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SUB-RIEMANNIAN LIMIT OF THE DIFFERENTIAL FORM SPECTRUM OF CONTACT MANIFOLDS

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1 Introduction

One natural approach to sub-Riemannian geometry lies in the study of the behavior of Riemannian objects in the sub-Riemannian limit. This consists of blowing up the metric transversely to the Carnot distribution. These metric spaces converge in the Gromov-Hausdorff sense to the sub-Riemannian ones [Gro].

However, very little is known about the convergence of some even basic and linear objects as the spectrum of the Laplacians on differential forms. We begin here this study in the contact case. We will see that the non-blowing parts of the spectrum of the Laplacians, $d+\delta$ and the signature operator $*d-d*$, concentrate and are described by their counterparts coming from the contact complex studied in [Ru]. In particular, an interesting infinite dimensional collapsing eigenvalues phenomenon occurs on middle degree forms. It corresponds to the special second order differential D of the contact complex. These spectrum convergences of unbounded operators are first studied in Theorems 3.5 and 3.6 through the convergence of their resolvents.

The techniques are much inspired from adiabatic limits as developed for example in [BeB], [BiL], [D], [MM]. Nevertheless, the algebraic and analytic situations here are quite different, in some sense opposite to the adiabatic case, where the unexploded directions need to be integrable and form a fibration. Anyway, this approach, pointed out by J.-M. Bismut, relies on some formal resemblances between the problems. Mainly, we will see in section 3, that the contact complex occurs as a natural spectral sequence in the sub-Riemannian blow up, just like the Leray spectral sequence of the fibration does in the adiabatic case.

These algebraic structures used to predict, in a formal power series sense at first, the different parts of the spectrum that blow up or collapse at different rates. The way to turn this into the actual convergence of the resolvents will rest here on the use of some L^2 a priori estimates. They will come from a sub-Riemannian Bochner technique developed in section 5. This one has to bypass three facts. First, the Riemannian curvature diverges here, and makes a Riemannian like Bochner formula unusable. Also, the presence of an infinite collapsing spectrum phenomenon in middle degrees precludes a classical first order Sobolev control there. Lastly, again in these degrees, the second order differential D, coming with the spectral sequence structure, can't directly come out and be dominated by the Riemannian Laplacian.

These basic resolvents' convergences of Laplacians or signature operator are then taken as a starting point for the study of global and local convergences of heat kernels and eta functions, for non-small times. These are shown to converge to their hypoelliptic contact counterparts in section 7.

The developed techniques, and most of the results, also apply to Galois coverings of compact contact manifolds. This can be used to study the asymptotic behavior of the heat kernel on forms on the Heisenberg groups, for large time. This problem is related to ours because, through the Heisenberg dilations, the large scale Riemannian geometry of this group looks like the local sub-Riemannian one. Here, the heat associated to the contact complex appears to be the self-similar limit of the Riemannian process, as stated in Theorem 7.14.

2 Notation and Heuristic Study

Let M be a contact manifold of dimension $2n+1$. We denote by H the contact field, that we will assume transversally oriented. Therefore, we can fix a contact form θ on M such that $H = \ker\theta$. The contact condition means that $d\theta$ restricted to H is a non-degenerate 2-form. We choose an (almost) complex structure J on H such that $d\theta(X, JY) = g_H(X,Y)$ is symmetric positive definite on H. There is no obstruction for doing this since we do not require J to be integrable: that is to induce an integrable CR structure. Also associated to θ is the transverse Reeb field T; it is the unique vector field satisfying $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$. We can now consider the family of metrics

$$g_\varepsilon = g_H + \tfrac{g_T}{\varepsilon^2}$$

where $g_H = d\theta(\cdot, J\,\cdot)$, $g_T = \theta^2$ and T is orthogonal to H. We plan to study the behavior of the Riemannian spectrum of (M, g_ε) in the sub-Riemannian limit $\varepsilon \to 0$.

The exterior algebra of M splits in horizontal and vertical forms, which we will denote by

$$\Omega^* M = \Omega^* H \oplus \theta \wedge \Omega^* H$$

where $\Omega^* H$ are forms vanishing on T. With respect to this decomposition, the exterior differential writes $d(\alpha_H + \theta \wedge \alpha_T) = (d_H \alpha_H + L\alpha_T) + \theta \wedge (\mathcal{L}_T \alpha_H - d_H \alpha_T)$, that is

$$d = \begin{pmatrix} d_H & L \\ \mathcal{L}_T & -d_H \end{pmatrix}$$

where $d_H = \Pi_{\Omega^* H} d$ is the horizontal part of d, $\mathcal{L}_T$ the Lie derivative along T and L is the operator on $\Omega^* H$ defined by $L\alpha = d\theta \wedge \alpha$. Observe that $\mathcal{L}_T$ and L both preserve horizontal and vertical forms. Before going ahead, we now work out a conjugation on the exterior algebra. This is the isometry from $(\Omega^*_m M, g_\varepsilon)$ to $(\Omega^*_m M, g_1)$ defined by

$$C_\varepsilon(\alpha_H + \theta \wedge \alpha_T) = \alpha_H + \varepsilon\, \theta \wedge \alpha_T\,.$$

We consider then $d_\varepsilon = C_\varepsilon d\, C_\varepsilon^{-1}$ and its adjoint $\delta_\varepsilon = (d_\varepsilon)^{*_{g_1}} = C_\varepsilon \delta^{g_\varepsilon} C_\varepsilon^{-1}$, such that we have $\Delta_\varepsilon = d_\varepsilon \delta_\varepsilon + \delta_\varepsilon d_\varepsilon = C_\varepsilon \Delta_{g_\varepsilon} C_\varepsilon^{-1}$, and for the Hodge star operator: $*_{g_1} = C_\varepsilon *_{g_\varepsilon} C_\varepsilon^{-1}$. The advantage of working with these operators is that they act on a fixed metric space, the spectrum being unchanged under conjugation.

Now, we have

$$\begin{cases} d_\varepsilon = \begin{pmatrix} d_H & \varepsilon^{-1} L \\ \varepsilon\, \mathcal{L}_T & -d_H \end{pmatrix} \qquad \delta_\varepsilon = \begin{pmatrix} \delta_H & \varepsilon\, \mathcal{L}_T^* \\ \varepsilon^{-1}\Lambda & -\delta_H \end{pmatrix} \\ *_1 = \begin{pmatrix} 0 & *_H \\ *_H(-1)^p & 0 \end{pmatrix} \\ \Delta_\varepsilon = \begin{pmatrix} \Delta_H + \varepsilon^2 \mathcal{L}_T^* \mathcal{L}_T + \varepsilon^{-2} L\Lambda & \varepsilon^{-1}[\delta_H, L] + \varepsilon[d_H, \mathcal{L}_T^*] \\ \varepsilon^{-1}[\Lambda, d_H] + \varepsilon[\mathcal{L}_T, \delta_H] & \Delta_H + \varepsilon^2 \mathcal{L}_T \mathcal{L}_T^* + \varepsilon^{-2} \Lambda L \end{pmatrix} \end{cases} \tag{1}$$

where Λ is the adjoint of L.

From the structure of these matrices, we can see the differences between the behavior of the spectrum of Δ_ε on functions and forms. Indeed, we have on functions $\Delta_\varepsilon = \Delta_H + \varepsilon^2 \mathcal{L}_T^* \mathcal{L}_T$. This is a *decreasing* ($\varepsilon \to 0$) family of positive elliptic operators dominating the hypoelliptic Kohn Laplacian Δ_H. The maxmin principle associated to subelliptic estimates shows then that the whole spectrum (eigenvalues and eigenfunctions) of Δ_ε converges towards the one of Δ_H (see [G]).

On the contrary, part of the spectrum on forms is divergent. For example, one has $\Delta_\varepsilon(\theta) = \frac{1}{n\varepsilon^2}\theta$ and also $\Delta_\varepsilon(d\theta) = d\,(\Delta_\varepsilon \theta) = \frac{1}{n\varepsilon^2}(d\theta)$. Of course, another part of the spectrum on 1-forms is still convergent, for we can take $\alpha_\varepsilon = d_\varepsilon f_\varepsilon$ where $\Delta_\varepsilon f_\varepsilon = \lambda_\varepsilon f_\varepsilon$ are convergent eigenfunctions. We observe also that these forms $\alpha_\varepsilon = d_H f_\varepsilon + \varepsilon\theta \wedge \mathcal{L}_T f_\varepsilon$ are asymptotically horizontal (for g_ε).

The aim of this is to introduce the heuristic remark that despite the divergent terms in Δ_ε, part of its spectrum actually converges and the corresponding eigenforms should concentrate on $E = \ker \left(\begin{smallmatrix} L\Lambda & 0 \\ 0 & \Lambda L \end{smallmatrix}\right)$ (the ε^{-2} term in Δ_ε). We note that since L is injective on $\Omega^p H$ for $p \le n-1$ (resp. surjective onto $\Omega^p H$ for $p \ge n+1$), one obtains that

$$\begin{aligned}
\text{for p} \le n\,,\ E \cap \Omega^p M &= \{\alpha \in \Omega^p M,\ \Lambda\alpha = 0\} \\
&= \Omega^p_0 H \qquad \text{(the primitive horizontal forms)} \\
\text{for } p > n\,,\ E \cap \Omega^p M &= \theta \wedge (\Omega^{p-1}_L H) \\
&= \{\alpha \in \theta \wedge (\Omega^{p-1} H),\ L\alpha = 0\}\,.
\end{aligned}$$

To complete these preliminary remarks, we now stress the fact that a special phenomenon of collapsing eigenvalues can occur in the middle dimensions spaces $\Omega^n M$ and $\Omega^{n+1} M$. We show this on some particular contact manifolds: we take M to be the boundary of a circular pseudo-convex domain of $\mathbb{C}^{n+1}$. Moreover, we suppose that the holomorphic S^1 action $z \to e^{i\varphi} z$ which preserves the CR structure of M is transverse to H. We can then fix a contact form θ such that $\theta(\partial/\partial\varphi) = 1$. Now, we consider the family of holomorphic $(n+1)$-forms

$$\alpha_P = P(z_1, \cdots, z_{n+1}) dz_1 \wedge \cdots \wedge dz_{n+1}\,,$$

where P is an homogeneous polynomial of degree k. Their restriction $\widetilde{\alpha_P}$ to M are closed vertical forms (because α_P restricted to H is of bidegree $(n+1, 0)$). So (1) gives $d_H(i_T \widetilde{\alpha_P}) = 0$. Moreover, we have

$$\begin{aligned}
*\alpha_P &= *_H(i_T \widetilde{\alpha_P}) \\
&= (-1)^{n(n+1)/2} J(i_T \widetilde{\alpha_P}) \qquad \text{(see [W, I, Thm. 2])} \\
&= (-1)^{n(n+1)/2} i^n (i_T \widetilde{\alpha_P})
\end{aligned}$$

so that finally

$$\begin{aligned}
(d_\varepsilon *)\alpha_P &= (-1)^{\frac{n(n+1)}{2}} i^n \varepsilon \mathcal{L}_T \widetilde{\alpha_P} \quad \text{by (1)} \\
&= (-1)^{\frac{n(n+1)}{2}} i^{n+1} \varepsilon (n+1+k) \widetilde{\alpha_P}\,.
\end{aligned}$$

That means the $\widetilde{\alpha_P}$ generate an infinite dimensional space of collapsing eigenforms for the signature operator (and of course for the Laplacian since $\Delta_\varepsilon(\widetilde{\alpha_P}) = \varepsilon^2 (n+1+k)^2 \widetilde{\alpha_P}$). This family of eigenvalues has been produced using an overabundance of strong geometric hypothesizes (embeddability in $\mathbb{C}^n$ and transverse S^1 CR action) although we a priori face a (sub-)Riemannian problem. One goal of this article is to understand whether this phenomenon survives to non-integrable J. There, holomorphic objects will disappear but small eigenvalues should remain.

3 The Contact Complex and the Sub-Riemannian Limit

The previous remarks lead us to think that the convergent part of the spectrum on forms concentrates on the subspace $E = \ker \left(\begin{smallmatrix} 0 & L \\ \Lambda & 0 \end{smallmatrix}\right)$ of $\Omega^* M$. It must therefore be described by differential operators acting on E and invariant under the anisotropic change of metric g_ε. There is a natural candidate for this: the so-called contact complex in [Ru].

We briefly recall its construction. One considers the differential ideal $\mathcal{I}^*$ generated by the contact forms, i.e. $\mathcal{I}^* = \{\theta \wedge \alpha + d\theta \wedge \beta\}$ and its annulator $\mathcal{J}^* = \{\alpha \mid \theta \wedge \alpha = d\theta \wedge \alpha = 0\} = E \cap \Omega^{p>n} M$. We have two induced complexes d_Q on $\Omega^* M/\mathcal{I}^*$ and $\mathcal{J}^*$ from the de Rham one, and a *second* order D from $\Omega^n M/\mathcal{I}^n$ in $\mathcal{J}^{n+1}$ defined the following way. For $\alpha \in \mathcal{I}^n$, let $D\alpha = d\widetilde{\alpha}$ where $\widetilde{\alpha}$ is any lift of α in $\Omega^n M$ such that $d\widetilde{\alpha}$ is a vertical form. D is easily seen to be well-defined and independent of θ. However, a choice of θ and of a partial complex structure J on H allows us to identify the quotient space $\Omega^* M/\mathcal{I}^*$ with the primitive horizontal forms $\Omega_0^* H = E \cap \Omega^{p \le n} M$. In this case one can take $\widetilde{\alpha} = \alpha_H^0 - \theta \wedge L^{-1} d_H \alpha_H^0$ where α_H^0 is the representative of α in $\Omega_0^* H$, and L^{-1} the inverse of L from $\Omega^{n-1} H$ to $\Omega^{n+1} H$. The formula for D reads then

$$D\alpha = \theta \wedge (\mathcal{L}_T + d_H L^{-1} d_H)\alpha_H^0 \,. \tag{2}$$

The main properties of this construction are:

Theorem 3.1 [Ru]**.** *The complex*

$$0 \to \mathbb{R} \to C^\infty M \xrightarrow{d_Q} \Omega_0^1 \xrightarrow{d_Q} H \cdots \Omega_0^n \xrightarrow{D} H \mathcal{J}^{n+1} \xrightarrow{d_Q} \cdots \mathcal{J}^{2n+1} \to 0$$

is a resolution and the associated Laplacians are hypoelliptic.

To enlighten both the construction and the role of this complex in our problem, we show that it can be derived from spectral sequence considerations (see P. Julg [J] for still another point of view on this complex). We have to find a filtration of the exterior algebra adapted to the contact sequence and the anisotropic blow-up of metric g_ε. We choose a basis $\theta_H^1, \cdots, \theta_H^{2n}$ of $\Omega^1 H$ and θ a contact form. We will say a (non-zero) form α is of contact-weight $N_H \alpha = p$ if it is in the linear span of $\theta_H^{I_1} \wedge \theta^{I_2}$ with $p = \operatorname{card}(I_1) + 2\operatorname{card}(I_2)$, $I_1 \subset \{1, \cdots, 2n\}$, $I_2 \subset \{1\}$. This weight depends on the choice of the basis but induces a *contact-intrinsic* filtration of the exterior algebra by $F^p\Omega = \{\alpha \mid N_H \alpha \ge p\}$. Equivalently a q-form is in $F^p\Omega$

iff it vanishes on all q-uples of vectors in $\otimes^{k_1} H \otimes^{k_2} TM$ with $k_1 + 2\,k_2 < p$. (And finally $F^p\Omega = \Omega^p M \oplus$ vertical forms of degree $p-1$). This filtration is stable under d, so that a spectral sequence can be associated to it (see [GrH]).

REMARK 3.2. This construction has a natural generalization in (regular) Carnot-Caratheodory geometry [Gro]. If a manifold is given a r-step bracket generating distribution H, one can intrinsically filter the forms using the d non-decreasing weight $N_{H_1} + 2\,N_{H_2} \cdots + r\,N_{H_r}$, where $H_1 = H$, $H_k = [H, H_{k-1}]$. (We hope to discuss some of the analytic properties of the associated spectral sequence elsewhere.)

It turns out that in our contact case this spectral sequence is merely equivalent to the contact complex as follows from the following straightforward proposition, stated in the standard notation of [GrH].

PROPOSITION 3.3.

1. $E_0^{p,q}$ $\left(= \frac{F^p\Omega^{p+q}}{F^{p+1}\Omega^{p+q}}\right) = 0$ *unless* $q = 0$ *or* -1
   $E_0^{p,0}$ = *horizontal (partial) p-forms*
   $E_0^{p,-1}$ = *vertical $(p-1)$-forms*
   $d_0 = \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right)$ *(induced by d on E_0)*
2. E_1 $(= \ker d_0 / \operatorname{Im} d_0) \simeq E$ = *primitive horizontal forms* $\oplus$ *coprimitive vertical forms*
   $d_1 = d_Q$ *(including the fact that $d_1 = d_Q = 0$ on $E_1^{n,0}$)*
3. E_2 $(= \ker d_Q / \operatorname{Im} d_Q) = \mathcal{H}_Q \simeq$ *de Rham cohomology except in degree n and $n+1$*
   $d_2 = 0$ *except in degree n where $d_2 = D$.*
4. $E_3 = E_2$ *except* $E_3^{n,0} = \ker D / \operatorname{Im} d_Q$ *and* $E_3^{n+2,-1} = \ker d_Q / \operatorname{Im} D$
5. $E_3 = E_4 = \cdots \simeq$ *de Rham cohomology.*

REMARK 3.4. Before going ahead, we insist on two particular features of this spectral sequence

- The first one is that $D = d_2$ is only a priori defined from $E_1^{n,0} \cap \ker d_Q$ into $E_1^{n+2,-1} / \operatorname{Im} d_Q$. But these spaces are actually $E_1^{n,0}$ $(= \Omega_0^n H)$ and $E_1^{n+2,-1}$ $(= \mathcal{J}^{n+1}$ by Theorem 3.1), because
$$d_Q : E_1^{n,0} \to E_1^{n+1,0} = 0$$
$$\text{and} \ \ d_Q : E_1^{n+1,-1} = 0 \to E_1^{n+2,-1}$$
- The reason why this spectral sequence really computes the cohomology of M and not only its graded part (see [GrH])
$$\operatorname{Gr}(H_{DR}) = \oplus_p \frac{F^p H_{DR}}{F^{p+1} H_{DR}}$$

is that they coincide! Indeed, this filtration degenerates in the following way:

$$\text{for } q \le n,\ F^p H^q_{DR} = \begin{cases} H^q_{DR} & \text{if } p \le q \\ \{0\} & \text{if } p > q \end{cases}$$

$$\text{for } q > n,\ F^p H^q_{DR} = \begin{cases} \{0\} & \text{if } p < q+1 \\ H^q_{DR} & \text{if } p \ge q+1\,. \end{cases}$$

This stems from the facts that there is no closed vertical form of degree $p \le n$ (due to the injectivity of $L : \Omega^{p-1}H \to \Omega^{p+1}H$), and that any form of degree $p > n$ is cohomological to a vertical one (due to the surjectivity of $L : \Omega^{p-2}H \to \Omega^p H$).

We now come back to our formula for d_ε

$$d_\varepsilon = \varepsilon^{-1}\begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d_H & 0 \\ 0 & -d_H \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ \mathcal{L}_T & 0 \end{pmatrix}$$

and observe that the components of this ε-splitting are *also* homogeneous with respect to the above contact weight N_H and associate filtration. We mean that $\left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right)$ preserves N_H, $\left(\begin{smallmatrix} d_H & 0 \\ 0 & -d_H \end{smallmatrix}\right)$ increases it by 1 and $\left(\begin{smallmatrix} 0 & 0 \\ \mathcal{L}_T & 0 \end{smallmatrix}\right)$ by 2.

This possibly explains the appearance of the associated spectral sequence (and the contact complex) in our problem. In fact, there are strong algebraic similarities with the adiabatic case. Recall that one considers there a family of metrics $g_\varepsilon = \varepsilon^{-2} g_H + g_V$ where, instead of our contact case, the unexploded V-directions need to form an integrable foliation with some *global* hypothesis on the leaves; they have to come from a compact fibration. One can then define the Leray spectral sequence associated with the d-stable filtration $F^p\Omega^{p+q}$ = linear span of $\theta_H^{I_1} \wedge \theta_V^{I_2}$ with $\mathrm{card}(I_1) \ge p$, which again are homogeneous under the above change of metric.[1]

A deep study of the adiabatic limit has been undertaken by many authors: Dai, Mazzeo-Melrose, Bismut and co-authors... They have shown, for various geometric operators (Laplacians, signature, Dolbeault) that the asymptotic behavior when $\varepsilon \to 0$ of the spectrum is encoded in the associated Leray spectral sequence. Roughly speaking, eigenforms associated to eigenvalues λ_ε of order less or equal ε^r converge in the nested spectral sequence's spaces E_r (rather a Hodge theoretic version of them). Moreover, the magnified spectrum $\varepsilon^{-r}\lambda_\varepsilon$ converges to the one of operators constructed from the successive differential d_r of the spectral sequence. This screening

[1] If the Reeb field induces a fibration one can take here $V = T$, and see that the adiabatic limit is the opposite of the sub-Riemannian one in this situation.

procedure converges to cohomology for a finite r with the spectral sequence degeneration.

This picture will apply in our sub-Riemannian problem. First of all we define the relevant Hodge versions of the quotient spectral spaces E_r of Prop. 3.3. Let

- $F_1 = \ker\left(\begin{smallmatrix} 0 & L \\ \Lambda & 0 \end{smallmatrix}\right) (= E)$: it is the candidate limit space for the convergent part of the spectrum.
- $F_2 = \ker(d_Q + \delta_Q) \subset F_1$ with the particularity that $d_Q = 0$ on $F_1^n = F_1 \cap \Omega^n M$ and $\delta_Q = 0$ on F_1^{n+1} (see Prop. 3.3): the limit space for de Rham cohomology, except in middle degrees.
- $F_3 = F_2 \cap (\Omega^n M + \Omega^{n+1} M)$ will be the limit space for the collapsing spectrum.

Now we define

- $P_\varepsilon = d_\varepsilon + \delta_\varepsilon$ with $\Delta_\varepsilon = P_\varepsilon^2 = d_\varepsilon \delta_\varepsilon + \delta_\varepsilon d_\varepsilon$
  the signature operator, when $\dim M = 4l - 1 = 2n + 1$, is
- $S_\varepsilon = (*d_\varepsilon - d_\varepsilon *)w$ acting on $\Omega^{even} M$ with $w = (-1)^p$ on $\Omega^{2p} M$.

Their contact counterparts are

- $P_Q = d_Q + \delta_Q$ and $\Delta_Q = P_Q^2 = d_Q \delta_Q + \delta_Q d_Q$ acting on F_1.
- $P_D = D + D^*$ acting on F_2 (with the convention, motivated by Prop. 3.3, that $D = 0$ outside degree n)
- $\Delta_D = D^* D + D D^*$ on F_2 ($\neq 0$ only on F_3) (a fourth order Laplacian)
  the contact-signature operators:
- $S_Q = (*d_Q - d_Q *)w$ on F_1^{even}
- $S_D = (-1)^l D *$ on F_2^{even} ($\neq 0$ only on F_2^{n+1}).

We can formulate two first theorems, describing respectively the non-collapsing and collapsing spectrum on a compact contact manifold endowed with the family of metrics g_ε. All convergences are relative to the norm of bounded operators in L^2. Π denotes orthogonal projection.

Theorem 3.5. *$\exists \lambda \in \mathbb{C}$ such that*

1. $\Pi_{F_3^\perp} (\lambda - P_\varepsilon)^{-1} \Pi_{F_3^\perp} \xrightarrow[\varepsilon \to 0]{} (\lambda - P_Q)^{-1} \Pi_{F_1 \cap F_3^\perp}$;
3. *the same holds with the signatures S_ε and S_Q instead;*
3. $\Pi_{F_3^\perp} (\lambda - \Delta_\varepsilon)^{-1} \Pi_{F_3^\perp} \xrightarrow[\varepsilon \to 0]{} (\lambda - \Delta_Q)^{-1} \Pi_{F_1 \cap F_3^\perp}$.

Theorem 3.6. *$\exists \lambda \in \mathbb{C}$ such that*

1. $\left(\lambda - \frac{P_\varepsilon}{\varepsilon}\right)^{-1} \xrightarrow[\varepsilon \to 0]{} (\lambda - P_D)^{-1} \Pi_{F_2}$;
2. *idem with $S_\varepsilon / \varepsilon$ and S_D;*
3. $\left(\lambda - \frac{\Delta_\varepsilon}{\varepsilon^2}\right)^{-1} \xrightarrow[\varepsilon \to 0]{} (\lambda - \Delta_D)^{-1} \Pi_{F_2}$.

One classical consequence (see [RS]) of convergence in the norm resolvent sense of self-adjoint operators is the convergence both of spectral projections (in particular of the eigenforms) and of the spectrum to the limit operator one's (over *bounded* intervals of $\mathbb{R}$).

4 Miscellaneous formulas

To prove Theorems 3.5 and 3.6, we need first to relate the horizontal H-operators to the quotiented Q-ones. Also, in view of formula (1), we have to take care of the actions of Δ_H, $\mathcal{L}_T$, $d_H, \dots$ with respect to the eigenspaces of $\left(\begin{smallmatrix} L\Lambda & 0 \\ 0 & \Lambda L \end{smallmatrix}\right)$ (which is the ε^{-2} term of Δ_ε). We recall these are the factors of the Lefschetz decomposition:

$$\Omega^* H = \Omega_0^* H \oplus L\Omega_0^* H \oplus \cdots \oplus L^n \Omega_0^* H.$$

where $\Omega_0^* H$ denotes the primitive horizontal forms, and we have

$$\begin{cases} \Lambda L = (k+1)(n-p-k)^+ \,\mathrm{Id} & \text{on } L^k \Omega_0^p H \\ [\Lambda, L] = (n-p)\,\mathrm{Id} & \text{on } \Omega^p H. \end{cases} \tag{3}$$

These formulas are purely algebraic (order 0), relying only on the existence on H of the Hermitian structure induced by the relation $g_H = d\theta(\,\cdot\,, J\,\cdot\,)$ with J almost complex structure (see [W]). We compute now the basic first and second order identities. We have

$$d_H^2 = -L\mathcal{L}_T \quad \text{and} \quad [d_H, L] = [d_H, \mathcal{L}_T] = [L, \mathcal{L}_T] = 0\,. \tag{4}$$

Proof. Expansion of $d^2 = \left(\begin{smallmatrix} d_H & \mathcal{L}_T \\ L & -d_H \end{smallmatrix}\right)^2 = 0$. □

Perhaps more surprising, the following key formula holds even for not integrable J. It can thus be called an *almost-contact-Kähler* identity

$$[\Lambda, d_H] = -\delta_H^J \ (= -J^{-1}\delta_H J)\,. \tag{5}$$

Proof. Weil's book ([W]) contains a proof of the Kähler identity $[\Lambda, d] = -\delta^J$ in the almost Kähler case: that is on symplectic manifolds with a compatible almost complex structure J and metric $g = \omega(\cdot, J\,\cdot)$. This first order formula depends only on the following facts:

- $d\omega = 0 \Longleftrightarrow [d, L] = 0$,
- $\delta = -*\,d*$,
- the above relation between g, ω and J.

In our situation, we have such a partial structure on H, taking

- $\omega = d\theta$,
- d_H and $\delta_H = -*_H\, d_H *_H$ (see [Ru]).

So, a proof “à la Weil” (without any connection and framing) applies here. □

REMARK 4.1. The fact that (4) holds without zero order term will only be crucial in the study of the collapsing spectrum, where we will need a Bochner like formula with controllable ‘error’ terms for Δ/ε^2.

We can express the action of d_H with respect to the Lefschetz decomposition. We have on $\Omega_0^p H$

$$\begin{cases} \delta_H = \delta_Q \\ d_H = d_Q - \frac{L}{n-p+1}\delta_Q^J \,. \end{cases} \tag{6}$$

Proof. The first formula comes from $[\Lambda, \delta_H] = 0$ (by (4)).

The second one is a refinement of Prop. 4 in [Ru] thanks to the new (5). □

We now come to second order relations[2]

$$\begin{cases} \mathcal{L}_T = \frac{1}{n-p}\delta_Q^J d_Q + \frac{1}{n-p+1} d_Q \delta_Q^J & \text{on } \Omega_0^p H \text{ with } p < n \\ \mathcal{L}_T = i_T D + d_Q \delta_Q^J & \text{on } \Omega_0^n H \,. \end{cases} \tag{7}$$

Proof. Using (6) and $[d_H, L] = 0$, one can expand $d_H^2 = -L\mathcal{L}_T$ (4). This leads to the first equation. Relation (2), $i_T D = \mathcal{L}_T - d_H L^{-1} d_H$ and (6) immediately give the second one ($d_Q = 0$ on $\Omega_0^n H$). □

Using (7) and $L = L^J$ (because $d\theta$ is a $(1,1)$ form), one can see that $\mathcal{L}_T$ preserves the Lefschetz decomposition and moreover, one has on the whole $\Omega^* H$:

$$\begin{cases} \mathcal{L}_T^* + \mathcal{L}_T^J = 0 \\ [\Lambda, \mathcal{L}_T] = 0 \end{cases} \tag{8}$$

We study Δ_H. On $\Omega_0^p H$ with $p \le n$, we have

$$\Delta_H = \Delta_Q + \frac{d_Q^J \delta_Q^J}{n-p+1} - \frac{L}{n-p+2}(\delta_Q \delta_Q^J + \delta_Q^J \delta_Q) \tag{9}$$

(where $\Delta_Q = d_Q \delta_Q$ on $\Omega_0^n H$).

Proof. This follows from $\Delta_H = d_H \delta_H + \delta_H d_H$ and (6). □

One consequence of (9) is the fact that Δ_H almost preserves the decomposition $\Omega^* H = \Omega_0^* H \oplus \operatorname{Im} L$. Indeed, $\delta_Q \delta_Q^J + \delta_Q^J \delta_Q$ is a first order operator, vanishing when J is integrable. This can be deduced from the decomposition

$$d_Q = \partial_Q + \bar{\partial_Q} + N + \bar{N}, \text{ where } N = \Pi_{\Omega_0^{p+2,q-1} H} d_Q \Pi_{\Omega_0^{p,q} H} \tag{10}$$

[2] All order references are given with respect to the natural contact and hypoelliptic weight: 1 for an horizontal derivative, 2 for a transversal one.

is an algebraic expression of the Nijenhuis tensor of J. So, we have shown the following, on $\Omega_0^p H = \Omega_0^p H \oplus (\operatorname{Im} L)^p$, $p \leq n$,

$$\Delta_H = \begin{pmatrix} \Delta_Q + \dfrac{d_Q^J \delta_Q^J}{n-p+1} & P_N^{(1)*} \\ P_N^{(1)} & \Pi_L \Delta_H \Pi_L \end{pmatrix}, \tag{11}$$

where $P_N^{(1)} = -\frac{L}{n-p+2}(\delta_Q \delta_Q^J + \delta_Q^J \delta_Q)$.

In fact, Δ_H almost preserves (modulo first order differential operators) the full Lefschetz decomposition. Even more, $[\Delta_H, L] =$ first order. This comes easily from (5) and decomposition of (4): $d_H^2 = -L\mathcal{L}_T$ with respect to the bigraduation of $\Omega^* H$. This also follows immediately from the fact that Δ_H is an almost scalar operator on $\Omega^{p,q} H$ with principal part a Folland-Stein operator (see [Ru, Prop. 2]):

$$\Delta_H = \Delta_K + i(p-q)\mathcal{L}_T + \text{first order}, \tag{12}$$

where $\Delta_K = -\sum_{i=1}^n (X_i^2 + Y_i^2)$, for a J-orthonormal base of H, is the Kohn Laplacian. Two other important properties of Δ_H also come from this.

- The first one is that Δ_H almost respects the complex bigrading of $\Omega^* H$. More precisely, the previous computations show that $\Delta_H = \Delta_H^J + P_J^{(1)}$ where $P_J^{(1)}$ is a first order operator vanishing when J is integrable and invariant under the transverse Reeb flow.
- Δ_H is hypoelliptic on $\Omega^{p,q} H$ when $|p-q| < n$.

In fact choosing a sub-Riemannian connection ∇ (like in [Ru], for example, but the horizontal part of the Levi-Cevita connection would also fit), one can obtain the following explicit a priori L^2 estimates on $\Omega^p M$ (resp. $\Omega_0^p H$) for $p < n$

$$\begin{cases} (\Delta_H \alpha, \alpha) + C\|\alpha\|^2 \geq \frac{1}{n}\|\nabla_H \alpha\|^2 \\ (\Delta_Q \alpha, \alpha) + C\|\alpha\|^2 \geq \dfrac{n-p}{n-p+2}(\Delta_H \alpha, \alpha) \end{cases} \tag{13}$$

where C is a constant depending only on the norm of the curvature of ∇ (including tensors like the Nijenhuis of J and its first horizontal covariant derivative, $\mathcal{L}_T J$ and horizontal curvature).

Proof. (See Prop. 6, 7 part I and Lemma 10 part II of [Ru]). The first control is the basic L^2 estimate for the Folland-Stein operators. This is obtained by writing

$$\mathcal{L}_T = \frac{1}{n}\sum_{i=1}^n [\nabla_{Y_i}, \nabla_{X_i}] + \text{first order},$$

so that,

$$\begin{aligned}|(\mathcal{L}_T\alpha,\alpha)| &\le \frac{2}{n}\sum_{i=1}^{n}\|\nabla_{Y_i}\alpha\|\|\nabla_{X_i}\alpha\| \\ &\le \tfrac{1}{n}(\nabla_H^*\nabla_H\alpha,\alpha) + \text{first order}\,.\end{aligned}$$

The second estimate relies on the use of auxiliary normalized Laplacians. Namely, (7) suggests to define, on $\Omega_0^p H$ for $p<n$, the following

$$\Delta_{Q_N} = \frac{1}{n-p}\delta_Q d_Q + \frac{1}{n-p+1}d_Q\delta_Q = \delta_{Q_N}d_{Q_N} + d_{Q_N}\delta_{Q_N}\,, \quad (14)$$

with $d_{Q_N} = \frac{1}{\sqrt{n-p}}d_Q$. The basic property of these Laplacians is that they nearly preserve J, because by (7),

$$\Delta_{Q_N} - \Delta_{Q_N}^J = (\mathcal{L}_T J) + P^{(1)}_{\text{Nijenhuis}} = \text{first order.} \quad (15)$$

Now, (9) can be rewritten, on $\Omega_0^p H$

$$\Delta_H = (n-p)\Delta_{Q_N} + d_{Q_N}\delta_{Q_N} + d^J_{Q_N}\delta^J_{Q_N} + P_N^{(1)}\,. \quad (16)$$

Together with $d^J_{Q_N}\delta^J_{Q_N} \le \Delta^J_{Q_N} = \Delta_{Q_N} + P_N^{(1)}$, we obtain that

$$\Delta_H + \text{first order} \le (n-p+2)\Delta_{Q_N} \le \frac{n-p+2}{n-p}\Delta_Q\,,$$

which gives the second control, the first order part being absorbed by Cauchy-Schwartz and loosing an arbitrarily small amount of $(\Delta_H\alpha,\alpha)$. □

Remark 4.2. These normalized Laplacians, that appear as a convenient tool here, also have the important feature that they nearly commute (modulo lower order terms) with all our algebra of operators: d_Q and δ_Q (of course), but also d_Q^J and δ_Q^J (because $\Delta_{Q_N} \simeq \Delta^J_{Q_N}$), except in middle degree.

Finally, we notice that the term $[\mathcal{L}_T,\delta_H]$ arising in formula (1) for Δ_ε is a first order (horizontal) operator. More precisely,

$$[\mathcal{L}_T,\delta_H] = -[J^{-1}(\mathcal{L}_T J),\delta_H]\,.$$

Proof. $[\mathcal{L}_T, d_H]=0$ (by (2)) implies $[\mathcal{L}_T^*,\delta_H]=0$. Then (8) gives

$$\mathcal{L}_T + \mathcal{L}_T^* = \mathcal{L}_T - \mathcal{L}_T^J = \mathcal{L}_T - J^{-1}\mathcal{L}_T(J\,\cdot\,) = -J^{-1}(\mathcal{L}_T J)$$

therefore,

$$[\mathcal{L}_T,\delta_H] = [\mathcal{L}_T + \mathcal{L}_T^*,\delta_H] = -[J^{-1}(\mathcal{L}_T J),\delta_H]\,. \qquad □$$

Collecting (2) and the formulas of this section lead finally to the following expression of Δ_ε in the splitting, for $p\le n$,

$$\Omega^p M = \Omega_0^p H \oplus \theta\wedge(\Omega_0^{p-1}H)\oplus(\operatorname{Im} L)^p \oplus \theta\wedge(\operatorname{Im} L)^{p-1}$$

$$\Delta_\varepsilon = D_{\varepsilon,H} + \varepsilon^{-2} D_L^2 + \varepsilon^2 D_T^2 + Q_N^{(1)} + \varepsilon Q_J^{(1)} \tag{17}$$

with

$$D_{\varepsilon,H} = \begin{pmatrix} \Delta_Q + \frac{1}{n-p+1} d_Q^J \delta_Q^J & -\frac{d_Q^J}{\varepsilon} & 0 & 0 \\ -\frac{\delta_Q^J}{\varepsilon} & \Delta_Q + \frac{1}{n-p+2} d_Q^J \delta_Q^J & -\frac{d_Q \Pi_0 \Lambda}{\varepsilon(n-p+2)} & 0 \\ 0 & -\frac{L\delta_Q}{\varepsilon(n-p+2)} & \Pi_L \Delta_H \Pi_L & -\frac{\Pi_L d_H^J \Pi_L}{\varepsilon} \\ 0 & 0 & -\frac{\Pi_L \delta_H^J \Pi_L}{\varepsilon} & \Pi_L \Delta_H \Pi_L \end{pmatrix}$$

and

$$D_L^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & n-p+1 & 0 & 0 \\ 0 & 0 & L\Lambda & 0 \\ 0 & 0 & 0 & \Lambda L \end{pmatrix} = L\Lambda \Pi_H + \Lambda L \Pi_T$$

$$D_T^2 = \begin{pmatrix} \mathcal{L}_T^* \mathcal{L}_T & 0 & 0 & 0 \\ 0 & \mathcal{L}_T \mathcal{L}_T^* & 0 & 0 \\ 0 & 0 & \mathcal{L}_T^* \mathcal{L}_T & 0 \\ 0 & 0 & 0 & \mathcal{L}_T \mathcal{L}_T^* \end{pmatrix} = \mathcal{L}_T^* \mathcal{L}_T \Pi_H + \mathcal{L}_T \mathcal{L}_T^* \Pi_T$$

$$Q_N^{(1)} = \begin{pmatrix} 0 & 0 & P_N^{(1)*} & 0 \\ 0 & 0 & 0 & P_N^{(1)*} \\ P_N^{(1)} & 0 & 0 & 0 \\ 0 & P_N^{(1)} & 0 & 0 \end{pmatrix} = \Pi_L \Delta_H \Pi_0 + \Pi_0 \Delta_H \Pi_L \quad \text{(by (11))}$$

$$Q_J^{(1)} = \Pi_T [\mathcal{L}_T, \delta_H] \Pi_H + \Pi_H [d_H, \mathcal{L}_T^*] \Pi_T \,,$$

where Π_H, Π_T, Π_0 and Π_L denote respectively the orthogonal projections on the horizontal, vertical, primitive forms and on $\operatorname{Im} L$. The reason for this choice of splitting will become clear in next section.

5 A Priori Estimates

We now briefly describe the method we will follow to prove the resolvent convergence Theorems 3.5 and 3.6. As already mentioned it is inspired by adiabatic techniques as developed by Bismut and co-authors in [BiL] and [BeB]. This consists here of two main steps. First, obtain a priori estimates of some Sobolev L^2 norm of α from $(\Delta_\varepsilon \alpha, \alpha)$ or $\big(\frac{\Delta_\varepsilon}{\varepsilon^2}\alpha, \alpha\big)$. Then, use them to justify some formal asymptotic inversion of $\lambda - P_\varepsilon$ or $\lambda - \frac{P_\varepsilon}{\varepsilon}$ in matrix form. By formal we mean treating the differential parts of the expression as a priori bounded ones. The associated spectral sequence will naturally arise, as in the adiabatic case, while computing the successive asymptotics of the resolvents $(\lambda - \varepsilon^{-r}P_\varepsilon)^{-1}$.

A well-known means to obtain a priori estimates from an expression $(\Delta\alpha, \alpha)$ is to use a Bochner type formula $\Delta = \nabla^*\nabla + \text{curvature}$. The classical Riemannian one can't apply here, because the curvature of the metrics g_ε diverges. In fact, one can show (see [Ru, II.5]) that the sectional curvatures have the following behavior (for X unitary $\in H$):

$$K_\varepsilon(X,T) \sim \tfrac{1}{4\varepsilon^2} \text{ and } K_\varepsilon(X,JX) \sim -\tfrac{3}{4\varepsilon^2}.$$

So that the Ricci curvature tensor is indefinite:

$$R_\varepsilon(T,T) \sim \tfrac{n}{2\varepsilon^2}\ ,\ R_\varepsilon(X,X) \sim -\tfrac{1}{2\varepsilon^2}\ .$$

Even the scalar curvature goes to $-\infty$ as $-\frac{n}{2\varepsilon^2}$. (Notice that the "universal" constants ruling the asymptotics are just the curvatures of the Heisenberg group. This is because our contact manifold viewed in the magnified metrics $\frac{g_\varepsilon}{\varepsilon^2} = \frac{g_H}{\varepsilon^2} + \frac{g_T}{\varepsilon^4}$ converges to its tangent cone, the Heisenberg group.)

We remark also that trying to use a more suited sub-Riemannian connection (preserving horizontality, like the one in [Ru, II] for example) instead of Levi Civita's could possibly be fruitful for estimates based on $(\Delta_\varepsilon\alpha, \alpha)$ but will introduce quite uncontrollable terms like $\big(\frac{\text{curvature}}{\varepsilon^2}\alpha, \alpha\big)$ while studying $\big(\frac{\Delta_\varepsilon}{\varepsilon^2}\alpha, \alpha\big)$ (for the collapsing spectrum). It's why we turn ourself towards a formula like (17) comparing the two geometric Laplacians Δ_ε and Δ_Q (instead of $\nabla^*\nabla$). Observe (17) has the characteristic that the curvature like "error terms" $Q^{(1)}$ are *only of non-diagonal type* (and we hope that the non-primitive and horizontal components collapse on the bounded spectrum).

Although formula (17) has a lot of positive diagonal terms, it still contains a divergent one of order 1. We will absorb it in a positive term, even if weakening the diagonal. This is the aim of the following Bochner type formula. We use the decomposition of Ω^pM described in (17) and note $\Pi_{H,0}$, $\Pi_{T,0}$, $\Pi_{H,L}$, $\Pi_{T,L}$ the projections on the different factors.

Lemma 5.1 (Sub-Riemannian Bochner formula). *One has on Ω^pM, for $p \le n$,*

$$\begin{aligned}\Delta_\varepsilon = \Delta_Q\Pi_{H,0} + \frac{K_\varepsilon^*K_\varepsilon}{n-p+1} + \left(\Delta_Q + \frac{1}{n{-}p{+}2}(d_Q^J\delta_Q^J - d_Q\delta_Q)\right)\Pi_{T,0} + L_\varepsilon^*L_\varepsilon \\ + \Pi_{H,L}\left(\Delta_H - \tfrac12 d_H^J\Pi_L\delta_H^J\right)\Pi_{H,L} + \tfrac{M_\varepsilon^*M_\varepsilon}{2} + \Pi_{T,L}\Delta_H\Pi_{T,L} \\ + \tfrac{1}{\varepsilon^2}P_L^2 + \varepsilon^2 D_T^2 + Q_N^{(1)} + \varepsilon Q_J^{(1)}\ .\end{aligned}$$

where

$$K_\varepsilon = \begin{pmatrix} \delta_Q^J & -\dfrac{n-p+1}{\varepsilon} & 0 & 0 \end{pmatrix}$$
$$L_\varepsilon = \begin{pmatrix} 0 & \dfrac{L\delta_Q}{n-p+2} & -\dfrac{1}{\varepsilon} & 0 \end{pmatrix}$$
$$M_\varepsilon = \begin{pmatrix} 0 & 0 & \Pi_L \delta_H^J \Pi_L & -\frac{2}{\varepsilon} \end{pmatrix}$$

and

$$P_L^2 = (L\Lambda - 1)\Pi_{H,L} + (\Lambda L - 2)\Pi_{T,L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & L\Lambda - 1 & 0 \\ 0 & 0 & 0 & \Lambda L - 2 \end{pmatrix}.$$

Proof. The principle of this decomposition is the absorption of the divergent non-diagonal terms of (17) by repeated applications of Cauchy-Schwartz, starting from the first (less positive diagonal term) to the fourth (more positive) factor in $\Omega^p M$. We observe that except for the two first terms (that will have strong geometric and analytic senses), the proposed decomposition of Δ_ε is not so canonical. We mean the non-diagonal terms could have been balanced by other combinations of diagonal ones, changing so the L_ε and M_ε. This does not matter as the goal of Lemma 5.1 is to obtain a priori estimates. □

We come to this. Recall that (section 3)

$F_1 =$ primitive horizontal forms $\oplus$ coprimitive ($\in \ker L$) vertical forms.

Definition 5.2. Let $\| \ \|_{1,\varepsilon}$ be the following Sobolev L^2 norm

- For $\alpha \in \Omega^p M, p \neq n, n+1$,
$$\|\alpha\|_{1,\varepsilon}^2 = \|\alpha\|_{1,H}^2 + \varepsilon^2 \|\alpha\|_{1,T}^2 + \tfrac{1}{\varepsilon^2}\|\Pi_{F_1^\perp}\alpha\|_0^2 + \|\alpha\|_0^2$$
- For $\alpha \in \Omega^n M$,
$$\|\alpha\|_{1,\varepsilon}^2 = \|\Pi_{F_1^\perp}\alpha\|_{1,H}^2 + \|\delta_Q \alpha_{H,0}\|_0^2 + \varepsilon^2\|\alpha\|_{1,T}^2 + \tfrac{1}{\varepsilon^2}\|\Pi_{(\ker\Lambda)^\perp}\alpha\|_0^2 + \|\delta_Q^J \alpha_{H,0} - \frac{\alpha_{T,0}}{\varepsilon}\|_0^2 + \|\alpha\|_0^2$$
- For $\alpha \in \Omega^{n+1} M$,
$$\|\alpha\|_{1,\varepsilon}^2 = \|*\alpha\|_{1,\varepsilon}^2 = \|\Pi_{F_1^\perp}\alpha\|_{1,H}^2 + \|d_Q \alpha_{T,\ker L}\|_0^2 + \varepsilon^2\|\alpha\|_{1,T}^2 + \tfrac{1}{\varepsilon^2}\|\Pi_{(\ker L)^\perp}\alpha\|_0^2 + \|\delta_Q i_T \alpha_{T,\ker L} - \tfrac{\Lambda}{\varepsilon}\alpha_{H,\ker L}\|_0^2 + \|\alpha\|_0^2$$

where the lower indices of the norms indicates the number of horizontal or transversal derivatives controlled in L^2.

These norms are adapted to our problem for we have

Proposition 5.3. *$\exists C_1, C_2 > 0$ such that, $\forall \varepsilon \in]0,1]$*

$$C_1\|\alpha\|^2_{1,\varepsilon} \le (\Delta_\varepsilon \alpha, \alpha)_0 + \|\alpha\|^2_0 \le C_2 \|\alpha\|^2_{1,\varepsilon}\,.$$

Proof. The right inequality amounts to the continuity of $P_\varepsilon = d_\varepsilon + \delta_\varepsilon$ from the Hilbert space $\mathcal{H}_{1,\varepsilon} = (\Omega^* M, \|\ \|_{1,\varepsilon})$ into $\mathcal{H}_0 = (\Omega^* M, \|\ \|_0) = L^2(\Omega^* M)$. This is clear outside the middle degrees from (1). The only point to check on $\Omega^n M$ is the continuity of

$$(\alpha^n_{H,0}, \alpha^n_{T,0}) \to \left(d_H \alpha^n_{H,0} + \tfrac{1}{\varepsilon} L \alpha^n_{T,0}\right)\,,$$

the upper matrix line of d_ε (see (1)), but this is precisely, by (6), $-L\big(\delta^J_Q \alpha^n_{H,0} - \frac{1}{\varepsilon}\alpha^n_{T,0}\big)$, included expression in definition of $\|\alpha^n\|_{1,\varepsilon}$ (this strange group is also nothing else but the $K_\varepsilon \alpha^n$ term of Lemma 5.1).

The left inequality comes of course from the Bochner type formula Lemma 5.1.

First, the so-called P^2_L is actually positive due to $L\Lambda \ge\ 2\mathrm{Id}$ on $\Omega^p H \cap \operatorname{Im} L$, and $\Lambda L \ge\ 4\mathrm{Id}$ on $\theta \wedge (\operatorname{Im} L)^{p-1}$ $(p \le n)$ as come from (3). Therefore

$$\frac{(P^2_L \alpha, \alpha)_0}{\varepsilon^2} \ge \frac{\|\Pi_{\operatorname{Im} L}\alpha\|^2_0}{\varepsilon^2}\,.$$

We now look at the four Laplacian like diagonal terms of Lemma 5.1. We have to check that each of them controls its respective part of $\|\alpha\|^2_{1,H}$.

- As concerns Δ_Q on $\Omega^p_0 H$, we know by (13) that, for $p < n$,

  $$(\Delta_Q \alpha_{H,0}, \alpha_{H,0})_0 + \|\alpha_{H,0}\|^2_0 \ge C\|\alpha_{H,0}\|^2_{1,H}$$

  Of course, $\Delta_Q = d_Q \delta_Q$ is not hypoelliptic on $\Omega^n H$, so that $(d_Q \delta_Q \alpha^n_{H,0}, \alpha^n_{H,0})_0$ only controls $\|\delta_Q \alpha\|^2_0$.
- On $\theta \wedge \Omega^{p-1} H$, we have, for $p \le n$,

  $$\Delta_Q + \frac{1}{n-p+2}(d^J_Q \delta^J_Q - d_Q \delta_Q) \ge \frac{n-p+1}{n-p+2}\Delta_Q$$

  controls $\|\alpha_{T,0}\|^2_{1,H}$ (modulo $C\|\alpha\|^2_0$) by (13).
- We have on $\Omega^p_{\operatorname{Im} L} H$, for $p \le n$,

  $$\Pi_L\left(\Delta_H - \tfrac{1}{2} d^J_H \Pi_L \delta^J_H\right)\Pi_L \ge \Pi_L\left(\Delta^J_H - \tfrac{1}{2} d^J_H \delta^J_H + \text{1st order}\right)\Pi_L\,,$$

  for $\Delta_H = \Delta^J_H + \text{1st order}$ (section 4),

  $$\begin{aligned} &\ge \Pi_L(\delta^J_H d^J_H + \tfrac{1}{2} d^J_H \delta^J_H + \text{1st order})\Pi_L \\ &\ge \tfrac{1}{2}\Pi_L(\Delta^J_H + \text{1st order})\Pi_L \\ &= \tfrac{1}{2}\Pi_L(\Delta_H + \text{1st order})\Pi_L\,, \end{aligned}$$

  controls $\|\alpha_{H,L}\|^2_{1,H}$ by (13) (extends for Δ_H on $\Omega^n_L H$ because $[\Delta_H, L]$ is first order (section 4)).
- On $\theta \wedge \Omega^{p-1}_L H$ with $p \le n$, Δ_H controls $\|\alpha_{T,L}\|^2_{1,H}$ by (13).

We now look at the "curvature terms" $Q_N^{(1)}$ and $\varepsilon Q_J^{(1)}$. They have no diagonal component from $\Omega_0^p H$ in itself by (17). One can therefore obtain, using Cauchy-Schwartz, the following control: $\exists K$ such that $\forall \varepsilon \in]0,1]$,

$$\big|(Q_N^{(1)}\alpha,\alpha)\big| + \varepsilon\big|(Q_J^{(1)}\alpha,\alpha)\big| \leq 2K\|\alpha\|_0\|\Pi_{F_1^\perp}\alpha\|_{1,H} \leq K\big(c^{-1}\|\alpha\|_0^2 + c\|\alpha\|_{1,H}^2\big)$$

for any arbitrarily small c.

Lastly, we still have to understand the domination of $\frac{1}{\varepsilon^2}\|\alpha_{T,0}\|_0^2$ for $p < n$. Thanks to previous work and Lemma 5.1, we now know that $(\Delta_\varepsilon\alpha,\alpha)_0 + \|\alpha\|_0^2$ controls $C\|\alpha\|_{1,H}^2$ and

$$\|K_\varepsilon\alpha\|_0^2 = \left\|\delta_Q^J\alpha_{H,0} - \frac{n-p+1}{\varepsilon}\alpha_{T,0}\right\|_0^2 \geq \frac{1}{\varepsilon^2}\|\alpha_{T,0}\|_0^2 - C'\|\alpha_{H,0}\|_{1,H}^2\,. \quad \square$$

Before going on, we remark that the deficiency of control of $\|\Pi_{F_3}\alpha\|_{1,H}^2$ by $(\Delta_\varepsilon\alpha,\alpha)_0$ in middle degree is unavoidable. The contrary would contradict, by the compact Sobolev embedding of $\mathcal{H}_{1,H}$ space in L^2, the infinite dimensional collapsing spectrum phenomenon we already observed in section 2.

This also precludes the possibility of independent controls of $\|\delta_Q^J\alpha_{H,0}\|_0$ and $\frac{1}{\varepsilon}\|\alpha_{T,0}\|_0$ instead of $\big\|\delta_Q^J\alpha_{H,0} - \frac{1}{\varepsilon}\alpha_{T,0}\big\|_0 = \|K_\varepsilon\alpha\|_0$ in $\|\alpha\|_{1,\varepsilon}$ for $\alpha \in \Omega^n M$. Or else, the control of $\|\delta_Q^J\alpha_{H,0}\|_0$ added to the already obtained one of $\|\delta_Q\alpha_{H,0}\|_0$ would give a domination of $\|\Pi_{(\Omega^{n,0}H\oplus\Omega^{0,n}H)^\perp}\alpha_{H,0}\|_{1,H}$, because the system (δ_Q,δ_Q^J) is hypoelliptic on $(\Omega^{n,0}H\oplus\Omega^{0,n}H)^\perp\cap\Omega_0^n H$ as comes from (11): $\Delta_H = d_Q\delta_Q + d_Q^J\delta_Q^J + 1\text{st order on } \Omega_0^n H$, and (12). This is still impossible since the collapsing spectrum should be described by the one of D^*D, not concentrated on $\Omega^{n,0}H\oplus\Omega^{0,n}H$. The conclusion is that the term $\big\|\delta_Q^J\alpha_{H,0} - \frac{n-p+1}{\varepsilon}\alpha_{T,0}\big\|_0 = \|K_\varepsilon\alpha\|_0$ arising in Lemma 5.1 and $\|\ \|_{1,\varepsilon}$ is analytically relevant (at least in middle degree for the moment).

It has also a geometric meaning. Namely, we observe there exist lifting maps r (implicit in [Ru]) from $\Omega_0^p H$ in $\Omega^p M$ for $p \leq n$, defined by

$$\begin{aligned} r(\alpha_{H,0}) &= \alpha_{H,0} - \theta\wedge L^{-1}d_H\alpha_{H,0} \text{ with } L^{-1} \text{ left inverse of } L\,, \\ &= \alpha_{H,0} + \theta\wedge\frac{1}{n-p+1}\delta_Q^J\alpha_{H,0} \quad \text{by (6) and } L^{-1} = (\Lambda L)^{-1}\Lambda\,. \end{aligned} \tag{18}$$

$r(\alpha_{H,0})$ is the unique vertical extension of $\alpha_{H,0}$ such that $d\,r(\alpha_{H,0})$ restricted to H is primitive.

A consequence of this definition is that $dr = rd_Q$ on $\Omega_0^p H$, $p < n$ ($dr(\alpha_{H,0})$ is a closed extension of $d_Q\alpha_{H,0}$ and so is $r(d_Q\alpha_{H,0})$) and by definition (section 3) $D = dr$ on $\Omega_0^n H$. This means that r, together with the projection $\Pi_{H,0}$ on $\Omega_0^* H$, realize an homotopy between the de Rham and contact complexes.

Now, we observe that

$$K_\varepsilon\alpha = 0 = \delta_Q^J\alpha_{H,0} - \frac{n-p+1}{\varepsilon}\alpha_{T,0} \quad\text{and}\quad P_L\alpha = 0 = \Pi_L\alpha\,,$$

are precisely the differential defining equations of $\operatorname{Im} r$ (through the conjugation $C_\varepsilon(\alpha_H + \theta\wedge\alpha_T) = \alpha_H + \varepsilon\theta\wedge\alpha_T$ we have done since section 2).

Remark 5.4. This space $\operatorname{Im} r$ has stronger geometric invariance it seems to possess at first sight. It is in fact a (C^1) *contact-invariant* space. This comes from its definition

$$\operatorname{Im} r = \{\alpha\in\Omega^pM,\ \alpha \text{ and } d\alpha \text{ restricted to } H \text{ are primitive}\}\,.$$

Indeed, following A. Weil ([W, corollary of Thm. I.3]), $\ker\Lambda = \ker L^{n-p+1}$ with $L = d\theta\wedge\cdot$ acting on H-partial p-forms. So this space does not depend on the chosen adapted J on H, neither of the choice of the contact form because $L\to fL$ on partial forms when $\theta\to f\theta$.

Coming back to Prop. 5.3, we now know we can't hope to control $\frac{1}{\varepsilon^2}\|\alpha_{T,0}\|_0^2$ with $(\Delta_\varepsilon\alpha,\alpha)_0$ in degree n, although we are aiming to prove that, even there, the bounded spectrum ($\varepsilon\to 0$) of Δ_ε concentrates on $F_1^n = \Omega_0^nH$. Even if loosing some speed of convergence, the component $\alpha_{T,0}$ is actually collapsing to 0.

Proposition 5.5. *$\exists C>0$ such that* $\forall\varepsilon\in\,]0,1]$, $\forall\alpha\in\Omega^nM$,

$$(\Delta_\varepsilon\alpha,\alpha)_0 + \|\alpha\|_0^2 \geq \tfrac{C}{\varepsilon}\|\alpha_{T,0}\|_0^2\,.$$

In order to prove this, we fix some notation

$\mathcal{H}_0 = (\Omega^*M, \|\ \|_0) = L^2(\Omega^*M)$

$\mathcal{H}_{1,H} = (\Omega^*M, \|\ \|_{1,H}) = L^2$ forms with 1 horizontal derivative in L^2

$\mathcal{H}_0$ will be our pivot space, that is we identify it with its dual $\mathcal{H}_0^*$. Let $\mathcal{H}_{1,H}^* = \mathcal{H}_{-1,H}$ denote the dual of $\mathcal{H}_{1,H}$. We have the dense inclusions

$$\mathcal{H}_{-1,H}\supset\mathcal{H}_0 = \mathcal{H}_0^*\supset\mathcal{H}_{1,H}$$
$$\text{with}\ \ \|\ \|_{-1,H}\leq\|\ \|_0\leq\|\ \|_{1,H}\,.$$

Recall that for $\alpha\in\mathcal{H}_0$, $\|\alpha\|_{-1,H} = \sup_{\|\beta\|_{1,H}\leq 1}(\alpha,\beta)_0$.

Now, Prop. 5.5 immediately stems from Prop. 5.3 and the following

Lemma 5.6. *$\exists C>0$, such that* $\forall\varepsilon\in\,]0,1]$, $\forall\alpha\in\Omega^nM$,

$$C\|\alpha\|_{1,\varepsilon}^2 \geq \tfrac{1}{\varepsilon^2}\|\alpha_{T,0}\|_{-1,H}^2 + \|\alpha_{T,0}\|_{1,H}^2 \geq \tfrac{1}{\varepsilon}\|\alpha_{T,0}\|_0^2\,.$$

Proof. The first inequality amounts to the observation that

$$\begin{aligned}\|\alpha\|_{1,\varepsilon}^2 &\geq \left\|\delta_Q^J\alpha_{H,0} - \tfrac{\alpha_{T,0}}{\varepsilon}\right\|_0^2 \geq \left\|\delta_Q^J\alpha_{H,0} - \tfrac{\alpha_{T,0}}{\varepsilon}\right\|_{-1,H}^2\\ &\geq \tfrac{1}{\varepsilon^2}\|\alpha_{T,0}\|_{-1,H}^2 - \|\delta_Q^J\alpha_{H,0}\|_{-1,H}^2\end{aligned}$$

with δ_Q^J continuous from $\mathcal{H}_0$ in $\mathcal{H}_{-1,H}$ since d_Q^J is from $\mathcal{H}_{1,H}$ in $\mathcal{H}_0$.

The second inequality is of well-known type in the elliptic context (Peetre's inequality). Here is a possible "elementary" proof in this hypoelliptic case. We choose a partially defined connection ∇_H and put, for $\alpha_{T,0} \in \theta \wedge \Omega_0^{n-1} H$,

$$\|\alpha\|_{1,H}^2 = \|\alpha\|_0^2 + \|\nabla_H \alpha\|_0^2 .$$

Denote by $\Pi_{[0,K]}$ (resp. $\Pi_{]K,+\infty[}$) the spectral projections associated to the spectral measures of $\nabla_H^* \nabla_H$ in $[0,K]$ and $]K,+\infty[$ for some $K \geq 0$.

- Obviously,
  $\|\Pi_{[0,K]}\alpha\|_{1,H}^2 = (\nabla_H^*\nabla_H \Pi_{[0,K]}\alpha, \Pi_{[0,K]}\alpha)_0 + \|\Pi_{[0,K]}\alpha\|_0^2 \leq (K+1)\|\alpha\|_0^2$
  and by duality, $(K+1)\|\alpha\|_{-1,H}^2 \geq \|\Pi_{[0,K]}\alpha\|_0^2$.
- We also have,
  $$\begin{aligned}\|\alpha\|_{1,H}^2 &\geq (\nabla_H^*\nabla_H \Pi_{]K,+\infty[}\alpha, \Pi_{]K,+\infty[}\alpha)_0 + \|\Pi_{]K,+\infty[}\alpha\|_0^2 \\ &\geq (K+1)\|\Pi_{]K,+\infty[}\alpha\|_0^2 .\end{aligned}$$

Adding these two inequalities with $K+1 = \varepsilon^{-1}$ gives the result. □

This completes the proof of the a priori estimates we will need for Theorem 3.5. We continue this section with the study of $\Delta_\varepsilon/\varepsilon^2$. We define the relevant norm here.

Definition 5.7.

- For $\alpha \in \Omega^p M$, $p \leq n$, let
  $$\begin{aligned}\|\alpha\|_{1',\varepsilon}^2 = \tfrac{1}{\varepsilon^2}(\Delta_Q \alpha_{H,0}, \alpha_{H,0})_0^2 + \tfrac{1}{\varepsilon^2}\|\Pi_{F_1^\perp}\alpha\|_{1,H}^2 + \|\alpha\|_{1,T}^2 + \tfrac{1}{\varepsilon^2}\|K_\varepsilon \alpha\|_0^2 \\ + \tfrac{1}{\varepsilon^4}\|\Pi_{(\ker\Lambda)}^\perp \alpha\|_0^2 + \|\alpha\|_0^2\end{aligned}$$
  where $K_\varepsilon \alpha = \delta_Q^J \alpha_{H,0} - \frac{(n-p+1)}{\varepsilon}\alpha_{T,0}$ (Lemma 5.1)
- $\|\alpha\|_{1',\varepsilon} = \|*\alpha\|_{1',\varepsilon}$ if $\alpha \in \Omega^p M$, $p \geq n+1$.

We will prove

Proposition 5.8. *$\exists C_1, C_2 > 0$, such that $\forall \alpha \in \Omega^p M$,*

$$C_1\|\alpha\|_{1',\varepsilon}^2 \leq \tfrac{1}{\varepsilon^2}(\Delta_\varepsilon \alpha, \alpha)_0 + \|\alpha\|_0^2 \leq C_2 \|\alpha\|_{1',\varepsilon}^2 .$$

Proof. Along the lines of proof of Prop. 5.3. We refer to it for some details.

The right inequality comes from the continuity of $\frac{P_\varepsilon}{\varepsilon} = \frac{d_\varepsilon + \delta_\varepsilon}{\varepsilon}$ from $\mathcal{H}_{1',\varepsilon} = (\Omega^* M, \| \ \|_{1',\varepsilon})$ into $\mathcal{H}_0 = L^2$. This follows from (2) and (6).

The left inequality is a use of the Bochner type formula Lemma 5.1. We still have

$$\tfrac{1}{\varepsilon^4}(P_L^2 \alpha, \alpha)_0^2 \geq \tfrac{1}{\varepsilon^4}\|\Pi_{(\ker\Lambda)^\perp}\alpha\|_0^2 .$$

This, associated to the diagonal terms in Lemma 5.1 gives the control of $\frac{1}{\varepsilon^2}\|\Pi_{(\ker\Lambda)^\perp}\alpha\|_{1,H}^2$. About $\frac{\alpha_{T,0}}{\varepsilon} \in \theta \wedge \Omega_0^{p-1} H$ we have two estimations.

- The first one is that the $(T,0)$ diagonal term of Lemma 5.1 satisfies
$$\begin{aligned}\tfrac{1}{\varepsilon^2}(\Delta_{T,0}\alpha,\alpha)_0 &= \tfrac{1}{\varepsilon^2}\big(\big(\Delta_Q + \tfrac{1}{n-p+2}(d_Q^J\delta_Q^J - d_Q\delta_Q)\big)\alpha_{T,0},\alpha_{T,0}\big)_0\\ &\geq \tfrac{1}{2\varepsilon^2}(\Delta_Q\alpha_{T,0},\alpha_{T,0})_0\,.\end{aligned}$$
- And also, following Lemma 5.6, $\exists C>0$ such that $\forall\varepsilon\leq 1$,
$$\begin{aligned}\tfrac{1}{\varepsilon^2}\|\alpha_{T,0}\|^2_{-1,H} &\leq \big\|\delta_Q^J\alpha_{H,0} - \tfrac{\alpha_{T,0}}{\varepsilon}\big\|_0^2 + C\|\alpha\|_0^2\\ &\leq \big\|\tfrac{K_\varepsilon\alpha}{\varepsilon}\big\|_0^2 + C\|\alpha\|_0^2\,.\end{aligned}$$

Now we have seen, in Lemma 5.6, that $\exists K>0$, $\forall\alpha\in\theta\wedge\Omega_0^{p-1}H$,
$$\|\alpha\|_0^2 \leq K^{-1}\|\alpha\|^2_{-1,H} + K\|\alpha\|^2_{1,H}$$
where one can take $\|\alpha\|^2_{1,H} = \|\alpha\|_0 + (\Delta_Q\alpha,\alpha)_0$ (due to positivity of Δ_Q and a priori estimate (13)). Using this with $K=1/2$ and $\alpha=\alpha_{T,0}/\varepsilon$ gives
$$\tfrac{1}{2\varepsilon^2}\|\alpha_{T,0}\|_0^2 \leq \tfrac{2}{\varepsilon^2}\|\alpha_{T,0}\|^2_{-1,H} + \tfrac{1}{2\varepsilon^2}(\Delta_Q\alpha_{T,0},\alpha_{T,0})_0^2\,,$$
and finally the control, for a $C>0$,
$$\tfrac{1}{4\varepsilon^2}\|\alpha_{T,0}\|^2_{1,H} \leq \Big(\tfrac{\Delta_{T,0}}{\varepsilon^2}\alpha,\alpha\Big)_0 + \big\|\tfrac{K_\varepsilon\alpha}{\varepsilon}\big\|_0^2 + C\|\alpha\|_0^2\,.$$

We are left with the domination of the (now not so small) curvature terms $\frac{1}{\varepsilon}(Q_J^{(1)}\alpha,\alpha)_0$ and $\frac{1}{\varepsilon^2}(Q_N^{(1)}\alpha,\alpha)_0$. In fact, collecting our estimations, we still know that there exists a constant C_1, depending only of the dimension of M, and C_2, depending of θ, J and M (by the norms of the sub-Riemannian curvatures) such that
$$\big(\tfrac{\Delta_\varepsilon}{\varepsilon^2}\alpha,\alpha\big)_0 \geq C_1\|\alpha\|^2_{1',\varepsilon} - C_2\|\alpha\|_0^2 - \tfrac{1}{\varepsilon}\big|(Q_J^{(1)}\alpha,\alpha)_0\big| - \tfrac{1}{\varepsilon^2}\big|(Q_N^{(1)}\alpha,\alpha)_0\big|\,. \tag{19}$$

$\frac{1}{\varepsilon}(Q_J^{(1)}\alpha,\alpha)_0$ is easily handled like in proof of Prop. 5.3: $\exists K>0$ such that
$$\begin{aligned}\tfrac{1}{\varepsilon}\big|(Q_J^{(1)}\alpha,\alpha)_0\big| &\leq 2K\|\alpha\|_0\Big\|\frac{\Pi_{F_1^\perp}\alpha}{\varepsilon}\Big\|_{1,H}\\ &\leq K\big(c^{-1}\|\alpha\|_0^2 + c\|\alpha\|^2_{1',\varepsilon}\big)\end{aligned}$$
for any arbitrary small constant c.

By (17), $Q_N^{(1)}$ is a first order operator exchanging $\ker\Lambda$ and $\operatorname{Im} L$, and preserving horizontality and verticality of forms. Thus, we can decompose
$$\tfrac{1}{\varepsilon^2}(Q_N^{(1)}\alpha,\alpha)_0 = \tfrac{2}{\varepsilon^2}(Q_N^{(1)}\alpha_{H,0},\alpha_{H,L})_0 + \tfrac{2}{\varepsilon^2}(Q_N^{(1)}\alpha_{T,0},\alpha_{T,L})_0\,.$$
- The second term is controlled by
$$K\varepsilon\,\big\|\tfrac{\alpha_{T,0}}{\varepsilon}\big\|_{1,H}\,\big\|\tfrac{\alpha_{T,L}}{\varepsilon^2}\big\|_0 \leq K\varepsilon\|\alpha\|^2_{1',\varepsilon}\,.$$
- The remaining one is more delicate because $\alpha_{H,0}$ does not collapse. One obtains,
$$\tfrac{1}{\varepsilon^2}\big|(Q_N^{(1)}\alpha_{H,0},\alpha_{H,L})\big| \leq \|Q_N^{(1)}\alpha_{H,0}\|_0\,\big\|\tfrac{\alpha_{H,L}}{\varepsilon^2}\big\|_0$$

$$\leq K_N\|\alpha_{H,0}\|_{1,H}\|\alpha\|_{1',\varepsilon} \tag{20}$$

where

$$K_N = \sup_{\|\alpha_{H,0}\|_{1,H}\leq 1} \|Q_N^{(1)}\alpha_{H,0}\| = |||Q_N^{(1)}\Pi_{H,0}|||_{(1,H),0}$$

ie. the norm of $Q_N^{(1)}$ as an operator from $\mathcal{H}_{1,H}\cap\Omega_0^pH$ in $\mathcal{H}_0\cap\Omega_L^pH$.

We need to control $\|\alpha_{H,0}\|_{1,H}$. The discussion breaks into two cases.

For $p<n$, one can put on Ω_0^pH (see (13))

$$\|\alpha_{H,0}\|_{1,H}^2 = (\Delta_Q\alpha_{H,0},\alpha_{H,0})_0^2 + \|\alpha\|_0^2\,.$$

So that, by definition of $\|\ \|_{1',\varepsilon}$, one has

$$\|\alpha_{H,0}\|_{1,H} \leq \varepsilon\|\alpha\|_{1',\varepsilon} + \|\alpha\|_0\,.$$

This, added in (20), concludes the proof of Prop. 5.8 in these degrees.

For $p=n$, $\Delta_Q = d_Q\delta_Q$ isn't hypoelliptic. We put then $\|\alpha\|_{1,H}^2 = \|\nabla_H\alpha\|_0^2 + \|\alpha\|_0^2$, where ∇_H is a partially defined connection. We will need the following control

Lemma 5.9. *$\exists C>0$ such that $\forall\alpha\in\Omega^nM$, $\forall\varepsilon<1$,*

$$\|\alpha_{H,0}\|_{1,H} \leq 2\|\alpha\|_{1',\varepsilon} + C\|\alpha\|_0\,.$$

Proof. Recall that on Ω^nM,

$$\begin{aligned}\|\alpha\|_{1',\varepsilon} &\geq \tfrac{1}{\varepsilon^2}\|\delta_Q\alpha_{H,0}\|_0^2 + \tfrac{1}{\varepsilon^2}\|\delta_Q^J\alpha_{H,0} + \tfrac{\alpha_{T,0}}{\varepsilon}\|_0^2 + \|\alpha\|_{1,T}^2\\ &\geq \|\delta_Q\alpha_{H,0}\|_0^2 + \|\delta_Q^J\alpha_{H,0}\|_0^2 + \|\alpha_{H,0}\|_{1,T}^2\,.\end{aligned}$$

Now we observe that this control is elliptic. More precisely, by (11) and (12), one has on $\Omega_0^{p,q}H$ with $p+q=n$,

$$\begin{aligned}d_Q\delta_Q + d_Q^J\delta_Q^J &= \Delta_H + \text{first order}\\ &= \nabla_H^*\nabla_H + i(p-q)\mathcal{L}_T + \text{first order}\ ,\end{aligned}$$

so that,

$$\begin{aligned}\|\delta_Q\alpha_{H,0}\|_0^2 + \|\delta_Q^J\alpha_{H,0}\|_0^2 + \|\mathcal{L}_T\alpha\|_0^2 &\geq \|\nabla_H\alpha\|_0^2 - n|(\mathcal{L}_T\alpha,\alpha)_0|\\ &\quad + \|\mathcal{L}_T\alpha\|_0^2 + (P_H^{(1)}\alpha,\alpha)_0\\ &\geq \tfrac12\|\nabla_H\alpha\|_0^2 + \tfrac12\|\mathcal{L}_T\alpha\|_0^2 - C\|\alpha\|_0^2\end{aligned}$$

for some constant C. □

This lemma associated with (20) gives us that

$$\begin{aligned}\tfrac{1}{\varepsilon^2}|(Q_N^{(1)}\alpha_{H,0},\alpha_{H,L})_0| &\leq K_N\big(2\|\alpha\|_{1',\varepsilon} + C\|\alpha\|_0\big)\|\alpha\|_{1',\varepsilon}\\ &\leq 2K_N\|\alpha\|_{1',\varepsilon}^2 + \tfrac{K_NC}{2}\big(c\|\alpha\|_{1',\varepsilon}^2 + \tfrac1c\|\alpha\|_0^2\big)\end{aligned}$$

for any arbitrary small c.

This estimation, together with (19) is seen to complete the proof of Prop. 5.8 in the case $2K_N < C_1$, but seems to be critical otherwise. The following rescaling argument shows we can always reduce to this case $2K_N < C_1$.

Indeed, consider the transformation $\theta \to k\theta$ for some $k > 0$ in the metrics $g_{\varepsilon,\theta} = d\theta(\cdot, J\cdot) + \frac{\theta^2}{\varepsilon^2}$. We have then

- $\|\alpha\|_0^2 \to k^{n+1-p}\|\alpha\|_0^2$ for $\alpha \in \Omega^p H$
- $\|\alpha\|_{1,H}^2 = \|\nabla_H\alpha\|_0^2 + \|\alpha\|_0^2 \to k^{n-p}\|\nabla_H\alpha\|_0^2 + k^{n-p+1}\|\alpha\|_0^2$
- $Q_N^{(1)} = \Pi_L\Delta_H\Pi_0 + \Pi_0\Delta_H\Pi_L \to k^{-1}Q_N^{(1)}$ because $\delta_H \to k^{-1}\delta_H$ and d_H is unchanged.

This shows that for $k \geq 1$,

$$K_{N,k\theta} = \sup \frac{\|Q_{N,k\theta}^{(1)}\alpha\|_{0,k\theta}}{\|\alpha\|_{(1,H),k\theta}} \leq \frac{1}{\sqrt{k}} K_{N,\theta}\,,$$

and therefore $2K_{N,k\theta} < C_1$ for k large enough (recall that following (19), C_1 is independent of θ). Proposition 5.8 is then proved for the family of metrics $g_{\varepsilon,k\theta}$ with $\varepsilon \to 0$. We finally observe that $g_{\varepsilon,k\theta} = kg_{\frac{\varepsilon}{\sqrt{k}},\theta}$ so that $\Delta_{\varepsilon,k\theta} = \frac{1}{k}\Delta_{\frac{\varepsilon}{\sqrt{k}},\theta}$, giving thus Prop. 5.8 for our original family $g_{\varepsilon',\theta}$ with $\varepsilon' = \varepsilon/\sqrt{k} \to 0$, at least for ε' small enough. Of course, the controls extend up to any positive constant, as can been seen with the help of the classical Bochner formula. Indeed, the Riemannian curvature stays bounded there, giving uniform a priori L^2 estimates. $\square$

Before leaving this section, we emphasize the geometric meaning of the controls of Prop. 5.8. We already knew from Prop. 5.3 and 5.5 that the bounded eigenforms of Δ_ε ($\varepsilon \to 0$) should concentrate on F_1, which geometrically interprets as the E_1 term of the contact spectral sequence of Prop. 3.3. Now, Prop. 5.8 tells us that bounded eigenforms for $\Delta_\varepsilon/\varepsilon^2$ (as harmonic forms for example) have to collapse (at least in weak-limit sense for the moment) on $F_2 = F_1 \cap \ker\Delta_Q$: the Hodge version of the E_2 term of the spectral sequence. Moreover, the controlled term $\frac{1}{\varepsilon}\|K_\varepsilon\alpha\|_0$ in $\|\alpha\|_{1',\varepsilon}$ shows that the weak limit of $C_\varepsilon^{-1}\alpha$ has to be in $\ker K_\varepsilon C_\varepsilon = \ker K_1$, that is, as already mentioned, in the image of the homotopy operator r from the contact to the de Rham complex. Now, we remember that $d_\varepsilon C_\varepsilon r = \varepsilon D$ on $\Omega_0^n H$, where D is the second order operator of the contact complex. Of course, this D wasn't directly controllable (in L^2 norm) by the first order $\frac{d_\varepsilon+\delta_\varepsilon}{\varepsilon}$, but nevertheless, indirectly appears here through the control of $\frac{1}{\varepsilon}\|K_\varepsilon\alpha\|_0$ and this homotopical formula.

6 Proofs of Theorems 3.5 and 3.6

We will do it here for $P_\varepsilon = d_\varepsilon + \delta_\varepsilon$. As will become clear, the proof would work equally for the signature operator $S_\varepsilon = (*d_\varepsilon - d_\varepsilon *)w$. The general method, widely used in the adiabatic setting (see [BeB], [BiL]) will rely on the previous a priori L^2 estimates and trying to solve, in a matrix form, the equations for $\lambda \notin \mathbb{R}$,

$$(\lambda - P_\varepsilon)\beta_\varepsilon = \alpha \quad \text{with } \alpha \in F_3^\perp$$
$$\text{and} \quad (\lambda - \frac{P_\varepsilon}{\varepsilon})\beta_\varepsilon = \alpha\,.$$

Recall that we have to work orthogonally to

$$F_3 = (\ker \delta_Q \cap F_1^n) \oplus (\ker d_Q \cap F_1^{n+1})$$

in the first case because it is the expected limit space for the collapsing spectrum of P_ε. (F_3 is a closed space in L^2 if δ_Q and d_Q are taken in distributional sense as continuous operators from $L^2 = \mathcal{H}_0$ in $(\mathcal{H}_{1,H})^* = \mathcal{H}_{-1,H}$.)

First, we need to decompose P_ε with respect to the splitting of $\Omega^* M$ in F_1 and $F_1^\perp$, paying some particular attention to the components in $F_3 \subset F_1^n \oplus F_1^{n+1}$, and even to $\theta \wedge (\Omega_0^{n-1} H) \oplus \Omega_{\ker L}^{n+1} H$, due to the unsplittable controlled combinations

$$K_\varepsilon^n = \delta_Q^J \Pi_{F_1^n} - \tfrac{i_T}{\varepsilon}\Pi_{\theta\wedge\Omega_0^{n-1}H} \quad \text{and} \quad K_\varepsilon^{n+1} = \delta_Q i_T \Pi_{F_1^{n+1}} - \tfrac{\Lambda}{\varepsilon}\Pi_{\Omega_{\ker L}^{n+1}H}\,.$$

We will therefore use the decompositions,

$$F_1 = F_1^{\rm tame} \oplus F_1^{\rm tough} \;\text{ and }\; F_1^\perp = F_1^{\perp,\rm tame} \oplus F_1^{\perp,\rm tough}\,,$$

with

$$F_1^{\rm tough} = F_1^n \oplus F_1^{n+1}\,, \quad F_1^{\rm tame} = F_1 \cap (F_1^{\rm tough})^\perp = \oplus_{p\neq n,n+1} F_1^p$$

and

$$F_1^{\perp,\rm tough} = (\theta \wedge \Omega_0^{n-1} H) \oplus \Omega_{\ker L}^{n+1} H\,,\; F_1^{\perp,\rm tame} = F_1^\perp \cap (F_1^{\perp,\rm tough})^\perp\,.$$

Note that tame and tough parts are respectively in $*$ (and $\wedge$) duality.

We start the decomposition of d_ε. We already know by (1) that we have in $\Omega^* M = \Omega^* H \oplus \theta \wedge \Omega^* H$,

$$d_\varepsilon = \begin{pmatrix} d_H & \varepsilon^{-1} L \\ \varepsilon \mathcal{L}_T & -d_H \end{pmatrix} = D_H + \varepsilon D_T + \varepsilon^{-1} D_L\,.$$

By (8), we have $[L, \mathcal{L}_T] = [\Lambda, \mathcal{L}_T] = 0$, so that

$$D_T\big((F_1^n)^\perp\big) \subset F_1^{\perp,\rm tame}\,,\; D_T(F_1^{n+1}) = 0 \;\text{ and } D_T(F_1^n) \subset F_1^{n+1}$$

where, thanks to (7),

$$D_T\alpha = \theta\wedge\mathcal{L}_T\alpha = D\alpha + \theta\wedge d_Q\delta_Q^J\alpha \quad\text{for } \alpha\in F_1^n = \Omega_0^n H\,.$$

From (6), $d_H(L^k\alpha) = L^k\big(d_Q\alpha - \frac{L}{n-p+1}\delta_Q^J\alpha\big)$ for $\alpha\in\Omega_0^pH = F_1^p$ $(p\le n)$, and by definition $d_Q(\theta\wedge\alpha) = d_\varepsilon(\theta\wedge\alpha) = D_H(\theta\wedge\alpha)$ for $\theta\wedge\alpha\in\theta\wedge\ker L = F_1^p$ $(p>n)$. So, we see that

$\Pi_{F_1}D_H\Pi_{F_1}{=}d_Q\,,\ \Pi_{F_1^n}D_H\Pi_{F_1^\perp}{=}0$ and $\Pi_{F_1^{n+1}}D_H\Pi_{F_1^\perp}{=}{-}\theta\wedge d_Qi_T\Pi_{\theta\wedge\Omega_0^{n-1}H}$.

Bringing all this together, we have found that in $F_1\oplus F_1^\perp$,

$$d_\varepsilon = \begin{pmatrix} d_Q+\varepsilon D & \Pi_{F_1^{\rm tame}}D_H\Pi_{F_1^\perp,{\rm tame}} \\ \Pi_{F_1^\perp}D_H\Pi_{F_1} & \Pi_{F_1^\perp}D_H\Pi_{F_1^\perp}+\varepsilon^{-1}D_L\end{pmatrix} + \varepsilon\theta\wedge d_QK_\varepsilon^n + \varepsilon\begin{pmatrix} 0 & 0 \\ \Pi_{F_1^\perp,{\rm tame}}D_T\Pi_{F_1^{\rm tame}} & D_T\end{pmatrix}.$$

Before taking the adjoint, we can put it in a slightly more convenient form, if we observe that,

$$\begin{aligned}\varepsilon\theta\wedge d_QK_\varepsilon^n &= \varepsilon(K_\varepsilon^{n+1})^*K_\varepsilon^n + LK_\varepsilon^n \\ &= \varepsilon(K_\varepsilon^{n+1})^*K_\varepsilon^n + L\left(\delta_Q^J\Pi_{F_1^n} - \tfrac{i_T}{\varepsilon}\Pi_{\theta\wedge\Omega_0^{n-1}H}\right) \\ &= \varepsilon(K_\varepsilon^{n+1})^*K_\varepsilon^n + \Pi_{F_1^\perp}D_H\Pi_{F_1^n} - \varepsilon^{-1}D_L\Pi_{F_1^\perp,{\rm tough}}\,.\end{aligned}$$

So that, using previous study, we get

$$d_\varepsilon = \begin{pmatrix} d_Q+\varepsilon D & \Pi_{F_1^{\rm tame}}(D_H+\varepsilon D_T)\Pi_{F_1^\perp,{\rm tame}} \\ \Pi_{F_1^\perp,{\rm tame}}(D_H+\varepsilon D_T)\Pi_{F_1^{\rm tame}} & \Pi_{F_1^\perp}(D_H+\varepsilon D_T)\Pi_{F_1^\perp}+\varepsilon^{-1}D_L\Pi_{F_1^\perp,{\rm tame}}\end{pmatrix} + \varepsilon(K_\varepsilon^{n+1})^*K_\varepsilon^n$$

and finally,

$$\begin{aligned}P_\varepsilon &= d_\varepsilon + \delta_\varepsilon \\ &= \begin{pmatrix} d_Q+\delta_Q+\varepsilon(D+D^*) & \Pi_{F_1^{\rm tame}}(P_H+\varepsilon P_T)\Pi_{F_1^\perp,{\rm tame}} \\ \Pi_{F_1^\perp,{\rm tame}}(P_H+\varepsilon P_T)\Pi_{F_1^{\rm tame}} & \Pi_{F_1^\perp}(P_H+\varepsilon P_T)\Pi_{F_1^\perp}+\varepsilon^{-1}\Pi_{F_1^\perp,{\rm tame}}P_L\Pi_{F_1^\perp,{\rm tame}}\end{pmatrix} \\ &\quad + \varepsilon\big((K_\varepsilon^{n+1})^*K_\varepsilon^n + (K_\varepsilon^n)^*K_\varepsilon^{n+1}\big) \qquad (21)\end{aligned}$$

where

$$P_H = D_H + D_H^*\,,\ P_T = D_T + D_T^* \ \text{ and }\ P_L = D_L + D_L^*\,.$$

Remarks 6.1. 1. We deliberately do not break the K_ε terms in matrix form because we still know we will not be able to control each component individually, whereas the entire combinations are controllable.

2. We mention here a technical difference with the adiabatic methods as developed by Bismut and others. Namely, the operator P_ε isn't a perturbation of a diagonal operator, nor converges to such.

We are now in a position to prove Theorems 3.5 and 3.6. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha \in F_3^\perp$. By self-adjointness of P_ε, $\beta_\varepsilon = (\lambda - P_\varepsilon)^{-1}\alpha$ exists, and one has by Prop. 5.3, for some $C > 0$,

$$\begin{aligned}\|\alpha\|_0^2 = \big\|(\lambda - P_\varepsilon)\beta_\varepsilon\big\|_0^2 &\geq |\operatorname{Im}\lambda|^2 \|\beta_\varepsilon\|_0^2 + \|P_\varepsilon \beta_\varepsilon\|_0^2 \\ &\geq C\|\beta_\varepsilon\|_{1,\varepsilon}^2 \,.\end{aligned}$$

Now, by Definition 5.2 and Lemma 5.6, $\|\beta_\varepsilon\|_{1,\varepsilon}^2$ controls $\frac{1}{\varepsilon^2}\|\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon\|_0^2$ and $\frac{1}{\varepsilon}\|\Pi_{F_1^{\perp,\mathrm{tough}}}\|_0^2$.

This gives us the uniform control

$$\|\Pi_{F_1^\perp}\beta_\varepsilon\|_0^2 = \big\|\Pi_{F_1^\perp}(\lambda - P_\varepsilon)^{-1}\alpha\big\|_0^2 \leq \varepsilon C^{-1}\|\alpha\|_0^2 \,,$$

so we can focus now on $\Pi_{F_1}\beta_\varepsilon$. For this, we use (21) to obtain the $\Pi_{F_1 \cap F_3^\perp}$ projection of the equation

$$(\lambda - P_\varepsilon)\beta_\varepsilon = \alpha \,.$$

It is

$$\begin{aligned}\big(\lambda - (d_Q + \delta_Q)\big)\Pi_{F_1\cap F_3^\perp}\beta_\varepsilon &- \Pi_{F_1^{\mathrm{tame}}}(P_H + \varepsilon P_T)\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon \\ &- \varepsilon \Pi_{F_1\cap F_3^\perp}\big((K_\varepsilon^{n+1})^* K_\varepsilon^n + (K_\varepsilon^n)^* K_\varepsilon^{n+1}\big)\beta_\varepsilon = \Pi_{F_1}\alpha \,,\end{aligned}$$

or equivalently,

$$\begin{aligned}(\lambda - P_Q)\Pi_{F_1\cap F_3^\perp}\beta_\varepsilon - \Pi_{F_1}\alpha &= \Pi_{F_1^{\mathrm{tame}}}(P_H + \varepsilon P_T)\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon \\ &\quad + \varepsilon \Pi_{F_1\cap F_3^\perp}(\theta \wedge d_Q K_\varepsilon^n + d_Q^J K_\varepsilon^{n+1})\beta\varepsilon \,. \qquad (22)\end{aligned}$$

We are led to compose this with the inverse of $(\lambda - P_Q)$ on $F_1 \cap F_3^\perp$. We gather here some analytic facts we will need about it and the inverse of $(\lambda - P_D)$ on F_3.

PROPOSITION 6.2. 1. *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $(\lambda - P_Q)$ (resp. $(\lambda - P_D)$) induces a bicontinuous bijection between the Hilbert spaces $F_1 \cap F_3^\perp \cap \mathcal{H}_{1,H}$ and $F_1 \cap F_3^\perp$ in L^2 (resp. $F_3 \cap \mathcal{H}_{2,H}$ and F_3).*

2. *P_Q (resp. P_D) extends to a self-adjoint operator on $F_1 \cap F_3^\perp$ (resp. F_3) in L^2 with domain $F_1 \cap F_3^\perp \cap \mathcal{H}_{1,H}$ (resp. $F_3 \cap \mathcal{H}_{2,H}$) and discrete spectrum without accumulation points.*

3. *$\Pi_{F_2 \cap F_1^{\mathrm{tame}}}$ is continuous from $\mathcal{H}_0$ $(= L^2)$ to any $\mathcal{H}_{k,H}$, $k \in \mathbb{N}$.*

4. *$\Pi_{F_1\cap F_3^\perp}$ extends continuously from $\mathcal{H}_{1,H}$ and $\mathcal{H}_{2,H}$ to themselves.*

Proof. We define $P = \Delta_Q + P_D$ on F_1, that is $P = d_Q\delta_Q + \delta_Q d_Q$ on F_1^{tame} and $P = \begin{pmatrix} d_Q\delta_Q & D^* \\ D & \delta_Q d_Q \end{pmatrix}$ on $F_1^{\mathrm{tough}} = F_1^n \oplus F_1^{n+1}$. By Theorem 3.1, it is a symmetric second order hypoelliptic operator. So, it follows classically that on a compact manifold, P extends to a self-adjoint operator on L^2 with domain $\mathcal{H}_{2,H}$. The assertions on P_D come then from the restrictions of P and its resolvent to the stable space $F_3 = (F_1^n \cap \ker \delta_Q) \oplus (F_1^{n+1} \cap \ker d_Q)$.

For the properties of P_Q, we consider

$$Q_\lambda = (\lambda + P_Q)(\lambda^2 - P)^{-1},$$

which induces a continuous map from $F_1 \cap F_3^\perp$ (in L^2) to $F_1 \cap F_3^\perp \cap \mathcal{H}_{1,H}$. It is a right inverse of $(\lambda - P_Q)$ on $F_1 \cap F_3^\perp$. We also observe that $Q_\lambda = (\lambda^2 - P)^{-1}(\lambda + P_Q)$ on $F_1 \cap F_3^\perp \cap \mathcal{H}_{1,H}$. This shows that $Q_\lambda(\lambda - P_Q) = \mathrm{Id}$ holds on $F_1 \cap F_3^\perp \cap \mathcal{H}_{2,H}$ and therefore extends continuously on all $F_1 \cap F_3^\perp \cap \mathcal{H}_{1,H}$. So we get the first point on P_Q, from which its self-adjointness is a classical consequence (see [RS, VIII]).

The third point comes from the fact that $F_2 \cap F_1^{\text{tame}}$ is the kernel of the hypoelliptic operator Δ_Q on F_1^{tame}. The statements about $\Pi_{F_1 \cap F_3^\perp}$ result from its L^2 boundedness and its commutation with P_Q and P_D. □

We apply this proposition to the expression (22), observing that

- P_H is a first order (horizontal) operator, so that $(\lambda - P_Q)^{-1}\Pi_{F_1^{\text{tame}}} P_H \Pi_{F_1^{\perp,\text{tame}}}$ is bounded from $\mathcal{H}_0 = L^2$ to itself.
- P_T must be viewed by (8) as a second order operator, therefore continuous from $\mathcal{H}_{1,H}$ to $\mathcal{H}_{-1,H}$.

We finally get, $\exists C > 0$ such that for $\alpha \in F_3^\perp$ and $\beta_\varepsilon = (\lambda - P_\varepsilon)^{-1}\alpha$,

$$\begin{aligned}
&\big\|\Pi_{F_1 \cap F_3^\perp}\beta_\varepsilon - (\lambda - P_Q)^{-1}\Pi_{F_1}\alpha\big\|_0 \\
&\qquad \le C\big(\|\Pi_{F_1^{\perp,\text{tame}}}\beta_\varepsilon\|_0 + \varepsilon\|K_\varepsilon\beta_\varepsilon\|_0 + \varepsilon\|\Pi_{F_1^{\perp,\text{tame}}}\beta_\varepsilon\|_{1,H}\big) \\
&\qquad \le C\varepsilon\|\alpha\|_0 \quad \text{by Definition 5.2 and Proposition 5.3.}
\end{aligned}$$

This completes the proof of the uniform (L^2) convergence,

$$\Pi_{F_3^\perp}(\lambda - P_\varepsilon)^{-1}\Pi_{F_3^\perp} \xrightarrow[\varepsilon\to 0]{} (\lambda - P_Q)^{-1}\Pi_{F_1 \cap F_3^\perp}.$$

Writing, by spectral theory,

$$(\lambda^2 - \Delta_\varepsilon)^{-1} = \tfrac{1}{2\lambda}\big((\lambda - P_\varepsilon)^{-1} + (\lambda + P_\varepsilon)^{-1}\big),$$

we also obtain the uniform convergence of the resolvents of Laplacians,

$$(\lambda^2 - \Delta_\varepsilon)^{-1} \xrightarrow[\varepsilon\to 0]{} (\lambda^2 - \Delta_Q)^{-1}\Pi_{F_1} \quad \text{on } \Omega^p M,\ p \ne n, n+1,$$

and

$$\Pi_{F_3^\perp}(\lambda^2 - \Delta_\varepsilon)^{-1}\Pi_{F_3^\perp} \xrightarrow[\varepsilon\to 0]{} (\lambda^2 - \Delta_Q)^{-1}\Pi_{F_1 \cap F_3^\perp} \quad \text{on } \Omega^n M \oplus \Omega^{n+1} M.$$

We now proceed with Theorem 3.6. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha \in \Omega^* M$, we consider the equation $\big(\lambda - \frac{P_\varepsilon}{\varepsilon}\big)\beta_\varepsilon = \alpha$. First of all, by Definition 5.7 and Proposition 5.8, we know that, for some $C > 0$,

$$\|\alpha\|_0^2 = \big\|\big(\lambda - \tfrac{P_\varepsilon}{\varepsilon}\big)\,\beta_\varepsilon\big\|_0^2 \ge |\operatorname{Im}\lambda|^2\|\beta_\varepsilon\|_0^2 + \big\|\tfrac{P_\varepsilon}{\varepsilon}\beta_\varepsilon\big\|_0^2$$

$$
\begin{aligned}
&\geq C\|\beta_\varepsilon\|^2_{1',\varepsilon}\\
&\geq C\Big(\tfrac{1}{\varepsilon^2}\left(\Delta_Q\Pi_{F_1}\beta_\varepsilon,\Pi_{F_1}\beta_\varepsilon\right)_0+\tfrac{1}{\varepsilon^4}\|\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon\|_0^2\\
&\qquad+\tfrac{1}{\varepsilon^2}\|\Pi_{F_1^{\perp,\mathrm{tough}}}\beta_\varepsilon\|_0^2\Big)\,.
\end{aligned}
$$

Also, from Prop. 6.2, 0 is isolated in the spectrum of P_Q on $F_1\cap F_3^\perp$. Therefore, $\exists C>0$ such that

$$
\tfrac{1}{\varepsilon^2}(\Delta_Q\Pi_{F_1}\beta_\varepsilon,\Pi_{F_1}\beta_\varepsilon)_0\geq\tfrac{C}{\varepsilon^2}\|\Pi_{F_2^\perp}\beta_\varepsilon\|_0^2\,,
$$

where we recall that $F_2=\ker\Delta_Q$. Thus,

$$
\Pi_{F_2^\perp}\left(\lambda-\tfrac{P_\varepsilon}{\varepsilon}\right)^{-1}\xrightarrow[\varepsilon\to0]{}0\quad\text{ uniformly in } L^2\,.
$$

We use (21) again to obtain the F_2 component of the equation $\left(\lambda-\frac{P_\varepsilon}{\varepsilon}\right)\beta_\varepsilon=\alpha$. It is

$$
\begin{aligned}
\big(\lambda-(D+D^*)\big)\Pi_{F_2}\beta_\varepsilon-\Pi_{F_2\cap F_1^{\mathrm{tame}}}\left(\tfrac{P_H}{\varepsilon}+P_T\right)\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon&\\
-\Pi_{F_3}(\theta\wedge d_QK_\varepsilon^n+d_Q^JK_\varepsilon^{n+1})\beta_\varepsilon&=\Pi_{F_2}\alpha\,,
\end{aligned}
$$

which in turn can be decomposed in $F_2\cap F_1^{\mathrm{tame}}$ and $F_3=F_2\cap F_1^{\mathrm{tough}}$ components, giving respectively,

$$
\begin{cases}\lambda\Pi_{F_2\cap F_1^{\mathrm{tame}}}\beta_\varepsilon-\Pi_{F_2\cap F_1^{\mathrm{tame}}}\left(\frac{P_H}{\varepsilon}+P_T\right)\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon=\Pi_{F_2\cap F_1^{\mathrm{tame}}}\alpha\\
(\lambda-P_D)\Pi_{F_3}\beta_\varepsilon-\Pi_{F_3}(\theta\wedge d_QK_\varepsilon^n+d_Q^JK_\varepsilon^{n+1})\beta_\varepsilon=\Pi_{F_3}\alpha\,.\end{cases}\tag{23}
$$

Using Prop. 6.2, we obtain that, for some $C>0$,

$$
\begin{aligned}
\|\lambda\Pi_{F_2\cap F_1^{\mathrm{tame}}}\beta_\varepsilon-\Pi_{F_2\cap F_1^{\mathrm{tame}}}\alpha\|_0&\leq\tfrac{1}{\varepsilon}\|\Pi_{F_1^{\perp,\mathrm{tame}}}\beta_\varepsilon\|_0\leq C\varepsilon\|\beta_\varepsilon\|_{1,\varepsilon}\\
&\leq C\varepsilon\|\alpha\|_0\text{ by Prop. 5.8 and Definition 5.7,}
\end{aligned}
$$

and also

$$
\begin{aligned}
\|\Pi_{F_3^\perp}\beta_\varepsilon-(\lambda-P_D)\Pi_{F_3}\alpha\|_0&\leq\|\Pi_{F_3}(\theta\wedge d_QK_\varepsilon^n+d_Q^JK_\varepsilon^{n+1})\beta_\varepsilon\|_{-2,H}\\
&\leq C\|K_\varepsilon\beta_\varepsilon\|_0\leq C\varepsilon\|\beta_\varepsilon\|_{1',\varepsilon}\\
&\leq C\varepsilon\|\alpha\|_0\ \text{ by Proposition 5.8.}
\end{aligned}
$$

So that finally, observing that $P_D=0$ on $F_3^\perp$, we have got

$$
\left(\lambda-\tfrac{P_\varepsilon}{\varepsilon}\right)^{-1}\xrightarrow[\varepsilon\to0]{}(\lambda-P_D)^{-1}\Pi_{F_2}\,,
$$

uniformly in L^2, and the corresponding result on Laplacians, by the same trick as above,

$$
\left(\lambda^2-\tfrac{\Delta_\varepsilon}{\varepsilon^2}\right)^{-1}=\tfrac{1}{2\lambda}\left(\left(\lambda-\tfrac{P_\varepsilon}{\varepsilon}\right)^{-1}+\left(\lambda+\tfrac{P_\varepsilon}{\varepsilon}\right)^{-1}\right)\xrightarrow[\varepsilon\to0]{}(\lambda^2-\Delta_D)^{-1}\Pi_{F_2}\,.
$$

7 On the Convergence of Heat Kernels

The heat operators $e^{-t\Delta}$ are of particular importance in Riemannian geometry. For instance, their global and local traces play a key role in the index theorem and analytic torsion theory. Also, eta invariants involve global and local traces like $\mathrm{Tr}(Pe^{-tP^2})$ for P Dirac operators.

We plan here to begin the study of the behavior of some of such traces in the sub-Riemannian limit of the metric, that is with respect to $g_\varepsilon = d\theta(\cdot, J\,\cdot\,) + \frac{\theta^2}{\varepsilon^2}$ when $\varepsilon \to 0$.

7.1 The global trace convergence. For P a self-adjoint operator on a compact manifold M with smooth kernel $K_P(x,y)$, the trace of P will be denote

$$\mathrm{Tr}_M(P) = \int_M \mathrm{Tr}(K_P(x,x))d\mathrm{vol} = \sum_{\mathrm{Spectrum}(P)} \lambda_i\,,$$

where $(\lambda_i)_{i\in\mathbb{N}}$ is the pure point spectrum of P.

Theorem 7.1. *Let M be a compact contact manifold endowed with the family of metrics g_ε. One has, uniformly for $t \geq t_0 > 0$,*

1. *on $\Omega^p M$ with $p \neq n, n+1$,*
$$\mathrm{Tr}_M(e^{-t\Delta_\varepsilon}) \xrightarrow[\varepsilon\to 0]{} \mathrm{Tr}_M(e^{-t\Delta_Q}) \quad (\Delta_Q \text{ acting on } F_1^p)$$
2. *on $\Omega^n M \oplus \Omega^{n+1} M$,*
$$\mathrm{Tr}_M(e^{-t\frac{\Delta_\varepsilon}{\varepsilon^2}}) \xrightarrow[\varepsilon\to 0]{} \mathrm{Tr}_M(e^{-t\Delta_D})$$
where $\Delta_D = D^*D + DD^*$ acts on $F_3 = (F_1^n \cap \ker \delta_Q) \oplus (F_1^{n+1} \cap \ker d_Q)$.
3. *Similarly, for the signature operator S_ε acting on $\Omega^{\mathrm{even}} M$, one has*
$$\mathrm{Tr}_M\left(\frac{S_\varepsilon}{\varepsilon} e^{-t\frac{\Delta_\varepsilon}{\varepsilon^2}}\right) \xrightarrow[\varepsilon\to 0]{} \mathrm{Tr}_M(S_D e^{-t\Delta_D})$$
where (section 3) $S_D = (-1)^l D*$ on F_3^{n+1} for $n = 2l+1$.

Proof. This follows easily from the resolvent convergence results (Theorems 3.5 and 3.6) and the developed Bochner technique. We consider the spectrum, arranged in increasing order, $(\lambda_i(\varepsilon))_{i\in\mathbb{N}}$ of Δ_ε (resp. $\Delta_\varepsilon/\varepsilon^2$ in middle degree). A classical consequence of norm resolvent convergence (see [RS]), is the convergence of the spectrum over bounded open intervals of $\mathbb{R}$. So that, for each $A > 0$ and not in $\mathrm{Spectrum}(\Delta_Q)$,

$$\mathrm{Spectrum}(\Delta_\varepsilon) \cap [0, A[\xrightarrow[\varepsilon\to 0]{} \mathrm{Spectrum}(\Delta_Q) \cap [0, A[$$

(resp. with $\Delta_\varepsilon \varepsilon^2$ and Δ_D). This implies, for the ith eigenvalue, $\lambda_i(\varepsilon) \xrightarrow[\varepsilon\to 0]{} \lambda_i(\Delta_Q)$ (resp. Δ_D).

We will be able to turn this into the convergence of the sum $\sum_{i\in\mathbb{N}} e^{-t\lambda_i(\varepsilon)}$ if we can find a fixed summable domination of it. But this comes from the Bochner technique. Namely, for $p \neq n, n+1$, Prop. 5.3 tell us that $\forall\alpha \in \Omega^p M$,

$$\big((\Delta_\varepsilon + \mathrm{Id})\alpha, \alpha\big) \geq C\|\alpha\|^2_{1,H} = (\Delta'_H\alpha, \alpha)$$

for an hypoelliptic Laplacian Δ'_H. So the maxmin principle gives that

$$\lambda_i(\varepsilon) \geq \lambda_i(\Delta'_H) - 1 \quad \text{and} \quad e^{-t\lambda_i(\varepsilon)} \leq e^{-t(\lambda_i(\Delta'_H)-1)},$$

whose sum is convergent by hypoellipticity of Δ'_H.

The corresponding domination in middle degree is elliptic as comes from Definition 5.7, Prop. 5.8 and Lemma 5.9,

$$\left(\left(\tfrac{\Delta_\varepsilon}{\varepsilon^2} + \mathrm{Id}\right)\alpha, \alpha\right)_0 \geq \|\alpha\|^2_{1,(H+T)}.$$

Lastly, the eta trace in 3 is controlled with

$$\begin{aligned}\left|\lambda_i\left(\tfrac{S_\varepsilon}{\varepsilon}\right) e^{-t\lambda_i^2(\frac{S_\varepsilon}{\varepsilon})}\right| &\leq \left(1 + \lambda_i^2\left(\tfrac{S_\varepsilon}{\varepsilon}\right)\right) e^{-t\lambda_i^2(\frac{S_\varepsilon}{\varepsilon})} = \left(1 + \lambda_i\left(\tfrac{\Delta_\varepsilon}{\varepsilon^2}\right)\right) e^{-t\lambda_i(\frac{\Delta_\varepsilon}{\varepsilon^2})} \\ &\leq Ce^{-\frac{t}{2}\lambda_i(\frac{\Delta_\varepsilon}{\varepsilon^2})}.\end{aligned}$$

□

7.2 The local convergence of heat kernels outside middle degrees. Let $K_\varepsilon(x,y)$ and $K_Q(x,y)$ be the respective kernels of $e^{-t\Delta_\varepsilon}$ and $\Pi_{F_1}e^{-t\Delta_Q}\Pi_{F_1}$. We will be interested now in the problem of the local convergence of K_ε when $\varepsilon \to 0$.

Theorem 7.2. *Let M be a compact contact manifold and $p \neq n, n+1$, then*

$$K_\varepsilon(x,y) \xrightarrow[\varepsilon\to 0]{} K_Q(x,y)$$

uniformly on M in C^k norm for any $k \in \mathbb{N}$.

For this, we have to improve the uniform L^2 convergence of $e^{-t\Delta_\varepsilon}$ towards $\Pi_{F_1}e^{-t\Delta_Q}\Pi_{F_1}$, which comes with Theorem 3.5 and functional calculus, to a uniform convergence at the kernel level. To perform this, we will follow a general method used in Bismut-Lebeau's work on Quillen's metrics (see [BiL, part XI, p. 165–179]).

Here is a sketch of it. First, we will extend the regularizing properties of the resolvent $(\lambda - \Delta_\varepsilon)^{-1}$ to higher order Sobolev spaces. This will be done by a commutator technique (adapted to the hypoelliptic context here). Then we will turn this into uniform regularizing properties of $e^{-t\Delta_\varepsilon}$ by using a contour integral such as $\int_\Gamma e^{-t\lambda}(\lambda - \Delta_\varepsilon)^{-k}d\lambda$, with k integer. This will give for k large enough a uniform L^2 control of K_ε, and for k even larger a sufficient Sobolev control to deduce the equicontinuity and uniform convergence of the kernels.

We begin this. The a priori estimate Prop. 5.3 and Lax-Milgram assure that $(\mathrm{Id} + \Delta_\varepsilon)$ induces a bicontinuous bijection from $\mathcal{H}_{1,\varepsilon}$ into $\mathcal{H}_{-1,\varepsilon}$ together with the following uniform norm control, for $\varepsilon \in]0,1]$,

$$\left\| (\mathrm{Id} + \Delta_\varepsilon)^{-1}\alpha \right\|_{1,\varepsilon} \leq C_1^{-1} \|\alpha\|_{-1,\varepsilon} \,. \tag{24}$$

To extend this to higher order regularizing properties, we introduce some families of Sobolev spaces modeled on $\| \ \|_{1,\varepsilon}$.

Definition 7.3. For $k \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha \in \Omega^p M$, let

$$\|\alpha\|_{k,\varepsilon} = \|\alpha\|_{k,H} + \tfrac{1}{\varepsilon}\|\Pi_{F_1^\perp}\alpha\|_{k-1,H} + \varepsilon\|T\alpha\|_{k-1,H} + \|\alpha\|_0 \,,$$

and $\mathcal{H}_{k,\varepsilon}$ be the corresponding Sobolev space (the (k,H) indices in norms denote the number of horizontal derivatives controlled in L^2).

The estimate (24) will be generalized the following way.

Lemma 7.4. *For $k \in \mathbb{N}$, $\exists C_k$ such that $\forall \varepsilon \in]0,1]$, $(\mathrm{Id} + \Delta_\varepsilon)^{-1}$ extends continuously from $\mathcal{H}_{2k-1,H}$ into $\mathcal{H}_{2k+1,\varepsilon}$, with the uniform operator's norms controls,*

$$\left\| (\mathrm{Id} + \Delta_\varepsilon)^{-1} \right\|_{(2k-1,H),(2k+1,\varepsilon)} \leq C_k \quad \textit{and}$$

$$\left\| (\mathrm{Id} + \Delta_\varepsilon)^{-1} \frac{\Pi_{F_1^\perp}}{\varepsilon} \right\|_{(2k,H),(2k+1,\varepsilon)} \leq C_k \,.$$

We observe that all this is clear for a fixed ε by elliptic regularity of Δ_ε. The problem is to handle its regularity degeneracy and divergences when $\varepsilon \to 0$. The commutator technique allows a quite explicit approach of this.

We cover the manifold M with small enough open sets such that on each of them we can choose orthonormal vector fields (for g_1): $X_1, X_2 = JX_1, \dots, X_{2n-1}, X_{2n} = JX_{2n-1}$ in H, and T (the Reeb field). Let $\theta^1, \dots, \theta^{2n}, \theta^0 = \theta$ be the dual base. For $X \in TM$, we note ∂_X the differential operator acting on forms defined by

$$\alpha = \sum_{I \subset [0,2n]} \alpha_I \theta^I \longmapsto \partial_X \alpha = \sum_I (X.\alpha_I)\theta_I \,.$$

We gather some useful facts about them. $P^{(k)}$ will denote linear differential operators of order k (in horizontal derivatives).

- $[\partial_X, L] = [\partial_X, \Lambda] = 0$ (because $L = \sum_{i=1}^n \theta^{2i-1} \wedge \theta^{2i} \wedge \cdot$ and Λ are constant coefficients algebraic operators)
- $[\partial_X, \partial_Y] = \partial_{[X,Y]}$
- $[\partial_X, d_H] = P^{(1)} + P^{(0)}\partial_T$ for X in H (by previous point and the local Cartan's formula $d_H = \sum_{i=1}^{2n} \partial_{X_i}(\theta_i \wedge \cdot) + P^{(0)}$)
- $[\partial_T, d_H] = P^{(1)}$ (here, $[T, X] \in H$ for all $X \in H$ because the Reeb field T preserves H)

- $[\partial_X, \mathcal{L}_T] = P^{(1)}$ for $X \in H$ (because $\mathcal{L}_T = \partial_T + P^{(0)}$)
- $[\partial_T, \mathcal{L}_T] = P^{(0)}$
- similar formulas hold with adjoint and J-conjugate (δ_H, $\mathcal{L}_T^*$, d_H^J) instead.

All this and (17) give easily the commutators we will need.

Proposition 7.5. *For $X, Y \in H$, one has*

$$[\partial^2_{XY}, \Delta_\varepsilon] = P^{(3)} + P^{(2)}\partial_T + \varepsilon(P^{(2)} + P^{(1)}\partial_T) + \varepsilon^2 P^{(2)}\partial_T \\ + \varepsilon^{-1}\Pi_{F_1^\perp}(P^{(2)} + P^{(1)}\partial_T) + \varepsilon^{-1}(P^{(2)} + P^{(1)}\partial_T)\Pi_{F_1^\perp}\,,$$

and,

$$[\partial_T, \Delta_\varepsilon] = P^{(2)} + \varepsilon P^{(1)} + \varepsilon^2 P^{(0)}\partial_T + \varepsilon^{-1}(\Pi_{F_1^\perp} P^{(1)} + P^{(1)}\Pi_{F_1^\perp})\,.$$

The first commutator does not seem so "good" because it is still of hypoelliptic order four. Even so, its high order part factorizes through ∂_T, which itself has good commutation properties (on the Heisenberg group, our local model, T is even in the center of the group).

Proof of Lemma 7.4. We begin now the proof of Lemma 7.4. The case $k = 0$ follows from (24). We suppose it is true for the rank k and proceed by recurrence. We use Proposition 7.5 to write for $X, Y \in H$,

$$\begin{aligned} \partial^2_{XY}(\mathrm{Id}+\Delta_\varepsilon)^{-1} &= (\mathrm{Id}+\Delta_\varepsilon)^{-1}\partial^2_{XY} - (\mathrm{Id}+\Delta_\varepsilon)^{-1}[\partial^2_{XY}, \Delta_\varepsilon](\mathrm{Id}+\Delta_\varepsilon)^{-1} \\ &= (\mathrm{Id}+\Delta_\varepsilon)^{-1}\partial^2_{XY} \\ &\quad - (\mathrm{Id}+\Delta_\varepsilon)^{-1}\big(P^{(3)} + \varepsilon(P^{(2)} + P^{(1)}\partial_T) + \varepsilon^2 P^{(2)}\partial_T \\ &\qquad + \varepsilon^{-1}(\Pi_{F_1^\perp}P^{(2)} + P^{(2)}\Pi_{F_1^\perp})\big)(\mathrm{Id}+\Delta_\varepsilon)^{-1} \\ &\quad - (\mathrm{Id}+\Delta_\varepsilon)^{-1}\big(P^{(2)} + \varepsilon^{-1}(\Pi_{F_1^\perp}P^{(1)} + P^{(1)}\Pi_{F_1^\perp})\big)\partial_T(\mathrm{Id}+\Delta_\varepsilon)^{-1} \end{aligned} \tag{25}$$

with,

$$\begin{aligned} \partial_T(\mathrm{Id}+\Delta_\varepsilon)^{-1} &= (\mathrm{Id}+\Delta_\varepsilon)^{-1}\partial_T - (\mathrm{Id}+\Delta_\varepsilon)^{-1}[\partial_T, \Delta_\varepsilon](\mathrm{Id}+\Delta_\varepsilon)^{-1} \\ &= (\mathrm{Id}+\Delta_\varepsilon)^{-1}\partial_T \\ &\quad - (\mathrm{Id}+\Delta_\varepsilon)^{-1}\big(P^{(2)} + \varepsilon P^{(1)} + \varepsilon^2 P^{(0)}\partial_T \\ &\qquad + \varepsilon^{-1}(\Pi_{F_1^\perp}P^{(1)} + P^{(1)}\Pi_{F_1^\perp})\big)(\mathrm{Id}+\Delta_\varepsilon)^{-1}\,. \end{aligned} \tag{26}$$

We use this to estimate the norm at rank $k+1$,

$$\big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+3,\varepsilon)} = \sum_{X,Y\in\{X_1,\dots,X_{2n}\}} \big\|\partial^2_{XY}(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+1,\varepsilon)} \\ + \big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+1,\varepsilon)}\,. \tag{27}$$

We have to check the continuity properties of each term of (25) (fixed multiplicative constants will be overviewed):

- $$\begin{aligned}\big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\partial^2_{XY}\alpha\big\|_{(2k+1,\varepsilon)} &\leq \|\partial^2_{XY}\alpha\|_{(2k-1,H)} \quad \text{by rank } k \text{ hypothesis}\\ &\leq \|\alpha\|_{(2k+1,H)}\,,\end{aligned}$$
- $$\begin{aligned}\big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}P^{(3)}(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+1,\varepsilon)} &\leq \big\|P^{(3)}(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k-1,H)}\\ &\leq \big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+2,H)}\,,\end{aligned}$$

and we use that $\|\ \|_{(2k+2,H)} \leq c\|\ \|_{(2k+3,H)} + c^{-1}\|\ \|_{(2k+1,H)}$ for any arbitrarily small c (which can be proved elementary, as in elliptic theory, by integration by parts and Cauchy-Schwartz:

$$\begin{aligned}\|\alpha\|^2_{(2k+2,H)} = (\nabla^{2k+2}_H\alpha, \nabla^{2k+2}_H)_0 &= (\nabla^{2k+3}_H\alpha, \nabla^{2k+1}_H\alpha)_0\\ &\leq \|\alpha\|_{(2k+3,H)}\|\alpha\|_{(2k+1,H)}\\ &\leq c^2\|\alpha\|^2_{(2k+3,H)} + c^{-2}\|\alpha\|^2_{(2k-1,H)} \quad).\end{aligned}$$

So that finally $c\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\|_{(2k+3,H)} \leq c\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\|_{(2k+3,\varepsilon)}$ can be absorbed in the left side of (27).

- $$\begin{aligned}&\big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}(\varepsilon(P^{(2)}+P^{(1)}\partial_T)+\varepsilon^2P^{(2)}\partial_T)(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+1,\varepsilon)}\\ &\quad\leq \big\|(\varepsilon P^{(2)}+P^{(1)}\varepsilon\partial_T+\varepsilon^2P^{(2)}\partial_T)(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k-1,H)}\\ &\quad\leq \|\alpha\|_{(2k-1,H)} + \varepsilon^2\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\|_{(2k+3,H)}\,,\end{aligned}$$

this last term being again absorbed in the left side of (27),

- $$\begin{aligned}&\Big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\Big(\frac{\Pi_{F_1^\perp}}{\varepsilon}P^{(1)}+P^{(1)}\frac{\Pi_{F_1^\perp}}{\varepsilon}\Big)(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\Big\|_{(2k+1,\varepsilon)}\\ &\quad\leq \big\|P^{(1)}(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k,H)} + \Big\|\frac{\Pi_{F_1^\perp}}{\varepsilon}(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\Big\|_{(2k,H)}\\ &\quad\leq \big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\alpha\big\|_{(2k+1,\varepsilon)} \leq \|\alpha\|_{(2k-1,H)}\,.\end{aligned}$$
- Using the same methods, we easily get that
  $$(\mathrm{Id}+\Delta_\varepsilon)^{-1}(P^{(2)}+\varepsilon^{-1}\Pi_{F_1^\perp}P^{(1)}+\varepsilon^{-1}P^{(1)}\Pi_{F_1^\perp})$$
  is continuous from $\mathcal{H}_{2k+1,\varepsilon}$ into itself, and that, thanks to (26), $\partial_T(\mathrm{Id}+\Delta_\varepsilon)^{-1}$ is from $\mathcal{H}_{2k+1,H}$ to $\mathcal{H}_{2k+1,\varepsilon}$. They can therefore be composed as in (25), so that we obtain finally the required boundedness of $\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\|_{(2k+1,H),(2k+3,\varepsilon)}$.

We are left with the control of $\big\|(\mathrm{Id}+\Delta_\varepsilon)^{-1}\frac{\Pi_{F_1^\perp}}{\varepsilon}\big\|_{(2k+2,H),(2k+3,\varepsilon)}$, which is the second part of our recurrence hypothesis at rank $k+1$. This can be handled in the same way as our previous estimation. We just have to compose the equations (25) and (26) by $\Pi_{F_1^\perp}/\varepsilon$ by the right, and commute it with ∂^2_{XY} in (25) and ∂_T in (26). All terms can then be controlled using the rank k hypothesis. □

In order to relate the heat operator $e^{-t\Delta_\varepsilon}$ to $(\lambda-\Delta_\varepsilon)^{-1}$ by a contour integral formula, we have to extend our resolvent estimates to λ's surrounding $\mathbb{R}^+$. Let therefore U be the domain of $\mathbb{C}$ defined by

$$U=\big\{\lambda\in\mathbb{C}\,,\ \Re\lambda\leq -1 \text{ or } |\Im\lambda|\geq 1\big\}\,.$$

We will need

LEMMA 7.6. *For $k\in\mathbb{N}$, $\exists C_k$ such that $\forall\varepsilon\in]0,1]$ and $\lambda\in U$, the following operator's norm control holds,*

$$\big\|(\lambda-\Delta_\varepsilon)^{-1}\big\|_{(2k-1,H),(2k+1,H)}\leq C_k\big(1+|\lambda|\big)^{2k+2}\,.$$

Proof. The case $\lambda=-1$ comes from Lemma 7.4. Moreover, we know that $(\lambda-\Delta_\varepsilon)^{-1}$ is related to $(\mathrm{Id}+\Delta_\varepsilon)^{-1}$ by the first resolvent equation (see [RS])

$$\text{(i)}\qquad (\lambda-\Delta_\varepsilon)^{-1}=-(\mathrm{Id}+\Delta_\varepsilon)^{-1}+(\lambda+1)(\mathrm{Id}+\Delta_\varepsilon)^{-1}(\lambda-\Delta_\varepsilon)^{-1}$$

$$\text{(ii)}\qquad =-(\mathrm{Id}+\Delta_\varepsilon)^{-1}+(\lambda+1)(\lambda-\Delta_\varepsilon)^{-1}(\mathrm{Id}+\Delta_\varepsilon)^{-1}\,.$$

To start the proof for $k=0$, we also observe that, Δ_ε being a positive self-adjoint operator, it satisfies for $\lambda\in U$,

$$\big\|(\lambda-\Delta_\varepsilon)^{-1}\alpha\big\|_0\leq\frac{\|\alpha\|_0}{\mathrm{dist}(\lambda,\mathrm{Spec}\Delta_\varepsilon)}\leq\|\alpha\|_0\,.$$

A first use of the resolvent equation (i) gives therefore,

$$\begin{aligned}\big\|(\lambda-\Delta_\varepsilon)^{-1}\alpha\big\|_{1,H}&\leq\big\|(\lambda-\Delta_\varepsilon)^{-1}\alpha\big\|_{1,\varepsilon}\\&\leq C\|\alpha\|_{-1,H}+\big(|\lambda|+1\big)\big\|(\lambda-\Delta_\varepsilon)^{-1}\alpha\big\|_{-1,H}\quad\text{by Lemma 7.4 or (24)}\\&\leq\big(C+|\lambda|+1\big)\|\alpha\|_0\,,\end{aligned}$$

which can be used again in (ii) to obtain

$$\big\|(\lambda-\Delta_\varepsilon)^{-1}\alpha\big\|_{1,H}\leq C\big(1+|\lambda|\big)^2\|\alpha\|_{-1,H}\,.$$

Lemma 7.6 is then proved by recurrence using (i) and Lemma 7.4. □

We can now complete the proof of Theorem 7.2. Setting $\Gamma=\partial U$, we have by the functional calculus

$$e^{-t\Delta_\varepsilon}=\frac{1}{2i\pi}\int_\Gamma e^{-t\lambda}(\lambda-\Delta_\varepsilon)^{-1}d\lambda\,,$$

but also, $\forall k\in\mathbb{N}^*$,

$$=\frac{(-t)^{k-1}}{2i\pi(k-1)!}\int_\Gamma e^{-t\lambda}(\lambda-\Delta_\varepsilon)^{-k}d\lambda\,.$$

This, together with Lemma 7.6, implies that $e^{-t\Delta_\varepsilon}$ is a uniformly bounded operator, when $\varepsilon \to 0$, from $\mathcal{H}_0$ into $\mathcal{H}_{k,H}$ for any $k \in \mathbb{N}$. The same is true with ordinary L^2 elliptic Sobolev spaces $\mathcal{H}_k$, because a transverse derivative is recovered by two horizontal ones. Therefore, if $\Delta = \mathrm{Id} + \nabla^*\nabla$ denotes an invertible elliptic Laplacian and $\|A\|_{HS} = \mathrm{Tr}(A^*A)^{1/2}$ the Hilbert-Schmidt's norm (see [RS]), one has $\forall l, m \in \mathbb{N}$,

$$\begin{aligned}\left\|\Delta_x^l \Delta_y^m K_\varepsilon(x,y)\right\|_{L^2} &= \|\Delta^l e^{-t\Delta_\varepsilon}\Delta^m\|_{HS}\\ &= \|\Delta^l e^{-t\Delta_\varepsilon}\Delta^{m+k}\Delta^{-k}\|_{HS} \quad \text{for any } k \in \mathbb{N},\\ &\leq \|\Delta^l e^{-t\Delta_\varepsilon}\Delta^{m+k}\|_{L^2}\|\Delta^{-k}\|_{HS}\,,\end{aligned}$$

with $\|\Delta^{-k}\|_{HS} < +\infty$ for k large enough, and $\Delta^l e^{-t\Delta_\varepsilon}\Delta^{m+k}$ uniformly bounded from L^2 in itself when $\varepsilon \to 0$ by Lemma 7.6 and duality.

The Sobolev inequalities give then a uniform control when $\varepsilon \to 0$ of K_ε and its derivatives in sup norm. This allows to improve the week convergence of K_ε towards K_Q (coming with the L^2 uniform convergence of $e^{-t\Delta_\varepsilon} \to \Pi_{F_1} e^{-t\Delta_Q}\Pi_{F_1}$) to a strong uniform one.

7.3 The local convergence of heat kernels in middle degrees. We now come to the the problem of the local convergence of the kernel $K_\varepsilon(x,y)$ of $e^{-t\frac{\Delta_\varepsilon}{\varepsilon^2}}$ on $\Omega^n M \oplus \Omega^{n+1}M$. We already know that it weakly converges towards the kernel $K_D(x,y)$ of $\Pi_{F_3}e^{-t\Delta_D}\Pi_{F_3}$, where $\Delta_D = D^*D + DD^*$ and $F_3 = (F_1^n \cap \ker \delta_Q) \oplus (F_1^{n+1} \cap \ker d_Q)$.

We want to improve this convergence using the same general methods as above. Although, thanks to Proposition 5.8 and Lemma 5.9, $\mathrm{Id} + \frac{\Delta_\varepsilon}{\varepsilon^2}$ now dominates an elliptic Laplacian, it contains more divergent terms. This makes finally the commutator technique more delicate to apply than previously. In particular, one important point in the proof of the regularizing properties of $(\mathrm{Id} + \Delta_\varepsilon)^{-1}$ was the uniform boundedness of $\frac{\Pi_{F_1^\perp}}{\varepsilon}(\mathrm{Id} + \Delta_\varepsilon)^{-1}$ and $(\mathrm{Id} + \Delta_\varepsilon)^{-1}\frac{\Pi_{F_1^\perp}}{\varepsilon}$ on higher order Sobolev spaces (Lemma 7.4). This was obtained with the use of basic operators, the "scalar" derivatives ∂_X, that had to preserve the components F_1 and $F_1^\perp$. The problem in middle degree will be similarly to find operators that respect the relevant splitting of forms, which is no longer of algebraic nature here.

Recall that we know there that F_1^n has to be decomposed further in $F_3^n = F_1^n \cap \ker \delta_Q$ and $F_1^n \cap \mathrm{Im}\, d_Q$ because $\left\|\frac{\Pi_{F_3^\perp}}{\varepsilon}\alpha\right\|_{1,H}$ is controlled whereas $\Pi_{F_3}\alpha$ does not collapse in $\|\ \|_{1',\varepsilon}$. This splitting of F_1^n implies in turn an anisotropic convergence of the component in $F_1^{\perp,\text{tough}} = \theta \wedge \Omega_0^{n-1}H$. Indeed, we know that

$$\begin{aligned}\frac{K_\varepsilon\alpha}{\varepsilon} &= \tfrac{1}{\varepsilon}\left(\delta_Q^J\alpha_{H,0} - \tfrac{\alpha_{T,0}}{\varepsilon}\right)\\ &= \tfrac{1}{\varepsilon}\delta_Q^J\Pi_{F_3^\perp}\alpha + \tfrac{1}{\varepsilon}\delta_Q^J\Pi_{F_3}\alpha - \tfrac{\alpha_{T,0}}{\varepsilon^2}\end{aligned}$$

is controlled, showing that $\Pi_{(\delta_Q^J(F_3))^\perp}\alpha_{T,0}$ should a priori collapse at speed ε^2 whereas its orthogonal part $\Pi_{\overline{\delta_Q^J(F_3)}}\alpha$ like ε only.

With this in mind, we now look for the possible operators that we can use in the commutator technique. The basic second order operator that preserves the decomposition $F_3 \oplus F_1^n \cap F_3^\perp$ is $d_Q\delta_Q$. It controls two derivatives of the $F_1^n \cap F_3^\perp$ component by hypoelliptic regularity. We have to complete it on $F_1^{\perp,\text{tough}} = \theta \wedge \Omega_0^{n-1}H$ in order that the whole nearly commutes with $K_\varepsilon/\varepsilon$. We have by (7)

$$\begin{aligned}\tfrac{1}{\varepsilon}\delta_Q^J(d_Q\delta_Q\Pi_{F_1^n}) &= \tfrac{1}{\varepsilon}\left(\mathcal{L}_T - \tfrac{1}{2}d_Q\delta_Q^J\right)\delta_Q\Pi_{F_1^n}\\ &= \tfrac{1}{\varepsilon}\big(\mathcal{L}_T\delta_Q + \tfrac{1}{2}d_Q\delta_Q\delta_Q^J + d_QP_{N(J)}^{(1)}\big)\Pi_{F_1^n}\,,\end{aligned}$$

where $P_{N(J)}^{(1)}$ is a first order operator function of the Nijenhuis tensor of J. Its non-vanishing seems to be an obstruction in this method because the term $d_QP_{N(J)}^{(1)}/\varepsilon$ is uncontrollable on F_3. So, we will suppose in all the following that the complex structure J is integrable (gives a CR structure). In this case, we have found that on $F_1^n \oplus \theta\wedge\Omega_0^{n-1}H$,

$$\frac{K_\varepsilon}{\varepsilon}\begin{pmatrix} d_Q\delta_Q & 0\\ 0 & \frac{1}{2}(d_Q\delta_Q + d_Q^J\delta_Q^J)\end{pmatrix} - \frac{1}{2}(d_Q\delta_Q + d_Q^J\delta_Q^J)\frac{K_\varepsilon}{\varepsilon} = \mathcal{L}_T\frac{\delta_Q}{\varepsilon}\Pi_{F_1^n}\,. \tag{28}$$

We will see that this $d_Q\delta_Q + d_Q^J\delta_Q^J$ controls the $\theta\wedge(\ker\delta_Q\cap\ker\delta_Q^J)^\perp$ component, which is contained in the above fast collapsing space $\theta\wedge(\delta_Q^J(\ker\delta_Q))^\perp$ when J is integrable. We therefore introduce

$$F_{3,T} = \theta\wedge(\ker\delta_Q\cap\ker\delta_Q^J)$$

the remaining part of $\theta\wedge\Omega_0^{n-1}H$ to be controlled. Before coming to this, we still have to complete our first regularizing operator to the rest $F_1^{\perp,\text{tame}} = \Omega^nM \ominus (F_1^n \oplus \theta\wedge\Omega_0^{n-1}H)$ of the algebra, so that the extension nearly commutes with the full $\Delta_\varepsilon/\varepsilon^2$. For this, we can use (16) and Remark 4.2, to write on $\Omega_0^{n-1}H$,

$$\tfrac{1}{2}(d_Q\delta_Q + d_Q^J\delta_Q^J) = d_{Q_N}\delta_{Q_N} + d_{Q_N}^J\delta_{Q_N}^J = \Delta_H - \Delta_{Q_N}\,. \tag{29}$$

Now $\Delta_H - \Delta_{Q_N}$ makes sense on all $\oplus_{p<n-1}\Omega_0^pH$ and, by (16), is hypoelliptic there. Then, we extend it to the full Lefschetz decomposition of $F_1^{\perp,\text{tame}}$, by requiring that it commutes with L. This choice has been induced by the fact that, thanks to (6), (15), (16) and Remark 4.2, $\Delta_H - \Delta_{Q_N}$ "nearly"

commutes with all our algebra of operators, d_Q, δ_Q, d^J_Q, δ^J_Q, outside F^n_1. Namely, one has on $(F^n_1)^\perp$, if J is integrable, the following commutation relations

$$\begin{cases} [\Delta_{Q_N}, \delta_Q] = 0 \\ [\Delta_{Q_N}, \delta^J_Q] = [\Delta_{Q_N} - \Delta^J_{Q_N}, \delta^J_Q] = P^{(1)}_{\mathcal{L}_T J} \\ [\Delta_H, \delta_Q] = [d^J_{Q_N}\delta^J_{Q_N}, \delta_Q] = \mathcal{L}^J_T \delta^J_Q \\ [\Delta_H, \delta^J_Q] = [\Delta_H - \Delta^J_H, \delta^J_Q] + [\Delta^J_H, \delta^J_Q] = P^{(1)}_{\mathcal{L}_T J} - \mathcal{L}_T \delta_Q\,. \end{cases} \tag{30}$$

Anyway, in order to be able to use this, and in view of (28), we still have to find a vertical derivative ∂_T which nicely commutes with $\Delta_\varepsilon/\varepsilon^2$. Again, in the previous section, this was easily done because our underlying splitting of forms in anisotropic components F_1 and $F_1^\perp$ was algebraic (define by a 0 order operator). Here, this ∂_T has to preserve both $F^n_3 = F^n_1 \cap \ker \delta_Q$, and commutes in a controllable way with $\frac{K_\varepsilon \alpha}{\varepsilon} = \frac{1}{\varepsilon}\big(\delta^J_Q \alpha_{H,0} - \frac{\alpha_{T,0}}{\varepsilon}\big)$. A natural, although very strong hypothesis, that allows this, is to suppose that the complex structure is invariant under the Reeb flow. In this case, one has by (8), $\mathcal{L}^*_T = -\mathcal{L}^J_T = -\mathcal{L}_T$, so that

$$[\mathcal{L}_T, \delta_Q] = 0 = [\mathcal{L}_T, \delta^J_Q] = [\mathcal{L}_T, d_Q] = [\mathcal{L}_T, d^J_Q]\,, \tag{31}$$

and finally $[\mathcal{L}_T, \Delta_\varepsilon/\varepsilon^2] = 0$, which can be seen directly because the Reeb flow preserves the metrics $g_\varepsilon = d\theta(\cdot, J\,\cdot\,) + \frac{\theta^2}{\varepsilon^2}$.

Lastly, we have to find a third operator in order to control the remaining components

$$F^n_3 = F^n_1 \cap \ker \delta_Q \quad \text{ and } \quad F^n_{3,T} = \theta \wedge (\ker \delta_Q \cap \ker \delta^J_Q)\,.$$

A second order one to try on F^n_3 is $*D$. Again, we have to find an admissible conjugate of it through K_ε. One has on F^n_1, by (1),(7) and [W],

$$\begin{aligned} *D &= *\theta \wedge (\mathcal{L}_T - d_Q \delta^J_Q) \\ &= (-1)^{\frac{n(n+1)}{2}} J(\mathcal{L}_T - d_Q\delta^J_Q)\,, \end{aligned}$$

so that,

$$\begin{aligned} \delta^J_Q\big((-1)^{\frac{n(n+1)}{2}} * D\big) &= \delta^J_Q\big((-1)^{\frac{n(n+1)}{2}} * D\big)\Pi_{F_3} \\ &= (\delta^J_Q J\mathcal{L}_T - \delta^J_Q J d_Q \delta^J_Q)\Pi_{F_3} \\ &= -J[\delta_Q, \mathcal{L}_T]\Pi_{F_3} + (J\delta_Q d_Q)\Pi_{F_{3,T}}\delta^J_Q \Pi_{F_3}\,, \end{aligned}$$

where $[\delta_Q, \mathcal{L}_T] = 0$ by (31), and on $F_{3,T}$

$$J\delta_Q d_Q \Pi_{F_{3,T}} = J\Delta_{Q_N}\Pi_{F_{3,T}} = \Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}} = \Pi_{F_{3,T}} J\delta_Q d_Q$$

because $\Delta_{Q_N} = \Delta^J_{Q_N}$ (by (15)) preserves $F_{3,T} = \theta \wedge (\ker \delta_Q \cap \ker \delta^J_Q)$ (that we will often confuse with $i_T F_{3,T}$). So, we have found that on F^n_1

$$\delta^J_Q\big((-1)^{\frac{n(n+1)}{2}} * D\big) = (\Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}})\delta^J_Q \Pi_{F_3}$$

$$= (\Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}})\delta_Q^J - \Pi_{F_{3,T}} J\delta_Q d_Q \delta_Q^J \Pi_{F_3^\perp}\,,$$

with by (7), $d_Q\delta_Q^J = \mathcal{L}_T$ on $\ker D \supset F_3^\perp$. Thus, we finally get the commutation relation

$$\delta_Q^J\big((-1)^{\frac{n(n+1)}{2}} * D\big) = (\Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}})\delta_Q^J - \Pi_{F_{3,T}} J\delta_Q \mathcal{L}_T \Pi_{F_3^\perp}\,, \quad (32)$$

and we have found an admissible counterpart of $*D$ on $\theta\wedge\Omega_0^{n-1}H$. Indeed, we have

$$\frac{K_\varepsilon}{\varepsilon}\begin{pmatrix}(-1)^{\frac{n(n+1)}{2}} * D & 0\\ 0 & \Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}}\end{pmatrix} = (\Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}})\frac{K_\varepsilon}{\varepsilon} - \Pi_{F_{3,T}} J\delta_{Q_N}\mathcal{L}_T\Pi_{F_3^\perp}\,. \quad (33)$$

We have completed our search of basic regular operators to apply in the commutator technique. Before coming to this, we need to collect some analytic information on $\Pi_{F_{3,T}}$. We first observe that $\Pi_{\ker\delta_Q\cap\ker\delta_Q^J} = \Pi_{\ker\partial_Q^*\cap\ker\overline{\partial}_Q^*}$. This is convenient because the projections $\Pi_{\ker\partial_Q^*}$ and $\Pi_{\ker\overline{\partial}_Q^*}$ commute. To see this, we define

$$\Delta_{\partial_{Q_N}} = \partial_{Q_N}^*\partial_{Q_N} + \partial_{Q_N}\partial_{Q_N}^* \quad\text{and}\quad \Delta_{\overline{\partial}_{Q_N}} = \overline{\partial}_{Q_N}^*\overline{\partial}_{Q_N} + \overline{\partial}_{Q_N}\overline{\partial}_{Q_N}^*\,.$$

It follows from (7), (10) and (14) that when J is integrable and invariant under T, one has on Ω_0^pH for $p<n$,

$$\begin{cases}\partial_{Q_N}^*\overline{\partial}_{Q_N} + \overline{\partial}_{Q_N}\partial_{Q_N}^* = \partial_{Q_N}^2 = \partial_{Q_N}\overline{\partial}_{Q_N} + \overline{\partial}_{Q_N}\partial_{Q_N} = 0\\ \Delta_{\partial_{Q_N}} + \Delta_{\overline{\partial}_{Q_N}} = \Delta_{Q_N} \quad\text{and}\quad \Delta_{\overline{\partial}_{Q_N}} - \Delta_{\partial_{Q_N}} = i\mathcal{L}_T\end{cases} \quad (34)$$

Now we will need the following facts.

LEMMA 7.7. 1. *$\Delta_{\partial_{Q_N}}$ (resp. $\Delta_{\overline{\partial}_{Q_N}}$) is hypoelliptic on $\Omega_0^{p,q}H$ for $1\le p\le n-1$ (resp. $1\le q\le n-1$).*

2. *$\Pi_{\ker\partial_{Q_N}^*}$ and its conjugate $\Pi_{\ker\overline{\partial}_{Q_N}^*}$ are continuous from any $\mathcal{H}_{k,H}$ to themselves.*

3. *One has $\Pi_{\ker\delta_Q\cap\ker\delta_Q^J} = \Pi_{\ker\partial_{Q_N}^*}\Pi_{\ker\overline{\partial}_{Q_N}^*} = \Pi_{\ker\overline{\partial}_{Q_N}^*}\Pi_{\ker\partial_{Q_N}^*}$.*

Proof. To see the first point, we use (16) and above (34) to write on $\Omega_0^{p,q}H$ with $p+q<n$,

$$\begin{aligned}\Delta_H &= (n-p-q)\Delta_{Q_N} + d_{Q_N}\delta_{Q_N} + d_{Q_N}^J\delta_{Q_N}^J\\ &\le (n-p-q+2)\Delta_{Q_N} = (n-p-q+2)(2\Delta_{\partial_{Q_N}} + i\mathcal{L}_T)\,,\end{aligned}$$

so that by (12)

$$\begin{aligned}2(n-p-q+2)\Delta_{\partial_{Q_N}} &\ge \Delta_H - i(n-p-q-2)\mathcal{L}_T\\ &= \Delta_K + i(2p-n-2)\mathcal{L}_T + \text{first order}\,,\end{aligned}$$

which is an hypoelliptic Folland-Stein operator for $1 < p < n$. This estimation is too crude for $p = 1$ but still gives anyway that $\Delta_{\partial_{Q_N}}$ controls the $H^{1,0}$ derivatives because

$$\Delta_K - in\mathcal{L}_T \simeq 2\Delta_{\partial_b} \simeq 2\nabla_H^{1,0*}\nabla_H^{1,0}.$$

This in turn gives that $\exists C$ such that $\overline{\partial}_{Q_N}\overline{\partial}^*_{Q_N} \leq C\Delta_{\partial_{Q_N}}$ because $\overline{\partial}^*_Q = \overline{\partial}^*_H$ is an expression of these $H^{1,0}$ derivatives. Lastly we can now refine the use of (16) to obtain

$$\begin{aligned}\Delta_H &= (n-1-q)\Delta_{Q_N} + 2\partial_{Q_N}\partial^*_{Q_N} + 2\overline{\partial}_{Q_N}\overline{\partial}^*_{Q_N}\\ &\leq (n-1-q)(2\Delta_{\partial_{Q_N}} + i\mathcal{L}_T) + 2(1+C)\Delta_{\partial_{Q_N}},\end{aligned}$$

which this time leads to an hypoelliptic control.

The other statements of Lemma 7.7 are then direct consequences of this first point and (34). □

Remark 7.8. The operators $\Delta_{\partial_{Q_N}}$ are not hypoelliptic on the remaining spaces $\Omega_0^{0,p}H$ since their principal part vanishes (at least) on forms with anti-CR components (and in fact more since these are quotiented versions of the more classical ∂_H ones).

We can now come to the commutator technique.

Proposition 7.9. *Let P_{F_3} and $P_{F_3^\perp}$ be defined on $\Omega^n M = F_1^n \oplus \theta \wedge \Omega_0^{n-1}H \oplus F_1^{\perp,tame}$ by (the power of i is for the symmetry)*

$$P_{F_3} = i^{n-1}\begin{pmatrix}(-1)^{\frac{n(n+1)}{2}} * D & 0 & 0\\ 0 & \Pi_{F_{3,T}} J\Delta_{Q_N}\Pi_{F_{3,T}} & 0\\ 0 & 0 & 0\end{pmatrix}$$

and

$$P_{F_3^\perp} = \begin{pmatrix} d_Q\delta_Q & 0 & 0\\ 0 & \Delta_H - \Delta_{Q_N} & 0\\ 0 & 0 & \Delta_H - \Delta_{Q_N}\end{pmatrix}.$$

1. *One has, if J is integrable and invariant under the Reeb flow,*

$$\left[P_{F_3}, \tfrac{\Delta_\varepsilon}{\varepsilon^2}\right] = -\tfrac{K_\varepsilon^*}{\varepsilon}P_H^{(1)}\mathcal{L}_T\tfrac{\Pi_{\mathrm{Im}\, d_Q}}{\varepsilon} + \tfrac{\Pi_{\mathrm{Im}\, d_Q}}{\varepsilon}P_H^{(1)}\mathcal{L}_T\tfrac{K_\varepsilon^*}{\varepsilon},$$

   and

$$\begin{aligned}[P_{F_3^\perp}, \tfrac{\Delta_\varepsilon}{\varepsilon^2}] = -\tfrac{K_\varepsilon^*}{\varepsilon}\mathcal{L}_T\tfrac{\delta_Q}{\varepsilon}\Pi_{F_1^n} + \tfrac{\Pi_{F_1^{\perp,tame}}}{\varepsilon^2}P_H^{(1)}\mathcal{L}_T\tfrac{\Pi_{F_1^\perp}}{\varepsilon}\\ + \tfrac{\Pi_{F_1^\perp}}{\varepsilon}P_H^{(2)}\mathcal{L}_T\tfrac{\Pi_{F_1^\perp}}{\varepsilon} - \text{adjoints.}\end{aligned}$$

2. *The system* $(P_{F_3}, P_{F_3^\perp})$ *is hypoelliptic of order 2, which means that* $\forall k \in \mathbb{N}$ *and* $\alpha \in \Omega^n M$

$$\|P_{F_3}\alpha\|_{k,H} + \|P_{F_3^\perp}\alpha\|_{k,H} + \|\alpha\|_{k,H} \geq \|\alpha\|_{k+2,H} .$$

Proof. The first point stems from formulas for Δ_ε as given in Lemma 5.1 and (17). The case of P_{F_3} follows then from (33). About $P_{F_3^\perp}$, we have first of all

$$\left[P_{F_3^\perp}, \tfrac{d_Q\delta_Q}{\varepsilon^2}\Pi_{F_1^n}\right] = 0 = [P_{F_3^\perp}, D_T^2] ,$$

and by (28) and (29)

$$\begin{aligned}\left[P_{F_3^\perp}, \tfrac{K_\varepsilon^* K_\varepsilon}{\varepsilon^2}\right] &= [P_{F_3^\perp}, K_\varepsilon^*]\tfrac{K_\varepsilon}{\varepsilon^2} + \tfrac{K_\varepsilon}{\varepsilon^2}[P_{F_3^\perp}, K_\varepsilon] \\ &= -\tfrac{K_\varepsilon^*}{\varepsilon}\mathcal{L}_T\tfrac{\delta_Q}{\varepsilon}\Pi_{F_1^n} - \text{adjoint}.\end{aligned}$$

Also, thanks to (28), we get that on $F_1^\perp$

$$[P_{F_3^\perp}, \delta_Q] = P_H^{(1)}\mathcal{L}_T = [P_{F_3^\perp}, \delta_Q^J] ,$$

and finally

$$\left[P_{F_3^\perp}, \text{non-diagonal terms of } \tfrac{\Delta_\varepsilon}{\varepsilon^2} \text{ on } F_1^\perp\right] = \frac{F_1^{\perp,\text{tame}}}{\varepsilon^2} P_H^{(1)}\mathcal{L}_T\frac{\Pi_{F_1^\perp}}{\varepsilon} - \text{adjoint},$$

together with

$$\begin{aligned}&\left[P_{F_3^\perp}, \text{second order diagonal terms of } \tfrac{\Delta_\varepsilon}{\varepsilon^2} \text{ on } F_1^\perp\right] \\ &\qquad = \frac{\Pi_{F_1^\perp}}{\varepsilon}P_H^{(2)}\mathcal{L}_T\frac{\Pi_{F_1^\perp}}{\varepsilon} - \text{adjoint}.\end{aligned}$$

The second point is clear outside $\theta \wedge \Omega_0^{n-1}H$ because $(d_Q\delta_Q, *D)$ is hypoelliptic on F_1^n, and $\Delta_H - \Delta_Q$ is on $\Omega_0^p H$ for $p < n-1$ by (16). On $\theta \wedge \Omega_0^{n-1}H = F_{3,T} \oplus F_{3,T}^\perp$, one has

$$\|P_{F_3}\alpha\|_{k,H} = \|J\Delta_{Q_N}\Pi_{F_{3,T}}\alpha\|_{k,H} \geq \|\Pi_{F_{3,T}}\alpha\|_{k+2,H} - \|\alpha\|_{k,H} ,$$

by hypoellipticity of Δ_{Q_N} and continuity of $\Pi_{F_{3,T}}$ (Lemma 7.12). About the remaining $F_{3,T}^\perp$, it can be orthogonally decomposed, thanks to Lemma 7.7 and (34) in the following way

$$F_{3,T}^\perp = \partial_Q(\ker\partial_Q^* \cap \ker\overline{\partial}_Q^*) \oplus \overline{\partial}_Q(\ker\partial_Q^* \cap \ker\overline{\partial}_Q^*) \oplus (\operatorname{Im}\partial_Q \cap \operatorname{Im}\overline{\partial}_Q) ,$$

on which $P_{F_{3,T}^\perp} = d_{Q_N}\delta_{Q_N} + d_{Q_N}^J\delta_{Q_N}^J = 2(\partial_{Q_N}\partial_{Q_N}^* + \overline{\partial}_{Q_N}\overline{\partial}_{Q_N}^*)$ acts as

$$P_{F_{3,T}^\perp} = \begin{pmatrix} 2\Delta_{\partial_{Q_N}} & 0 & 0 \\ 0 & 2\Delta_{\overline{\partial}_{Q_N}} & 0 \\ 0 & 0 & 2\Delta_{Q_N}\end{pmatrix}$$

completing the hypoelliptic control. □

We can now state the refinement of Theorem 7.1 under the geometric assumptions which we have been leaded to.

Theorem 7.10. *Let M be a compact contact manifold endowed with the family of metrics g_ε, and such that the complex structure is integrable and invariant under the Reeb flow. Then one has uniformly for $t \geq t_0 > 0$,*

1. *on $\Omega^p M$ with $p = n$ or $n+1$,*
$$e^{-t\frac{\Delta_\varepsilon}{\varepsilon^2}} \xrightarrow[\varepsilon \to 0]{} e^{-t\Delta_D}\Pi_{F_3}$$
2. *for the signature operator S_ε acting on $\Omega^{\mathrm{even}}M$,*
$$\frac{S_\varepsilon}{\varepsilon} e^{-t\frac{\Delta_\varepsilon}{\varepsilon^2}} \xrightarrow[\varepsilon \to 0]{} S_D e^{-t\Delta_D} ,$$

smoothly at the kernel level.

Remark 7.11. It seems unlikely such strong geometric hypothesizes are really necessary to pass from the global statement, we already obtained in Theorem 7.1, to this local one. In index theory, some geometric price has sometimes to be paid to do this, but only for small time problems, which is not the case here!

The proof will follow the lines of the previous section and we refer to it for more details. We introduce higher order L^2 Sobolev spaces and extend the regularizing properties of the resolvent on them.

Lemma 7.12. *For $k \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha \in \Omega^n M$, let*
$$\|\alpha\|_{k,\varepsilon} = \|\alpha\|_{k,H} + \tfrac{1}{\varepsilon}\|\Pi_{F_3^\perp}\alpha\|_{k,H} + \tfrac{1}{\varepsilon}\|K_\varepsilon\alpha\|_{k-1,H} + \tfrac{1}{\varepsilon^2}\|\Pi_{F_1^{\perp,\mathrm{tame}}}\alpha\|_{k-1,H} .$$
Then $\big(\mathrm{Id} + \frac{\Delta_\varepsilon}{\varepsilon^2}\big)^{-1}$ extends continuously from $\mathcal{H}_{2k-1,H}$ to $\mathcal{H}_{2k+1,\varepsilon}$ with the uniform boundedness, when $\varepsilon \to 0$, of the following operators
$$\begin{cases} \big(\mathrm{Id} + \frac{\Delta_\varepsilon}{\varepsilon^2}\big)^{-1} & : \quad \mathcal{H}_{2k-1,H} \to \mathcal{H}_{2k+1,\varepsilon} \\ \big(\mathrm{Id} + \frac{\Delta_\varepsilon}{\varepsilon^2}\big)^{-1} \frac{\Pi_{F_3^\perp}}{\varepsilon} & : \quad \mathcal{H}_{2k-1,H} \to \mathcal{H}_{2k+1,\varepsilon} \\ \big(\mathrm{Id} + \frac{\Delta_\varepsilon}{\varepsilon^2}\big)^{-1} \frac{K_\varepsilon^*}{\varepsilon} & : \quad \mathcal{H}_{2k,H} \to \mathcal{H}_{2k+1,\varepsilon} \\ \big(\mathrm{Id} + \frac{\Delta_\varepsilon}{\varepsilon^2}\big)^{-1} \frac{\Pi_{F_1^{\perp,tame}}}{\varepsilon^2} & : \quad \mathcal{H}_{2k,H} \to \mathcal{H}_{2k+1,\varepsilon} \end{cases}$$

Proof. We can proceed by recurrence as in Lemma 7.4. The case $k = 0$ follows from the basic L^2 a priori control of Proposition 5.8. The recurrence then uses the commutator method applied with P_{F_3}, $P_{F_3^\perp}$ and $\mathcal{L}_T$, as Proposition 7.9 allows it. □

The proof of the first point of Theorem 7.10 is then completed in the same manner as in previous section. Lastly, the statement for the signature operator comes from the obtained uniform smoothing properties of $e^{-t\frac{\Delta_\varepsilon}{\varepsilon^2}}$ and the continuity of $S_\varepsilon/\varepsilon$ from L^2 to $\mathcal{H}_{-1',\varepsilon}$ as seen in Prop. 5.8.

7.4 On the large scale behavior of the heat kernels on forms on the Heisenberg groups. Although we have been concerned so far with compact contact manifolds, we now observe that the techniques developed can also been used in the following non-compact situation, namely, the study of the asymptotic behavior of the heat operators on forms on the Heisenberg groups for large times. The first thing to remark for this is that most of the convergence theorems obtained here for compact contact manifolds M also hold on their Galois coverings $\Gamma \to \widetilde{M} \to M$ (like their universal covering spaces with $\Gamma = \pi_1 M$).

PROPOSITION 7.13. 1. *The resolvent convergence Theorems 3.5 and 3.6 hold on $\widetilde{M}$ except that the collapsing of the F_2 component in 3.6, instead of being uniform in L^2, is only weak on $\widetilde{M}$ and strong on compact sets of $\widetilde{M}$.*

2. *The local convergence theorems of heat kernels 7.2 and 7.10 hold on compact sets of $\widetilde{M} \times \widetilde{M}$.*

Proof. The basic L^2 estimates of section 5 rely on the Bochner technique and still apply here. The only changes in the proofs in section 6 thus depend on Proposition 6.2. It is still true on $\widetilde{M}$, at the exception of the statement on the discreteness of the spectra.

Indeed, the only point to check here is the essential self-adjointness of the operator $P = \Delta_Q + P_D$ considered in the proof of Proposition 6.2. It can be obtained using the same general methods as described for elliptic operators by Atiyah in [A]. Namely, P being a maximally hypoelliptic operator of order 2, it satisfies basic estimates

$$\|P\alpha\|_0 + \|\alpha\|_0 \geq C\|\alpha\|_{2,H}$$

for compactly supported forms in a fixed compact set of $\widetilde{M}$. The fact is that, as in the elliptic case of [A], these estimates give a global one by Γ-translating and adding them using a Γ-invariant partition of unity.

Now, in the proofs of section 6, the discreteness of the spectrum of Δ_Q was only used to obtain the uniform control

$$\|\beta_\varepsilon\|^2_{1',\varepsilon} \geq \tfrac{1}{\varepsilon^2}(\Delta_Q\beta_\varepsilon, \beta_\varepsilon)_0 \geq \tfrac{C}{\varepsilon^2}\|\Pi_{F_2^\perp}\beta_\varepsilon\|_0^2 \,,$$

where $\beta_\varepsilon = \left(\lambda - \frac{P_\varepsilon}{\varepsilon}\right)^{-1}\alpha$. That gave the uniform collapsing of $\Pi_{F_2^\perp}$ component in Theorem 3.6. Instead of this, we only get here that

$$(\Delta_Q\beta_\varepsilon, \beta_\varepsilon)_0 \leq \varepsilon^2\|\beta_\varepsilon\|^2_{1',\varepsilon} \leq \varepsilon^2\|\alpha\|_0^2 \,.$$

Therefore, if we decompose $\Pi_{F_2^\perp}\beta_\varepsilon$ with respect to the spectral spaces $E(]0,\varepsilon])$ and $E(]\varepsilon,\infty[)$ associated to Δ_Q, we obtain that,

$$\|\Pi_{E(]\varepsilon,\infty[)}\beta_\varepsilon\|_0^2 \leq \tfrac{1}{\varepsilon}(\Delta_Q\beta_\varepsilon, \beta_\varepsilon)_0 \leq \varepsilon\|\alpha\|_0^2 \,.$$

So, this component is still uniformly collapsing, whereas we only have that

$$(\Pi_{E(]0,\varepsilon])}\beta_\varepsilon,\gamma)_0 = (\beta_\varepsilon,\Pi_{E(]0,\varepsilon])}\gamma)_0 \to 0\,,$$

for all $\gamma \in L^2$ by the spectral theorem. Anyway, we conserve a uniform first order Sobolev control of β_ε through its $(1',\varepsilon)$ norm, and this still implies its strong convergence to 0, on compact sets. This completes the proof of the first point of the proposition. The remaining one is clear since the methods of section 7.2 and 7.3 still apply here, again in restriction to compact sets, to take profit of the Sobolev controls of the kernels. □

We now come to the Heisenberg group H^{2n+1}. We recall that its Lie algebra $\mathfrak{h}^{2n+1}$ is generated by X_i, Y_i, $1 \leq i \leq n$ and T in the center, with the commutation relations $[Y_i, X_j] = \delta_{ij}T$. We endowed H^{2n+1} with a left invariant adapted metric $g_1 = d\theta(\cdot, J\,\cdot\,) + \theta^2$ with θ dual to T and $JX_i = Y_i$. It is well known that these groups admit compact quotients $M = H^{2n+1}/\Gamma$ (which are non-trivial circle bundles over a torus), thus the previous proposition will apply to them. The link with our large scale Riemannian problem is provided by the parabolic dilations h_ε. These are automorphisms of H^{2n+1} acting on $\mathfrak{h}^{2n+1}$ by $h_\varepsilon = \varepsilon\,\mathrm{Id}$ on $H = \mathrm{Vect}\,(X_i,\,Y_i)$ and $h_\varepsilon T = \varepsilon^2 T$. Moreover, h_ε induces an isometry between the spaces (H^{2n+1}, g_1) and $(H^{2n+1}, g_\varepsilon/\varepsilon^2)$, so that

$$h_\varepsilon^*\Delta_{g_1} = \Delta_{g_\varepsilon/\varepsilon^2} = \varepsilon^2\Delta_{g_\varepsilon}\,.$$

We recall also that, since section 2, we work with the conjugate operator $\Delta_\varepsilon = C_\varepsilon\Delta_{g_\varepsilon}C_\varepsilon^{-1}$, where C_ε is the point-wise map defined on Ω^*M by $C_\varepsilon(\alpha_H + \theta\wedge\alpha_T) = \alpha_H + \varepsilon\theta\wedge\alpha_T$. Lastly, if P is an operator on Ω^*H^{2n+1}, the kernel of h_ε^*P transforms like

$$K_{h_\varepsilon^*P}(x,y) = \varepsilon^{-2n-2}C_\varepsilon^{-1}K_P(h_\varepsilon^{-1}x, h_\varepsilon^{-1}y)C_\varepsilon\,,$$

when expressed in a fixed volume form, and with tangent spaces identified through translations. Thus, by Proposition 7.13, we finally obtain the asymptotic behavior at large time of the heat kernels on forms on H^{2n+1}.

Theorem 7.14. *One has on Ω^pH^{2n+1} for $p \neq n, n+1$*

$$\varepsilon^{-2n-2}K_{e^{-\frac{t}{\varepsilon^2}\Delta}}(h_\varepsilon^{-1}x, h_\varepsilon^{-1}y) \xrightarrow[\varepsilon\to 0]{} K_{e^{-t\Delta_Q}}(x,y)\,,$$

and on $\Omega^nH^{2n+1}\oplus\Omega^{n+1}H^{2n+1}$

$$\varepsilon^{-2n-2}K_{e^{-\frac{t}{\varepsilon^4}\Delta}}(h_\varepsilon^{-1}x, h_\varepsilon^{-1}y) \xrightarrow[\varepsilon\to 0]{} K_{e^{-t\Delta_D}}(x,y)\,,$$

uniformly on compact sets of $H^{2n+1}\times H^{2n+1}$ and non-small t.

Thus, heat on forms “at infinity” of the Heisenberg groups converges to the contact complex ones, its higher order degeneracy as a spectral sequence, in middle degrees, being directly reflected in the lower speed of diffusion there. This is of course consistent with a time-dilation rescaling, the contact Laplacians being homogeneous under h_λ. Namely, one has

$$h_\lambda^* \Delta_{Q,g_1} = \Delta_{Q,\frac{g_\lambda}{\lambda^2}} = \lambda^2 \Delta_{Q,g_1} \quad \text{and} \quad h_\lambda^* \Delta_{D,g_1} = \lambda^4 \Delta_{D,g_1}\,,$$

so that,

$$h_\lambda^*(e^{-t\Delta_Q}) = e^{-t\lambda^2 \Delta_Q} \quad \text{and} \quad h_\lambda^*(e^{-t\Delta_D}) = e^{-t\lambda^4 \Delta_D}.$$

We now mention that this result gives, using von Neumann's Γ-dimension, some numerical information on the spectrum of Δ near 0. Recall (see [A]) that associated to the Γ-cocompact action are the notions of Γ-trace of operators and the related Γ-dimension of closed Γ-invariant spaces. Briefly, if P is a positive Γ-equivariant Hermitian operator acting on some Hilbert space H which is a free Γ-module, its Γ-trace is defined by $\mathrm{Tr}_\Gamma(P) = \sum_{i\in I}(Pe_i, e_i)$ where one has chosen one vector e_i by Γ-orbit of a Γ-equivariant Hilbert base of H. If V is a closed Γ-invariant subspace in H, its Γ-dimension is $\dim_\Gamma(V) = \mathrm{Tr}_\Gamma(\Pi_V)$, where Π_V is the orthogonal projection on V. Lastly, in the case we consider where P acts on $\Omega^* M$ with smooth kernel one has $\mathrm{Tr}_\Gamma(P) = \int_{\mathcal{F}} \mathrm{Tr}(K_P(x,x))d\,\mathrm{vol}$, $\mathcal{F}$ being a fundamental domain of the Γ-action. Moreover, since our operators are left invariant on H^{2n+1}, this formula simplifies further to $\mathrm{Tr}_\Gamma(P) = \mathrm{Tr}(K(e,e))\mathrm{vol}(\mathcal{F})$. Thus, the previous theorem leads to the following

Corollary 7.15. *For Δ acting on $\Omega^p H^{2n+1}$, let $\theta_\Delta(t) = \mathrm{Tr}_\Gamma(e^{-t\Delta})$ (this is also the Laplace transform of the spectrum distribution function of Δ, $\theta_\Delta(t) = \int e^{-t\lambda} dN_\Gamma(\lambda)$ with $N_\Gamma(\lambda) = \dim_\Gamma(E_\Delta([0,\lambda]))$ (see [GroS])). One has, when $\varepsilon \to 0$*

$$\theta_\Delta(\varepsilon^{-2}) \sim \begin{cases} \varepsilon^{2n+2}\theta_{\Delta_Q}(1) & \text{if } p \neq n, n+1 \\ \varepsilon^{n+1}\theta_{\Delta_D}(1) & \text{otherwise.} \end{cases}$$

Therefore, the decay exponent of heat on p-forms is $\alpha_p = n+1$ if $p \neq n, n+1$ and $\frac{n+1}{2}$ otherwise.

These numbers α_p (or sometimes twice these numbers) are called the Novikov-Shubin numbers, of $M = H^{2n+1}/\Gamma$ here, and have some geometric interest since Gromov and Shubin proved in [GroS] their homotopical invariance (see also [BlMW]). Lott in [L], computed them on H^3 using representation theory, and also gave (sharp) bounds on the higher dimensional Heisenberg groups. Recently, and independently of this work, Schubert announced in [S] that he found them in all dimensions, using again representation theory and algebraic methods (however, it seems there is a gap in this proof). The proof we gave here has some geometric flavour. But still, as homotopical invariants, they deserve an "homotopical" approach.

A direct one is indeed possible. (I am grateful to P. Pansu for his unyielding and finally communicative conviction about this).

Another computation of α_p. Recall that in sections 2 and 3 we observed that the construction of the contact complex, and in fact the proof of its local exactness in [Ru], come with explicit homotopy operators, the lifting maps r in (18), towards the de Rham complex. They are (first order) operators from $\Omega_0^p H$ to $\Omega^p M$ satisfying $dr = rd_Q$ for $p < n$ and $dr = D$ for $p = n$. Above degree n, the contact complex is naturally embedded as a subcomplex of de Rham's one, so we can extend r by the inclusion map here. Reversely, we have a natural projection Π from $\Omega^* M$ to $\Omega^* M/\mathcal{I} \simeq \Omega_0^* M$ until degree n and satisfying (by definition) $\Pi d = d_Q \Pi$. Again, this map Π has an extension from $\Omega^* M$ to $\mathcal{J}^*$ in higher degrees. To define this, let $L_R^{-1} = \Lambda(L\Lambda)^{-1}$ be the right inverse of L (exists since L is surjective in the degrees we consider), and define $h = \theta \wedge L_R^{-1} \Pi_H$ (that is $h = \left(\begin{smallmatrix} 0 & 0 \\ L_R^{-1} & 0 \end{smallmatrix}\right)$ in $\Omega^* M = \Omega^* H \oplus \theta \wedge \Omega^* H$. Note also that in some sense h is d_0^{-1}, the inverse of the zero order term of d by Prop. 3.3). It is easy to check that $\Pi = \mathrm{Id} - hd - dh$ maps $\Omega^* M$ to $J^* = \theta \wedge \ker L$. In fact Π and r induce an homotopy equivalence between the de Rham and contact complexes since they are seen to satisfy $\Pi r = \mathrm{Id}$ and $r\Pi = \mathrm{Id} - hd - dh$ (where one uses the left, instead of right, inverse of L in degree $\leq n$).

Now, one would like to use the fundamental result of Gromov and Shubin in [GroS] that boundedly homotopy equivalent Hilbert complexes have the same Novikov-Shubin numbers since, as we already observed, the ones of the contact complex are clear on the Heisenberg group by its homogeneity through the parabolic dilations. Our problem here is that the first order maps Π and r are unbounded in L^2, but it is bypassed if, instead of working with the full de Rham complex, one restricts or projects it to some spectral space $E([0, \lambda])$ associated to Δ (and the same on the contact complex). Indeed, the projection $\Pi_{E([0,\lambda])}$ is a regularizing map and induces a bounded homotopy equivalence between the de Rham and cut-off de Rham complexes, because $\mathrm{Id} - \Pi_{E([0,\lambda])} = \Pi_{E(]\lambda,\infty[)} = dH + Hd$ where $H = \delta \Delta^{-1} \Pi_{E(]\lambda,\infty[)}$ is bounded. □

Remark 7.16. We would like to mention, still about these α_p, that this work also suggests a third natural approach, relying only on the Bochner formula and related basic estimates of section 4, together with the following variational principle to estimate the distribution function of Laplacians.

Namely, one has (see [GroS]),

$$N_\Gamma(\lambda) \quad (= \dim_\Gamma(E_\Delta([0,\lambda]))) = \sup_{L\in\mathcal{L}_\lambda} \dim_\Gamma(L)\,,$$

where $\mathcal{L}_\lambda$ is the set of all Γ-invariant closed spaces such that $(\Delta\alpha,\alpha) \leq \lambda\|\alpha\|^2$ for all $\alpha \in L$. One can check (left to the tireless reader) that the injection in the Bochner formula of projection and lifting of basic relevant spectral spaces leads rather easily to another computation of the α_p, and even, outside middle degrees, to the full equivalent

$$N_{\Gamma,\Delta}(\lambda) \underset{\lambda\to 0}{\sim} N_{\Gamma,\Delta_Q}(\lambda) = \lambda^{n+1} N_{\Gamma,\Delta_Q}(1)\,.$$

As a conclusion of this work, we think that it can suggest two directions of investigations. The first question raised, mimicking a famous one, is "What do we hear at the infinite of the nilpotent groups?" We mean by this to study the Laplacians and the limit of heat and wave propagation on forms at large scale on these groups. As we have just seen here, the answer to this question for the Heisenberg group is more or less "the contact complex". A clue for more general groups is certainly provided by the existence, we mentioned in section 3, of a spectral sequence naturally generalizing the contact complex in Carnot-Caratheodory geometry.

The other direction is to study further the behavior, in the contact case, of global spectral Riemannian invariants (like eta invariant or analytic torsion) in the limit we considered. Although the short time problems are out of scope of this work, we have seen that some hypoelliptic eta and zeta functions naturally come out that should be related to the finite part of their Riemannian counterparts in the sub-Riemannian limit. What we could really expect in this problem is still quite unclear to us, but an efficient analytic tool to investigate it could probably be the extended Heisenberg pseudo-differential calculus, recently developed by Epstein, Mendoza and Melrose in [EMM], and containing both the elliptic and hypoelliptic Heisenberg calculi (see [BG]).

References

[A] M.F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Astérisque, 32-33 (1976), 43–72.

[BG] R. Beals, P. Greiner, Calculus on Heisenberg Manifolds, Annals of Math. Studies 119, Princeton University Press, Princeton, NJ, 1988.

[BeB] A. Berthomieu, J.-M. Bismut, Quillen metrics and higher analytic torsion form, J. für die Reine und Angew. Math., Band 457 (1994), 85–184.

[BiL] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Pub. Math. IHES 74 (1991), 1–298.

[BlMW] J. Block, V. Mathai, S. Weinberger, Homotopy invariance of Novikov-Shubin invariants and L^2 Betti numbers, Proceedings of the AMS 125:12 (1997), 3757–3762.

[D] X. Dai, Adiabatic limits, non-multiplicativity of signature and Leray spectral sequence, J.A.M.S. 4 (1991), 265–321.

[EMM] C.L. Epstein, R.B. Melrose, G. Mendoza, Homology and the Heisenberg algebra, in preparation.

[G] Z. Ge, Collapsing Riemannian metrics to Carnot-Caratheodory metrics and Laplacians to sub-Laplacians, Canadian Journal of Math. 45 (1993), 537–553.

[GrH] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, 1978.

[Gro] M. Gromov, Carnot-Caratheodory spaces seen from within, in "Sub-Riemannian Geometry" (A. Bellaïche, J.-J. Risler, eds.), Birkhäuser Progress in Math. 144 (1996), 79–323.

[GroS] M. Gromov, M.A. Shubin, Von Neumann spectra near zero, Geometric and Functional Analysis 1:4 (1991), 375–404.

[J] P. Julg, Complexe de Rumin, suite spectrale de Forman et cohomologie L^2 des espaces symétriques de rang 1, C.R.Acad. Sci. Paris 320 (1995), 451–456.

[L] J. Lott, Heat kernels on covering spaces and topological invariants, J. of Diff. Geo. 35 (1992), 471–510.

[MM] R. Mazzeo, R.B. Melrose, The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration, J. of Diff. Geo. 31 (1990), 185–213.

[RS] M. Reed, B. Simon, Methods of Modern Mathematical Physics, volume 1, Functional Analysis, Academic Press, 1972.

[Ru] M. Rumin, Formes différentielles sur les variétés de contact, J. of Diff. Geo. 39 (1994), 281–330.

[S] L. Schubert, The Laplacian on p-forms on the Heisenberg group, preprint, University of Adelaïde, Australia. Math DG/9807148.

[W] A. Weil, Introduction à l'étude des variétés kählériennes, Hermann Paris, 1958.

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