ideal of $\mathbb{R}^n$ is $\mathcal{R}(\mathbb{R}^n) = \bigoplus_{p\geq 2} F_p$. It is generated by

$$F_2 = \operatorname{span}\{[X,Y]_A = XY - YX \mid X, Y \in H = \mathfrak{g}_1 = \mathbb{R}^n\}.$$

Given the canonical basis $\{e_i\}$ of $\mathbb{Z}^n$, the elementary relations

$$r_{ij} = (e_i, e_j) = e_i e_j e_i^{-1} e_j^{-1},$$

describing closed rectangles in the Cayley graph of $\mathbb{Z}^n$, lift to

$$\begin{aligned} \widetilde{r}_{ij} &= X_i * X_j * (-X_i) * (-X_j) \\ &= [X_i, X_j]_A + \cdots \quad \text{by} \quad (24). \end{aligned}$$

Hence $\widetilde{r}_{ij}$ are relations of order 2 (or quadratic) and their directions are the previous infinitesimal relations $D(r_{ij}) = [X_i, X_j]_A$ in F_2.

• The Heisenberg group of dimension $2n+1$, denoted by $\mathbf{H}^{2n+1}$, can be defined as $\mathbb{R}^{2n+1} = H \times \mathbb{R}$ with the product

$$(x,t) * (x',t') = (x + x', t + t' + \frac{1}{2}\omega(x,x')),$$

where ω is a non-degenerated skew-symmetric two-form on $H \simeq \mathbb{R}^{2n}$. The corresponding Lie bracket on $\mathfrak{h}^{2n+1} = H \oplus \mathbb{R}T$ is given by

$$[X + tT, X' + t'T] = \omega(X, X')T. \tag{26}$$

Hence $\mathbf{H}^{2n+1}$ is a 2-step Carnot group whose Lie algebra $\mathfrak{h}^{2n+1}$ is generated by H. Given a reduction basis $\{X_i, Y_i\}$ of ω in H, one gets the defining brackets

$$[X_i, X_j] = [Y_i, Y_j] = [H, T] = 0 \quad \text{and} \quad [X_i, Y_j] = \delta_{ij} T. \tag{27}$$

Let us see that, with respect to the free Lie algebra $\mathcal{F}(H)$, the infinitesimal relation ideal $\mathcal{R}(\mathbf{H}^{2n+1})$ is generated by elements of order 3 for $n = 1$, but only 2 for $n \geq 2$.

$\mathbf{H}^3$ has no quadratic relation. Indeed $T = [X,Y]$ is not a relation, but rather a notation, with respect to $\mathcal{F}(H) = \mathcal{F}(X,Y)$. In comparison all order 3 brackets $[X,[X,Y]]$ and $[Y,[X,Y]]$ vanish in $\mathfrak{h}^3$ and lift in $\mathcal{F}(X,Y)$ as $[X,[X,Y]]_A$ and $[Y,[X,Y]]_A$ spanning F_3. Therefore $\mathcal{R}(\mathbf{H}^3) = \bigoplus_{p\geq 3} F_p$.

In contrary for $\mathbf{H}^{2n+1}$, (27) gives us a lot of true quadratic relations

$$[X_i, X_j] = [Y_i, Y_j] = 0 \quad \text{and} \quad [X_i, Y_j] = 0 \text{ if } i \neq j.$$

But they are also 'hidden' ones, namely

$$[X_i, Y_i] - [X_j, Y_j] = 0 \quad \text{for} \quad i \neq j.$$

Calling T the common value of $[X_i, Y_i]$ in $\mathfrak{h}^{2n+1}$, one recovers easily the missing defining brackets $[H, T] = 0$ in (27). Indeed, given i, we can choose a $j \neq i$, then using Jacobi identity

$$[X_i, T] = [X_i, [X_j, Y_j]] = -[Y_j, [X_i, X_j]] - [X_j, [Y_j, X_i]] = 0.$$

This gives that, for $n \geq 2$, $\mathbf{H}^{2n+1}$ can be quadratically presented, meaning that $\mathcal{R}(\mathbf{H}^{2n+1})$ is generated by elements of order 2, namely

$$[X_i, X_j]_A \ , \ [Y_i, Y_j]_A \ , \ [X_i, Y_j]_A \quad (\text{if } i \neq j), \tag{28}$$

$$[X_i, Y_i]_A - [X_j, Y_j]_A. \tag{29}$$

In fact we can see that the quadratic relations we gave span the hyperplane $\ker \omega$ in $F_2 = \Lambda^2 H$, and finally

$$\mathcal{R}(\mathbf{H}^{2n+1}) = (\ker \omega) \bigoplus_{p \geq 3} F_p. \tag{30}$$

We now study the discrete viewpoint. The Heisenberg groups $\mathbf{H}^{2n+1}$ admit discrete cocompact groups

$$\mathbf{H}^{2n+1}_{\mathbb{Z}} = \left\{ \sum_{i=1}^{n} x_i X_i + y_i Y_i + tT/2 \mid x_i, y_i, t \in \mathbb{Z} \right\}.$$

Given the horizontal generating set $X_i, Y_i \in H$, one gets again different types of possible elementary relations in the Cayley graphs of $\mathbf{H}^{2n+1}_{\mathbb{Z}}$.

For $n = 1$ a choice may be $r_1 = (X, (X, Y))$ and $r_2 = (Y, (X, Y))$. They correspond to closed loops in the Cayley graph that look like a figure 8 as seen in figure 2. Their directions are the order 3 previous infinitesimal relations $[X, [X, Y]]_A$ and $[X, [X, Y]]_A$. These loops span a zero area at order 2, but still have an order 3 moment.

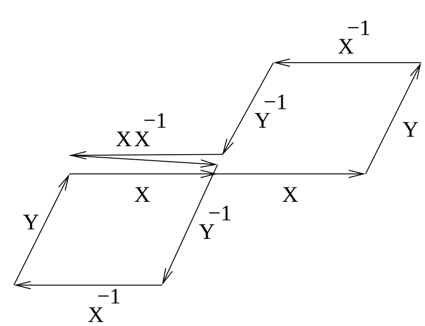


Figure 2: $(X, (X, Y))$ in $\mathbf{H}^3$.

For $n \geq 2$, one can choose again relations (X_i, X_j), (Y_i, Y_j), (X_i, Y_j) $(i \neq j)$, describing rectangles in the Cayley graph, and whose directions are the order 2 Lie relations in (28). We have to complete the list by adding

$$r_{ij} = (X_i, Y_i)(X_j, Y_j)^{-1},$$

which now look like a twisted figure 8 in the Cayley graph, see figure 3. Using (24), they are still relations of order 2, with directions $[X_i, Y_i]_A - [X_j, Y_j]_A$ as in (29).

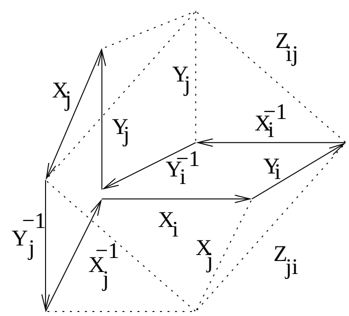


Figure 3: $(X_i, Y_i)(X_j, Y_j)^{-1}$ and its horizontal filling.

To complete the picture, we remark that the directions $D(r_{ij}) = [X_i, Y_i]_A - [X_j, Y_j]_A$ are not pure in $F_2 = \Lambda^2 H$, and therefore are not directions of plane loops (staying in an horizontal plane in H). Anyway one can find a presentation of $\mathbf{H}^{2n+1}_{\mathbb{Z}}$ using only planar relations.

Namely, we can add the generators $Z_{ij} = X_i^{-1} * Y_j$ with $i \neq j$ to the previous ones X_i, Y_i. They are also horizontal (in $H = \mathfrak{g}_1$) since $Z_{ij} = -X_i + Y_j$ as comes from $[X_i, Y_j] = 0$. Then, as shown in figure 3, r_{ij} can be horizontally filled in this extended Cayley graph, using the flat triangles (X_i, Y_j, Z_{ij}) and the horizontal rectangles spanned by the commuting Z_{ij}.

At the Lie level, the existence of this 'horizontal' Cayley polyhedra for $\mathbf{H}^{2n+1}_{\mathbb{Z}}$ is reflected by the fact that in (30),

$$\mathcal{R}_2(\mathbf{H}^{2n+1}) = \ker \omega \subset F_2 \simeq \Lambda^2 H,$$

is *spanned* by its pure forms $X \wedge Y$. This is not automatically satisfied for general quadratically presented Carnot groups, so that such groups don't always admit 'flat' Cayley polyhedras. This subtle matter enters in the geometric problem of horizontally filling horizontal loops (see §6.2), but not at the cohomological level of the presentation complex we are studying.

4.3 The asymptotic of $d_1^{\Gamma_\varepsilon}$

From relations to differential operators. We now return to the asymptotic holonomy problem. A priori, in the circulation formula (22) :

$$d_1^{\Gamma_\varepsilon} \alpha(r_\varepsilon) = \oint_{r_\varepsilon} \alpha,$$

the form α need only to be a discrete function on the horizontal edges of the shrinking Cayley graph of Γ_ε. However in order to estimate this sum, we will assume that α actually comes from a smooth horizontal one form on G. We note

$$\Omega^1 H = C^\infty(G, \Lambda^1 H^*)$$

this space of smooth partial one forms on G.

We would like to use the direction $D(r)$. Recall that it belongs to the free associative algebra $\mathcal{A}(H)$ (and even to the free Lie algebra $\mathcal{F}(H)$). There is a canonical mean to transform an element $P \in \mathcal{A}(H)$ into an operator on $\Omega^1 H$.

This follows from the remark that, given any basis $\{X_1, \cdots, X_n\}$ of H, a polynomial $P \in \mathcal{A}(H)$ uniquely factorizes in

$$P = c + \sum_{i=1}^{n} P_i X_i \,, \tag{31}$$

with c scalar and $P_i \in \mathcal{A}(H)$. We can then define a differential operator $i_H(P)$ acting on $\Omega^1 H$ by

$$i_H(P)\alpha = \sum_{i=1}^{n} P_i.\alpha(X_i). \tag{32}$$

More invariantly, using the splitting $\mathcal{A}(H) = \mathbb{R}1 \oplus \mathcal{A}(H) \otimes H$, we first define

$$i_H : \mathcal{A}(H) \to \mathcal{A}(H) \otimes (\Lambda^1 H^*)'$$

by

$$i_H(1) = 0 \quad \text{and} \quad i_H(PX) = P \otimes \operatorname{int}(X) \text{ for } X \in H,$$

where $\operatorname{int}(X)\alpha = \alpha(X)$. Then we view $\mathcal{A}(H) \otimes (\Lambda^1 H^*)'$ as differential operators acting on $\Omega^1 H = C^\infty(G) \otimes \Lambda^1 H^*$.

From the definition we note that

$$i_H(P) \circ d_H = P \tag{33}$$

for any $P \in \mathcal{A}(H)$ without constant term, in particular for $P \in \mathcal{F}(H)$. Recall that d_H is the horizontal part of the differential on functions. We have also

$$i_H(PQ) = P i_H(Q) \tag{34}$$

for any $P, Q \in \mathcal{A}(H)$ such that Q has no constant term.

Given a partial one-form $\alpha \in \Omega^1 H$, the components $i_H(P)\alpha$ have a simple geometric meaning when put all together. Let

$$\widetilde{\alpha} \in \Omega^1 \mathcal{F}(H) = C^\infty(G, \Lambda^1 \mathcal{F}(H)^*)$$

be defined on $P \in \mathcal{F}(H)$ by

$$\widetilde{\alpha}(P) = i_H(P)\alpha \,. \tag{35}$$

Proposition 4.4. *$\widetilde{\alpha}$ is the unique closed extension of α in $\Omega^1 \mathcal{F}(H)$.*

Proof. For $X \in H$ one has $\widetilde{\alpha}(X) = i_H(X)\alpha = \alpha(X)$ so that $\widetilde{\alpha}$ extends α.

Also, given P and $Q \in \mathcal{F}(H)$ one has

$$\begin{aligned} d\widetilde{\alpha}(P,Q) &= P\widetilde{\alpha}(Q) - Q\widetilde{\alpha}(P) - \widetilde{\alpha}([P,Q]) \\ &= Pi_H(Q)\alpha - Qi_H(P)\alpha - i_H([P,Q])\alpha \\ &= i_H(PQ - QP - [P,Q])\alpha \quad \text{by} \quad (34), \\ &= 0. \end{aligned}$$

Lastly, a closed form $\beta \in \Omega^1\mathcal{F}(H)$ satisfies $\beta([P,Q]) = P\beta(Q) - Q\beta(P)$ and is thus determined by its restriction to the bracket generating H. □

Each individual component $\widetilde{\alpha}(P)$ may be considered as the "infinitesimal holonomy" of α in the direction P. More precisely we can express the asymptotic of $d_1^{\Gamma_\varepsilon}$ along a shrinking relation r_ε.

Proposition 4.5. *Let $\alpha \in \Omega^1 H$. Then for $\varepsilon \to 0$,*

$$\begin{aligned} d_1^{\Gamma_\varepsilon}\alpha(r_\varepsilon) &= \varepsilon^{d(r)}\widetilde{\alpha}(D(r)) + O(\varepsilon^{d(r)+1}) \\ &= \varepsilon^{d(r)} i_H(D(r))\alpha + O(\varepsilon^{d(r)+1}) \end{aligned}$$

where $d(r)$ is the degree of r and $D(r)$ its direction (see Definition 4.3).

Proof. Let $\widetilde{\alpha}$ be the closed extension of α, and $\widetilde{r}_\varepsilon$ be the lifting in $\mathcal{F}(H)$ of the loop r_ε (using Baker-Campbell-Hausdorff formula). Since r_ε is horizontal and $\widetilde{\alpha} = \alpha$ on H, one has

$$d_1^{\Gamma_\varepsilon}\alpha(r_\varepsilon) = \oint_{r_\varepsilon} \alpha = \int_{\widetilde{r}_\varepsilon} \widetilde{\alpha}.$$

The form $\widetilde{\alpha}$ being closed, this last integral only depends on the ends of the path, and is therefore the same as on the straight line tangent to $\widetilde{r}_\varepsilon \in \mathcal{R}(\mathfrak{g})$, that is

$$\begin{aligned} d_1^{\Gamma_\varepsilon}\alpha(r_\varepsilon) &= \widetilde{\alpha}(\widetilde{r}_\varepsilon) \\ &= \varepsilon^{d(r)}\widetilde{\alpha}(D(r)) + O(\varepsilon^{d(r)+1}), \end{aligned}$$

since $\widetilde{r}_\varepsilon = \varepsilon^{d(r)}D(r) + O(\varepsilon^{d(r)+1})$ by (25).

Of course these computations have to be taken in the sense of jets on the infinite dimensional $\mathcal{F}(H)$. Anyway this asymptotic can also be obtained staying in finite dimensional groups. Given an $n > d(r)$, one can restrict $\widetilde{\alpha}$ on the 'free' n-step nilpotent group, whose Lie algebra is $\mathcal{F}(H)/F_{n+1}$. The extension $\widetilde{\alpha}_n$ is now only closed at order n, hence its use in the previous computations gives the same asymptotic at order $d(r)$.

□

Examples, continued. • On $\mathbb{Z}^n \subset \mathbb{R}^n$, the discrete relations $r = (X,Y)$ have directions $D(r) = [X,Y]_A = XY - YX$, for which by (32)

$$i_H(D(r))\alpha = X\alpha(Y) - Y\alpha(X) = d\alpha(X,Y), \tag{36}$$

giving the comforting

$$d_1^{(\varepsilon\mathbb{Z})^n}\alpha(r_\varepsilon) = \oint_{\text{rectangle}(\varepsilon X,\varepsilon Y)} \alpha \sim \varepsilon^2 d\alpha(X,Y).$$

• In the same spirit, the Heisenberg groups $\mathbf{H}^{2n+1}$ and their discrete cocompact $\Gamma = \mathbf{H}_{\mathbb{Z}}^{2n+1}$ are quadratically presented for $n \geq 2$. We therefore find again quadratic holonomy and first order $i_H(D(r))$. For instance the twisted 8

$$r_{ij} = (X_i, Y_i)(X_j, Y_j)^{-1}$$

of figure 3, gives

$$D(r_{ij}) = [X_i, Y_i]_A - [X_j, Y_j]_A = X_iY_i - Y_iX_i - X_jY_j + Y_jX_j$$

so that

$$d_1^{\Gamma_\varepsilon}\alpha(h_\varepsilon r_{ij}) \sim \varepsilon^2\big(X_i\alpha(Y_i) - Y_i\alpha(X_i) - X_j\alpha(Y_j) + Y_j\alpha(X_j)\big).$$

• For the 3 dimensional Heisenberg group $\mathbf{H}^3$ and its cocompact $\mathbf{H}_{\mathbb{Z}}^3$, we have cubical relations $r_1 = (X,(X,Y))$ and $r_2 = (Y,(X,Y))$ leading to

$$\begin{cases} D(r_1) = [X,[X,Y]]_A = X(XY - YX) - [X,Y]X \\ D(r_2) = [Y,[X,Y]]_A = Y(XY - YX) - [X,Y]Y, \end{cases} \tag{37}$$

so that by (32),

$$\begin{cases} i_H(D(r_1))\alpha = X(X\alpha(Y) - Y\alpha(X)) - T\alpha(X) \\ i_H(D(r_2))\alpha = Y(X\alpha(Y) - Y\alpha(X)) - T\alpha(Y), \end{cases} \tag{38}$$

where $T = [X,Y]$. This gives us the first term, in ε^3, for the holonomy of α along the shrinking 8-loops $h_\varepsilon r_1$ and $h_\varepsilon r_2$ seen in figure 2. Observe that the limit of the 'rescaled differential' $\varepsilon^{-2}d_1^{\Gamma_\varepsilon}$ is now given by a *second order* differential operator, to be compared to the function case, where $\varepsilon^{-1}d_0^{\Gamma_\varepsilon}$ always leads to a first order d_H. That can be taken as a hint that, at large scale, d_1^{Γ} should more behave like a second order operator rather than a first order one.

Remarks 4.6. Note that in the computation of $i_H(P)$, like in (38), one doesn't need to fully develop P, but only the relevant part giving the tails of the monomials, as due to (34).

We remark also that, since the $i_H(P)$ are used in Proposition 4.5 as differential operators on G, one can identify in the final expression the A-bracket with the one on $\mathfrak{g} = \mathcal{F}(H)/\mathcal{R}(\mathfrak{g})$. For instance $T = [X,Y]$ in (38), may be seen as $[X,Y]_{\mathfrak{h}_3} \in \mathfrak{h}_3$.

4.4 The infinitesimal presentation complex

Summary. We sum up what has been seen. Given a cocompact discrete Γ, horizontally generated in a Carnot Lie group G, the simplicial presentation complex of the shrinking $\Gamma_\varepsilon = h_\varepsilon \Gamma$

$$C(\Gamma_\varepsilon) \xrightarrow{d_0^{\Gamma_\varepsilon}} C(S_\varepsilon) \xrightarrow{d_1^{\Gamma_\varepsilon}} C(R_\varepsilon) \tag{39}$$

rescales, when restricted on smooth traces, towards an *infinitesimal presentation complex*

$$C^\infty(G) \xrightarrow{d_H} \Omega^1 H \xrightarrow{d_{\mathcal{R}}} \Omega^1 \mathcal{R}(\mathfrak{g}). \tag{40}$$

where

- d_H is the horizontal part of the differential d on functions,
- $(d_{\mathcal{R}}\alpha)(P) = \widetilde{\alpha}(P) = i_H(P).\alpha$ for any infinitesimal relation $P \in \mathcal{R}(\mathfrak{g})$ and horizontal one form α.

We gather some features of this construction.

• The property $d_{\mathcal{R}} \circ d_H = 0$ can be seen either as a limit of the corresponding fact on the presentation complex, or using (33) :

$$(d_{\mathcal{R}} d_H f)(P) = i_H(P) d_H f = Pf = 0,$$

since f, being a function on $\mathfrak{g} = \mathcal{F}(H)/\mathcal{R}(\mathfrak{g})$, is invariant along $P \in \mathcal{R}(\mathfrak{g})$.

• *Exactness.* One has $\ker d_H =$ constants and $\ker d_{\mathcal{R}} = \operatorname{Im} d_H$.

Proof. If $f \in C^\infty(G)$ is such that $d_H f = 0$, then $df = 0$ along all brackets of H, that spans $\mathfrak{g}$.

If $d_{\mathcal{R}}\alpha = 0$, then the closed extension $\widetilde{\alpha}$ of α vanishes on $\mathcal{R}(\mathfrak{g})$. Hence $\widetilde{\alpha}$ is a closed one form on G itself. Then there exists f on G such that $\widetilde{\alpha} = df$, and in particular $\alpha = d_H f$. □

• Actually, by Proposition 4.5, the only components of $d_{\mathcal{R}}$ that appear in the limit of (39) are $(d_{\mathcal{R}}\alpha)(D(r_i))$ for the finite set of directions $D(r_i)$ of the chosen elementary relations r_i of Γ. As already observed, these $D(r_i)$ generate the ideal $\mathcal{R}(\mathfrak{g})$. This implies that all the components of $d_{\mathcal{R}}\alpha$ are determined by these $(d_{\mathcal{R}}\alpha)(D(r_i))$.

Indeed, given $X \in H$, $P \in \mathcal{R}(\mathfrak{g})$ and $\alpha \in \Omega^1 H$, one has

$$\begin{aligned} d_{\mathcal{R}}\alpha([X,P]) &= \widetilde{\alpha}([X,P]) \\ &= X\widetilde{\alpha}(P) - P\widetilde{\alpha}(X) \quad \text{for } \widetilde{\alpha} \text{ is closed,} \\ &= X \cdot d_{\mathcal{R}}\alpha(P) \end{aligned} \tag{41}$$

since $\widetilde{\alpha}(X) = \alpha(X)$, being a function on $\mathfrak{g}$, is constant along $P \in \mathcal{R}(\mathfrak{g})$.

Staying on G. Thanks to the previous remark and Hopf's relation, one can replace the last space $\Omega^1\mathcal{R}(\mathfrak{g})$ in (40) by a more convenient bundle on G.

Recall that, in the Lie algebra setting, the second homology group $H_2(\mathfrak{g},\mathbb{R})$ is defined as follows. The Lie bracket on $\mathfrak{g}$ induces a complex

$$\Lambda^3\mathfrak{g} \xrightarrow{\partial_{\mathfrak{g}}} \Lambda^2\mathfrak{g} \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}$$

with

$$\begin{cases} \partial_{\mathfrak{g}}(X\wedge Y\wedge Z) = X\wedge[Y,Z]_{\mathfrak{g}} + Y\wedge[Z,X]_{\mathfrak{g}} + Z\wedge[X,Y]_{\mathfrak{g}} \\ \qquad \partial_{\mathfrak{g}}(X\wedge Y) = [X,Y]_{\mathfrak{g}} \end{cases}$$

leading to define

$$H_2(\mathfrak{g},\mathbb{R}) = \ker\partial_{\mathfrak{g}} / \operatorname{Im}\partial_{\mathfrak{g}}.$$

Moreover, since $\mathfrak{g} = \mathcal{F}(H)/\mathcal{R}(\mathfrak{g})$ here, one can consider the canonical map

$$\widetilde{\partial} : H_2(\mathfrak{g},\mathbb{R}) \longrightarrow \mathcal{R}_c(\mathfrak{g}) = \mathcal{R}(\mathfrak{g})/[\mathcal{F}(H),\mathcal{R}(\mathfrak{g})]$$

defined by

$$\widetilde{\partial}\big(\sum a_{ij}X_i\wedge X_j \ / \ \operatorname{Im}\partial\big) = \sum a_{ij}[\widetilde{X}_i,\widetilde{X}_j]_{\mathcal{A}} \ / \ [\mathcal{F}(H),\mathcal{R}(\mathfrak{g})]$$

for *any* choice of lifts $\widetilde{X}_i$ of X_i in $\mathcal{F}(H)$.

Hopf's formula [4, 36], in this setting, states that $\widetilde{\partial}$ is an isomorphism. Therefore, given any supplementary subspace V of $[\mathcal{F}(H),\mathcal{R}(\mathfrak{g})]$ in $\mathcal{R}(\mathfrak{g})$, one can project the map $d_{\mathcal{R}}$ by defining

$$d_V : \Omega^1 H \to C^\infty(H^2(\mathfrak{g}))$$

such that for $Y\in H_2(\mathfrak{g},\mathbb{R})$

$$(d_V\alpha)(Y) = (d_{\mathcal{R}}\alpha)(\Pi_V\widetilde{\partial}Y),$$

where Π_V is the projection of $\mathcal{R}(\mathfrak{g})$ on V along $[\mathcal{F}(H),\mathcal{R}(\mathfrak{g})]$.

The reduction of the complex (40) given by

$$C^\infty(G) \xrightarrow{d_H} \Omega^1 H \xrightarrow{d_V} C^\infty(H^2(\mathfrak{g})), \tag{42}$$

is still a resolution, since V generates the ideal $\mathcal{R}(\mathfrak{g})$, but now *depends* on the choice of V.

For instance, if some cocompact $\Gamma\subset G$ is given, one can take for V the subspace of $\mathcal{R}(\mathfrak{g})$ spanned by the directions of chosen elementary relations of Γ.

Also, it may happens that some invariant (with respect to automorphisms of G) choice of V may be done. That's the case for Carnot groups which are homogeneously presented. That means that the graded relation ideal

$$\mathcal{R}(\mathfrak{g}) = \bigoplus_{d\geq d_{\min}} \mathcal{R}_d(\mathfrak{g})$$

is generated by its elements of lowest degree, so that we can take

$$V = \mathcal{R}_{d_{\min}}(\mathfrak{g}).$$

The differential d_V is not invariant in general. However, it is a convenient reduction of the canonical $d_{\mathcal{R}} : \Omega^1 H \to \Omega^1\mathcal{R}(\mathfrak{g})$, using a bundle over G of *minimal* possible dimension $\dim H^2(\mathfrak{g})$.

Connection with d. Even if d_V may be an operator of high order (equals to the maximal order of generating relations of $G-1$), it is closely related to the standard first order d, but now restricted to some particular space of forms and directions.

Namely, pick some lifting map $X \to \overline{X}$ from $\mathfrak{g}$ into $\mathcal{F}(H)$, and extend it from $\Lambda^2\mathfrak{g}$ into $\Lambda^2\mathcal{F}(H)$. Choose also a subspace $\mathcal{H}_2 \subset \ker \partial_{\mathfrak{g}} \cap \Lambda^2\mathfrak{g}$ isomorphic to $H_2(\mathfrak{g}, \mathbb{R})$ (as for instance $\partial_{\mathfrak{g}} + \partial_{\mathfrak{g}}^*$ harmonic vectors relatively to a given metric on G). Then $V = \partial_{\mathcal{F}}(\overline{\mathcal{H}_2})$ is supplementary to $[H, \mathcal{R}(\mathfrak{g})]$ in $\mathcal{R}(\mathfrak{g})$ by Hopf's formula.

Proposition 4.7. *For $\alpha \in \Omega^1 H$, let $\overline{\alpha}$ be the one form on G defined by*

$$\overline{\alpha}(X) = \widetilde{\alpha}(\overline{X})$$

Then one has $d_V\alpha = d\overline{\alpha}$, in restriction *to $\mathcal{H}_2$.*

Proof. Let $Y = \sum a_{ij} X_i \wedge X_j \in \mathcal{H}_2$. Then since $\partial_{\mathfrak{g}} Y = \sum a_{ij}[X_i, X_j]_{\mathfrak{g}} = 0$, Cartan's formula gives

$$\begin{aligned} d\overline{\alpha}(Y) &= \sum a_{ij}\big(X_i\overline{\alpha}(X_j) - X_j\overline{\alpha}(X_i)\big) \\ &= \sum a_{ij}\big(\overline{X}_i\widetilde{\alpha}(\overline{X}_j) - \overline{X}_i\widetilde{\alpha}(\overline{X}_j)\big) \\ &= \widetilde{\alpha}\big(\sum a_{ij}[\overline{X}_i, \overline{X}_j]_{\mathcal{A}}\big) \quad \text{since} \quad d\widetilde{\alpha} = 0, \\ &= \widetilde{\alpha}(\partial_{\mathcal{F}}\overline{Y}) = d_V\alpha(Y). \end{aligned}$$

□

Observe that more generally one obtains on $\Lambda^2 H$

$$d\overline{\alpha}(Y) = \widetilde{\alpha}\big(\partial_F\overline{Y} - \overline{\partial_{\mathfrak{g}}Y}\big), \tag{43}$$

so that in particular $d\overline{\alpha}$ vanishes on the kernel of the curvature map

$$R : Y \in \Lambda^2\mathfrak{g} \longrightarrow \partial_F\overline{Y} - \overline{\partial_{\mathfrak{g}}Y} \in \mathcal{R}(\mathfrak{g}).$$

We finally point out that one can make some choice of lifting $\mathfrak{g} \to \mathcal{F}(H)$ that allows to characterize and compute $\overline{\alpha}$, and finally $d_V|_{\mathcal{H}_2}$, while staying on G, without referring to $\widetilde{\alpha}$ and the free Lie algebra $\mathcal{F}(H)$.

By Definition 4.1 we have $\mathfrak{g}_{k+1} = [\mathfrak{g}_1, \mathfrak{g}_k]_{\mathfrak{g}}$ and in particular

$$\partial_g : \mathfrak{g}_1 \wedge \mathfrak{g}_k \subset \Lambda^2_{k+1}\mathfrak{g} \longrightarrow \mathfrak{g}_{k+1}$$

is surjective. We can choose then a subspace $W_{k+1} \subset \Lambda^2_{k+1}\mathfrak{g}$ such that ∂_g induces an isomorphism between W_{k+1} and $\mathfrak{g}_{k+1}$. A convenient choice may be given by $W = \operatorname{Im} \partial_{\mathfrak{g}}^*$ if some metric is fixed.

Now we can define a lifting map step by step, starting with $\overline{X} = X$ for $X \in \mathfrak{g}_1 = H \subset \mathcal{F}(H)$, and satisfying $\overline{\partial_{\mathfrak{g}}Y} = \partial_{\mathcal{F}}\overline{Y}$ for $Y \in W_{k+1}$. Then the one form $\overline{\alpha}$ on G, introduced in Proposition 4.7, satisfies the following properties.

Proposition 4.8. *$\overline{\alpha} \in \Omega^1 G$ is the unique extension of the horizontal $\alpha \in \Omega^1 H$, such that $d\overline{\alpha} = 0$ on W. It can be computed step by step on $\mathfrak{g}_{k+1}$ using*

$$\overline{\alpha}(\partial_{\mathfrak{g}} Y) = \sum a_{ij}\big(X_i \overline{\alpha}(X_j) - X_j \overline{\alpha}(X_i)\big) \tag{44}$$

for $Y = \sum a_{ij} X_i \wedge X_j \in W_{k+1}$.

Proof. The vanishing of $d\overline{\alpha}$ on W comes from (43) and the construction of the lifting. Then (44) is just a rewriting of this property, showing in particular the required uniqueness. □

This aspects of d_V will be convenient to generalize to any degree on the large class of Carnot-Carathéodory manifolds in §4.

Examples. • $G = \mathbb{R}^n$ is quadratically presented and one can take $V = F_2 \simeq \Lambda^2 \mathbb{R}^n$. Of course $H_2(\mathbb{R}^n, \mathbb{R}) = \Lambda^2 \mathbb{R}^n$, and by (36), the complex (42) is just (the beginning of) de Rham's one.

• We know that the Heisenberg group $\mathbf{H}^3$ is cubically presented, with

$$\mathcal{R}_3(\mathfrak{h}^3) = F_3 = \operatorname{span}([X,[X,Y]]_A, [Y,[X,Y]]_A).$$

One sees easily that $H_2(\mathfrak{h}^3, \mathbb{R}) = H \wedge T$, where $T = [X,Y]$. Dually, $H^2(\mathfrak{h}^3, \mathbb{R})$ identifies with the vertical 2-forms $\theta \wedge H^*$ (where $\ker \theta = H$). Taking $V = F_3$ the map d_V is given by (38), namely

$$\begin{cases} d_V \alpha(X \wedge T) = X(X\alpha(Y) - Y\alpha(X)) - T\alpha(X) \\ d_V \alpha(Y \wedge T) = Y(X\alpha(Y) - Y\alpha(X)) - T\alpha(Y). \end{cases}$$

As stated in Proposition 4.7, we observe that in restriction to $H \wedge T$, one has $d_V \alpha = d\overline{\alpha}$, where $\overline{\alpha}$ is the extension of α to $\mathfrak{g}$ such that

$$\overline{\alpha}(T) = X\alpha(Y) - Y\alpha(X) = \widetilde{\alpha}([X,Y]_{\mathcal{A}}).$$

From Proposition 4.8, it is the unique extension of α such that $d\overline{\alpha}(X \wedge Y) = 0$. Note that this is an invariant condition here, due to uniqueness of possible choices of $\mathcal{H}_2 = H \wedge T$ and $W = \Lambda^2 H$.

• By (30), the higher Heisenberg groups $\mathbf{H}^{2n+1}$ are quadratically presented for $n \geq 2$ with

$$\mathcal{R}_2(\mathfrak{h}^{2n+1}) = \ker \omega,$$

where $\omega \in \Lambda^2 H^*$ is the non-degenerate form defining $\mathbf{H}^{2n+1}$ as in (26). One finds that $H_2(\mathfrak{h}^{2n+1}, \mathbb{R}) = \ker \omega \cap \Lambda^2 H$. Dually, $H^2(\mathfrak{h}^{2n+1}, \mathbb{R})$ identifies with the quotient space $\Lambda^2 H^* / \mathbb{R}\omega$ of horizontal two forms modulo ω. Given $V = \ker \omega$, and

$$Y = \sum a_{ij} X_i \wedge X_j \in H_2(\mathfrak{h}^{2n+1}, \mathbb{R}) = \ker \omega \cap \Lambda^2 H,$$

one gets that

$$\Pi_V \widetilde{\partial} Y = \sum a_{ij} [X_i, X_j]_{\mathcal{A}} = \sum a_{ij} (X_i X_j - X_j X_i),$$

so that

$$\begin{aligned} d_V\alpha(Y) &= (d_{\mathcal{R}}\alpha)(\Pi_V\widetilde{\partial}Y) = i_H(\Pi_V\widetilde{\partial}Y)\alpha \\ &= \sum a_{ij}(X_i\alpha(X_j) - X_j\alpha(X_i))\alpha \quad \text{by} \quad (32), \\ &= d\overline{\alpha}(Y), \end{aligned}$$

for even *any* vertical extension $\overline{\alpha}$ of α here.

This points out the fact that d_V is also given by the action of the standard d modulo the differential ideal $\mathcal{I}$ generated by vertical one forms.

Indeed let θ be the one form defined by $\theta(T) = 1$ and $\theta = 0$ on H. By (26), $\omega = -d\theta$, hence

$$\mathcal{I} = \{\theta \wedge \alpha + \omega \wedge \beta\},$$

giving the isomorphisms

$$\Omega^1 H \simeq \Omega^1 G/\mathcal{I}^1 \xrightarrow{d_V \simeq d} C^\infty(H^2(\mathfrak{g})) \simeq \Omega^2 G/\mathcal{I}^2. \tag{45}$$

From this viewpoint, it is clear that such a quotiented differential can be invariantly defined on *contact manifolds*. These are manifolds M endowed with a codimension one subbundle $H \subset TM$ such that, given locally (any) one form θ satisfying $\ker\theta = H$, then $\omega = d\theta$ is non-degenerate on H. (The ideal $\mathcal{I}$ is independent of the choice of such a θ called *contact form*.)

• Consider now the following example G, called Engel's group. It is the (unique) three-step four dimensional Lie group, such that $\mathfrak{g}$ is generated by $H = \mathbb{R}(X,Y)$ with the defining brackets

$$\begin{cases} [X,Y] = Z\ , \quad [X,Z] = T\ , \\ [Y,Z] = [X,T] = [Y,T] = [Z,T] = 0. \end{cases} \tag{46}$$

With respect to the free Lie algebra $\mathcal{F}(H)$ anyway, the first two brackets are notations, while only the two relations $[Y,Z] = [X,T] = 0$ are needed, since then

$$\begin{aligned} &[Y,T] = [Y,[X,Z]] = [Z,[X,Y]] + [X,[Y,Z]] = 0, \\ &[Z,T] = [[X,Y],T] = [[X,T],Y] + [[T,Y],X] = 0. \end{aligned}$$

That means that the infinitesimal relations ideal $\mathcal{R}(\mathfrak{g})$ is generated by

$$r_1 = [Y,[X,Y]]_{\mathcal{A}} \quad \text{and} \quad r_2 = [X,[X,[X,Y]]]_{\mathcal{A}}, \tag{47}$$

so that we can take $V = \mathbb{R}(r_1, r_2)$. This choice is not canonical this time. Indeed the map $X \to X + cY$ and $Y \to Y$ induces an isomorphism of G, which preserves r_1, but replaces r_2 by $r_2 + [2cX + c^2Y, r_1]$.

Using Hopf's relation, or a direct (co)homological computation, one sees also that $Y \wedge Z$ and $X \wedge T$ give a (non canonical) choice representing the quotient $H_2(\mathfrak{g},\mathbb{R}) = \ker\partial/\operatorname{Im}\partial$. One can readily compute the differential here. Namely developing (47) with remarks 4.6 gives

$$\begin{aligned} r_1 &= Y(XY - YX) - [X,Y]Y \\ r_2 &= X\big(X(XY - YX) - [X,Y]X\big) - [X,[X,Y]]X \end{aligned}$$

so that by (32) and (34),

$$d_V\alpha(Y\wedge Z) = i_H(r_1)\alpha = Y(X\alpha(Y) - Y\alpha(X)) - Z\alpha(Y) \tag{48}$$

and

$$\begin{aligned} d_V\alpha(X\wedge T) &= i_H(r_2)\alpha \\ &= X\big(X(X\alpha(Y) - Y\alpha(X)) - Z\alpha(X)\big) - T\alpha(X). \end{aligned} \tag{49}$$

Again as given by Proposition 4.7, we see that in restriction to $Y\wedge Z$ and $X\wedge T$ (representing $H_2(\mathfrak{g},\mathbb{R})$), $d_V\alpha = d\overline{\alpha}$ for the extension $\overline{\alpha}$ of α to $\mathfrak{g}$ defined by

$$\begin{cases} \overline{\alpha}(Z) = X\alpha(Y) - Y\alpha(X) = \widetilde{\alpha}([X,Y]_{\mathcal{A}}) \\ \overline{\alpha}(T) = X\overline{\alpha}(Z) - Z\alpha(X) = \widetilde{\alpha}([X,Z]_{\mathcal{A}}), \end{cases} \tag{50}$$

and which by Proposition 4.8 is also the unique extension of α such that

$$d\overline{\alpha}(X\wedge Y) = d\overline{\alpha}(X\wedge Z) = 0.$$

5 Extension to Carnot-Carathéodory spaces

5.1 Carnot-Carathéodory geometry

Definition. The previous construction can be adapted to a class of manifolds whose tangent space is “modelled” on Carnot groups.

A *Carnot-Carathéodory* (or C-C) structure on a smooth manifold M is, by definition, a bracket generating subbundle H of the tangent bundle TM. This gives an increasing filtration of TM by distributions

$$H_{k+1} = [H, H_k] \quad \text{with} \quad H_r = TM \tag{51}$$

for some minimal number of steps r. All C-C structures will be assumed *regular* here, meaning that the H_k have constant dimensions over each point of M. These H_k can then be seen as subbundles of TM.

To each point $x_0 \in M$ is associated a tangent Carnot Lie group G_{x_0} in the following way. The Lie bracket induces a quotient map

$$[\ ,\ ]_0 : H_k/H_{k-1} \times H_p/H_{p-1} \to H_{k+p}/H_{k+p-1},$$

which turns out to be a zero order (algebraic) operator here since

$$[X, fY] = f[X,Y] + (X.f)Y = f[X,Y] \mod H_{k+p-1}.$$

Therefore, given any $x_0 \in M$, $[\ ,\ ]_0$ defines at x_0 a Lie algebra structure on the graded tangent space at x_0

$$\mathfrak{g}_{x_0} = \mathrm{Gr}(T_{x_0}M) = \bigoplus_{k=1}^{r} H_{k,x_0}/H_{k-1,x_0}.$$

By Definition 4.1, this Lie algebra defines a Carnot group G_{x_0} since it is graded, nilpotent and generated by its first layer H. Details on the relationships between C-C structures and their tangent Carnot groups may be found for instance in [2, 32].

Examples. • Any Carnot group is a C-C manifold everywhere tangent to itself!

• The trivial C-C structure is $H = TM$, giving $[\ ,\ ]_0 = 0$ and $\mathfrak{g}_{x_0} = \mathbb{R}^n = T_{x_0}M$, meaning that the tangent group is the tangent space.

• *Contact structures* have already been defined in the previous section. This is the special instance of codimension one C-C structure H where, for any choice of one form θ with $\ker\theta = H$, one has $\omega = d\theta$ non-degenerate on H.

Given locally any $T \in TM \setminus H$, and a contact form θ with $\theta(T) = 1$, one sees that the tangent bracket $[\ ,\ ]_0 : H \times H \to \mathbb{R}T$ is $[X,Y]_0 = -d\theta(X,Y)T$. This implies by (26) that a contact structure is everywhere tangent in the previous sense to the Heisenberg group $\mathbf{H}^{2n+1}$, with $\dim M = 2n+1$.

Even more here, by a classical result of Darboux, a contact structure can be locally *embedded* into an Heisenberg group. But this is a very particular case since in general, a C-C structure with a constant tangent group can't be embedded in it.

• A k-dimensional distribution H^k in $\mathbb{R}^n$ or TM^n is generically locally bracket generating, and thus defines a C-C structure. For instance the case $k = 2p$ and $n = 2p+1$ corresponds to the previous contact structures. A generic plane distribution H^2 in $\mathbb{R}^4$ leads to a tangent 3-step 4-dimensional Carnot group. It is the Engel's group we considered in §4.4. Many other examples are given in [32].

5.2 Retracting de Rham complex

Filtrations. From Propositions 4.7 and 4.8, the infinitesimal presentation complex (42) of a Carnot group, reduced to be some components of the standard differential, but acting on a particular family of one forms. Counterparts of these structures exist on C-C manifolds.

We will follow closely presentations given in [38, 40].

Firstly, the *increasing* sequence of bundles H_k in TM gives rise to a natural *decreasing* filtration on p-forms by forms in $\Lambda^p_{(k)}T^*M$ vanishing on all p-vectors of $\bigotimes_{i=1}^{p} H_{k_i}$ such that $\sum_{i=1}^{p} k_i < k$. If we see vectors in H_k as being of weight $\leq k$, then forms in $\Lambda^*_{(k)}T^*M$ are of weight $\geq k$.

By Cartan's formula and (51), each $\Omega^*_{(k)}M = C^\infty(M, \Lambda^*_{(k)}T^*M)$ is preserved by d. We get in particular that de Rham's complex is filtered by these $\Omega^*_{(k)}M$, and one can consider the quotiented differential d_0 induced from d on

$$\Omega^*_k M = \Omega^*_{(k)}M / \Omega^*_{(k+1)}M. \tag{52}$$

Cartan's formula again gives on $\Omega^p_k M$ that

$$d_0\alpha(X_1, \cdots, X_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \alpha([X_i, X_j]_0, X_1, \cdots, \widehat{X}_{i,j}, \cdots, X_{p+1}),$$

is a 0-order (algebraic) operator, with $[\ ,\ ]_0$ the Lie algebra bracket on the tangent Lie algebra $\mathfrak{g}_{x_0}$ of the C-C structure. This d_0 is the Lie algebra coboundary on $\Lambda^*\mathfrak{g}_{x_0}^*$. It can be seen as de Rham differential acting on left invariant forms on G, or also as dual maps to the boundaries $\partial_{\mathfrak{g}_{x_0}}$ already met. Its cohomology $\ker d_0/\operatorname{Im} d_0$ is by definition the Lie algebra cohomology $H^*(\mathfrak{g}_{x_0},\mathbb{R})$, dual to the homology introduced in §4.4.

We will note $H^*(\mathfrak{g}_{x_0},\mathbb{R}) = E_0^*$, in reference to spectral sequence techniques (see eg [20, Chapter 3.5]). Indeed, this E_0^* is really the first space arising in the spectral sequence associated to the natural filtered complex $(\Omega^*_{(k)}M, d)$.

In order to get bundles on M, we will suppose that the C-C structure has some extra regularity hypothesis (always satisfied at least on an open dense set).

Definition 5.1. A C-C structure is called E_0-regular if each E_0^p has constant dimension.

In that case we can consider the bundle, still called E_0, of smooth sections of these E_{0,x_0}. We note that, since

$$H^1(\mathfrak{g}_{x_0},\mathbb{R}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* = \mathfrak{g}_1^* = \Lambda^1 H^*,$$

one has $E_0^1 = \Omega^1 H$, while $E_0^2 = C^\infty(M, H^2(\mathfrak{g}_{x_0},\mathbb{R}))$. They correspond, in this varying tangent group situation, to the two bundles of the infinitesimal presentation complex (42).

Extra choices. We would like now to adapt Propositions 4.7 and 4.8 to our setting. We have to describe some relevant spaces of "true" forms on which we could restrict de Rham's complex while staying a resolution. Such a result is achieved when applying an homotopical equivalence $r = \mathrm{Id} - Ad - dA$.

As in these propositions we first have to fix some choices of spaces representing the quotient spaces, $\mathfrak{g}_k = H_k/H_{k-1}$, $E_0 = H^*(\mathfrak{g}_{x_0},\mathbb{R})$ and $T^*M/\ker d_0$. Therefore we choose

- V_k such that $V_1 = H$ and $H_{k+1} = H_k \oplus V_{k+1}$,
- $\mathcal{E}_0$ such that $\ker d_0 = \operatorname{Im} d_0 \oplus \mathcal{E}_0$,
- W such that $\Lambda^*T^*M = \ker d_0 \oplus W$.

Here d_0 is viewed as acting on *true* forms on M, as allowed by the choice of $V_k \simeq \mathfrak{g}_k$ that fixes the weight of vectors and forms. Of course, if some metric is available on M, one can take orthogonal spaces as supplementaries :

$$V_{k+1} = H_{k+1} \cap H_k^\perp\ ,\ \mathcal{E}_0 = \ker \delta_0 \cap \ker d_0 = \mathcal{H}(\mathfrak{g}_{x_0},\mathbb{R})\ ,\ W = \operatorname{Im} \delta_0\,, \qquad (53)$$

for $\delta_0 = d_0^*$ adjoint of d_0.

Remarks 5.2. • There are no C-C invariant such choices in general (depending only on $H \subset TM$). Anyway, for the problems we are dealing with here, one breaks the invariance sooner or later, when introducing a metric and use adjoints.

• Anyway, there may be non (completely) invariant choices which, in some particular situations, like contact geometry, quaternionic contact geometry (see §5.3), and maybe more or less flat parabolic geometries (?), finally lead to invariant operators.

We observe that d_0, when seen as acting on $\Omega^* M$ is actually the component of d which preserves the weight of forms :

$$d = d_0 + d_1 + \cdots + d_r \tag{54}$$

where d_k increases the weight by k. The fact that d_0 is an *algebraic* operator allows us to partially inverse it. Let d_0^{-1} be defined by

$$d_0^{-1} d_0 = \mathrm{Id} \text{ on } W \quad \text{and} \quad d_0^{-1} = 0 \text{ on } W \oplus \mathcal{E}_0,$$

giving

$$\begin{aligned} d_0^{-1} d_0 = \Pi_{W/\ker d_0} \ , \quad d_0 d_0^{-1} = \Pi_{\mathrm{Im}\, d_0/\mathcal{E}_0 \oplus W} \ , \\ \mathrm{Id} - d_0^{-1} d_0 - d_0 d_0^{-1} = \Pi_{\mathcal{E}_0/\mathrm{Im}\, d_0 \oplus W}. \end{aligned} \tag{55}$$

An homotopy. We can now define a retraction of de Rham complex by

$$r = \mathrm{Id} - d_0^{-1} d - d d_0^{-1}. \tag{56}$$

This is by definition an homotopical equivalence, that (non-strictly) increases the weight of forms. By (54) and (55), the component r_0 of r preserving the weight is $\Pi_{\mathcal{E}_0/\mathrm{Im}\, d_0 \oplus W}$, the projection onto $\mathcal{E}_0$ relatively to $\mathrm{Im}\, d_0 \oplus W$.

In order to retract de Rham complex on the minimal possible space of forms, we can iterate r. The basic fact is that these r^k do stabilize for k large enough to a map $\Pi_{E/F}$, which has to be both an homotopical equivalence, and a projection onto a sub-complex (E, d) along another (F, d).

The following lemma is useful to identify E and F.

Lemma 5.3. *The map $d_0^{-1} d$ induces an isomorphism from W into itself, whose inverse is a differential operator P.*

Proof. On W, one can write

$$d_0^{-1} d = d_0^{-1} d_0 + d_0^{-1}(d - d_0) = \mathrm{Id} + N, \tag{57}$$

where $N = d_0^{-1}(d - d_0)$ is a nilpotent differential operator since it strictly increases the weight.

Then $P = \sum_{k=0}^{\text{max weight}} (-1)^k N^k$ is the required inverse. □

This lemma points out the fact that, when restricted to W, de Rham differential itself has a left inverse $Q = P d_0^{-1}$, meaning that $Qd = \mathrm{Id}$ on W. Thus this subspace W can be cut out from de Rham complex, using the homotopical equivalence $\mathrm{Id} - Qd$. One can also get rid of dW and identify the remaining space.

Theorem 5.4. *[38] Let (M,H) be a E_0-regular C-C manifold with the above structures and notations.*

1. *De Rham complex $(\Omega^* M, d)$ splits in the direct sum of two sub-complexes*

$$E = \ker d_0^{-1} \cap \ker(d_0^{-1} d) = \{\alpha \in \mathcal{E}_0 \oplus W \mid d\alpha \in \mathcal{E}_0 \oplus W\}$$
$$F = \operatorname{Im} d_0^{-1} + \operatorname{Im} dd_0^{-1} = W + dW.$$

   The projection $\Pi_{E/F}$, onto E along F, is an homotopical equivalence given by $\operatorname{Id} - Qd - dQ$, with $Q = Pd_0^{-1}$ as above.

2. *The retractions r^k stabilize to $\Pi_{E/F}$.*

3. *Let $\Pi_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0/\operatorname{Im} d_0 \oplus W}$ and $\Pi_E = \Pi_{E/F}$. One has*

$$\Pi_{\mathcal{E}_0}\Pi_E = \operatorname{Id} \text{ on } \mathcal{E}_0 \quad \text{and} \quad \Pi_E \Pi_{\mathcal{E}_0} = \operatorname{Id} \text{ on } E.$$

   In particular, the complex (E,d) is conjugated to the complex $(\mathcal{E}_0, d_c)$ with $d_c = \Pi_{\mathcal{E}_0} d \Pi_E \Pi_{\mathcal{E}_0}$.

These complexes are gathered in the following commutative diagram :

$$\begin{array}{ccc}
\Omega^* M = E \oplus F & \xrightarrow{\quad d \quad} & \Omega^* M = E \oplus F \\
\Pi_E \downarrow \uparrow i & & \Pi_E \downarrow \uparrow i \\
E & \xrightarrow{\qquad d \qquad} & E \\
\Pi_{\mathcal{E}_0} \downarrow \uparrow \Pi_E & & \Pi_{\mathcal{E}_0} \downarrow \uparrow \Pi_E \\
\mathcal{E}_0 & \xrightarrow{\qquad d_c \qquad} & \mathcal{E}_0
\end{array}$$

Proof. 1. We consider $\Pi = \operatorname{Id} - Qd - dQ$ and recognize it as a projection.

One has $Q = Pd_0^{-1} = 0$ on W. Moreover $Qd = \operatorname{Id}$ on W by Lemma 5.3. Thus $\Pi = 0$ on W, but also on $F = W + dW$ since $\Pi d = d\Pi$, and finally $F \subset \ker \Pi$.

Reversely, $\operatorname{Im} Q \subset W$ and $\operatorname{Im} dQ = \operatorname{Im} dPd_0^{-1} \subset dW$ by construction. This gives that $\ker \Pi \subset \operatorname{Im}(Qd + dQ) \subset W + dW = F$ and the equality $\ker \Pi = F$.

About $\operatorname{Im} \Pi$, since $d_0^{-1} Q = 0$, we have

$$d_0^{-1}\Pi = d_0^{-1} - d_0^{-1} dQ = d_0^{-1} - (d_0^{-1} dP) d_0^{-1} = 0$$

by definition of P. We have then also $d_0^{-1} d\Pi = d_0^{-1} \Pi d = 0$, and thus $\operatorname{Im} \Pi \subset E$. Lastly since $Q = Pd_0^{-1} = 0$ on $\ker d_0^{-1}$, we have $dQ + Qd = 0$ on $E = \ker d_0^{-1} \cap \ker d_0^{-1} d$ so that $\Pi_E = \operatorname{Id}$ on E, and the conclusion $\Pi = \Pi_{E/F}$.

2. One has directly $r = \operatorname{Id}$ on E from the definitions.

By (57), we have $r = \operatorname{Id} - d_0^{-1} d = -N$ on W, with $N = d_0^{-1}(d - d_0)$ nilpotent. Therefore $r^n = 0$ on W for n large enough, but also on dW and $F = W + dW$ since $r^n d = dr^n$.

3. Since $\mathcal{E}_0 \subset \ker Q = \ker Pd_0^{-1}$ and $\operatorname{Im} Q \subset W \subset \ker \Pi_{\mathcal{E}_0}$, we have that

$$\Pi_{\mathcal{E}_0}\Pi_E\Pi_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0}(\operatorname{Id} - Qd - dQ)\Pi_{\mathcal{E}_0} = \Pi_{\mathcal{E}_0}.$$

Lastly, we have $\Pi_E = 0$ on $W \subset F$. Therefore, we have $\Pi_E \Pi_{\mathcal{E}_0} = \Pi_E = \operatorname{Id}$ on $E \subset \ker d_0^{-1} = W \oplus \mathcal{E}_0$. □

For convenience, this construction will be referred as “Carnot complex” in the sequel, also we emphasize *it is* indeed de Rham complex, but restricted to a particular subspace of forms.

Comparison to the presentation complex. In the case of M being a Carnot group G, the two first steps

$$C^\infty(G) \xrightarrow{d_c} \mathcal{E}_0^1 \xrightarrow{d_c} \mathcal{E}_0^2$$

of the previous construction are actually equivalent to the infinitesimal presentation complex, as considered in propositions 4.7 and 4.8

$$C^\infty(G) \xrightarrow{d_H} \Omega^1 H \xrightarrow{d_V} C^\infty(G, \mathcal{H}_2^*).$$

Indeed, we have already observed that $\mathcal{E}_0^1 = \Omega^1 H$. Also by definition $\mathcal{E}_0^2 = C^\infty(G, \mathcal{H}^2(\mathfrak{g}))$, where $\mathcal{H}^2(\mathfrak{g})$ is a choice of subspace of $\Lambda^2\mathfrak{g}^*$ representing the cohomology, dual to a $\mathcal{H}_2(\mathfrak{g}) \subset \Lambda^2\mathfrak{g}$ representing homology.

Now we show that $d_c = d_V$. Given $\alpha \in \Omega^1 H$, $\Pi_E\alpha$ is an extension of α to $\Omega^1 G$, since when restricted to H,

$$\Pi_E\alpha = \Pi_{\mathcal{E}_0}\Pi_E\alpha = \alpha.$$

By definition of E, we have $d(\Pi_E\alpha) \in W \oplus \mathcal{H}^2(\mathfrak{g})$ which is supplementary to $\operatorname{Im} d_0$ in $\Lambda^2\mathfrak{g}^*$. This condition is equivalent to $d(\Pi_E\alpha) = 0$ on W', dual space to $W \oplus \mathcal{H}^2(\mathfrak{g})$, and which is supplementary to $\ker \partial_{\mathfrak{g}}$ in $\Lambda^2\mathfrak{g}$. Therefore, by proposition 4.8, $\Pi_E\alpha$ coincides with the special lifting $\overline{\alpha}$ described there. And by proposition 4.7, one has finally on $\mathcal{H}_2(\mathfrak{g})$,

$$d_V\alpha = d\overline{\alpha} = d\Pi_E\alpha = d_c\alpha.$$

Liftings and spectral sequence. The previous remark that for $\alpha \in \mathcal{E}_0^1 = \Omega^1 H$, $\Pi_E\alpha$ may be seen as a particular extension of α in $\Omega^1 M$, is true in any degree, as comes again from the relation $\Pi_{\mathcal{E}_0}\Pi_E = \operatorname{Id}$ on $\mathcal{E}_0$. In the spirit of Proposition 4.8, we have the following characterization :

Proposition 5.5. *For $\alpha \in \mathcal{E}_0$, $\Pi_E\alpha$ is the unique extension $\overline{\alpha}$ of α modulo W such that $d_0^{-1}d\overline{\alpha} = 0$.*

Proof. On $\mathcal{E}_0$, Π_E reduces to $\operatorname{Id} - Qd$ since $dQ = dPd_0^{-1} = 0$ here. Moreover $\operatorname{Im} Q \subset \operatorname{Im} P \subset W$ by construction. Therefore $\Pi_E\alpha \in \alpha + W$ is an extension satisfying the conditions.

If $\overline{\alpha}$ is another one, then $d_0^{-1}dw = 0$ with $w = \overline{\alpha} - \Pi_E\alpha \in W$, and $w = 0$ by lemma 5.3. □

In practice, computation of this extension may be done by iterating r which reduces to $\operatorname{Id} - d_0^{-1}d$ on $\ker d_0^{-1}$, to be compared with (44).

This viewpoint on liftings over $\mathcal{E}_0$ is also related to the natural spectral sequence associated with the filtration by weight $(\Omega^*_{(k)}M, d)$ of de Rham complex.

More precisely fix a p and an $\alpha \in E^p_{0,k}$ of *maximal* possible weight k. Then given any lifting $\overline{\alpha}$ of α in $\Omega^p_{(k)}M$, the class in $E^{p+1}_{0,k'}$ of the component of *minimal* weight k' of $d_c\overline{\alpha}$ is easily seen to be invariantly defined, independant of choices of supplementaries and $\overline{\alpha}$. It is indeed the $(k'-k)^{\text{th}}$ differential of α arising in the spectral sequence, and giving the first obstruction to finding a closed extension $\overline{\alpha}$, as seen by diagram chasing. (If not working with p-forms α of maximal possible weight in E^p_0, quotients appear in the spectral sequence differentials.)

In particular, if the d_0-cohomology bundles E_0 have *pure* weights in degrees p and $p+1$, then $d_c : \mathcal{E}^p_0 \to \mathcal{E}^{p+1}_0$ actually represents a C-C invariant operator $[d_c] : E^p_0 \to E^{p+1}_0$.

This happens for instance for all degrees in contact geometry as we will see in the next section. In general this happens between one and two forms iff the tangent group is homogeneously presented. If not, it may be observed that in constrast to general spectral sequences, components of high order of differentials are always defined in the Carnot complex, and not only on kernels of lower order ones. In the case of a fixed Carnot group, we recall that following §4.4, d_c interprets as giving the infinitesimal holonomy of an horizontal $\alpha \in \Omega^1 H$ along infinitesimal relations in $\mathcal{R}(\mathfrak{g})$. Note that in this setting, looking for the holonomy along any relation, whatever its order, actually makes sense (and can indeed be obtained from Proposition 4.5), without requiring the vanishing of holonomies along relations of lower order.

Duality. Although the definitions of E and F seem to break it, Hodge-$*$ duality is preserved, if the choices of supplementaries are done like in (53) with respect to a metric.

Proposition 5.6. *[38, 40]*

1. $*\delta_0 = (-1)^{k+1} d_0 *$ *on* $\mathcal{E}^k_0$ *and* $*$ *preserves* $\mathcal{E}_0$.
2. $*E = F^\perp$ *or equivalently* $\int_M E \wedge F = 0$. *The formal adjoint* Π^*_E *of* Π_E *is* $*^{-1}\Pi_E * = \Pi_{F^\perp/E^\perp}$.
3. $*\delta_E = (-1)^{k+1} d_E *$ *on* $\Omega^k M$ *for* $d_E = d\Pi_E$. *Similarly* $*\delta_c = (-1)^{k+1} d_c *$ *on* $\mathcal{E}^k_0$.

Proof. 1. Such pointwise duality formulas hold on unimodular Lie algebras, in particular on the nilpotent tangent $\mathfrak{g}_{x_0} \simeq T_{x_0}M$.

2. Since $E = \ker \delta_0 \cap \ker \delta_0 d$, then

$$(*E)^\perp = (\ker d_0 \cap \ker d_0 \delta)^\perp = \operatorname{Im} \delta_0 + \operatorname{Im} d\delta_0 = F.$$

Therefore $\Pi^*_E = \Pi^*_{E/F} = \Pi_{F^\perp/E^\perp} = \Pi_{*E/*F} = *^{-1}(\Pi_{E/F})*$. $(*^2 = \pm 1)$

3. From $*\delta = (-1)^{k+1} d*$ and $d_E = d\Pi_E = \Pi_E d$, we get

$$*\delta_E = *\Pi^*_E \delta = \Pi_E * \delta = (-1)^{k+1}\Pi_E d* = (-1)^{k+1} d_E * .$$

Lastly from $d_c = \Pi_{\mathcal{E}_0} d_E \Pi_{\mathcal{E}_0}$, we get

$$*\delta_c = \Pi_{\mathcal{E}_0} * \delta_E \Pi_{\mathcal{E}_0} = (-1)^{k+1} \Pi_{\mathcal{E}_0} d_E * \Pi_{\mathcal{E}_0} = (-1)^{k+1} d_c *,$$

using that $\Pi_{\mathcal{E}_0}$ is both an orthogonal and $*$ self-adjoint projection.

□

5.3 Some examples

Trivial C-C structure. Here $H = TM$ with $\mathfrak{g} = \mathbb{R}^n$ and $d_0 = 0$. Therefore $F = \{0\}$, $E = \Omega^* M = \mathcal{E}_0$, and $d_c = d_E = d$ is de Rham complex (not restricted).

Contact manifolds. Let (M^{2n+1}, H) be a contact manifold. We know from §5.1, that the tangent structure is the Heisenberg group $G_{x_0} = \mathbf{H}^{2n+1}$. We first compute d_0.

Given a contact form $\theta \in \Omega^1 M$ (with $\ker \theta = H$), one has a natural transversal T, called a Reeb field, such that $\theta(T) = 1$ and $i_T d\theta = 0$. Now $\Omega^* M$ splits in horizontal $\Omega^* H = \ker i_T$ and vertical forms $\theta \wedge \Omega^* H$. For $\alpha = \alpha_H + \theta \wedge \alpha_T$, we have

$$d\alpha = d_H \alpha_H + d\theta \wedge \alpha_T + \theta \wedge (\mathcal{L}_T \alpha_H + d_H \alpha_H),$$

and the component that preserves the C-C weight is seen to be (the algebraic)

$$d_0 \alpha = d\theta \wedge \alpha_T.$$

Since $d\theta$ is non-degenerate on H, we find that

$$E_0^k = H^k(\mathfrak{g}) = \begin{cases} \Lambda^k H^* / \operatorname{Im} L & \text{if } k \leq n, \\ \theta \wedge \ker L & \text{if } k > n, \end{cases} \tag{58}$$

where $L : \Lambda^k H^* \to \Lambda^{k+2} H^*$ is defined by $L\alpha = d\theta \wedge \alpha$.

Note that these E_0^k have pure C-C weights for all k, namely k if $k \leq n$ and $k+1$ if $k > n$. As previously observed, this implies that the differentials in the d_c-complex actually come from the (contact invariant) ones given by filtering de Rham complex (see [39, §3] for details). In particular, one gets a second order differential on E_0^n, given by the *usual* formula for second order differential in spectral sequences

$$d_c^n = d_2 - d_1 d_0^{-1} d_1 = \theta \wedge (\mathcal{L}_T - d_H L^{-1} d_H).$$

This is also de Rham differential of the extension

$$\Pi_E \alpha = r(\alpha) = \alpha - d_0^{-1} d\alpha = \alpha - \theta \wedge L^{-1} d_H \alpha,$$

if α is seen (lift) in $\mathcal{E}_0^n = \ker i_T \cap (\operatorname{Im} L)^\perp$, using a choice of θ and a metric.

We describe now some contact invariant choices of the sub-complexes E and F used in Theorem 5.4, to split the (true) differential forms $\Omega^* M = E \oplus F$. Recall that

$$E = \{\alpha \in V \mid d\alpha \in V\} \quad \text{and} \quad F = W + dW, \tag{59}$$

where W is supplementary to $\ker d_0$ and $V(\simeq W \oplus \mathcal{E}_0)$ to $\operatorname{Im} d_0$. Given any contact form θ and any almost complex structure J on H such that

$$g_H = d\theta(\cdot\,, J\cdot)$$

is a metric on H, we can form the adjoint $\Lambda : \Lambda^* H^* \to \Lambda^* H^*$ of the above multiplication L. By (58), one can take in degree k

$$V^k = \begin{cases} \{\alpha \in \Omega^k M \mid \Lambda\alpha_H = 0\} & \text{if} \quad k \leq n, \\ \{\alpha = \theta \wedge \beta\} = \text{vertical forms} & \text{if} \quad k \geq n+1, \end{cases} \tag{60}$$

where α_H is (now) the restriction of α to H. This is independent of the choice of θ, by conformal invariance of L, but even of J, since from classical properties of L (see e.g [44] or [39, Sec. 4]) one has

$$\ker \Lambda = \ker L^{n-k+1} \quad \text{on} \quad \Lambda^k H^*. \tag{61}$$

This choice of V, giving E by (59), is therefore *contact invariant*. One gets finally that :

- for $k \geq n+1$,

$$\begin{aligned} E^k &= \{\text{vertical forms } \alpha \text{ with } d\alpha \text{ vertical}\} \\ &= \{\alpha = \theta \wedge \beta \mid d\theta \wedge \beta = 0 \text{ on } H\} \\ &= \mathcal{E}_0^k \quad \text{by} \quad (58), \end{aligned} \tag{62}$$

- and for $k \leq n$,

$$\begin{aligned} E^k &= \{\alpha \in \Omega^k M \mid \Lambda\alpha_H = \Lambda(d\alpha)_H = 0\} \\ &= \{\alpha \in \Omega^k M \mid \theta \wedge L^{n-k+1}\alpha = \theta \wedge L^{n-k} d\alpha = 0\}, \end{aligned}$$

  with in particular $d(E^n) \subset$ closed vertical forms $\subset E^{n+1}$, as needed.

About W (leading to F), one can take in degree k

$$W^k = \begin{cases} \text{vertical forms} & \text{if} \quad k \leq n, \\ \{\theta \wedge \operatorname{Im} \Lambda\} & \text{if} \quad k \geq n+1, \end{cases}$$

which is again a contact invariant choice, since by duality from (61)

$$\operatorname{Im} \Lambda = \operatorname{Im} L^{n-k+1} \quad \text{on} \quad \Lambda^{2n-k} H^*.$$

To get more geometric feeling on E, let us describe the k-dimensional submanifolds N^k of M whose associated intersection currents $I(N^k)$ lie in E. That means N^k is seen as a distribution acting on $\Omega^k M$ by

$$\langle N^k, \alpha \rangle = \int_{N^k} \alpha = \int_M I(N^k) \wedge \alpha,$$

and we are looking for those ones satisfying $I(N^k) \in E^{2n+1-k}$. Since by Stokes

$$dI(N^k) = (-1)^{k+1} I(\partial N^k),$$

and $E = V \cap d^{-1}(V)$, this condition is equivalent to

$$I(N^k) \in V^{2n+1-k} \quad \text{and} \quad I(\partial N^k) \in V^{2n-k}. \tag{63}$$

We recall that, from its definition, $I(N^k) = \alpha(N^k)\mu_N$, where

- μ_N is the superficial measure on N,
- $\alpha(N^k)$ is the conormal volume form to N^k, that is the unitary oriented section of $\Lambda^{\max}(TN^k)^c$, where the conormal bundle $(TN^k)^c$ is the space of one forms on TM vanishing on TN^k. Given a metric, one has also $\alpha(N^k) = * \, d\,\mathrm{vol}_{TN^k}$ (up to sign).

Putting this together with (60) and (63), one sees that :

- for $k \leq n$,

$$I(N^k) \in V^{2n+1-k} \Leftrightarrow \alpha(N^k) \text{ is vertical} \Leftrightarrow TN^k \subset H.$$

Therefore such (smooth) N^k, and their boundaries ∂N^k, are necessarily *horizontal* submanifolds, meaning that their tangent spaces lie in the contact distribution H.

Note that by integrability of TN^k, this implies moreover that TN^k is a *Legendrian* distribution of H, i.e. $d\theta = 0$ on TN^k. (Showing that such manifolds don't exist for $k > n$.) One can check that this last condition on TN^k translates, for the conormal volume $\alpha(N^k)$ (and $I(N^k)$), into $d\theta \wedge \alpha(N^k) = 0$, which is one defining equation of E^k in (62).

- For $k \geq n+1$, one sees using a metric (as defined above) that

$$\begin{aligned} I(N^k) \in V^{2n+1-k} &\Longleftrightarrow \Lambda\alpha(N^k) = 0 \quad \text{on} \quad H, \\ &\Longleftrightarrow d\theta \wedge d\mathrm{vol}_{TN^k \cap H} = 0, \\ &\Longleftrightarrow d\theta = 0 \quad \text{on} \quad (TN^k)^\perp \cap H, \end{aligned}$$

meaning that, this time, the distribution $(TN^k)^\perp \cap H$ is Legendrian. Such manifolds N^k can then be called *co-Legendrian*. Again, this is a contact notion, since V^{2n+1-k} (and V^{2n-k}) can be given contact invariant definitions like in (61).

In fact, the previous Legendrian condition on $(TN^k)^\perp \cap H$, is easily seen to be equivalent to the following invariant one : the restriction to H of the conormal bundle $(TN^k)^c$ is Legendrian with respect to the dual symplectic form ω^*, induced on H^* from $\omega = d\theta$ on H.

We observe also that for $k = n+1$, a co-Legendrian N^{n+1} has the property that $(TN^{n+1}) \cap H$ is (generically) a n-dimensional subbundle of H, and is therefore itself Legendrian, being orthogonal to a Legendrian one in middle dimension. Therefore, such a N^{n+1} is foliated by (integrable) n-dimensional horizontal submanifolds. Then the condition $\partial(N^{n+1}) \in V^n$ means that $\partial(N^{n+1})$ consists in some of these leaves.

The conclusion of this discussion is that, at the level of currents induced by submanifolds, the complex (E, d) is dual to these families of co-Legendrian (in dim $\geq n+1$) and Legendrian manifolds (in dim $\leq n$), with the standard boundary operator ∂ relating them.

Since we know by Theorem 5.4 that, at the linear level of forms, one can retract de Rham complex on (E, d), it looks tentative trying to represent homology by the previous special manifolds. This much harder non-linear problem is studied, at least at the level of horizontal submanifolds, in Gromov's work [21].

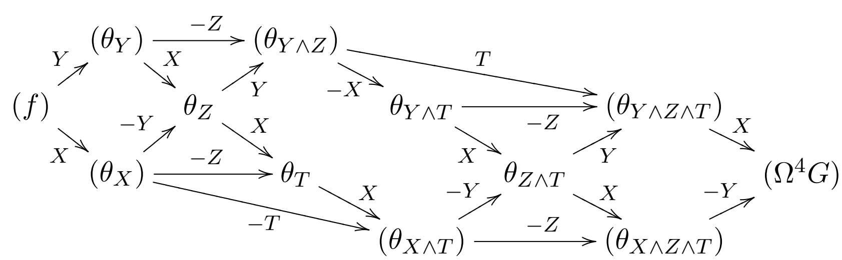
Engel's structures. Recall that an Engel's structure is given by a three-step two-plane field H in TM^4. It's tangent group has been described in §4.4. Let θ_X, θ_Y, θ_Z, $\theta_T \in \Lambda^1\mathfrak{g}^*$ be dual to $(X, Y) \in H = \mathfrak{g}_1$, $Z \in \mathfrak{g}_2$ and $T \in \mathfrak{g}_3$ as used in (46). Then, d_0 is given by

$$d_0\theta_X = d_0\theta_Y = 0 \ , \quad d_0\theta_Z = -\theta_X \wedge \theta_Y \quad \text{and} \quad d_0\theta_T = -\theta_X \wedge \theta_Z,$$

from which we get

$$\begin{cases} E_0^1 = \mathfrak{g}_1^* = \mathrm{span}(\theta_X, \theta_Y), \\ E_0^2 = \mathrm{span}\,(\theta_Y \wedge \theta_Z, \theta_X \wedge \theta_T) = *E_0^2, \\ E_0^3 = *E_0^1 = \mathrm{span}\,(\theta_Y \wedge \theta_Z \wedge \theta_T, \theta_X \wedge \theta_Z \wedge \theta_T) \\ E_0^4 = \Lambda^4\mathfrak{g}^*. \end{cases}$$

In degree one, $d_c = d_V$ and Π_E have been computed *on* G in (48), (49) and (50). The missing differentials can be obtained using $*$-duality. The results are summarized in the following diagram, adding all possible travels between points gives the various components of d_c (within braces) and the liftings Π_E (without braces)



(64)

A general Engel's structure $H \subset TM^4$ is not equivalent to its tangent G, meaning that one can't find vectors fields X, Y, Z, $T \in TM^4$ satisfying the structure brackets (46) of G. Anyway by definition of $[\ ,\ ]_0$, brackets agree up to vectors of lower C-C weights. That means that, for suitable local choices of vectors fields in TM^4, the differentials in the corresponding d_c-complex on M^4 will agree with the ones on G in (64), up to lower order terms. This is a general feature of the construction.

Quaternionic contact geometry. Let us consider now some attractive C-C structure arising in various asymptotic analytic and geometric problems on quaternion-Kähler manifolds (like in [3, 26]).

We first describe the Carnot group Q^{4n+3}, called quaternionic Heisenberg group, to which this particular C-C structure is tangent. Let $H \simeq \mathbb{R}^{4n}$ and an oriented $V \simeq \mathbb{R}^3$ be endowed with scalar products. Suppose given a linear map $J : V^* \to \mathrm{End}(H)$ such that for some, and then all, direct orthonormal base $(\theta_1, \theta_2, \theta_3)$ of V^*, $J_i = J(\theta_i)$ defines a quaternionic structure on H. That means the J_i are complex structures on H satisfying the imaginary quaternions commutation relations $J_1J_2 = -J_2J_1 = J_3$.

Consider then the 2-step $4n+3$ dimensional Carnot group Q^{4n+3} whose Lie algebra $\mathfrak{q}^{4n+3} = H \oplus V$ has a bracket $[\ ,\ ]_0 : H \times H \to V$ defined by

$$\theta([X, Y]_0) = -\langle J(\theta)X, Y\rangle_H.$$

Dually, the curvature $d_0 : V^* \to \Lambda^2 H^*$ of Q^{4n+3} is given by

$$(d_0\theta)(X, Y) = \langle J(\theta)X, Y\rangle_H. \tag{65}$$

Geometrically this group (of Heisenberg-type [27, 12]) arises in the Iwasawa decomposition of the rank 1 semi-simple Lie group $\mathrm{Sp}(n+1, 1)$.

Definition 5.7. Given a a $(4n+3)$-dimensional manifold M, a codimension 3 C-C structure $H \subset TM$ is called a quaternionic contact structure if its tangent Carnot group $\mathfrak{g}_{x_0}$ is everywhere isomorphic to Q^{4n+3}.

For instance the sphere at infinity S^{4n+3} of the quaternionic hyperbolic space $\mathbf{H}^{n+1}_{\mathbb{H}}$ possesses such a structure, since it is indeed a (one point) compactification of Q^{4n+3} itself. More interesting, this geometric structure has some flexibility for there exist many non locally conformally flat examples.

For $n \geq 2$, this is a consequence of works by C. Le Brun [28] and O. Biquard [3], that the previous flat quaternionic contact structure on S^{4n+3} admits an infinite dimensional space of deformations (asymptotic to deformations of the quaternionic-Kähler metric on $\mathbf{H}^{n+1}_{\mathbb{H}}$).

The 7-dimensional case is simpler, since being a quaternionic contact structure is even an open condition for a 4-dimensional distribution $H \subset TM^7$.

Namely (see e.g [32]), we first note that in (65), the three dimensional $\mathrm{Im}\, d_0$ is the subspace of self-dual forms $\Lambda^2_+ H^*$, and the metric on $V^* \simeq \mathrm{Im}\, d_0$ is induced by the intersection form $q(\omega) d\mathrm{vol}_H = \omega^2$. Now a general 4-dimensional 2-step distribution $H \subset TM^7$ is said *elliptic* if, at every point x_0, q is positive definite on the 3-dimensional $L = \mathrm{Im}\, d_0 \subset \Lambda^2 H^*$, where $d_0 : (TM/H)^* \to \Lambda^2 H^*$ is again the curvature of the tangent Carnot group $\mathfrak{g}_{x_0}$. In that case, it is a classical algebraic fact that one can find a unique conformal class of metric on H (and an induced one on $V = TM/H$), such that $\mathrm{Im}\, d_0 = \Lambda^2_+ H^*$ and d_0 is given by (65), showing finally that G_{x_0} is isomorphic to Q^7.

We note also that, in any dimension, the conformal class of the metric on H is actually determined by the quaternionic contact distribution H itself. This is due to the fact that isomorphisms of Q^{4n+3} conformally preserves the 4-fundamental form $\Omega = \sum_{i=1}^{3} (d_0\theta_i)^2$, and following [41, Lemma 9.1], are up to dilations and translations, induced by $Sp(n)Sp(1) \subset SO(H)$.

The C-C weights of the cohomology groups E_0 of Q^{4n+3} have been computed for instance in [25]. On p-forms,

$$E_0^p \quad \text{has weight(s)} \quad \begin{cases} p & \text{for} \quad p \leq n, \\ p \text{ and } p+1 & \text{for} \quad n+1 \leq p \leq 2n \\ 2n+2 & \text{for} \quad p = 2n+2, \end{cases} \tag{66}$$

and weights complementary to $N(Q^{4n+3}) = 4n+6$, in complementary degrees.

In dimension 7 (for $n=1$), the pattern of the Carnot complex is the following, with the degree of forms along x and (weight − degree) along y :

$$\begin{array}{cccccccccccc}
 & & & & & & & \mathcal{E}^5_{0,8} & \xrightarrow{d_c} & \mathcal{E}^6_0 & \xrightarrow{d_c} & \Omega^7 M \\
 & & & & & & & \nearrow^{d_c} & & \nearrow_{d_c} & & \\
 & & & & & & \mathcal{E}^4_0 & \xrightarrow{d_c} & \mathcal{E}^5_{0,7} & & & \\
 & & & & & & \nearrow^{d_c} & & & & & \\
 & & & \mathcal{E}^2_{0,3} & \xrightarrow{d_c} & \mathcal{E}^3_0 & & & & & & \\
 & & & \nearrow^{d_c} & & \nearrow_{d_c} & & & & & & \\
C^\infty(M) & \xrightarrow{d_H} & \Omega^1 H & \xrightarrow{d^-} & \Omega^2_- H & & & & & & &
\end{array} \tag{67}$$

where $\Omega^2_- H = \mathcal{E}^2_{0,2} \simeq \Omega^2 H / \operatorname{Im} d_0$ is the space of anti-self dual partial 2-forms on the 4-dimensional H. We can see that the bottom line looks like the half-signature complex on H, except H is not integrable here.

We now discuss some invariance properties of this construction. Together with analytic features developed in [38, 40], they have been used by Pierre Julg in its proof of Baum-Connes conjecture for $\mathrm{Sp}(n,1)$ in [26]. Although the techniques we'll follow look different, these invariance results are certainly closed to general constructions proposed by Čap, Slovák and Souček in [5], extending Bernstein-Gelfand-Gelfand sequences in parabolic geometries.

We first describe some (family of) transversal spaces T to H that we need to fix supplementaries. Let again V^* be the space of vertical forms (i.e. vanishing on H), on which we have our (conformal class of) metric. Given $X \in TM$, we can define $\theta_X \in V^*$ by $\theta(X) = \langle \theta_X, \theta \rangle_{V^*}$. Mimicking the definition of Reeb field in contact geometry, we consider the quadratic form $Q : TM \to \Omega^1 H$ defined by

$$Q(X) = d(\|\theta_X\|^2) + 2 i_X d\theta_X \quad \text{restricted to } H.$$

This is a tensor since

$$\begin{aligned} Q(fX) &= d(f^2 \|\theta_X\|^2) + 2 f i_X d(f\theta_X) \\ &= f^2 Q(X) + \|\theta_X\|^2 d(f^2) + 2f(X.f)\theta_X - 2\theta_X(X) f df \\ &= f^2 Q(X) \quad \text{on} \quad H. \end{aligned}$$

Its symmetric bilinear form B satisfies, for $(X,h) \in TM \times H$,

$$\begin{aligned} B(X,h) &= i_h d_0 \theta_X \\ &= (J(\theta_X)h)^{\#_{g_H}} \quad \text{by} \quad (65). \end{aligned}$$

Hence it induces, for $X \notin H$ fixed, an isomorphism from H into $\Lambda^1 H^*$. Therefore, the quadric $Q^{-1}(0)$ splits into $H \cup T$, where T is a *cone* transverse to H and isomorphic to TM/H.

It turns out that this "Reeb cone" T is automatically a *vector space* if $\dim M \geq 11$ (i.e. for $n \geq 2$). Actually, this T is easily seen to coincide, in these dimensions, with the vertical vector space associated to the connection constructed by O. Biquard in [3, II.1]. In dimension 7, a connection has also been given by D. Duchemin in [16]. In this dimension, the flatness of T is a priori a non-vacuous condition on the quaternionic contact structure. Such 7-dimensional quaternionic contact structures, with flat Reeb cone, will be called integrable in the sequel.

In any case this T depends on the choice of a metric on H, within its invariant conformal class. We compute its variation.

Proposition 5.8. *If $g_H \to g'_H = e^f g_H$, then $T \to T' = (\mathrm{Id} + \Delta)(T)$ with*

$$\Delta(X) = -J(\theta_X)\nabla_H f,$$

where $\nabla_H f = (d_H f)^{\#_{g_H}}$ is the horizontal gradient of f.

Proof. From their definitions, if $g'_H = e^f g_H$, then

$$J' = e^{-f} J \ , \quad g'_{V^*} = e^{-2f} g_{V^*} \quad \text{and} \quad \theta'_X = e^{2f}\theta_X,$$

from which one finds

$$Q'(X) = e^{2f}(Q(X) - 2\|\theta_X\|^2 d_H f).$$

Hence if $Q(X) = 0$, then $Q'(X + h) = 0$ for some $h \in H$ iff

$$B(X, h) = \|\theta_X\|^2 d_H f \iff h = -J(\theta_X)\nabla_H f,$$

since $B(X, h) = (J(\theta_X)h)^{\#_{g_H}}$ and $J^2(\theta) = -\|\theta\|^2\,\mathrm{Id}$. □

Remark 5.9. This shows in particular, as stated above, that the 'flatness' of the Reeb cone only depends on the quaternionic structure H : because Δ is a linear map. For instance T is flat on the sphere S^{4n+3}, since it is locally equivalent to the group model Q^{4n+3}, for which T is $\mathfrak{q}_2 = [H, H]_0$.

Using these T, one can now extend (or lift) partially defined forms α into 'true' ones $\overline{\alpha}$, by requiring their vanishing on vectors of higher C-C weight. These extensions are not invariant, but induce invariant choices of $\ker \delta_0$, $\mathrm{Im}\,\delta_0$ and sub-complexes E and F in Theorem 5.4. This follows from the lemma.

Lemma 5.10. *Let (M, H) be a quaternionic-contact structure, assumed integrable if $\dim(M) = 7$. Then, given two conformal g_H and g'_H, the induced variation Δ of vertical extensions of partial forms satisfies*

$$\Delta(\ker \delta_0) \subset \mathrm{Im}\,\delta_0.$$

Proof. As in (52), we note $\Omega^p_k M$ the space of partially defined p-forms on p-vectors of weight $\leq k$. If $g'_H = e^f g_H$, we show that for $\alpha \in \Omega^p_k M$, its variation of vertical extension to $\Omega^p_{k+1} M$ is given by

$$\Delta\alpha = \delta_0(d_H f \wedge \alpha) + d_H f \wedge \delta_0 \alpha. \tag{68}$$

• We first check it on horizontal forms $\Omega^p_p M = \Omega^p H$. Fix orthonormal bases $(T_j)_{1\leq j\leq 3}$ of T, dual to vertical one forms $(\theta_j)_{1\leq j\leq 3}$. Given $\alpha = \sum_{j=1}^3 \theta_j \wedge \alpha_j$ with $\alpha_j \in \Omega^{p-1} H \subset \ker d_0$, one has $d_0\alpha = \sum_{j=1}^3 d_0\theta_j \wedge \alpha_j$, so that

$$d_0 = \sum_{j=1}^3 d_0\theta_j \wedge i_{T_j} \quad \text{on} \quad \Omega^p_{p+1} M.$$

Therefore, for any $X \in H$, one has on $\Omega^p_{p+1} M$,

$$d_0 i_X + i_X d_0 = \sum_j (i_X d_0 \theta_j) \wedge i_{T_j} = \sum_j (J(\theta_j)X)^\# \wedge i_{T_j}.$$

Taking adjoints and $X = \nabla_H f$, gives for $\alpha \in \Omega^p H$,

$$\begin{aligned} \delta_0(d_H f \wedge \alpha) + d_H f \wedge \delta_0 \alpha &= \sum_j \theta_j \wedge i_{J(\theta_j)\nabla_H f} \alpha \\ &= \alpha(J(\theta_\cdot)\nabla_H f, \cdots) \\ &= -i_{\Delta(\cdot)}\alpha = \Delta\alpha, \end{aligned}$$

as comes from Proposition 5.8.

• To get (68) in general, we observe that for vertical one forms θ,

$$\Delta(\theta \wedge \alpha) = \theta \wedge \Delta\alpha,$$

since $\theta \wedge \cdot$ commutes with vertical extension, while

$$\delta_0(\theta \wedge \alpha) = -\theta \wedge \delta_0 \alpha,$$

which is dual to $d_0 i_T + i_T d_0 = 0$ $(= \mathrm{ad}_{\mathfrak{q}}(T))$ for vertical T. □

As already mentioned, Lemma 5.10 gives that the splitting of de Rham complex into the two sub-complexes

$$E = \ker \delta_0 \cap \ker \delta_0 d \quad \text{and} \quad F = \mathrm{Im}\, \delta_0 + \mathrm{Im}\, d\delta_0$$

is invariant, meaning it depends only on the (integrable) quaternionic contact structure H. That's not the case of the reduction of (E, d) to $(\mathcal{E}_0, d_c)$, as presented in Theorem 5.4, because

$$\mathcal{E}_0 = \ker d_0 \cap \ker \delta_0,$$

where $\ker d_0$ is not invariant, when seen as acting on 'true' forms like in (54).

Anyway, one can correct this by adapting the construction. Let replace $\mathcal{E}_0$ by its graded version

$$\mathcal{E}_0^{\mathrm{gr}} = \bigoplus_k \mathcal{E}_{0,k}^{\mathrm{gr}} = \bigoplus_k \big(\ker d_{0,k} \cap \ker \delta_{0,k}\big),$$

where $d_{0,k}$ is now the (invariant) quotient action of d on the graded (by weight) exterior algebra $\Omega_k^* M = \Omega_{(k)}^* M / \Omega_{(k+1)}^* M$, as defined in (52). Then, since by Lemma 5.10 vertical extensions are defined up to $\operatorname{Im} \delta_0 \subset F$, their projections on E along F are unique. In particular we get an invariant lifting map

$$\Pi_{\mathcal{E}_0^{\mathrm{gr}} \to E} : \mathcal{E}_0^{\mathrm{gr}} \to E.$$

In the opposite direction, we also have a natural projection

$$\Pi_{E \to \mathcal{E}_0^{\mathrm{gr}}} : E \to \mathcal{E}_0^{\mathrm{gr}}.$$

Indeed, one can extract successive homogeneous components of elements of $E \subset \ker \delta_0$, which again are defined up to $\operatorname{Im} \delta_0$ by Lemma 5.10. These components therefore represent unique elements in $\mathcal{E}_0^{\mathrm{gr}} \simeq (\ker \delta_0 / \operatorname{Im} \delta_0)^{\mathrm{gr}}$.

As in Theorem 5.4, these maps $\Pi_{\mathcal{E}_0^{\mathrm{gr}} \to E}$ and $\Pi_{E \to \mathcal{E}_0^{\mathrm{gr}}}$ are inverse to each other. One can then define a conjugated complex $(\mathcal{E}_0^{\mathrm{gr}}, d_c)$ to (E, d), by considering

$$d_c = \Pi_{E \to \mathcal{E}_0^{\mathrm{gr}}} d \Pi_{\mathcal{E}_0^{\mathrm{gr}} \to E}.$$

The advantage of this last reduction is that $\mathcal{E}_0^{\mathrm{gr}}$ is a (now invariant) vector bundle, while E consists in forms satisfying some differential equations. This feature was useful in Julg's work [26].

We close here this series of examples, and return to our primary study of large time heat decays on Carnot groups. Recall that, in §4, we were led to considering these differential complexes, by looking at the discrete presentation complex of lattices at large scale. The link was rather formal anyway, and we still have to check it is analytically relevant.

6 Back to spectral problems

6.1 Algebraic pinching of heat decay

Carnot complex and near-cohomology. Let (M, H) be any E_0-regular C-C manifold. We know by Theorem 5.4 that, from the topological viewpoint, one can retract de Rham complex on the sub-complex (E, d), this one being itself conjugated to $(\mathcal{E}_0, d_c)$. One can then use any of these homotopy equivalent complexes to express the cohomology of M.

One can extend this principle to near-cohomology using the general ideas presented in §3.

Theorem 6.1. *[38, 40] Let (M, H) be a compact E_0-regular C-C manifold and $\widetilde{M}$ some Galois covering. Then, de Rham complex, (E, d) and $(\mathcal{E}_0, d_c)$, have isomorphic near-cohomologies on $\widetilde{M}$. In particular, they have the same Novikov-Shubin exponents (twice the large time heat decay exponents on* $(\ker d)^\perp$ *by §2.2).*

Proof. The proof is straightforward, using the notions introduced in §3. By Theorem 3.12, it suffices to show that these complexes are homotopy equivalent, in the Hilbertian sense of Definition 3.1.

We first describe the underlying Hilbert complexes here. We work respectively in "true" L^2 forms on $\widetilde{M}$ for de Rham complex, L^2 sections of the bundle $\mathcal{E}_0$ for d_c, and the L^2 closure $\overline{E}$ of $E \cap C_0^\infty(\Omega^* M)$ for (E, d).

Starting with smooth compactly supported forms as initial domains, one then closes the differentials (i.e. their graphs) in these Hilbert spaces. This is possible because the adjoints of these differential operators are also densely defined (at least on C_0^∞).

The basic homotopies Π_E and $\Pi_{\mathcal{E}_0}\Pi_E$, between de Rham complex, (E, d) and $(\mathcal{E}_0, d_c)$, are not bounded in L^2, being differential operators. They can't thus be used directly in Definition 3.1. Anyway, one can first cut-out high frequencies, irrelevant in the near-zero spectral problem we are dealing with. Namely, let $E_\Delta(1)$ be the spectral projector associated to $[0, 1]$ by de Rham Laplacian Δ. One has

$$E_\Delta(1) = \mathrm{Id} - E_\Delta(]1, +\infty[) = \mathrm{Id} - Bd - dB$$

where $B = \delta\Delta^{-1}E_\Delta(]1, +\infty[)$ is bounded in L^2 by the spectral theorem. Thus it induces a bounded homotopical equivalence between de Rham and the cut-off de Rham complex on $\mathcal{E}_\Delta(1) = \mathrm{Im}\, E_\Delta(1)$. Now, by elliptic regularity of de Rham Laplacian, any (Γ-invariant) differential operator is bounded on $\mathcal{E}_\Delta(1)$. In particular the previous maps Π_E and $\Pi_{\mathcal{E}_0}\Pi_E$ provide the required bounded homotopies on this cut-off de Rham complex.

□

Previous result claims that for near-cohomology study, one can mod out F, keeping only E. One can give some geometric flavour to this statement. Consider the near-cohomology cones $C_d(\varepsilon)$ of the cut-off de Rham complex $(\mathcal{E}_\Delta(1), d)$. Recall that by (17)

$$C_d(\varepsilon) = \{\alpha \in [\mathcal{E}_\Delta(1)] = \mathcal{E}_\Delta(1)/\ker d \mid \|d\alpha\| \leq \varepsilon\|\alpha\|\}.$$

The splitting of de Rham complex into $E \oplus F$, induces a splitting $[E] \oplus [F]$ of forms modulo $\ker d$. Moreover, by Theorem 5.4, one has $\Pi_F = Qd + dQ$, where Q becomes bounded as before, when restricted to $\mathcal{E}_\Delta(1)$. Therefore, one gets for $\alpha \in C_d(\varepsilon)$,

$$\|[\Pi_F\alpha]\| = \|[Qd\alpha]\| \leq C\|d\alpha\| \leq C\varepsilon\|\alpha\|,$$

meaning that, when $\varepsilon \to 0$, the near-cohomology cones $C_d(\varepsilon)$ of the cut-off de Rham complex are actually shrinking around $[E]$ relatively to $[F]$ (inside cones of slope $\leq C\varepsilon$). This is suggested in figure 4.

Theorem 6.1 is probably not very useful in the problem of studying Novikov-Shubin numbers on general (C-C) manifolds. Actually we will only apply it on nil-manifolds.

More precisely, let G be a rational Carnot Lie group (see §4.1), and consider the quotient $M = G/\Gamma$ where Γ is a discrete cocompact group in G. By

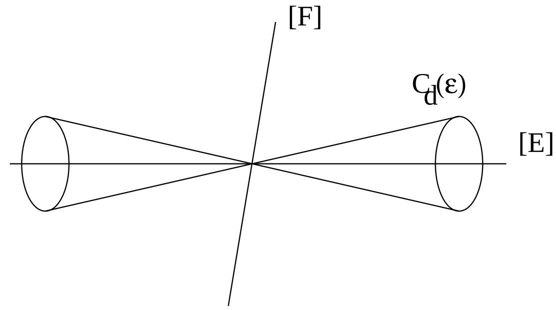


Figure 4: The shrinking of near-cohomology cones $C_d(\varepsilon)$.

contractibility of G, one has $\pi_1(M) = \Gamma$, so that by Theorem 2.6 and Corollary 2.7, de Rham complex on the smooth group G may be used to compute the two first Novikov-Shubin exponents $\alpha_0(\Gamma)$ and $\alpha_1(\Gamma)$. Then by Theorem 6.1, these exponents are the same as for the complex $(\mathcal{E}_0, d_c)$. The advantage of this last one is its better behaviour through the natural dilations h_ε available on G.

Indeed, recall that the first stages of the construction are

$$C^\infty(G) \xrightarrow{d_H} \Omega^1 H \xrightarrow{d_c} \mathcal{E}_0^2. \tag{69}$$

The first map d_H is the differentiation along H, the first strata of $\mathfrak{g}$, and is thus an homogeneous operator of order one with respect to the dilations h_ε. For the second map $\mathcal{E}_0^2 = L^2(G, H^2(\mathfrak{g}, \mathbb{R}))$ may have components of different homogeneities, pinched by the order of generating relations of $\mathfrak{g}$ relatively to the free Lie algebra $\mathcal{F}(H)$, as comes from Hopf's relation (see §4.3 and 4.4).

Dilations and Γ-dimension. To take profit of the previous remarks we have first to check the behaviour of Γ-dimension and trace (see §3.2) under dilations. They actually behave like densities on G.

Proposition 6.2. *Let L be a Γ-invariant subspace of L^2-differential forms on G. Then for $n \in \mathbb{N}$,*

$$\dim_\Gamma(h_n^* L) = \dim_{h_n\Gamma}(L) = n^{N(G)} \dim_\Gamma(L),$$

where $N(G) = \displaystyle\sum_{i=1}^r i\dim(\mathfrak{g}_i)$ *is the growth of G.*

Remark 6.3. In this statement we use dilations with integer coefficients, and a lattice Γ horizontally generated, in order to have $h_n(\Gamma) \subset \Gamma$, and L also $h_n\Gamma$-invariant. Another 'continuous' approach is possible, for the G-invariant operators we are actually dealing with (see next proposition).

Proof. Given an initial (invariant) metric g on G, the map h_n induces an *isometry* between $L \subset L^2_g(G, \Lambda^* G)$ and $h_n^* L \subset L^2_{h_n^* g}(G, \Lambda^* G)$, that conjugates respectively the $h_n\Gamma$ and Γ actions. Then by definition of $\dim_\Gamma$ (see §3.2), we have

$$\dim_{\Gamma, h_n^* g}(h_n^* L) = \dim_{h_n\Gamma, g}(L).$$

Moreover, by the general invariance result Proposition 3.9, $\dim_\Gamma$ actually does not depend on the choice of Γ-invariant metrics on G and the bundle, and hence

$$\dim_{\Gamma,h_n^*g}(h_n^*L) = \dim_{\Gamma,g}(h_n^*L) = \dim_\Gamma(h_n^*L).$$

as needed.

The second equality is a particular case of the multiplicativity of $\dim_\Gamma$ under finite coverings, here

$$\Gamma/h_n\Gamma \to M_n = G/h_n\Gamma \to M = G/\Gamma.$$

If (e_i) is an Hilbert base of L and $\mathcal{F} \subset G$ a fundamental domain of the Γ action, we have by (20) and (21) that

$$\dim_\Gamma L = \sum_i \int_{\mathcal{F}} \|e_i(x)\|^2 dx.$$

A fundamental domain for $h_n\Gamma$ consists in $\mathrm{card}(\Gamma/h_n\Gamma)$ copies of $\mathcal{F}$, so that

$$\dim_{h_n\Gamma} L = \mathrm{card}(\Gamma/h_n\Gamma)\dim_\Gamma L.$$

Lastly, we recall that h_n acts by multiplication by n^k on $\mathfrak{g}_k$, hence

$$\mathrm{card}(\Gamma/h_n\Gamma) = \mathrm{vol}(h_n\mathcal{F})/\mathrm{vol}(\mathcal{F}) = n^{N(G)},$$

with $N(G) = \sum_k k\dim(\mathfrak{g}_k)$. □

The previous result actually holds for any dilation and lattice Γ, if the operators and spaces are G-invariant.

Proposition 6.4. *Let P be a positive G-invariant operator acting on L^2-sections of a G-invariant vector bundle V. If P has a finite Γ-trace for some lattice Γ, it has for any, and its kernel K_P is a bounded continuous function with*

$$\tau_\Gamma(P) = \mathrm{vol}(G/\Gamma)\,\mathrm{Tr}(K_P(e,e)),$$

where Tr *is the trace on* $\mathrm{End}(V_e)$.

We recover in particular the dependency of Γ-trace and Γ-dimension in $\mathrm{vol}(G/\Gamma)$.

Proof. From (20), P is Γ-trace class iff its square root S is such that $S\chi_{\mathcal{F}}$ is an Hilbert-Schmidt operator, that is iff the kernel K_S of S is in $L^2(G\times\mathcal{F})$. Moreover, we have

$$\tau_\Gamma(P) = \|S\chi_{\mathcal{F}}\|_{HS}^2 = \int_{G\times\mathcal{F}} \|K_S(x,y)\|^2 dxdy. \tag{70}$$

For (left) invariant operators, one has $K_P(x,y) = k_P(y^{-1}x)$, and P acts on $L^2(G,V)$ by convolution as

$$Pf(x) = (k_P * f)(x) = \int_G k_P(y^{-1}x)f(y)dy.$$

By (70), we see that

$$\tau_\Gamma(P) = \|S\chi_{\mathcal{F}}\|_{HS}^2 = \mathrm{vol}(\mathcal{F})\|k_S\|_2^2. \tag{71}$$

In particular, $\tau_\Gamma(P) < +\infty$ iff $k_S \in L^2(G, \mathrm{End}(V))$, in which case

$$k_P = k_S * k_S \in L^2(G) * L^2(G)$$

is bounded and continuous on G (and vanishes at ∞). Then, by (70) and hermitian symmetry of k_S (i.e. $k_S(x^{-1}) = k_S(x)^*$), we obtain

$$\begin{aligned}\tau_\Gamma(P) &= \mathrm{vol}(\mathcal{F})\|k_S\|_2^2\\ &= \mathrm{vol}(\mathcal{F})\,\mathrm{Tr}\big((k_S * k_S)(e)\big)\\ &= \mathrm{vol}(G/\Gamma)\,\mathrm{Tr}(k_P(e)).\end{aligned}$$

□

Proposition 6.4 points out the fact that, given a Lie group G and a lattice $\Gamma \subset G$ (i.e. Γ discrete and $\mathrm{vol}(G/\Gamma) < +\infty$), the Novikov-Shubin numbers $\alpha_p(M)$ of the covering $G \to M = G/\Gamma$ actually depend only on G, but not on the lattice Γ. The common values for all Γ, denoted by $\alpha_p(G)$ in the sequel, are indeed either infinite, or given by the decays of kernels at the origin e of G-invariant spectral projectors $E_{\delta d}(]0, \lambda^2])$, also twice the heat kernel decays on $(\ker d)^\perp$ (see §2.2).

Application to α_0. We can now use the homogeneity of d_H in (69) to recover the value of $\alpha_0(\Gamma)$ for lattices in a Carnot Lie group G.

Namely, one has $\Delta_H h_\varepsilon = \varepsilon^2 h_\varepsilon \Delta_H$, so that the spectral spaces $\mathcal{E}_{\Delta_H}(\lambda)$ of Δ_H rescale as follows

$$h_\varepsilon(\mathcal{E}_{\Delta_H}(\lambda)) = \mathcal{E}_{\Delta_H}(\varepsilon^2\lambda). \tag{72}$$

Therefore, by Propositions 6.2 and 6.4, the spectral density function of Δ_H is homogeneous,

$$F_{\Delta_H}(\varepsilon^2) = \dim_\Gamma(\mathcal{E}_{\Delta_H}(\varepsilon^2)) = \varepsilon^{N(G)} F_{\Delta_H}(1). \tag{73}$$

Hence we recover Varopoulos' result, for these groups Γ, that

$$\alpha_0(\Gamma) = \lim_{\varepsilon\to 0} \frac{\ln F_{\Delta_H}(\varepsilon^2)}{\ln \varepsilon} = N(G),$$

relating twice the large time heat decay on functions to the growth of G (also the growth of any cocompact Γ).

Pinching of α_1. The same kind of discussion applies to the next exponent $\alpha_1(\Gamma) = \alpha_1(G)$, except that now $d_c : \Omega^1 H \to \mathcal{E}_0^2$ may be polyhomogeneous.

We recall that, by Hopf's relation, $E_0^2 \simeq H^2(\mathfrak{g}, \mathbb{R})$ is isomorphic to the space $V = \mathcal{R}(g)/[\mathcal{R}(g), H]$ of defining relations of $\mathfrak{g} = \mathcal{F}(H)/\mathcal{R}(\mathfrak{g})$ (with respect to the free Lie algebra $\mathcal{F}(H)$).

This leads to the following pinching of $\alpha_1(G)$, for rational Carnot group G, i.e. those admitting a lattice Γ.

Theorem 6.5. *[38, 40] If Γ is a lattice in a Carnot group G then $\alpha_1(\Gamma) = \alpha_1(G)$ satisfies*

$$r_{\min} - 1 \le \beta_1(G) = \frac{N(G)}{\alpha_1(G)} \le r_{\max} - 1,$$

where $r_{\min}$ and $r_{\max}$ are the minimal and maximal order of defining relations of G, also minimal and maximal weights of $H^2(\mathfrak{g}, \mathbb{R})$.

Hence the higher order defining relations of G are, the slower heat decays on one forms, with respect to the growth of G. Notice that if G is a r-step groups (i.e. $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$), then generating relations of G are necessarily of order pinched between 2 and $r + 1$, and we obtain in general that

$$1 \le \beta_1(G) \le r. \tag{74}$$

Thus heat on one forms never decays quicker than on function, in $t^{-N(G)/2}$, and never slower that $t^{-N(G)/2r}$. Examples of each type will be given in §6.2.

Proof. In general the differential d_c splits in homogeneous components

$$d_c = d_c^{r_{\min}-1} + \cdots + d_c^{r_{\max}-1},$$

and we don't have anymore that the spectral spaces of Δ_c properly rescale under h_ε, like in (72) for the case of functions. We consider instead the action of h_ε on the near-cohomology cones (see (17))

$$C_{d_c}(\lambda) = \{\alpha \in L^2(G, \Lambda^1 H)/\ker d_c \mid \|d_c\alpha\| \le \lambda \|\alpha\|\}.$$

Namely, we have

$$\|d_c(h_k^*\alpha)\| = \|h_k^* d_c \alpha\| \le \begin{cases} k^{r_{\min}} \|d_c\alpha\| & \text{if} \quad k \le 1, \\ k^{r_{\max}} \|d_c\alpha\| & \text{if} \quad k \ge 1, \end{cases}$$

leading to

$$h_k(C_{d_c}(\lambda)) \subset \begin{cases} C_{d_c}(\lambda k^{r_{\min}-1}) & \text{if} \quad k \le 1, \\ C_{d_c}(\lambda k^{r_{\max}-1}) & \text{if} \quad k \ge 1, \end{cases}$$

and finally to the following rescaling, for $\varepsilon \le 1$

$$C_{d_c}(\lambda \varepsilon^{r_{\max}-1}) \subset h_\varepsilon(C_{d_c}(\lambda)) \subset C_{d_c}(\lambda \varepsilon^{r_{\min-1}}). \tag{75}$$

By Lemma 3.11, we can recover the spectral density function of $\delta_c d_c$ on $(\ker d_c)^\perp$, using all Γ-invariant subspaces in the cones $C_{d_c}(\lambda)$, namely

$$F_{\delta_c d_c}(\lambda^2) = \sup_{L \subset C_{d_c}(\lambda)} \dim_\Gamma L.$$

Putting this together with (75) and Propositions 6.2–6.4, we get the pinching

$$F_{\delta_c d_c}(\varepsilon^{2(r_{\max-1})}) \le \varepsilon^{N(G)} F_{\delta_c d_c}(1) \le F_{\delta_c d_c}(\varepsilon^{2(r_{\min-1})}),$$

and finally

$$\varepsilon^{N(G)/(r_{\min-1})} F_{\delta_c d_c}(1) \leq F_{\delta_c d_c}(\varepsilon^2) \leq \varepsilon^{N(G)/(r_{\max-1})} F_{\delta_c d_c}(1),$$

giving the pinching of $\alpha_1(\Gamma) = \liminf_{\varepsilon\to 0} \dfrac{\ln F_{\delta_c d_c}(\varepsilon^2)}{\ln \varepsilon}$.

Notice that the spectral function for d_c is finite, since by Theorems 3.12 and 6.1, its dilatational class is the same as for de Rham complex, or even to an ℓ^2-simplicial complex by Theorem 2.6. (Finiteness is automatic at the discrete simplicial level, while for de Rham complex, it is a direct consequence of the ellipticity of the Laplacian, see e.g. [1].) □

Extensions of Theorem 6.5. We gather some developments around Theorem 6.5.

• This theorem gives a pinching of the asymptotic heat decay on one forms, on *rational* Carnot Lie groups, i.e. for those admitting a lattice Γ. It actually makes sense, and stays true for non rational Carnot Lie groups. One way to prove it is to notice that, even for non rational Carnot group G, there exists a discrete $Z \subset G$ and a relatively compact $D \subset G$ such that

$$Z^{-1} = Z \quad \text{and} \quad G = \bigsqcup_{z\in Z} zD \quad \text{(disjoint union)}.$$

(One takes $Z = \exp(\mathcal{Z})$ for an additive integral lattice in $\mathfrak{g}$, see [40, §3.2.2] for details.) This allows to 'discretize' $L^2(G) = \ell^2(Z) \otimes L^2(D)$ and adapt the basic Propositions 3.8 and 3.9 to this situation.

This technique also raises the following question :

Even, if Z is not a group, one can take a simplicial complex given by the nerve of an open covering $G = \bigcup_{z\in Z} z\Omega$, where $D \subset \Omega$ open. Is it true that $\alpha_1(G)$ still gives the asymptotic heat decay on discrete 1-cochains of this now non-periodic simplicial complex?

• Another viewpoint on non-rational G is to define a G-trace and dimension along the lines of Proposition 6.4.

Briefly, let P be a *positive* G-invariant operator acting on L^2 sections of a G-invariant bundle V, and $D \subset G$ be such that $0 < \mu(D) < +\infty$. Then the G-trace of P is defined by

$$\begin{aligned}\tau_G(P) &= \mu(D)^{-1} \mathrm{Trace}(\chi_D P \chi_D) \\ &= \mu(D)^{-1} \sum_i (\chi_D P \chi_D e_i, e_i),\end{aligned}$$

for any Hilbertian basis (e_i) of $L^2(G,V)$. This G-trace doesn't depend on D. Indeed if S is the positive square root of P, one has also

$$\begin{aligned}\tau_G(P) &= \mu(D)^{-1} \sum_i \|S\chi_D e_i\|_2^2 \qquad (76)\\ &= \mu(D)^{-1} \|S\chi_D\|_{HS}^2 \quad \text{(Hilbert − Schmidt norm)} \\ &= \mu(D)^{-1} \|K_S \chi_D\|_2^2 \quad (L^2 - \text{norm}) \\ &= \|k_S\|_2^2,\end{aligned}$$

where $K_S(x,y) = k_S(y^{-1}x)$ is the kernel of the G-invariant S (to be compared with (71)). In the case $\tau_G(P)$ is finite, then $k_P = k_S * k_S$ is bounded and continuous on G, and we have also

$$\tau_G(P) = \mathrm{Tr}_{V_e}(k_P(e)).$$

Following Propositions 3.8 and 3.9, the basic point to check about τ_G is the following property, valid on *unimodular* Lie groups (like nilpotent ones).

Proposition 6.6. *If P is a (not necessarily positive) G-invariant bounded operator, then*

$$\tau_G(P^*P) = \tau_G(PP^*).$$

Proof. Let $P = U|P|$ be the polar decomposition of P, then $P^*P = |P|^2$ and by (76)

$$\begin{aligned}
\tau_G(P^*P) &= \mu(A)^{-1} \sum_i \||P|\chi_A e_i\|_2^2 \\
&= \mu(A)^{-1} \sum_i \|U|P|\chi_A e_i\|_2^2 \quad (U \text{ partial isometry}) \\
&= \mu(A)^{-1} \|P\chi_A\|_{HS}^2 = \|k_P\|_2^2.
\end{aligned}$$

Finally, since $k_{P^*}(x) = k_P(x^{-1})^*$, one has $\|k_{P^*}\|_2 = \|k_P\|_2$ on unimodular groups, and the result.

□

This gives another mean to extend Theorem 6.5 on non-rational Carnot groups, replacing the Γ by G-trace.

• In another direction, one can obtain pinchings of higher Novikov-Shubin exponent $\alpha_k(G)$ on k-forms, *if* $E_0^k \simeq H^k(\mathfrak{g}, \mathbb{R})$ is of homogeneous weight w_k. One get then

$$\min(1, w_{k+1}^{\min} - w_k) \leq \frac{N(G)}{\alpha_k(G)} \leq w_{k+1}^{\max} - w_k\,. \tag{77}$$

The proof is the same as for Theorem 6.5. The homogeneity condition on $H^k(\mathfrak{g}, \mathbb{R})$ is not automatically satisfied for $k \geq 2$. It allows to control the action of h_ε on norms in $(\ker d_c)^\perp$ (needed in this proof).

6.2 Examples

The algebraic pinching of $\alpha_1(G)$ is sharp if the group is presented by relations of same order. We start with such examples.

Heisenberg groups. For instance we have seen in §4.2 that the Heisenberg groups $\mathbf{H}^{2n+1}$ are quadratically presented for $n \geq 1$ and cubically for $n = 1$. Therefore

$$\beta_1(\mathbf{H}^{2n+1}) = \begin{cases} 1 & \text{for} \quad n \geq 2 \\ 2 & \text{for} \quad n = 1, \end{cases}$$

and hence

$$\alpha_1(\mathbf{H}^{2n+1}) = \begin{cases} N(\mathbf{H}^{2n+1}) = 2n+2 & \text{for} \quad n \geq 2 \\ N(\mathbf{H}^3)/2 = 2 & \text{for} \quad n = 1. \end{cases}$$

In particular heat on one-forms of $\mathbf{H}^3$ only decays as $1/t$ when $t \to +\infty$, half its speed in $1/t^2$ on functions. Therefore this asymptotic spectral invariant α_1 actually distinguishes $\mathbf{H}^3$ from $\mathbb{R}^4$ (or the discrete groups $\mathbf{H}^3_{\mathbb{Z}}$ from $\mathbb{Z}^4$) although they have the same growth $N(G) = 4$.

In order to distinguish (with the asymptotic spectra) higher dimensional $\mathbf{H}^{2n+1}_{\mathbb{Z}}$ from $\mathbb{Z}^{2n+2}$, one has to consider the higher Novikov-Shubin exponent α_n, about the spectrum between n and $(n+1)$-cocycles (discrete or forms). Indeed, by (77) and the cohomological computations (58), one has

$$\beta_k(\mathbf{H}^{2n+1}) = 1 \quad \text{for } k \neq n \quad \text{and} \quad \beta_n(\mathbf{H}^{2n+1}) = 2,$$

and hence

$$\begin{cases} \alpha_k(\mathbf{H}^{2n+1}) = 2n+2 = \alpha_k(\mathbb{R}^{2n+2}) & \text{for} \quad k \leq n-1 \\ \alpha_n(\mathbf{H}^{2n+1}) = n+1 = \alpha_n(\mathbb{R}^{2n+2})/2. \end{cases}$$

Geometrically, this coincidence of the first exponents is reflected by the fact that one can find simplicial triangulations of both groups using *horizontal* simplexes up to topological dimension n, hence with the same homogeneous (Hausdorff) dimensions, while *vertical* $(n+1)$-dimensional simplexes (with Hausdorff dim $= n+2$) are needed on $\mathbf{H}^{2n+1}$. Notice that (77) actually relates the exponents α_k to the homogeneous dimension gap between the cohomology groups $H^{k+1}(\mathfrak{g}, \mathbb{R})$ and $H^k(\mathfrak{g}, \mathbb{R})$.

Quadratically presented groups and Dehn function. In fact quadratically presented 2-step groups are very common if

$$\dim(\mathfrak{g}_2) \leq \dim(\mathfrak{g}_1)/2 - 1. \tag{78}$$

Indeed, one can show that, within this bound, a Zariski open dense set of brackets

$$[\ ,\ ]_0 : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2$$

give quadratically presented Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see [40, Prop. 4.1]). Even more by Gromov's work [21, §4.2 A"] 2-step groups satisfying (78) are generically "quadratically fillable", meaning that an horizontal closed curve γ can be filled by a surface of area $\leq K\text{length}(\gamma)^2$.

In fact β_1 can always be compared with another asymptotic invariant of finitely presented discrete groups Γ called Dehn filling exponent. Recall it is the smallest λ such that any trivial word of length n can be factorized using Kn^λ elementary relations (equivalently, any closed loop in a Cayley graph of Γ can be filled by Kn^λ elementary 2-cells). The following inequalities hold on general r-step Carnot groups Γ

$$2 \leq \beta_1(\Gamma) + 1 \leq \max \text{weight}(H^2(\mathfrak{g}, \mathbb{R})) \leq \text{Dehn}(\Gamma) \leq r+1, \tag{79}$$

and the above case corresponds to equality between the first four terms.

Proof. The two first inequalities comes from Theorem 6.5 and (74), while the last one has been proved by C. Pittet in [35]. We look at the third one.

Given $\omega \in H^2(\mathfrak{g}, \mathbb{R})$ of weight $N(\omega)$, one can find a closed polygonal curve γ in the generators of Γ such that

$$\int_S \omega = C \neq 0$$

for one (and then any) surface S bounding γ. Indeed by §4.2 and 4.3, one can pick a γ whose direction in $\mathcal{R}(\mathfrak{g})$ is not in $\ker \alpha$, where $\alpha \in (\mathcal{R}(g)/[H, \mathcal{R}(g)])^*$ represents ω in Hopf's relation.

Now if S_n is any simplicial surface filling the dilated loop $h_n\gamma$, one has

$$\begin{aligned} |C| = |\int_S \omega| =& |\int_{S_n} h_n^* \omega| = |\int_{S_n} n^{-N(\omega)} \omega| \\ &\leq n^{-N(\omega)} \|\omega\|_\infty \text{Area}(S_n) \end{aligned}$$

so that $\text{Area}(S_n) \geq K \,\text{length}(h_n\gamma)^{N(\omega)}$ as needed. □

Remark 6.7. It is not true in general that $\beta_1(\Gamma) + 1 = \text{Dehn}(\Gamma)$. In fact these two exponents behave differently under products since one finds that

$$\text{Dehn}(\Gamma_1 \times \Gamma_2) = \max(\text{Dehn}(\Gamma_1), \text{Dehn}(\Gamma_2)),$$

while

$$\alpha_1(\Gamma_1 \times \Gamma_2) = \min\big(\alpha_1(\Gamma_1) + \alpha_0(\Gamma_2), \alpha_1(\Gamma_2) + \alpha_0(\Gamma_1)\big).$$

In particular $\alpha_1(\mathbb{R}^n \times \mathbf{H}^3) = n + 2$ so that

$$\beta_1(\mathbb{R}^n \times \mathbf{H}^3) = \frac{n+4}{n+2} \quad \text{while} \quad \text{Dehn}(\mathbb{R}^n \times \mathbf{H}^3) = 3.$$

We leave this as exercises for this Winter School.

One can go beyond the bound (78) staying quadratically presented. Let $H = \mathbb{O} \simeq \mathbb{R}^8$ be the octonions (or Cayley numbers). We define an 'octonionic contact group' G^{15} as follows. Given an imaginary octonion $v \in V = \text{Im}\,\mathbb{O}$, we note $J(v)$ the left multiplication by v on H. Then G^{15} is the 15-dimensional H-type group whose Lie algebra $\mathfrak{g} = H \oplus V^*$ is define as in (65) by

$$(d_0 v)(X, Y) = \langle J(v)X, Y\rangle_H$$

This group is quadratically presented. Indeed, following [41, (12.11)] the map

$$\begin{aligned} d_0 : \Lambda^2_3\, \mathfrak{g}^* = V \wedge H^* &\longrightarrow \Lambda^3 H^* \\ v \wedge \alpha &\longmapsto d_0 v \wedge \alpha \end{aligned} \tag{80}$$

is injective (even an isomorphism) and thus the cohomology group $H^2(\mathfrak{g}, \mathbb{R})$ has no component of weight 3.

Notice that, although it is quadratically presented, this group doesn't possess any integrable Legendrian plane $(X, Y) \in H$ (corresponding to 'pure' relations $[X, Y]_0 = 0$). In particular it doesn't enter Gromov's family of known quadratically fillable groups (with a quadratic Dehn function), and its Dehn exponent seems unknown. (Though there is no C^1 Legendrian surface, quite irregular 'crumpled' surfaces of Hausdorff dimension 2 might exist anyway?)

Once obtained such an example, we can take products staying in the quadratic presentation class : because taking products only adds quadratic relations

$$[X_1, X_2] = 0 \quad \text{for} \quad (X_1, X_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

We get therefore a $15n$-dimensional quadratically presented 2-step Carnot groups with $\dim H = 8n$. The map (80) being algebraic, we obtain in fact that a *generic* (in the Zariski sense) $15n$-dimensional Carnot groups with $n_1 = \dim(H) = 8n$ is quadratically presented. (Otherwise stated a generic $8n$-dimensional distribution $H \subset \mathbb{R}^{15n}$ gives rise to a quadratically presented Carnot group.) Such examples raises the following problem.
Question : Is it possible to significantly improve the bound

$$n_2 = \dim(\mathfrak{g}_2) \leq \frac{7n_1}{8},$$

while keeping a quadratic presentation? This is a purely algebraic question on finding effective bounds under which (80) stays injective. Note that (80) only implies the much larger bound

$$n_2 \leq \frac{(n_1 - 1)(n_1 - 2)}{6},$$

but d_0 is a very special linear map.

In another direction, we mention there exists (a few) examples of quadratically presented groups of arbitrary high steps (see [40, §4.1]). These examples (due to S. Chen [9], J. Carlson and J. Toledo [6]) show that β_1 (= 1 here) can be much smaller than the number of steps r.

Higher weights. We now describe some examples with relations of higher weights.

Given k and r in $\mathbb{N}^*$, we note $F_{k,r}$ the r-step 'free' nilpotent group over $\mathbb{R}^k$. This is the Lie group whose Lie algebra is the quotient of the free Lie algebra $\mathcal{F}(\mathbb{R}^k)$ by elements of weight $\geq r+1$. Notice that relations of $F_{k,r}$ are generated by all elements of weight $r+1$ in $\mathcal{F}(\mathbb{R}^k)$, so that Theorem 6.5 gives

$$\beta_1(F_{k,r}) = r,$$

the maximum possible value for r-step groups.

In fact β_1 is 'generically' close to r. Given k and n, by a *generic* Carnot group G with $\dim(G) = n$ and $\dim(\mathfrak{g}_1) = k$, we mean a group associated to a Zariski open dense set of jets of k-dimensional distributions H in $\mathbb{R}^n$.

Proposition 6.8. *[40, Prop. 4.4] Let $n(k,r) = \dim(F_{k,r})$. Generically, one has*

$$r - 1 \leq \beta_1(G) \leq r \quad \text{if} \quad n(k,r) \leq n < n(k, r+1). \tag{81}$$

Hence the generic pinching is much sharper than the general one (74). This follows from the remark that, within these dimensions bounds, G is generically a r-step group which doesn't contains any relation of weight $\leq r-1$. Then (81) comes from Theorem 6.5. (So this is not an improvement of Theorem 6.5, but just the observation that the pinching given there is generically quite sharp and close to the maximal possible value.)

In the opposite direction we now give an example of Carnot groups arising in semi-simple geometry.

For $n \geq 4$, let $T_n \subset SL(n,\mathbb{R})$ be the nilpotent group of upper triangular matrices (Id + strictly upper). This is a $(n-1)$-step Carnot group whose Lie algebra is generated by the elementary matrices $X_i = E_{i,i+1}$ for $1 \leq i \leq n-1$. By Kostant and Cartier's works (see e.g. [7, 14]) the relations of T_n are quadratic and cubical

$$\begin{cases} [X_i, X_j] = 0 \quad \text{for} \quad |j-i| > 1, \\ [X_i,[X_i,X_{i+1}]] = [X_{i+1},[X_i,X_{i+1}]] = 0 \end{cases} \tag{82}$$

(Notice that T_3 is the 3-dimensional Heisenberg group and is cubically presented by the last relations.) These T_n give examples of increasing number of steps with

$$1 \leq \beta_1(T_n) \leq 2$$

anyway, in contrast to the generic case (81).

These groups give examples of mixed homogeneity, and estimating more precisely β_1 in that case looks delicate in general. Let us show however that

$$\beta_1(T_4) = 2,$$

as if T_4 were purely cubically presented.

Proof. From the results of §4.3 and 4.4, the two relations

$$[X_1, X_3] = [X_1,[X_1,X_2]] = 0$$

given in (82), translate into the following components of $d_c\alpha$ for $\alpha \in \Omega^1 H$

$$d_c\alpha(X_1,X_3) = i_H([X_1,X_3])\alpha = X_1\alpha(X_3) - X_3\alpha(X_1),$$

and

$$\begin{aligned} d_c\alpha(X_1,Y_1) &= i_H([X_1,[X_1,X_2]])\alpha \\ &= X_1(X_1\alpha(X_2) - X_2\alpha(X_1)) - Y_1\alpha(X_1), \end{aligned}$$

where $Y_1 = [X_1,X_2]$. In particular for $\alpha = f\theta_{X_2}$, one sees that $d_c\alpha$ is actually of weight 3, and non zero if $X_1^2.f \neq 0$. Then the near-cohomology cone $C_{d_c}(\lambda)$ contains a non-vanishing sub-cone

$$C'(\lambda) = \{\alpha \in L^2(\Lambda^1 H^*)/\ker d_c \mid d_c\alpha \text{ of weight} \geq 3 \text{ and } \|d_c\alpha\| \leq \lambda\|\alpha\|\},$$

which now rescales quadratically through the dilations :

$$h_\varepsilon(C'(\lambda)) = C'(\lambda\varepsilon^2).$$

Following the proof of Theorem 6.5, this leads to the homogeneity of the spectral density function

$$F_{\delta_c d_c}(\varepsilon^2) \asymp \varepsilon^{N(G)/2},$$

giving the result. □

Thus it happens here that the component of low degree of d_c is so degenerated analytically that the asymptotic spectral behaviour is determined by the relations of higher weight.

Remark 6.9. It looks possible (but the proof is not fully checked yet) that this behaviour occurs each time the system $\delta_c\alpha = d_c\alpha = 0$ is under-determined, when restricted to relations of weight $\leq n$, meaning that

$$\dim\big(H^2_{\text{weight}\leq n}(\mathfrak{g}, \mathbb{R})\big) < \dim(H) - 1 \Longrightarrow \beta_1(G) \geq n - 1?$$

In the opposite direction, it also happens that the components of low degree of d_c 'dominate' analytically the others. For instance let $G = T_n/H$ be any quotient of the triangular group T_n by a normal subgroup H generated by elements of weight ≥ 4. Relations of G are generated by those of T_n, at most cubical, and the generators of $\mathfrak{h}$, that can be of order $n - 1$. One can show anyway that

$$\beta_1(G) \leq 2,$$

as if the added relations of high order were 'inaudible' in our asymptotic spectral problem.

This result relies on analytic properties of the operators, derived from hypoellipticity. They are developed in [40, Section 5] and provide some tools for other geometric applications. Many open problems in Carnot-Carathéodory geometry, mixing various fields, may be found in [21].

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