TOPOLOGICAL AND DYNAMICAL ASPECTS OF SOME SPECTRAL INVARIANTS OF CONTACT MANIFOLDS WITH CIRCLE ACTION

MICHEL RUMIN

Abstract. We study analytic torsion and eta like invariants on CR contact manifolds of any dimension admitting a transverse circle action, and equipped with a unitary representation. We show that, when defined using the spectrum of relevant operators arising in this geometry, the spectral series involved can been interpreted in their whole, both from a topological viewpoint, and as purely dynamical functions of the Reeb flow.

Tous les chemins mènent à Rome.

1. Introduction

This paper deals with the study of some spectral series associated to geometric invariants on particular compact contact CR manifolds M. Those who admit a transverse locally free circle action.

That means the generator T of this action is the Reeb field of an invariant contact form θ and preserves an integrable complex structure J on $H = \ker\theta$. In that case, the orbifold $N = M/\mathbb{S}^1$ appears to be a Kähler V-manifold in the sense of Satake [12]. It is a topological space endowed with a smooth open dense Kähler structure, corresponding to the generically free orbit, and a finite number of singular conical points, corresponding to the exceptional fibers.

Independently of this circle action, we also equip M with a unitary representation $\rho : \pi_1(M) \to U(d)$. This broadens the framework to twisted spectral invariants associated to the flat bundles of these representations, and provides us with dynamical data using the holonomies induced by the closed orbits of the Reeb flow.

1.1. Around the contact analytic torsion.

We will be concerned with spectral series associated to two typical spectral invariants. The first one is the 'contact' analytic torsion, as defined in [19]. This analytic determinant is associated to the contact de Rham complex $(\mathcal{E}^*, d_Q)$, a hypoelliptic complex, homotopic to the usual Hodge-de Rham one, but benefiting from better contact homogeneity when rescaling θ in $k\theta$. See Section 2.1 for a presentation of this construction. This resolution of constants starts on functions with $d_Q = d_H$, the usual differential, but restricted to the horizontal vectors H in the contact distribution. The price for this however, is the appearance of a second order differential $D = d_Q : \mathcal{E}^n \to \mathcal{E}^{n+1}$ in 'middle degree', with $\dim N = 2n + 1$. In order to

Date: October 22, 2024.

2020 *Mathematics Subject Classification.* 58J52, 58J28, 32V05, 32V20, 11M36, 37C30.

Key words and phrases. analytic torsion, eta invariant, contact complex, Sasakian manifold, trace formula.

preserve homogeneity, this in turns leads to using *fourth-order* Laplacians $\mathbf{\Delta}_Q$ in all degrees; see Section 2.3. These self-adjoint operators are hypoelliptic. They possess discrete spectrum and smooth heat kernels on compact contact manifolds.

This allows to consider our first spectral series which is related to the analytic torsion of the contact complex. In the Riemannian setting, the analytic torsion was introduced by Ray and Singer in [16] as an infinite dimensional analogue of the Reidemeister-Franz torsion of a finite dimensional complexes. It is defined by an appropriate combination of analytic determinants of the Hodge-de Rham Laplacians using their zeta functions.

In [19], the authors proposed to adapt the construction on contact manifolds. Starting from heat kernels, one considers for $t > 0$

$$\vartheta(t) = \sum_{k=0}^{n} (-1)^k (n+1-k) \operatorname{Tr}(e^{-t\mathbf{\Delta}_Q} \mid \mathcal{E}^k). \tag{1}$$

This particular combination leads to the definition of the analytic torsion of the contact complex. Briefly, taking Mellin transform leads to zeta functions

$$\zeta(\mathbf{\Delta}_Q)(s) = \operatorname{Tr}^*(\mathbf{\Delta}_Q^{-s}) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \operatorname{Tr}^*(e^{-t\mathbf{\Delta}_Q}) t^{s-1} dt\,,$$

where Tr^* denotes the trace over the non zero spectrum of $\mathbf{\Delta}_Q$. These functions are well defined for $\operatorname{Re}(s)$ large and meromorphic with (at worst) simple poles occurring at $s \in S = \{\frac{n+1-j}{2} \mid j \in \mathbb{N}\} \setminus (-\mathbb{N})$; see e.g. [19, Section 3.1] for references. Following [19], we define then the contact torsion zeta function

$$Z(s) = \sum_{k=0}^{n} (-1)^k (n+1-k) \zeta(\mathbf{\Delta}_Q)(s)\,. \tag{2}$$

Then, the analytic torsion of the contact complex is defined by

$$T_Q(M, \rho) = \exp\bigl(-\frac{1}{2} Z'(0)\bigr)\,. \tag{3}$$

It is shown in [19] that it coincides with Ray-Singer analytic torsion on three dimensional CR Seifert manifolds. An explicit formula is given in this case. On general contact manifolds, Albin and Quan proved in [1] that the Riemannian and contact analytic torsions differ by integral of (unknown) local terms.

Our first main results relate this ϑ series to three other expressions, one using topological data, another to an explicit geometric sum, and the last one to dynamical properties of the Reeb flow.

1.2. **The heat analytic torsion as an index series.**

We summarise the main steps toward the topological expression. As we shall see, it turns out that the spectrum in the combination of trace in ϑ is highly symmetric. Much of the plus and minus contributions cancel each other out, except on a simple residual spectrum we describe.

Let $\Omega^* H$ denotes the bundle of horizontal forms on M with coefficients in V, the flat bundle associated to the representation $\rho : \pi_1(M) \to U(d)$ and consider the horizontal part of the differential, d_H, acting on $\Omega^* H$ and the operator

$$D_H = d_H + \delta_H \,.$$

It exchanges $\Omega^{ev} H$ and $\Omega^{odd} H$. Let $\mathcal{H} = \ker D_H$. We will show that

$$\vartheta(t) = \mathrm{Tr}(e^{tT^2} \mid \mathcal{H}^{ev}) - \mathrm{Tr}(e^{tT^2} \mid \mathcal{H}^{odd}).$$

As we shall see in Section 3.1, the space $\mathcal{H}$ is infinite dimensional, and contains forms build using CR (holomorphic) functions and their conjugate. We can however split it into finite dimensional peaces using the spectrum of the Reeb flow T. Fourier decomposition along the circle orbits, gives

$$V = \bigoplus_{\lambda \in \mathrm{Spec}(iT)} V_\lambda \,, \tag{4}$$

where each component V_λ can be seen as a V-bundle over the orbifold $N = M/\mathbb{S}^1$. Then the spectral ϑ finally reduces to a renormalised index series

$$\vartheta(t) = \vartheta^{top}(t) = \sum_{\lambda \in \mathrm{Spec}(iT)} \mathrm{ind}(D_H^{ev} \mid V_\lambda) e^{-t\lambda^2} \,, \tag{5}$$

with $D_H^{ev} = D_H : \Omega^{ev} H \to \Omega^{odd} H$. The index terms can be explicitly computed using Kawasaki's index formula for V-manifold; see Section 3.2. This will link ϑ to two other expressions, one using explicit geometric data over N and the other as a dynamical series over all closed orbits.

1.3. Geometric and dynamical viewpoints on the heat analytic torsion.

We now turn to the geometric and dynamical aspects of the series ϑ. We first would like to express it using data over the orbifold N. In our case, its smooth part corresponds to the generic closed primitive orbits f, and its finite set of singular conical points are associated to exceptional orbits f_i of order α_i.

Recall that from the unitary representation $\rho : \pi_1(M) \to U(d)$, each $\gamma \in \pi_1(M)$ induces an holonomy map $\rho(\gamma)$ and a character value

$$\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma)) \,.$$

For any closed orbit we shall also need its algebraic length $\ell(\gamma) = \int_\gamma \theta$. Note that since $i_T d\theta = 0$, this length is always constant through orbit deformation in contact geometry.

At last, the geometric expression will also relies on some weight on the character. For $\gamma = f$ or f_i, we define $V_\gamma^x = \ker(\rho(\gamma) - e^{2i\pi x}\,\mathrm{id})$ and

$$\chi_\rho^\theta(\gamma)(t) = \sum_{e^{2i\pi x} \in \mathrm{Spec}\,\rho(\gamma)} \dim V_\gamma^x \, \theta(x, 4\pi^2 t/\ell(\gamma)^2) \tag{6}$$

where

$$\theta(x,t) = \sum_{n \in \mathbb{Z}} e^{-t(n+x)^2}$$

is a Jacobi theta function. As we shall see, this weighted character is related to the average of the holonomies of closed random loops on the circle γ.

Starting from the topological series (5), we will show the following explicit geometric expression

$$\vartheta = \vartheta^{geo} = \chi(N^*)\chi_\rho^\theta(f) + \sum_i \chi_\rho^\theta(f_i)\,, \tag{7}$$

where $\chi(N^*)$ is the Euler characteristic of the punctured manifold $N^* = N \setminus \cup_i\{p_i\}$.

This will eventually lead to a Selberg-type trace formula giving an expression of ϑ using data over the free homotopy classes γ of *all* the closed orbits of the Reeb flow together with their inverse, meaning the closed orbits of the reverse flow. These are powers of the primitive closed orbits f and f_i. For these classes let

$$e(\gamma) = \begin{cases} \ell(f)\chi(N) & \text{if } \gamma \sim f^n \quad n \in \mathbb{Z}\,, \\ \ell(f_i) & \text{if } \gamma \sim f_i^n \quad n \not\equiv 0 \mod \alpha_i\,, \end{cases}$$

where $\chi(N) = \chi(N^*) + \sum_i \frac{1}{\alpha_i}$ is the rational Euler class of the orbifold $N = M/\mathbb{S}^1$.

Generalising a result obtained in dimension 3 in [19], we shall prove that

$$\vartheta(t) = \vartheta^{dyn}(t) = \frac{1}{\sqrt{4\pi t}} \sum_\gamma \chi_\rho(\gamma) e(\gamma) e^{-\ell(\gamma)^2/4t}\,. \tag{8}$$

where the sum ranges over the homotopy classes of closed orbits of the flow, together with their inverses and the constant loop. All trace formulae given here are invariant under the rescaling $\theta \mapsto k\theta$ and $t \mapsto k^2 t$, which is specific to the use of the contact de Rham complex instead of the usual Riemannian one. It holds without particular assumption on the curvature or symmetry of the Kähler orbifold N.

As we shall see in Section 3.5, these identities have counterparts using zeta type spectral functions instead of heat ones. They will lead in Corollary 3.7 to explicit Lefschetz type and dynamical formulae for the analytic contact torsion.

We now turn to the second spectral series we will be concerned in.

1.4. Around the eta invariant in the contact setting.

In Riemannian geometry of dimension $4k-1$, the eta invariant is defined using the odd signature operator

$$S = (-1)^k (*d + d*) w$$

acting (for convenience here) on odd forms, with $*$ the Hodge star and $w = (-1)^p$ on $\Omega^{2p-1}M$; see [4, p. 63]. This operator is self-adjoint and $S^2 = \Delta$ is Hodge–de Rham Laplacian. The eta invariant is given by the value at $s = 0$ of the meromorphic function

$$\eta(S)(s) = \operatorname{Tr}(S|S|^{-2s-1}) = \frac{1}{\Gamma(s+1/2)} \int_0^{+\infty} \operatorname{Tr}(\sqrt{t} S e^{-t\Delta}) t^{s-1} dt.$$

As S maps $\Omega^{2p-1}M$ to $\Omega^{2(2k-p)-1}M \oplus \Omega^{2(2k-p+1)-1}M$, only forms in 'middle degree' $\Omega^{2k-1}M$ contribute to the trace in $\eta(S)$, so that

$$\eta(S)(s) = \frac{1}{\Gamma(s+1/2)} \int_0^{+\infty} \operatorname{Tr}(\sqrt{t}(*d) e^{-t\Delta} \mid \Omega^{2k-1}M) t^{s-1} dt\,.$$

Now, it has been shown that $\eta(S)(0)$ is related to the eta invariant of its contact second order counterpart $*D$ acting on $\mathcal{E}^{2k-1}$; see [6] for the three dimensional Seifert case and Albin-Quan's more recent work [1, §6] on general contact manifolds. The difference is given by the integral of (unknown) universal curvature polynomial. Although it captures the eta invariant, the operator $*D$ itself has not good analytic properties, due to its infinite dimensional kernel. It is not hypoelliptic. It needs to be completed by some extra term like in the signature Riemannian operator S. Possible choices could be $P = *D \pm d_Q\delta_Q$ on $\mathcal{E}^{2k-1}$. One has $P^2 = \mathbf{\Delta}_Q$ and $\eta(P) = \eta(*D) \pm \zeta(d_Q\delta_Q)$, where the zeta series of the positive operator $d_Q\delta_Q$ contributes to an alternating sum of cohomological dimensions up to some local term at $s = 0$. In higher dimension however, these choices of 'extensions' of $*D$ don't seem to be the most natural ones in terms of spectral symmetry.

We will consider instead the operator defined by

$$S_Q = \begin{cases} *D + (d_Q + \delta_Q)\sigma\delta_Q & \text{on } \mathcal{E}^{2k-1} \\ (d_Q + \delta_Q)\sigma(d_Q + \delta_Q) & \text{on } \bigoplus_{1\le p\le k-1}\mathcal{E}^{2k-1-2p} \end{cases} \tag{9}$$

where $\sigma = (-1)^p J$ on $\mathcal{E}^{2p}$ and $J = i^{a-b}$ on forms $\mathcal{E}^{a,b}$ of bidegre (a, b) with respect to the complex structure. We shall see in Proposition 4.2 that when the CR structure has a transverse symmetry, $S_Q^2 = \mathbf{\Delta_Q}$ and still

$$\eta(S_Q) = \eta(*D) + \sum \pm\, \zeta(\Delta_Q)$$

leading again to adding cohomological dimensions and local terms at $s = 0$. The advantage of this choice of signature operator lies in its extra symmetry with respect to $\sigma = (-1)^p J$ on $\mathcal{E}^{2p-1}$. It splits into

$$S_Q = \sigma T + P$$

here $\sigma P = -P\sigma$ while $\sigma T = T\sigma$ and $TP = PT$, so that the spectrum of S_Q is symmetric except on (an infinite dimensional space) $\ker P = \mathcal{H}_S$ on which $S_Q = \sigma T$.

1.5. **The contact eta trace as topological and dynamical series.**

As in the previous case of the analytic torsion, the spectral series involved in $\eta(S_Q)$ have both closed topological and dynamical expression. Let

$$\vartheta_S(t) = \mathrm{Tr}(\sqrt{t}S_Q e^{-t\mathbf{\Delta}_Q})\,. \tag{10}$$

The domain of S_Q

$$\mathcal{E}_S = \bigoplus_{1\le p\le k} \mathcal{E}^{2p-1}$$

splits into $\mathcal{E}_S^+ \oplus \mathcal{E}_S^-$ with respect to the involution $\tau = i\sigma$, as does $\mathcal{H}_S = \mathcal{H}_S^+ \oplus \mathcal{H}_S^-$. The operator P exchanges this splitting and we set $P^+ = P : \mathcal{E}_S^+ \to \mathcal{E}_S^-$. By the previous discussion $\vartheta_S(t)$ reduces on $\mathcal{H}_S$ and we have that

$$\vartheta_S(t) = -\,\mathrm{Tr}(i\sqrt{t}Te^{tT^2} \mid \mathcal{H}_S^+) + \mathrm{Tr}(i\sqrt{t}Te^{tT^2} \mid \mathcal{H}_S^-)\,.$$

Using the same splitting of V through the circle action as in (4), we will finally get

$$\vartheta_S(t) = \vartheta_S^{top}(t) = -\sqrt{t}\sum_{\lambda\in\mathrm{Spec}(iT)} \mathrm{ind}(P^+ \mid V_\lambda)\lambda e^{-t\lambda^2}\,,$$

where, as we shall see in Section 4.1, the index there is the signature of the (non flat) bundle V_λ over the orbifold $N = M/\mathbb{S}^1$. This is the first interpretation of ϑ_S as an index series. This leads to an explicit geometric expression using Kawasaki's index formula.

We now turn to the link with dynamical data. The objective is to single out the contribution of each closed orbit like in the analytic torsion case (8). Let

$$\mathbf{c} = c_1(L) = -\frac{d\theta}{2\pi}$$

be the first Chern class of M seen as the circle bundle of a complex line bundle over $N = M/\mathbf{S}^1$. Let $\mathcal{L}(N)$ be Hirzebruch L-genus of the (smooth part of the) orbit space N.

We shall see in Section 4.2 that

$$\vartheta_S(t) = \vartheta_S^{dyn}(t) = \frac{1}{\sqrt{4\pi}} \sum_\gamma \chi_\rho(\gamma)\sigma(\gamma)(t),$$

where powers of the generic orbit $\gamma = f^n$ contribute to

$$\sigma(\gamma)(t) = \frac{i\ell(f)}{2t} \left\langle (\ell(\gamma) + i\mathbf{c}) e^{-(\ell(\gamma)+i\mathbf{c})^2/4t} \wedge \mathcal{L}(N), [N_{smooth}] \right\rangle,$$

while powers of the singular orbits $\gamma = f_i^k$ with $k \not\equiv 0 \mod \alpha_i$ contribute to

$$\sigma(\gamma)(t) = \frac{i\ell(f_i)}{2t} \ell(\gamma) e^{-\ell(\gamma)^2/4t} \nu(\gamma)$$

with

$$\nu(\gamma) = i(-1)^k \prod_{j=1}^{2k-1} \cot(\theta_j/2)\,.$$

Here θ_j are the angles of the action of γ on the horizontal space H. Following Atiyah-Bott's work [3], $\nu(\gamma)$ arises in the Lefschetz fixed point formula for the signature operator.

This will eventually lead to explicit expressions of the twisted eta invariant $\eta(S_Q)(0)$ in terms of these topological and dynamical data; see Sections 4.3 and 4.4.

2. Review of basic constructions and miscellaneous formulae

To make the paper as self contained as possible we will start by discussing the contact de Rham complex because it plays an important role here. We will also review miscellaneous formulae around it. Much of this material can be found in other places; see e.g. [17, 18, 19]

Let M be a smooth manifold of dimension $2n+1$. A $2n$-dimensional sub-bundle $H \subset TM$ is a *contact distribution* if a 1-form θ such that $H = \ker\theta$ satisfies the non integrability condition $\theta \wedge d\theta^n \neq 0$. Such a form is called a *contact form*. Associated to a choice of θ is the transverse *Reeb field* T; it is the unique vector field satisfying $\theta(T) = 1$ and $\mathcal{L}_T\theta = i_T d\theta = 0$, where $\mathcal{L}_T$ is Lie derivative along T.

The exterior algebra of M splits into horizontal and vertical forms

$$\Omega^* M = \Omega^* H \oplus \theta \wedge \Omega^* H$$

where $\Omega^* H$ are forms vanishing on T. The exterior differential d on $\Omega^* M$ writes then

$$d(\alpha_H + \theta \wedge \alpha_T) = (d_H\alpha_H + d\theta \wedge \alpha_T) + \theta \wedge (T\alpha_H - d_H\alpha_T)$$

using the notation $T = \mathcal{L}_T$ on forms, that is in matrix form

$$d = \begin{pmatrix} d_H & L \\ T & -d_H \end{pmatrix}, \tag{11}$$

where $d_H = \Pi_{\Omega^* H} d$ is the horizontal part of d (that skips the differential along T), and $L\alpha = d\theta \wedge \alpha$. From $d^2 = 0$, one gets

$$d_H^2 = -LT \ , \quad [L, T] = 0 = [L, d_H]\,. \tag{12}$$

Note that d_H is not a complex, and moreover that the splitting of $\Omega^* M$ and d_H depends on the choice of a contact form θ. It is possible to construct another sequence of operators that avoid this.

2.1. **The contact complex.**

Let $\mathcal{I}^*$ be the ideal in $\Omega^* M$ generated by θ and $d\theta$

$$\mathcal{I}^* = \{\alpha \in \Omega^* M \mid \alpha = \theta \wedge \beta + d\theta \wedge \gamma\}\,,$$

and $\mathcal{J}^*$ its annihilator

$$\mathcal{J}^* = \{\alpha \in \Omega^* M \mid \theta \wedge \alpha = d\theta \wedge \alpha = 0\}\,.$$

They are independent on the choice of contact form and stable under d. From e.g. [20], L is injective on $\Omega^k H$ for $k \leq n-1$ and surjective onto $\Omega^k H$ for $k \geq n+1$. Hence $\mathcal{I}^k = \Omega^k M$ for $k \geq n+1$ and $\mathcal{J}^k = 0$ for $k \leq n+1$. Then de Rham exterior differential induces two Quotiented complexes

$$\Omega^0 M \xrightarrow{d_Q} \Omega^1 M/\mathcal{I}^1 \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega^n M/\mathcal{I}^n$$

and

$$\mathcal{J}^{n+1} \xrightarrow{d_Q} \mathcal{J}^{n+2} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \mathcal{J}^{2n+1}.$$

These can be joined using the following:

Lemma 2.1. [17, p. 286] *Let $\alpha \in \Omega^n M/\{\text{vertical forms}\}$. Then there exists a unique lift $\overline{\alpha}$ of α in $\Omega^n M$ such that $d\overline{\alpha} \in \mathcal{J}^{n+1}$. Moreover $d\overline{\alpha} = 0$ if $\alpha = d\theta \wedge \beta$.*

One defines then $D : \Omega^n M/\mathcal{I}^n \to \mathcal{J}^{n+1}$ by $D\alpha = d\overline{\alpha}$. Note that D is a *second order* operator, taking T as a second order one in our contact setting by (12). Given a choice of contact form, the formula for D reads

$$D\alpha = d(\alpha_H - \theta \wedge L^{-1} d_H \alpha_H) = \theta \wedge (T + d_H L^{-1} d_H)\alpha_H\,, \tag{13}$$

if α_H is the representative of α in $\Omega^n H$. The so-called contact complex is then

$$\Omega^0 M \xrightarrow{d_Q} \Omega^1 M/\mathcal{I}^1 \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega^n M/\mathcal{I}^n \xrightarrow{D} \mathcal{J}^{n+1} \xrightarrow{d_Q} \mathcal{J}^{n+2} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \mathcal{J}^{2n+1}.$$

We have :

Proposition 2.2. [17, p. 286] *The contact complex is a resolution of the constant sheaf and hence its cohomology coincides with de Rham cohomology of M. Moreover the canonical projections $\pi : \Omega^k M \to \Omega^k M/\mathcal{I}^k$ for $k \leq n$ and injections $i : \mathcal{J}^k \to \Omega^k M$ for $k \geq n+1$ induce isomorphism between the two cohomologies.*

The arguments being purely local, these results also apply on twisted version of the complexes with a flat bundle V, as coming from a representation $\rho : \pi_1(M) \to U(d)$.

Using a complex structure J on H such that $d\theta(\cdot, J\cdot)$ is Hermitian positive definite, one defines a Riemannian metric on M

$$g = d\theta(\cdot, J\cdot) + \theta^2 .$$

Let then $\Lambda = L^*$ be the adjoint of $L : \Omega^k H \to \Omega^{k+2} H$ where $L\alpha = d\theta \wedge \alpha$, and $\Omega_0^* H = \ker \Lambda$ be the bundle of primitive horizontal forms. We will identify in the sequel the quotient spaces $\Omega^k M/\mathcal{I}^k$ in the lower-half of the contact complex with $\Omega_0^k H$. Let

$$\mathcal{E}^k = \begin{cases} \Omega_0^k H & \text{if } k \leq n \\ \mathcal{J}^k & \text{if } k \geq n+1. \end{cases} \tag{14}$$

be the definition spaces of the contact complex in this identification. Note that Hodge star operator $*$ exchanges $\mathcal{E}^k$ and $\mathcal{E}^{2n+1-k}$.

2.2. **Miscellaneous formulae.**

We gather now some useful identities; see e.g. [18, Section 4] for more details. The first ones are similar to basic formulae from Kählerian geometry, see [20]. At the algebraic level, it holds on the Hermitian space H that

$$[\Lambda, L] = n - p \quad \text{on } \Omega^p H . \tag{15}$$

Moreover, following [20, Thm. 3] for instance, $\Omega^* H$ splits under the Lefschetz decomposition

$$\Omega^* H = \bigoplus_{0 \leq k \leq q \leq n} L^k \Omega_0^{n-q} H = \bigoplus_{0 \leq k \leq q \leq n} L^k \mathcal{E}^{n-q} . \tag{16}$$

At the level of first order operators one has

$$[\Lambda, d_H] = -\delta_H^J . \tag{17}$$

where $\delta_H^J = J^{-1} \delta_H J$ and $J\alpha(X_1, \cdots, X_p) = \alpha(JX_1, \cdots, JX_p)$ on $\Omega^p H$.

This leads to the action of d_H with respect to Lefschetz decomposition. Thanks to (15) and (17), it holds on $\Omega_0^p H$ that

$$d_H = d_Q - \frac{L}{n-p+1} \delta_Q^J , \tag{18}$$

with the convention here that $d_Q = 0$ on $\Omega_0^n H$. This extends on $L^k \Omega_0^p H$ using $[L, d_H] = 0$ by (12).

We now come to second order relations on the contact complex. From (12) (13) and (18), one gets:

$$T = \begin{cases} \frac{1}{n-p} \delta_Q^J d_Q + \frac{1}{n-p+1} d_Q \delta_Q^J & \text{on } \mathcal{E}^p \quad \text{for } p < n \\ i_T D + d_Q \delta_Q^J & \text{on } \mathcal{E}^n . \end{cases} \tag{19}$$

In order to get rid of the multiplicative coefficients in formulae as above, we will normalise the differentials d_Q as in [18, p. 418]. Namely on $\mathcal{E}^p$ for $p < n$, we shall use from now on

$$\frac{1}{\sqrt{n-p}} d_Q \quad \text{instead of} \quad d_Q . \tag{20}$$

We will keep the same notation d_Q for this normalised differential in the sequel since we will only use them. Hence (19) reads now

$$T = \begin{cases} \delta_Q^J d_Q + d_Q \delta_Q^J & \text{on } \mathcal{E}^p \text{ for } p < n \\ i_T D + d_Q \delta_Q^J & \text{on } \mathcal{E}^n\,. \end{cases} \tag{21}$$

Using this and $L = L^J$, $d\theta$ being a $(1,1)$ form, one deduces that on $\Omega^* H$

$$T^* = -T^J \quad \text{and} \quad [\Lambda, T] = 0\,. \tag{22}$$

So far, all identities here hold for any calibrated complex structure J, i.e. satisfying that $d\theta(\cdot, J\cdot)$ is positive Hermitian. In the sequel, we will assume moreover that J is *integrable*, meaning that $[H^{1,0}, H^{1,0}] \subset H^{1,0}$. In that case both d_H and d_Q split into two components

$$d_H = \partial_H + \overline{\partial}_H \quad \text{and} \quad d_Q = \partial_Q + \overline{\partial}_Q\,, \tag{23}$$

where $\partial_{H,Q}$ increases the bidegree by $(1,0)$ and $\overline{\partial}_{H,Q}$ by $(0,1)$. Developing $d_Q^2 = 0$ on $\mathcal{E}^p$ for $p \leq n-2$, first gives

$$\partial_Q^2 = \overline{\partial}_Q^2 = 0 = \partial_Q \overline{\partial}_Q + \overline{\partial}_Q \partial_Q = d_Q d_Q^J + d_Q^J d_Q\,. \tag{24}$$

We can also get other second order relations between the Q-differentials by developing (21) on $\mathcal{E}^p$ with $p < n$. This gives

$$\begin{cases} \Delta_{\overline{\partial}_Q} - \Delta_{\partial_Q} = iT^{0,0} \\ \overline{\partial}_Q^* \partial_Q + \partial_Q \overline{\partial}_Q^* = iT^{1,-1} \\ \partial_Q^* \overline{\partial}_Q + \overline{\partial}_Q \partial_Q^* = -iT^{-1,1} \end{cases} \tag{25}$$

where

$$\Delta_{\overline{\partial}_Q} = \overline{\partial}_Q^* \overline{\partial}_Q + \overline{\partial}_Q \overline{\partial}_Q^* \quad \text{and} \quad \Delta_{\partial_Q} = \partial_Q^* \partial_Q + \partial_Q \partial_Q^*\,.$$

At this point we note that

$$T - T^J = -J^{-1}(\mathcal{L}_T J) = (1+i)T^{1,-1} + (1-i)T^{-1,1}$$

is a zero order algebraic operator that vanishes when the Reeb flow preserves the complex structure, thus the metric. In conclusion we have the following:

Proposition 2.3. [18, p. 418] *Suppose that the complex structure on a CR contact manifold is integrable and preserved by the Reeb flow.*

Then it holds on $\mathcal{E}^p$ for $p < n$ that the second order Q-Laplacian $\Delta_Q = d_Q \delta_Q + \delta_Q d_Q$ commutes with J and writes

$$\Delta_Q = \Delta_{\partial_Q} + \Delta_{\overline{\partial}_Q} \quad \text{with} \quad \Delta_{\overline{\partial}_Q} - \Delta_{\partial_Q} = iT\,. \tag{26}$$

Remark 2.4. We recall that these results hold with the renormalised differentials as defined in (20).

2.3. The middle degree case.

At this point we still miss a Q-Laplacian in middle degree. The differential $D : \mathcal{E}^n \to \mathcal{E}^{n+1}$ here is second order. Hence a starting expression for a positive Laplacian is the fourth order D^*D. A natural way to complete it, is to set on $\mathcal{E}^n$

$$\boldsymbol{\Delta}_Q = (d_Q\delta_Q)^2 + D^*D\,. \tag{27}$$

An important feature of this choice lies in its commuting property with J, as in the lower degree case.

Proposition 2.5. [17, p. 312] *$\boldsymbol{\Delta}_{\mathbf{Q}}$ preserves the bidegree of forms in $\mathcal{E}^n$ when the complex structure is integrable and invariant through the Reeb flow.*

Proof. From (22) one has $T^* = -T$ when $\mathcal{L}_T J = 0$, and then by (19)

$$\begin{aligned} D^*D &= (i_T D)^*(i_T D) = (-T - d_Q^J\delta_Q)(T - d_Q\delta_Q^J) \\ &= -T^2 + T(-d_Q^J\delta_Q + d_Q\delta_Q^J) + d_Q^J\delta_Q d_Q\delta_Q^J\,. \end{aligned}$$

There T and $-d_Q^J\delta_Q + d_Q\delta_Q^J$ commute with J, whereas by Proposition 2.3

$$\begin{aligned} d_Q^J\delta_Q d_Q\delta_Q^J &= d_Q^J(\Delta_Q - d_Q\delta_Q)\delta_Q^J = d_Q^J\Delta_Q^J\delta_Q^J - d_Q^J d_Q\delta_Q\delta_Q^J \\ &= (d_Q^J\delta_Q^J)^2 - d_Q^J d_Q\delta_Q\delta_Q^J\,. \end{aligned}$$

The last term preserves the bidgree by (24). Finally adding $(d_Q\delta_Q)^2$ gives that $\boldsymbol{\Delta}_Q$ commutes with J. Note that $\boldsymbol{\Delta}_Q$ has no $(2,-2)$ (and $(-2,2)$) component neither since it can come only from combination of type

$$\partial_Q\overline{\partial}_Q^*\partial_Q\overline{\partial}_Q^* = -\partial_Q\partial_Q\overline{\partial}_Q^*\overline{\partial}_Q^* = 0\,,$$

by (25) and (24). □

In order to get fourth order Laplacians for all degrees we shall define

$$\boldsymbol{\Delta}_Q = \Delta_Q^2 = (d_Q\delta_Q)^2 + (\delta_Q d_Q)^2 \quad \text{on } \mathcal{E}^p \text{ for } p < n\,. \tag{28}$$

We extend these operators on all Ω^*H using the Lefschetz decomposition (16) and requiring that

$$L\boldsymbol{\Delta}_Q = \boldsymbol{\Delta}_Q L\,.$$

In this way, the Laplacian $\boldsymbol{\Delta}_Q$ commutes with all the algebra of operators we face here: L, J, d_Q, d_H and their adjoints. It will play the role of a "Casimir" operator in our situation.

To conclude this part, we mention the main analytic result on the Q-Laplacian.

Theorem 2.6. [17, p. 290] *The Q-Laplacians Δ_Q and $\boldsymbol{\Delta}_Q$ are maximally hypoelliptic on any compact contact manifold.*

Roughly speaking, that means that these operators control as much horizontal derivatives as possible: two for Δ_Q and four for $\boldsymbol{\Delta}_Q$. See for instance the discussion in [19, Section 3.1] for a presentation of main properties and references about this analytic notion. A consequence for this paper is that on compact contact manifolds, the Q-Laplacians are self-adjoint and possess pure point spectrum. Moreover the associated heat kernels $e^{-t\boldsymbol{\Delta}_Q}$ are smooth hence trace class in this setting.

3. Results on the contact analytic torsion

We now state our main result on the torsion function. We first define the contact manifolds we will be concerned in.

Definition 3.1. Let M be a compact contact manifold of any dimension. We shall say that M is a *CR Seifert manifold* if it admits a locally free circle action generated by a Reeb field T that preserves moreover an integrable complex (CR) structure J on H.

The orbit space $N = M/\mathbf{S^1}$ of such a manifold inherits a structure of a Kählerian orbifold with cyclic quotient singularities; see e.g. [10, p. 52]. Note that these manifold are particular cases of Sasakian manifolds, called quasi-regular, whose Reeb field have closed orbits; see [15].

We also endow M with a unitary representation $\rho : \pi_1(M) \to U(d)$ (that can be trivial). This possibly allows to twist the contact complex by taking values in the associated flat bundle V. That will highlight the role of each individual homotopy class of closed orbit in the dynamical expression.

Using notations introduced in Sections 1.2 and 1.3 we recall that the initial heat torsion function is spectral and given by (1)

$$\vartheta(t) = \sum_{k=0}^{n} (-1)^k (n+1-k) \operatorname{Tr}(e^{-t\mathbf{\Delta}_Q} \mid \mathcal{E}^k).$$

We also define the topological theta function

$$\vartheta^{top}(t) = \sum_{\lambda \in \operatorname{Spec}(iT)} \operatorname{ind}(D_H^{ev} \mid V_\lambda) e^{-t\lambda^2},$$

the geometric expression

$$\vartheta^{geo} = \chi(N^*)\chi_\rho^\theta(f) + \sum_i \chi_\rho^\theta(f_i),$$

and the dynamical series

$$\vartheta^{dyn}(t) = \frac{1}{\sqrt{4\pi t}} \sum_\gamma \chi_\rho(\gamma) e(\gamma) e^{-\ell(\gamma)^2/4t}.$$

We shall prove the following.

Theorem 3.2. *Let M be a CR Seifert manifold of any dimension endowed with a unitary representation. Then it holds that*

$$\vartheta = \vartheta^{top} = \vartheta^{geo} = \vartheta^{dyn}.$$

3.1. From spectral to topological torsion series.

We start with the proof of the first identity $\vartheta = \vartheta^{top}$. Recall that by Lefschetz decomposition (16)

$$\Omega^* H = \bigoplus_{0 \le p \le q \le n} L^p \mathcal{E}^{n-q}.$$

Therefore Ω^*H contains $n+1-k$ copies of each $\mathcal{E}^k$ for $0 \le k \le n$. This is the multiplicity of $\mathcal{E}^k$ in ϑ. Moreover, since $L = d\theta \wedge \cdot$ preserves the parity, the parity of forms in Ω^*H and their components in $\mathcal{E}^*$ coincides. Thus the plus and minus parts of ϑ combine to give

$$\vartheta(t) = \operatorname{Tr}(e^{-t\mathbf{\Delta}_Q} \mid \Omega^{ev}H) - \operatorname{Tr}(e^{-t\mathbf{\Delta}_Q} \mid \Omega^{odd}H)\,.$$

Now $D_H = d_H + \delta_H$ exchanges $\Omega^{ev}H$ and $\Omega^{odd}H$ and commutes with $\mathbf{\Delta}_Q$ by Section 2.5. Hence, setting $\mathcal{H} = \ker D_H$, one has

$$\vartheta(t) = \operatorname{Tr}(e^{-t\mathbf{\Delta}_Q} \mid \mathcal{H}^{ev}) - \operatorname{Tr}(e^{-t\mathbf{\Delta}_Q} \mid \mathcal{H}^{odd})\,.$$

We need the following Lemma to identify the residual spectrum on $\mathcal{H}$.

Lemma 3.3. *On CR Seifert manifolds, one has $\mathbf{\Delta}_Q = -T^2$ on $\mathcal{H} = \ker D_H$.*

Note that $\mathcal{H} = \ker D_H$ should not be confused with $\ker \Delta_H = \ker d_H \cap \ker \delta_H$ since d_H is not a complex by (12). The next proposition will lead us to a useful description of $\mathcal{H}$.

Proposition 3.4. *Let $P = L + \Lambda$ acting on Ω^*H and $U = e^{i\pi P/4}$. Then on a contact manifold with an integrable J, it holds that*

$$\begin{cases} [P, D_H] = d_H^J - \delta_H^J \\ [P, d_H^J - \delta_H^J] = D_H \end{cases}$$

and

$$U^{-1} D_H U = \sqrt{2}(\overline{\partial}_H + \overline{\partial}_H^*)\,.$$

Proof. The first identities come from (12) and (17) giving $[L, d_H] = 0$ and $[\Lambda, d_H] = -\delta_H^J$ and conjugated relations. Therefore for any angle φ

$$\begin{aligned} e^{i\varphi\,\mathrm{ad}(P)} D_H &= \cos\varphi D_H + i \sin\varphi (d_H^J - \delta_H^J) \\ &= \mathrm{Ad}(e^{i\varphi P}) D_H = e^{i\varphi P} D_H e^{-i\varphi P}\,. \end{aligned}$$

This yields the last identity using $\varphi = -\pi/4$ and the splitting of $d_H = \partial_H + \overline{\partial}_H$ for integrable J by (23). □

Let $\mathcal{H}_{\overline{\partial}_H} = \ker(\overline{\partial}_H + \overline{\partial}_H^*)$. By the previous proposition

$$\mathcal{H} = \ker D_H = U(\mathcal{H}_{\overline{\partial}_H})\,,$$

with the advantage that on CR Seifert manifolds $\overline{\partial}_H^2 = 0$ from (12). Hence one has also $\mathcal{H}_{\overline{\partial}_H} = \ker \Delta_{\overline{\partial}_H}$ with

$$\Delta_{\overline{\partial}_H} = \overline{\partial}_H \overline{\partial}_H^* + \overline{\partial}_H^* \overline{\partial}_H\,.$$

Now by definition $\Delta_{\overline{\partial}_H}$ preserves the bi-degree of forms, but also the spaces $L^k\Omega_0^p H$ in the Lefschetz decomposition. This is due to the commutation relation

$$\Delta_{\overline{\partial}_H} L = L(\Delta_{\overline{\partial}_H} - iT)$$

that comes from (17) when $\mathcal{L}_T J = 0$. Indeed, from

$$[L, \overline{\partial}_H] = 0 \quad \text{and} \quad [L, \overline{\partial}_H^*] = -i\partial_H$$

one gets

$$\Delta_{\overline{\partial}_H} L - L\Delta_{\overline{\partial}_H} = i(\partial_H \overline{\partial}_H + \overline{\partial}_H \partial_H) = -iLT$$

by (12). This leads eventually to the splitting of $\mathcal{H}_{\overline{\partial}_H}$ into forms of pure bidegree and type in the Lefschetz decomposition. Using (18) we first find.

Proposition 3.5. *On a CR Seifert manifold, one has*

$$\ker \overline{\partial}_H = \begin{cases} \ker \overline{\partial}_Q \cap \ker \partial_Q^* & \text{on } L^k \mathcal{E}^{n-q} \text{ for } 0 \le k < q \le n \\ \ker \partial_Q^* & \text{on } L^q \mathcal{E}^{n-q} \end{cases}$$

and

$$\ker \overline{\partial}_H^* = \begin{cases} \ker \overline{\partial}_Q^* \cap \ker \partial_Q & \text{on } L^k \mathcal{E}^{n-q} \text{ for } 1 \le k \le q \le n \\ \ker \overline{\partial}_Q^* & \text{on } \mathcal{E}^p \text{ for } 0 \le p \le n\,. \end{cases}$$

In conclusion

$$\mathcal{H}_{\overline{\partial}_H} = \begin{cases} \ker \Delta_Q & \text{on } L^k \mathcal{E}^{n-q} \text{ for } 1 \le k < q \le n \\ \ker \overline{\partial}_Q \cap \ker \partial_Q^* \cap \ker \overline{\partial}_Q^* & \text{on } \mathcal{E}^p \text{ for } p < n \\ \ker \partial_Q^* \cap \ker \overline{\partial}_Q^* \cap \ker \partial_Q & \text{on } L^q \mathcal{E}^{n-q} \text{ for } 0 < q \le n \\ \ker \partial_Q^* \cap \ker \overline{\partial}_Q^* & \text{on } \mathcal{E}^n\,. \end{cases}$$

This leads to the following results on $\mathcal{H}_{\overline{\partial}_H}$:

- On $L^k \mathcal{E}^{n-q}$ for $0 \le k < q \le n$, $\Delta_Q = \Delta_Q^J = \mathcal{L}_T = 0$ by (26),
- On $\mathcal{E}^p$ for $p < n$, $\Delta_{\overline{\partial}_Q} = 0$, so that by (26) $\Delta_Q = -iT$ and $\mathbf{\Delta}_Q = \Delta_Q^2 = -T^2$,
- On $L^q \mathcal{E}^{n-q}$ for $0 < q \le n$, $\Delta_{\partial_Q} = 0$, so that $\Delta_Q = iT$ and $\mathbf{\Delta}_Q = \Delta_Q^2 = -T^2$,
- On $\mathcal{E}^n$, $\delta_Q = 0 = \delta_Q^J = 0$, so that $i_T D = T$ and $\mathbf{\Delta}_Q = -T^2$.

This yields Lemma 3.3 because the isometry $U = e^{i\frac{\pi}{4}(L+\Lambda)}$ mapping $\mathcal{H}_{\overline{\partial}_H}$ to $\mathcal{H} = \ker D_H$ in Proposition 3.4 commutes with $\mathbf{\Delta}_Q$ and T hence preserves the equation $\mathbf{\Delta}_Q = -T^2$.

To conclude with this study of $\mathcal{H}$ we observe that it is infinite dimensional since it contains images by U of CR functions (satisfying $\overline{\partial}_H f = 0$).

We have obtained that

$$\vartheta(t) = \operatorname{Tr}(e^{tT^2} \mid \mathcal{H}^{ev}) - \operatorname{Tr}(e^{tT^2} \mid \mathcal{H}^{odd})\,. \tag{29}$$

Note that both traces converge since they are parts of traces of the heat operators $e^{-t\mathbf{\Delta}_Q}$. The next step will be to split this as a series of index using Fourier decomposition along the circle action by T. We see this briefly and refer to [19, Section 4.1] for a more detailed discussion.

The flat bundle V over M associated to the unitary representation $\rho : \pi_1(M) \to U(d)$ is the quotient of the trivial bundle $\widetilde{M} \times \mathbb{C}^N$ by the deck transformations $\gamma.(m, v) = (\tau(\gamma)m, \rho(\gamma)v)$. The circle action φ_t induced by T on M may be lifted to V by parallel transport using the flat connection ∇^ρ. On V we don't have a circle action since

$$\varphi_{2\pi} = \rho(f)^{-1}$$

where $f = \varphi_{[0,2\pi]}(m)$ is the generic closed primitive orbit. However we can split V into irreducible representations V^x on which

$$\rho(f) = e^{2i\pi x}$$

with $x \in]0,1]$ by convention. We recover a circle action on V^x by setting

$$\psi_t = e^{itx}\varphi_t \,.$$

We can still perform a Fourier decomposition of sections $s \in \mathbf{V}^x = \Gamma(M, V^x)$. One has

$$s = \sum_{n\in\mathbb{Z}} \pi_n s \quad \text{with} \quad \pi_n s = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\psi_t(s)dt \,.$$

Since $\psi_t(\pi_n s) = e^{int}\pi_n s$ one gets that $\nabla^\rho_T(\pi_n s) = i(n-x)\pi_n s$ and the spectrum of $iT = i\nabla^\rho_T$ on $\mathbf{V}^x$ is the shifted $x + \mathbb{Z}$. For $\lambda = x + n$ we shall note

$$\mathbf{V}_\lambda = \pi_{-n}(\mathbf{V}^x) = \mathbf{V}^x \cap \{iT = \lambda\}. \tag{30}$$

Since T commutes with all our geometric operators here as D_H, we can split the V-valued bundle

$$\mathcal{H} = \ker D_H = \bigoplus_{\lambda\in \mathrm{Spec}(iT)} \mathcal{H} \cap \mathbf{V}_\lambda \,.$$

Therefore (29) eventually reads

$$\begin{aligned}\vartheta(t) &= \sum_{\lambda\in\mathrm{Spec}(iT)} \big(\dim(\mathcal{H}^{ev}\cap \mathbf{V}_\lambda) - \dim(\mathcal{H}^{odd}\cap\mathbf{V}_\lambda)\big)e^{-t\lambda^2} \\ &= \sum_{\lambda\in\mathrm{Spec}(iT)} \mathrm{ind}(D^{ev}_H \mid \mathbf{V}_\lambda)e^{-t\lambda^2} = \vartheta^{top}(t)\,,\end{aligned}$$

with $D^{ev}_H : \Omega^{ev}H \to \Omega^{odd}H$. This is the topological version of the ϑ series. We shall see that these index can be interpreted as coming from operators and bundles over the orbifold $N = M/\mathbf{S}^1$

3.2. **Index computations.**

The index in the previous formula can be computed using the index theorem for V-(orbi)bundles over V-manifolds (orbifolds) as developed by Kawasaki in [12]. One has first to interpret the index of D^{ev}_H on $\mathbf{V}_\lambda$ as the index of an operator over the orbifold $N = M/\mathbf{S}^1$ acting on a V-bundle.

We follow the discussion in [19, section 5.2]. We first introduce the relevant V-(orbi)bundle here. Given a point $p \in N$, we consider the vector space $V_\lambda(p)$ of sections of V^x along the orbit $\mathbf{S}^1(p)$ in M satisfying $iTs = \lambda s$. Call V_λ this family of spaces. One has by definition

$$\mathbf{V}_\lambda = \Gamma(N, V_\lambda)\,,$$

and V_λ is a genuine vector bundle of dimension $\dim V^x$ over the generic non singular points of N since the circle action is locally free.

We now describe the orbifold structure of V_λ for $\lambda = x + n$ near a singular point p_j corresponding to an exceptional closed primitive orbit f_j of order α_j. Locally over p_j, the bundle V^x is isomorphic to the quotient of the trivial bundle $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}^{\dim V^x}$ by the deck transformation

$$F_j : (p,t,v) \mapsto (M_j(p), t + 2\pi/\alpha_j, \rho(f_j)v)$$

where $M_j \in U(d)$ generates a cyclic group of order α_j. Now let $\widetilde{V}_\lambda$ be the trivial bundle over $\mathbb{C}^n$ whose fiber over p consists in functions $s_p : \mathbb{R} \to \mathbb{C}^{\dim V_x}$ satisfying $s_p(t) = e^{-i\lambda t}s_p(0)$.

Since $(iT)s_p = \lambda s_p$, one sees that a section $s : p \mapsto s_p$ of $\widetilde{V}_\lambda$ near $0 \in \mathbb{C}^n$ goes down to $\mathbf{V}_\lambda$ if it is invariant by the deck transform F_j above. This means that $(p, s_p(0))$ is invariant by

$$F_{j,\lambda} : (p, v) \mapsto (M_j(p), e^{-2i\pi\lambda/\alpha_j}\rho(f_j)v)\,. \tag{31}$$

Since $f_j^{\alpha_j} \sim f$ in $\pi_1(M)$, one has $\rho(f_j)^{\alpha_j} = \rho(f) = e^{2i\pi x}$ and $F_{j,\lambda}^{\alpha_j} = \mathrm{id}$.

This shows that V_λ is a V-bundle (orbi-bundle) over N since locally over p_j it is the quotient of $\widetilde{V}_\lambda$ by the finite group $\Gamma_j \simeq \mathbb{Z}/\alpha_j\mathbb{Z}$ generated by $F_{j,\lambda}$.

At this point we can identify $\mathrm{ind}(D_H^{ev} \mid \mathbf{V}_\lambda)$ as being the index of the Dirac operator D_H^{ev} acting on sections of the V-bundle V_λ over N. Note that even if d_H is not a complex, the operator D_H is elliptic on N as $D_H^2 = \Delta_H - LT + T\Lambda = \Delta_H$ up to order 0 terms on V_λ. More concretely from Proposition 3.4, D_H is unitarily conjugated to $\bar{\partial\!\!\!/}_H = \overline{\partial}_H + \overline{\partial}_H^*$ hence

$$\begin{aligned}\mathrm{ind}(D_H^{ev} \mid V_\lambda) &= \mathrm{ind}(\bar{\partial\!\!\!/}_H \mid \Omega^{ev}H \otimes V_\lambda)\\ &= \sum_p (-1)^p \chi_{\overline{\partial}_H}(N, \Omega^{p,0}H \otimes V_\lambda)\end{aligned}$$

where $\chi_{\overline{\partial}_H}(N, \Omega^{p,0}H \otimes V_\lambda)$ is the holomorphic Euler characteristic of the V-bundle $\Omega^{p,0}H \otimes V_\lambda$. Recall that $\overline{\partial}_H^2 = 0$ on CR Seifert manifolds by (12). From this discussion, one can compute these index using Kawasaki's Riemann–Roch theorem for complex V-manifolds [12]. It reads here

$$\mathrm{ind}(D_H^{ev} \mid V_\lambda) = \langle \mathrm{ch}(V_\lambda) \wedge e(N), [N] \rangle_{orb}\,. \tag{32}$$

This pairing on the orbifold N splits into a usual smooth contribution over the generic orbits of the characteristic classes and an average of equivariant classes over the finite exceptional orbits; see [12, 3]. Let $N^* = N \setminus \cup_j \{p_j\}$ where $p_j = \pi(f_j)$ are the singular points. One gets first

$$\langle \mathrm{ch}(V_\lambda) \wedge e(N), [N] \rangle_{orb} = \int_{N^*} \mathrm{ch}(V_\lambda) \wedge e(N) + \sum_j \frac{1}{\alpha_j} \sum_{k=1}^{\alpha_j - 1} \mathrm{Tr}(F_{j,\lambda}^k \mid \widetilde{V}_\lambda)\nu(f_j^k)\,.$$

Here $\mathrm{ch}(V_\lambda)$ is the Chern character of V_λ as seen from N, $e(N)$ the Euler class of N and $\nu(f_j^k)$ is the Lefschetz index of the Poincaré return map along f_j^k. Since $df_j^k \in U(n)$, one has $\nu(f_j^k) = 1$. Also $e(N)$ being a top order class, the index formula reduces to

$$\langle \mathrm{ch}(V_\lambda) \wedge e(N), [N] \rangle_{orb} = \dim(V^x)\chi(N^*) + \sum_j \dim\ker(F_{j,\lambda} - \mathrm{id} \mid \widetilde{V}_\lambda)\,, \tag{33}$$

where $\chi(N^*)$ is the Euler characteristic of the punctured manifold N^* and

$$\chi(N) = \int_{N^*} e(N) = \chi(N^*) + \sum_j \frac{1}{\alpha_j}$$

is the rational Euler characteristic of the orbifold N.

3.3. End of proof of the geometric formula for ϑ.

We complete now the proof of the geometric identity

$$\vartheta = \vartheta^{geo} = \chi(N^*)\chi^\theta_\rho(f) + \sum_j \chi^\theta_\rho(f_j)\,.$$

From (32), (33) and the index series formula for ϑ we know that

$$\vartheta(t) = \sum_{\lambda\in \mathrm{Spec}(iT)} \big(\dim(V^x)\chi(N^*) + \sum_j \dim\ker(F_{j,\lambda} - \mathrm{id} \mid \widetilde{V}_\lambda)\big)e^{-t\lambda^2}\,, \tag{34}$$

where we recall that $\mathrm{Spec}(iT)$ splits into $\mathbb{Z}+x$ on $V^x = \ker(\rho(f) - e^{2i\pi x}\,\mathrm{id})$. Hence the smooth contribution already reads

$$\chi(N^*)\sum_{e^{2i\pi x}\in \mathrm{Spec}(\rho(f))} \dim V^x \sum_{n\in\mathbb{Z}} e^{-t(n+x)^2} = \chi(N^*)\chi^\theta_\rho(f)(t)\,,$$

with χ^θ_ρ as defined in (6). In order to prove (7) it remains to evaluate the sums over the exceptional orbits.

From (31) one has $F_{j,\lambda} = e^{-2i\pi\lambda/\alpha_j}\rho(f_j)$ on $\widetilde{V}_\lambda$ at $p_j = \pi(f_j)$. Therefore $F_{j,\lambda} = \mathrm{id}$ if $\rho(f_j) = e^{2i\pi\lambda/\alpha_j}$. Moreover since $\rho(f_j)^{\alpha_j} = \rho(f)$, the spectrum of $\rho(f_j)$ on V^x consists in complex numbers $e^{2i\pi x_{j,k}}$ with $\alpha_j x_{j,k} = x + n_{j,k}$ for some integers $n_{j,k}$. Hence if $\lambda = x+n$ one finds that

$$e^{2i\pi x_{j,k}} = e^{2i\pi\lambda/\alpha_j} \Leftrightarrow x_{j,k} \equiv \frac{x+n}{\alpha_j}\mathrm{mod}\ \mathbb{Z} \Leftrightarrow n \equiv n_{j,k} \ \mathrm{mod}\ \alpha_j\mathbb{Z}\,,$$

so that $\lambda = \alpha_j x_{j,k} + \alpha_j p$ with $p\in\mathbb{Z}$. Let $V^{x_{j,k}}_{f_j} = \ker(\rho(f_j) - e^{2i\pi x_{j,k}}\,\mathrm{id})$. The contribution of f_j to (34) reads

$$\begin{aligned}
&\sum_{\lambda\in\mathrm{Spec}(iT)} \dim\ker(F_{j,\lambda} - \mathrm{id} \mid \widetilde{V}_\lambda)e^{-t\lambda^2}\\
&\qquad = \sum_{e^{2i\pi x_{j,k}}\in\mathrm{Spec}(\rho(f_j))} \dim V^{x_{j,k}}_{f_j} \sum_{p\in\mathbb{Z}} e^{-t\alpha_i^2(x_{j,k}+p)^2}\\
&\qquad = \sum_{e^{2i\pi x_{j,k}}\in\mathrm{Spec}(\rho(f_j))} \dim V^{x_{j,k}}_{f_j}\theta(x_{j,k},\alpha_j^2 t) = \chi^\theta_\rho(f_j)(t)\,.
\end{aligned}$$

Note that we replace α_j by $2\pi/\ell(f_j)$ in the definition of $\chi^\theta_\rho(f_j)(t)$ in order to preserve its homogeneity in the rescaling $\theta\mapsto c^2\theta$ and $\ell(f_j)\mapsto c\ell(f_j)$.

3.4. ϑ from the dynamical viewpoint.

It remains to link the geometric formula for ϑ to its dynamical one ϑ^{dyn}. This is based on the usual Poisson formula relating the Gaussian and Jacobi theta function. It reads

$$\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{-t(n+x)^2} = \sqrt{\frac{\pi}{t}}\sum_{n\in\mathbb{Z}} e^{2i\pi nx}e^{-\pi^2n^2/t}\,. \tag{35}$$

This leads to the following interpretation of $\chi_\rho^\theta(\gamma)(t)$ for $\gamma = f$ and f_i. One gets

$$\chi_\rho^\theta(\gamma)(t) = \frac{\ell(\gamma)}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \chi_\rho(\gamma^n) e^{-\ell^2(\gamma^n)/4t}\,. \tag{36}$$

Thus $\chi_\rho^\theta(\gamma)(t)$ is a pondered sum of holonomies of ρ along all the closed curves γ^n. More precisely, we recall that the function

$$k_t(y) = \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}$$

is the heat kernel on $\mathbb{R}$. Now, given a random (Brownian) curve $y(s)$ on the circle orbit γ, it is closed at time t if its lift $\widetilde{y}$ in $\mathbb{R}$ satisfies $\widetilde{y}(t) - \widetilde{y}(0) = n\ell(\gamma)$. In such a case, the rotation index of $y_{[0,t]}$ is n. Therefore its holonomy along ρ is $\chi_\rho(\gamma^n)$. Moreover the probability density of such a displacement is $k_t(\ell(\gamma^n)) = \frac{1}{\sqrt{4\pi t}} e^{-\ell^2(\gamma^n)/4t}$, that arises in (36).

Using (36) in the geometric expression of ϑ gives

$$\begin{aligned}\sqrt{4\pi t}\,\vartheta(t) = \chi(N^*)\ell(f) &\sum_{n\in\mathbb{Z}} \chi_\rho(f^n) e^{-t\ell^2(f^n)/4t} \\ &+ \sum_i \ell(f_i) \sum_{n\in\mathbb{Z}} \chi_\rho(f_i^n) e^{-t\ell^2(f_i^n)/4t}\,.\end{aligned}$$

Now for $p \in \mathbb{Z}$, $f_i^{\alpha_i p} \sim f^p$ in $\pi_1(M)$ and $\ell(f_i) = \ell(f)/\alpha_i$. Thus using $\chi(N) = \chi(N^*) + \sum_i \frac{1}{\alpha_i}$ one finds that

$$\begin{aligned}\sqrt{4\pi t}\,\vartheta(t) =& \chi(N)\ell(f) \sum_{n\in\mathbb{Z}} \chi_\rho(f^n) e^{-t\ell^2(f^n)/4t} \\ &+ \sum_i \ell(f_i) \sum_{n\notin\alpha_i\mathbb{Z}} \chi_\rho(f_i^n) e^{-t\ell^2(f_i^n)/4t}\,,\end{aligned}$$

which gives the required dynamical series (8) over all homotopy classes of closed orbits, their inverse, and the constant loop.

3.5. **Zeta functions viewpoint.**

We now turn to identities on the contact analytic torsion from the viewpoint of zeta functions. On the spectral side, let

$$Z(s) = \sum_{k=0}^{n} (-1)^k (n+1-k) \zeta(\mathbf{\Delta}_Q \mid \mathcal{E}^k)(s)\,,$$

and on the dynamical side

$$Z^{dyn}(s) = \sum_{\gamma\neq 0} \chi_\rho(\gamma) e(\gamma) |\ell(\gamma)|^{2s-1}\,,$$

where γ ranges over the homotopy classes of non trivial closed orbits together with their inverse.

On the geometric side at last we consider

$$Z^{geo}(s) = \chi(N^*) \chi_\rho^\zeta(f)(s) + \sum_i \chi_\rho^\zeta(f_i)(s) \tag{37}$$

with

$$\mathrm{order}(\gamma)^{2s}\chi_\rho^\zeta(\gamma)(s) = \sum_{e^{2i\pi x}\in \mathrm{Spec}^*(\rho(\gamma))} \dim V_\gamma^x(\zeta(2s,x)+\zeta(2s,1-x)) \\ + 2\zeta(2s)\dim V_\gamma^1$$

where $V_\gamma^x = \ker(\rho(\gamma) - e^{2i\pi x}\,\mathrm{id})$, ζ is Riemann zeta function and

$$\zeta(s,x) = \sum_{n\geq 0}\frac{1}{(n+x)^s} \tag{38}$$

is Hurwitz zeta function.

Theorem 3.6. *On CR Seifert manifolds of any dimension the functions Z and Z^{geo} are meromorphic and equal, with a simple pole at $s = 1/2$.*

The function Z^{dyn} is meromorphic with a simple pole at $s = 0$, and one has

$$\Gamma(s)Z(s) = \frac{2^{-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2}-s)Z^{dyn}(s)\,. \tag{39}$$

Moreover

$$\begin{cases} Z(0) = \mathrm{Res}_0(Z^{dyn}) = -\chi'(M,\rho) \\ \mathrm{Res}_{1/2}(Z) = -\dfrac{1}{2\pi}Z^{dyn}(1/2) = \chi(N)\dim V\,. \end{cases}$$

with

$$\chi'(M,\rho) = \sum_{k=0}^{n}(-1)^k(n+1-k)\dim H^k(M,\rho)\,,$$

using the cohomology groups of the flat bundle V over M.

This extends results obtained in [19] in the three dimensional case. Note also that Kitaoka computed in [13] the spectral series $Z(s)$ on Lens spaces endowed with their symmetric metric and found it reduces to a Hurwitz zeta function.

3.6. **Proof of Theorem 3.6.**

We know from the spectral expression of ϑ that

$$\vartheta(t) = \sum_{k=0}^{n}(-1)^k(n+1-k)\dim\ker(\mathbf{\Delta}_Q \mid \mathcal{E}^k) + 0(e^{-Ct})$$

for some $C > 0$ when $t \to +\infty$. According to Proposition 2.2 the limit constant is the topological number $\chi'(M,\rho)$.

From $\vartheta = \vartheta^{geo}$, one gets using $t \to +\infty$ that

$$\chi'(M,\rho) = \chi(N^*)\dim\ker(\rho(f)-\mathrm{id}) + \sum_i \dim\ker(\rho(f_i)-\mathrm{id})\,. \tag{40}$$

Whereas when $t \to 0^+$, the dynamical expression yields

$$\vartheta(t) = \sqrt{\frac{\pi}{t}}\chi(N)\dim V + O(e^{-C/t})\,.$$

Let consider the function

$$\vartheta_0(t) = \vartheta(t) - c_0\mathbf{1}_{[1,+\infty[}(t) - \frac{c_{-1/2}}{\sqrt{t}}\mathbf{1}_{]0,1]}(t)$$

with

$$c_0 = \chi'(M,\rho) \quad \text{and} \quad c_{-1/2} = \sqrt{\pi}\chi(N)\dim V\,.$$

From the previous discussion, the integral

$$I(s) = \int_0^{+\infty} \vartheta_0(t)t^{s-1}dt$$

coming from Mellin transform defines a holomorphic function over $\mathbb{C}$. Writing

$$\vartheta_0(t) = \vartheta(t) - c_0 + (c_0 - \frac{c_{-1/2}}{\sqrt{t}})\mathbf{1}_{]0,1]}(t)$$

one finds first for $\Re(s) > 1/2$ that

$$I(s) = \Gamma(s)Z(s) + \frac{c_0}{s} - \frac{c_{-1/2}}{s-1/2} = \Gamma(s)Z^{geo}(s) + \frac{c_0}{s} - \frac{c_{-1/2}}{s-1/2}\,.$$

Using

$$\vartheta_0(t) = \vartheta^{dyn}(t) - \frac{c_{-1/2}}{\sqrt{t}} + (\frac{c_{-1/2}}{\sqrt{t}} - c_0)\mathbf{1}_{]1,+\infty]}(t)$$

yields for $\Re(s) < 0$ that

$$I(s) = \frac{2^{-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2} - s)Z^{dyn}(s) + \frac{c_0}{s} - \frac{c_{-1/2}}{s-1/2}\,.$$

This proves Theorem 3.6.

3.7. The contact analytic torsion from the Lefschetz and dynamical viewpoints.

Recall that from (3) the contact analytic torsion is defined by $T_Q(M,\rho) = \exp(-Z'(0)/2)$. We shall now compute it using the previous formulae. This extends results obtained in [19] in the three dimensional case.

Corollary 3.7. *Let M be a CR Seifert manifold of any dimension endowed with a unitary representation $\rho : \pi_1(M) \to U(N)$. Let*

$$V_f^1 = \ker(\rho(f) - \mathrm{id}) \quad \text{and} \quad V_{f_i}^1 = \ker(\rho(f_i) - \mathrm{id})\,,$$

and denote by $\rho(f)^\perp$ and $\rho(f_i)^\perp$ the restriction of these holonomies to respectively $(V_f^1)^\perp$ and $(V_{f_i}^1)^\perp$.

Then it holds that

$$T_Q(M,\rho) = (2\pi)^{\chi'(M,\rho)}|\det(\rho(f)^\perp - \mathrm{id})|^{\chi(N^*)}\prod_i \frac{|\det(\rho(f_i)^\perp - \mathrm{id})|}{\alpha_i^{\dim(V_{f_i}^1)}}\,.$$

Moreover one has

$$-2\ln(T_Q(M,\rho)) = Z'(0) = \lim_{s\to 0}\big(Z^{dyn}(s) + \frac{\chi'(M,\rho)}{s}\big)\,.$$

The first expression coincides with that found, via topological methods, for the Reidemeister–Franz torsion by Fried [11, p. 198], in the case of an *acyclic representation*, i.e. $H^*(M,\rho)=0$. Therefore, the contact analytic torsion also coincides with the (Riemannian) Ray–Singer analytic torsion in that case, from works of Cheeger and Müller [8, 14]. The only new factor for general representations is the cohomological term $(2\pi)^{\chi'(M,\rho)}$.

The dynamical expression for the analytic torsion also extends a result proved by Fried in the acyclic case on Seifert manifolds [10, 11]. We have there

$$Z'(0)=Z^{dyn}(0)=\sum_{\gamma\neq 0}\chi_\rho(\gamma)e(\gamma)/|\ell(\gamma)|\,,$$

which is known as the total twisted Fuller measure of periodic orbits. It has a formal invariance by deformation of the flow, as long as the orbit periods stay bounded; see [10, Section 4].

Proof of Corollary 3.7.

Since $Z=Z^{geo}$ is an explicit function by Theorem 3.6, we compute $(Z^{geo})'(0)$. We follow the proof in [19, p. 771] that we recall for completeness.

From [21, p. 271], one has for $x\in]0,1[$, $\zeta(0,x)=\frac{1}{2}-x$, hence $\zeta(0,x)+\zeta(0,1-x)=0$. Also by Lerch's formula $\partial_s\zeta(s,x)_{s=0}=\ln\Gamma(x)-\frac{1}{2}\ln(2\pi)$, it holds that $\zeta'(0)=-\frac{1}{2}\ln(2\pi)$ and

$$\begin{aligned}\partial_s\zeta(s,x)+\partial_s\zeta(s,1-x)&=\ln(\Gamma(x)\Gamma(1-x)/2\pi)=-\ln(2\sin(\pi x))\\&=-\ln|1-e^{2i\pi x}|\,.\end{aligned}$$

Hence from (37), one finds that

$$\begin{aligned}\chi^\zeta_\rho(f)'(0)&=-2\sum_{e^{2i\pi x}\in\operatorname{Spec}\rho(\gamma)^\perp}\dim V^x_f\ln|1-e^{2i\pi x}|+\dim V^1_f\ln 2\pi\\&=-2\ln|\det(\rho(f)^\perp-\mathrm{id})|-2\dim V^1_f\ln(2\pi)\,,\end{aligned}$$

and for the exceptional fibers,

$$\chi^\zeta_\rho(f_i)'(0)+2\ln(\alpha_i)\chi^\zeta_\rho(f_i)(0)=-2\ln|\det(\rho(f_i)^\perp-\mathrm{id})|-2\dim V^1_{f_i}\ln(2\pi)\,.$$

Summing these results using the formula (37) for Z^{geo} and (40) gives the first result for $T_Q(M,\rho)$.

For the second one, it is useful to observe that

$$2\Gamma(2s)\cos(\pi s)Z=Z^{dyn}\,,\tag{41}$$

as comes by multiplying (39) by $\Gamma(s+1/2)$ and using the classical identities

$$\Gamma(s)\Gamma(s+\frac{1}{2})=2^{1-2s}\sqrt{\pi}\Gamma(2s)\quad\text{and}\quad\Gamma(s+\frac{1}{2})\Gamma(s-\frac{1}{2})=\frac{\pi}{\cos\pi s}\,.$$

One knows from Theorem 3.6 that $Z(0)=-\chi'(M,\rho)$. Hence developing (41) when $s\to 0$ gives the dynamical formula for $T_Q(M,\rho)$. □

4. Results on the contact eta trace.

We now turn to the study of the contact eta invariant and trace as introduced in Section 1.4. Following (9) we consider the contact signature operator on contact manifolds of dimension $2n+1=4k-1$

$$S_Q = \begin{cases} *D + (d_Q+\delta_Q)\sigma\delta_Q & \text{on } \mathcal{E}^{2k-1} \\ (d_Q+\delta_Q)\sigma(d_Q+\delta_Q) & \text{on } \bigoplus_{1\le p\le k-1}\mathcal{E}^{2k-1-2p} \end{cases}$$

where $\sigma=(-1)^pJ$ on $\mathcal{E}^{2p}$ and $\mathcal{E}^{2p-1}$. As claimed the advantage of this choice over others lies in the following algebraic properties.

Proposition 4.1.

- *The operator S_Q is symmetric on any contact manifold of dimension $4k-1$ endowed with a calibrated complex structure J.*
- *If moreover J is integrable and invariant by T, then it holds that $S_Q^2=\mathbf{\Delta}_Q$ and*

$$S_Q=\sigma T+P \text{ with } P\sigma=-\sigma P \text{ and } [T,\sigma]=[P,T]=0\,.$$

- *The non-zero spectrum of S_Q splits as follows*

$$\mathrm{Spec}^*(S_Q)=\mathrm{Spec}^*(*D)\bigsqcup_{0\le p\le k-1}\mathrm{Spec}^*\big(\sigma\Delta_Q\mid\mathcal{E}^{2p}\big)\,.$$

Proof. • The first statement follows easily from the definition and the facts that by (19)

$$(-1)^k*D=J(i_TD)=JT-Jd_QJ^{-1}\delta_QJ$$

with $J^*=(-1)^pJ$ on $\mathcal{E}^p$ and $T^*=-T^J$ on calibrated J by (22). Note that using the σ symmetry this writes

$$*D=\sigma T-\sigma^{-1}d_Q\sigma\delta_Q\sigma\,. \tag{42}$$

• By Proposition 2.3 one knows that Δ_Q commutes with σ on $\mathcal{E}^p$ for $p<n=2k-1$ on CR manifolds with transverse symmetry. Then using that the sequence (d_Q,D) is a complex, one finds on $\mathcal{E}^{2k-1-2p}$ for $p\ge 2$ that

$$\begin{aligned} S_Q^2 &= (d_Q+\delta_Q)\sigma(d_Q+\delta_Q)^2\sigma(d_Q+\delta_Q)\\ &=(d_Q+\delta_Q)\sigma\Delta_Q\sigma(d_Q+\delta_Q)=(d_Q+\delta_Q)\Delta_Q(d_Q+\delta_Q)\\ &=\Delta_Q^2=\mathbf{\Delta}_Q\,. \end{aligned}$$

In degree $2k-3$

$$\begin{aligned} S_Q^2 &= (*D+(d_Q+\delta_Q)\sigma\delta_Q)(d_Q\sigma d_Q)\\ &\quad+(d_Q+\delta_Q)\sigma(d_Q+\delta_Q)((d_Q+\delta_Q)\sigma\delta_Q+\delta_Q\sigma d_Q)\\ &=(d_Q+\delta_Q)\sigma\Delta_Q\sigma d_Q+(d_Q+\delta_Q)\sigma\Delta_Q\sigma\delta_Q\\ &=(d_Q+\delta_Q)\Delta_Qd_Q+(d_Q+\delta_Q)\Delta_Q\delta_Q=\Delta_Q^2=\mathbf{\Delta}_Q \end{aligned}$$

In degree $2k-1$

$$\begin{aligned} S_Q^2 &= (*D)^2 + ((d_Q+\delta_Q)\sigma\delta_Q)(d_Q\sigma\delta_Q) + (d_Q+\delta_Q)\sigma(d_Q+\delta_Q)\delta_Q\sigma\delta_Q \\ &= D^*D + (d_Q+\delta_Q)\sigma\Delta_Q\sigma\delta_Q \\ &= D^*D + (d_Q+\delta_Q)\Delta_Q\delta_Q = D^*D + (d_Q\delta_Q)^2 = \mathbf{\Delta}_Q\,. \end{aligned}$$

We determine the invariant part $(S_Q)^\sigma$ of S_Q through σ. One observes first from (24) that $d_Q\sigma d_Q$ adds $(1,1)$ to the bidegree, hence preserves J and anti-commutes with σ. The same holds for $\delta_Q\sigma\delta_Q$. Therefore

$$(S_Q)^\sigma = \begin{cases} (*D + d_Q\sigma\delta_Q)^\sigma & \text{on } \mathcal{E}^{2k-1} \\ (d_Q\sigma\delta_Q + \delta_Q\sigma d_Q)^\sigma & \text{on } \bigoplus_{1\le p\le k-1}\mathcal{E}^{2k-1-2p}\,. \end{cases} \tag{43}$$

Then (42) gives in degree $2k-1$ that

$$(S_Q)^\sigma = (\sigma T - \sigma^{-1}d_Q\sigma\delta_Q\sigma + d_Q\sigma\delta_Q)^\sigma = \sigma T\,.$$

In degree $2k-1-2p$ with $p\ge 1$, we know from (21) that $T = \delta_Q^J d_Q + d_Q\delta_Q^J$, thus

$$\sigma T = \delta_Q\sigma d_Q + \sigma^{-1}d_Q\sigma\delta_Q\sigma\,.$$

By (43), we look for

$$(d_Q\sigma\delta_Q + \delta_Q\sigma d_Q)^\sigma = (\sigma T - \sigma^{-1}d_Q\sigma\delta_Q\sigma + d_Q\sigma\delta_Q)^\sigma = \sigma T\,,$$

as needed.

• We come back to the proof of Proposition. To relate the spectrum of S_Q to the one of $*D$ we first observe that $S_Q = *D$ on $\mathcal{E}_Q^n = \mathcal{E}^n\cap\ker\delta_Q$. Then

$$\mathrm{Spec}^*(S_Q) = \mathrm{Spec}^*(*D)\bigsqcup\mathrm{Spec}^*(S_Q \mid (\mathcal{E}_Q^n)^\perp)\,.$$

Moreover on $(\mathcal{E}_Q^n)^\perp$, $(d_Q+\delta_Q)S_Q = \Delta_Q\sigma(d_Q+\delta_Q)$, with the convention that $d_Q=0$ in degree n. Therefore

$$\mathrm{Spec}^*(S_Q \mid (\mathcal{E}_Q^n)^\perp) = \bigsqcup_{0\le p\le k-1}\mathrm{Spec}^*\big(\sigma\Delta_Q \mid \mathcal{E}^{2p}\big)\,.$$

□

From this, we can compare the eta trace functions of S_Q and $*D$. It holds on CR Seifert manifolds that

$$\eta(S_Q)(s) = \eta(*D)(s) + \sum_{0\le p\le q}\eta(\sigma\Delta_Q \mid \mathcal{E}^{2p})(s)\,. \tag{44}$$

Moreover $\sigma\Delta_Q$ splits through bidegree and

$$\eta(\sigma\Delta_Q \mid \mathcal{E}^{2p})(s) = \sum_{a+b=2p}(-1)^p i^{a-b}\zeta(\Delta_Q \mid \mathcal{E}^{a,b})(s)\,.$$

For positive hypoelliptic operators like Δ_Q, one knows that

$$\zeta(\Delta_Q)(0) + \dim\ker\Delta_Q$$

is the constant term in the development of $e^{-t\Delta_Q}$ and is given by the integral over M of some universal polynomial of local invariant of the pseudo-hermitian metric; see [19, Section

3.1] or [1] for discussion and references. Moreover the kernels of Δ_Q are isomorphic to the cohomology groups $H^*(M,\rho)$. Hence we finally get the following

Proposition 4.2. *On a CR Seifert manifold of given dimension $4k-1$, it holds that*

$$\eta(S_Q)(0) - \eta(*D)(0) + \sum_{0\le a+b=2p\le 2(k-1)} (-1)^p i^{a-b} \dim H^{a,b}(M,\rho)$$

is the integral over M of some universal polynomial of local invariants of the metric.

Remarks 4.3. In dimension three with a trivial representation ρ, $H^0(M,\rho)=\mathbb{R}$ is given by the constant functions. Hence the cohomological sum is 1 and $\eta(S_Q)(0)-\eta(*D)(0)$ is *not* given by a local term a priori. The two eta invariants are not equivalent up to local terms.

Note also that in dimension 3, one has $S_Q = *D + d_Q\delta_Q$, as studied for instance in previous work [6], but the expression differs in higher dimensions.

The advantage of working with S_Q instead of $*D$ or $*D \pm d_Q\delta_Q$ in general lies in its spectral symmetry with respect to σ.

4.1. The torsion contact trace from the topological viewpoint.

We start the study of the eta trace spectral series involved in the eta invariant S_Q. From (10) it reads

$$\vartheta_S(t) = \mathrm{Tr}(\sqrt{t} S_Q e^{-t\mathbf{\Delta}_Q})\,.$$

We complete arguments already sketched in Section 1.5. From Proposition 4.1, one has $\sigma P S_Q = -S_Q \sigma P$ with $\sigma P = -P\sigma$. Therefore the spectrum of $S_Q = P + \sigma T$ is symmetric with respect to zero, except on $\mathcal{H}_S = \ker P$ and the eta trace ϑ_S retracts on it with $S_Q = \sigma T$. Hence

$$\mathbf{\Delta}_Q = S_Q^2 = -T^2 \quad \text{on} \quad \mathcal{H}_S\,.$$

One can split the domain $\mathcal{E}_S$ of S_Q and $\mathcal{H}_S$ with respect to the action of the involution $\tau = i\sigma = \pm 1$. We have

$$\vartheta_S(t) = -\sqrt{t}(\mathrm{Tr}(iTe^{tT^2} \mid \mathcal{H}_S^+) - \mathrm{Tr}(iTe^{tT^2} \mid \mathcal{H}_S^-))\,.$$

On CR Seifert manifolds, the V-valued forms in $\mathcal{H}_S$ split under Fourier decomposition as in Sections 3.1–3.2. Hence we obtain the identity.

Theorem 4.4. *One has on CR Seifert manifolds*

$$\vartheta_S(t) = \vartheta_S^{top}(t) = -\sqrt{t} \sum_{\lambda\,\mathrm{Spec}(iT)} \mathrm{ind}(P^+ \mid V_\lambda)\lambda e^{-t\lambda^2}\,,$$

where $P^+ = P : \mathcal{E}_S^+ \to \mathcal{E}_S^-$.

We shall now compute the index involved.

4.2. **From the topological to the geometric expression for ϑ_S.**

Let $*_H$ denotes the $*$ operator on the Hermitian space H. From [20, Thm. 2] one has

$$*_H L^r\alpha = L^r((-1)^{p(p+1)/2}J\alpha) = L^r(\sigma\alpha)$$

on $\Omega_0^p H = \mathcal{E}^p$ when $p+2r=n$. That means that the involution $\tau = i\sigma$ on $\mathcal{E}_S$ is conjugated to $\tau_H = i*_H$ on $\Omega^n H$ through the Lefschetz decomposition

$$\mathcal{E}_S \simeq \Omega^n H \quad \text{with} \quad \mathcal{E}_S^{\tau=\pm1} \simeq (\Omega^n H)^{\tau_H=\pm1}\,.$$

From these isomorphisms $P^+ : \Omega^{n,+}H \to \Omega^{n,-}H$. Hence its elliptic symbol class as seen from N is associated to the signature index on N; see [5, Section 6]. Let then $\mathcal{L}(N)_{orb}$ denotes the Hirzebruch L-genus of the orbifold N. As in Section 3.2 we have by Kawasaki index theorem and [5]

$$\mathrm{ind}(P^+ \mid V_\lambda) = \langle \mathrm{ch}(V_\lambda)\wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb}\,,$$

and $\vartheta_S(t)$ can be written

$$\vartheta_S(t) = -\langle \mathrm{ch}(V)^\theta_{odd}(t)\wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb} \tag{45}$$

using the notation

$$\mathrm{ch}(V)^\theta_{odd}(t) = \sum_{\lambda\in \mathrm{Spec}(iT)} \sqrt{t}\lambda\, \mathrm{ch}(V_\lambda) e^{-t\lambda^2}\,. \tag{46}$$

We need to study this θ-regularised Chern character of V as seen from N. It is also useful to consider its even version

$$\mathrm{ch}(V)^\theta_{ev}(t) = \sum_{\lambda\in \mathrm{Spec}(iT)} \mathrm{ch}(V_\lambda) e^{-t\lambda^2}\,. \tag{47}$$

Note that its zero degree part already showed up in (32) in the study of the topological series for the torsion function, that writes

$$\vartheta^{top}(t) = \langle \mathrm{ch}(V)^\theta_{ev}(t)\wedge e(N)_{orb}, [N]\rangle_{orb}\,.$$

From (45) we need now to compute these differential forms in their whole.

To compute the Chern character of V_λ, we use the following twisted connection

$$\nabla^\lambda = \nabla^\rho + i\lambda\theta\,,$$

where ∇^ρ is the flat connection on V. Since $\nabla^\lambda_T s = 0$ on sections of $\mathbf{V}_\lambda$, this connection goes down on M as a connection on V_λ. Its curvature form is given for $X, Y \in H$ by

$$\begin{aligned} R_{\nabla^\lambda}(X,Y) &= \nabla^\lambda_X\nabla^\lambda_Y - \nabla^\lambda_Y\nabla^\lambda_X - \nabla^\lambda_{[X,Y]}\\ &= R_{\nabla^\rho}(X,Y) + i\lambda d\theta(X,Y)\\ &= i\lambda d\theta(X,Y)\,. \end{aligned}$$

Hence the bundle $(V_\lambda, \nabla_\lambda)$ has curvature form $\Omega_\lambda = i\lambda d\theta\otimes \mathrm{id}_{V_\lambda}$. Let then

$$\mathbf{c} = -\frac{d\theta}{2\pi}\,.$$

Note that $\mathbf{c} = c_1(L)$ is the first Chern class of the line bundle L over N whose circle bundle is M. Using the V-bundle structure of V_λ as given in (31), one obtains that

$$\mathrm{ch}(V_\lambda) = \mathrm{Tr}(e^{\frac{i\Omega_\lambda}{2\pi}}) = \dim V^x e^{\lambda \mathbf{c}}$$

over smooth points of N, while the equivariant Chern character at f_j^r is

$$\mathrm{ch}(V_\lambda)(f_j^r) = e^{-2i\pi r\lambda/\alpha_j} \chi_\rho(f_j^r \mid V^x)\,.$$

This leads to the geometrical and dynamical expressions for the θ-regularised Chern characters $\mathrm{ch}(V)^\theta_{ev}$ and $\mathrm{ch}(V)^\theta_{odd}$ using the classical Jacobi theta function $\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{-t(n+x)^2}$ and Poisson formula (35).

Proposition 4.5. • *Over smooth orbits it holds that*

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{ev}(t) &= \sum_{\lambda\in\mathrm{Spec}(iT)} \dim V^x e^{\lambda\mathbf{c} - t\lambda^2} \\
&= \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} \dim V^x e^{\mathbf{c}^2/4t}\theta(x - \frac{\mathbf{c}}{2t}, t) \\
&= \frac{\ell(f)}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \chi_\rho(f^n) e^{-(\ell(f^n)+i\mathbf{c})^2/4t}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{odd}(t) &= \sqrt{t} \sum_{\lambda\in\mathrm{Spec}(iT)} \dim V^x \lambda e^{\lambda\mathbf{c} - t\lambda^2} \\
&= \sqrt{t}\frac{d}{d\mathbf{c}}\,\mathrm{ch}(V)^\theta_{ev}(t) \\
&= -\frac{i\ell(f)}{4t\sqrt{\pi}} \sum_{n\in\mathbb{Z}} \chi_\rho(f^n)(\ell(f^n) + i\mathbf{c}) e^{-(\ell(f^n)+i\mathbf{c})^2/4t},
\end{aligned}$$

in terms of formal derivation as a polynomial in $\mathbf{c}$.

• *At a singular orbit* f_j^r*, the equivariant* θ*-Chern characters writes*

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{ev}(f_j^r)(t) &= \sum_{\lambda\in\mathrm{Spec}(iT)} \chi_\rho(f_j^r \mid V^x) e^{-2i\pi r\lambda/\alpha_j - t\lambda^2} \\
&= \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} \chi_\rho(f_j^r \mid V^x) e^{-\pi^2 r^2/t\alpha_j^2}\theta(x + i\frac{\pi r}{t\alpha_j}, t) \\
&= \frac{\alpha_j \ell(f_j)}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \chi_\rho(f_j^{r+n\alpha_j}) e^{-\ell(f_j^{r+n\alpha_i})^2/4t}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ch}(V)^\theta_{odd}(f_j^r)(t) &= \sqrt{t} \sum_{\lambda\in\mathrm{Spec}(iT)} \chi_\rho(f_j^r \mid V^x)\lambda e^{-2i\pi r\lambda/\alpha_j - t\lambda^2} \\
&= -\frac{i\alpha_j\ell(f_j)}{4t\sqrt{\pi}} \sum_{n\in\mathbb{Z}} \chi_\rho(f_j^{r+n\alpha_j})\ell(f_j^{r+n\alpha_j}) e^{-\ell(f_j^{r+n\alpha_j})^2/4t}\,.
\end{aligned}$$

Note the similar expressions for regular and discrete orbits. All the homotopy classes of closed orbits of the circle action, together with their opposite, enter only once in the various contributions. We also observe that the spectral–dynamical duality in our Selberg–type trace formulae actually shows up at the level of these θ-regularised Chern characters, and only depends on the classical Poisson formula for the heat kernel on the circle.

We can use this into the index series (45)

$$\vartheta_S(t) = -\langle \mathrm{ch}(V)^\theta_{odd}(t) \wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb}\,.$$

We recall that from Kawasaki and Atiyah–Bott works [12] and [3, Section 6], this orbifold pairing decomposes into a classical smooth integral of characteristic classes and a discrete singular sum

$$\begin{aligned}\langle \mathrm{ch}(V)^\theta_{odd}(t) \wedge \mathcal{L}(N)_{orb}, [N]\rangle_{orb} &= \langle\, \mathrm{ch}(V)^\theta_{odd} \wedge \mathcal{L}(N), [N_{smooth}]\rangle \\ &\quad + \sum_i \frac{1}{\alpha_i} \sum_{r=1}^{\alpha_i - 1} \mathrm{ch}(V)^\theta_{odd}(f_i^r)\nu(f_i^r)\,,\end{aligned} \tag{48}$$

where, for $\gamma = f_j^n$ with $n \not\equiv 0 \mod \alpha_j$,

$$\nu(\gamma) = i(-1)^k \prod_{m=1}^{2k-1} \cot(\theta_m(\gamma)/2)$$

for angles $\theta_m(\gamma)$ associated to the unitary return map $d\tau_\gamma$ along γ in H_{p_j} and $\dim N = 4k-2$. Note that $e^{2i\pi\theta_m(f_j)}$ are primitive α_i-th roots of unity since $f_i^{\alpha_i} = f$ generates a locally free circle action.

Using Proposition 4.5, we obtain the dynamical expression for the contact eta function.

Theorem 4.6. *On CR Seifert manifolds, one has*

$$\vartheta_S(t) = \vartheta_S^{dyn}(t) = \frac{1}{\sqrt{4\pi}} \sum_\gamma \chi_\rho(\gamma)\sigma(\gamma)(t)$$

where powers of the generic orbit $\gamma = f^n$, with $n \in \mathbb{Z}$, contribute to

$$\sigma(\gamma)(t) = \frac{i\ell(f)}{2t} \left\langle (\ell(\gamma) + i\mathbf{c}) e^{-(\ell(\gamma)+i\mathbf{c})^2/4t} \wedge \mathcal{L}(N), [N_{smooth}] \right\rangle,$$

while powers of the singular orbits $\gamma = f_j^p$, with $p \not\equiv 0 \mod \alpha_j$, contribute to

$$\sigma(\gamma)(t) = \frac{i\ell(f_j)}{2t} \ell(\gamma) e^{-\ell(\gamma)^2/4t}\, \nu(\gamma)\,.$$

Remark 4.7. We observe that $\sigma(\gamma)(t)$ is real. Indeed for the smooth component, only the odd part in $\mathbf{c}$ contributes to it since $\mathcal{L}(N)$ has components in degrees $4p$ while $\dim N = 4k-2$. Then $\sigma(\gamma^{-1})(t) = \sigma(\gamma)(t)$ so that $\vartheta_S(t)$ is real on unitary representations.

4.3. Applications to the contact eta function $\eta(S_Q)(s)$.

In order to get applications to the contact eta invariant we need now to study our theta–regularised contact eta function ϑ_S from its "zeta" viewpoint $\eta(S_Q)(s)$.

We first study the odd zeta regularised Chern character of V i.e. the series

$$\mathrm{ch}(V)^{\zeta}_{odd}(s) = \sum_{\lambda\in\mathrm{Spec}^*(iT)} \frac{\lambda\,\mathrm{ch}(V_\lambda)}{|\lambda|^{2s+1}} = \sum_{\lambda\in\mathrm{Spec}^*(iT)} \dim V^x \frac{\lambda e^{\lambda\mathbf{c}}}{|\lambda|^{2s+1}}$$

We proceed by taking Mellin transform of $\mathrm{ch}(V)^{\theta}_{odd}(t)$. From Proposition 4.5 its smooth part reads

$$\mathrm{ch}(V)^{\theta}_{odd}(t) = \sqrt{t} \sum_{\lambda\in\mathrm{Spec}(iT)} \dim V^x \lambda e^{\lambda\mathbf{c}-t\lambda^2}$$

$$= \sum_{\lambda\in\mathrm{Spec}^*(iT)} \dim V^x \sum_{p=0}^{2k-1} \frac{\mathbf{c}^p}{p!}\lambda^{p+1}\sqrt{t}e^{-t\lambda^2},$$

with $\dim N = 4k-2$. Then one finds by dominated convergence and direct computation that for $\Re(s) > k$ the following integral converges and

$$I(s) = \int_0^{+\infty} \mathrm{ch}(V)^{\theta}_{odd}(t)t^{s-1}dt = \Gamma(s+\frac{1}{2}) \sum_{p=0}^{2k-1} \frac{\mathbf{c}^p}{p!} \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} Z(p,s,x)$$

with

$$Z(p,s,x) = \begin{cases} \dim V^x(\zeta(2s-p,x) + (-1)^{p+1}\zeta(2s-p,1-x)) & \text{if } 0<x<1 \\ \dim V^1(1+(-1)^{p+1})\zeta(2s-p) & \text{if } x=1 \end{cases} \tag{49}$$

where $\zeta(s,x)$ is Hurwitz zeta function (38). Therefore $I(s)/\Gamma(s+\frac{1}{2})$ defines a meromorphic function with possible simple poles at $s=\frac{j}{2}$ for $j\in[[1,2k]]$. Hence we get that the series

$$\mathrm{ch}(V)^{\zeta}_{odd}(s) = \sum_{\lambda\in\mathrm{Spec}(iT)*} \lambda\frac{\mathrm{ch}(V_\lambda)}{|\lambda|^{2s+1}} = \frac{I(s)}{\Gamma(s+\frac{1}{2})} \tag{50}$$

$$= \sum_{p=0}^{2k-1} \frac{\mathbf{c}^p}{p!} \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} Z(p,s,x)$$

are well defined and meromorphic on the same domain.

On the discrete part $\mathrm{ch}(V)^{\theta}_{odd}(f^r_j)$ as given in Proposition 4.5, one finds that for $\Re(s) > 1/2$

$$\mathrm{ch}(V)^{\zeta}_{odd}(f^r_j)(s) = \sum_{e^{2i\pi x}\in\mathrm{Spec}(\rho(f))} \chi_\rho(f^r_j \mid V^x) Z(f^r_j,s,x) \tag{51}$$

with

$$Z(f^r_j,s,x) = \tag{52}$$

$$\begin{cases} e^{-2i\pi rx/\alpha_j}L(e^{-2i\pi r/\alpha_j},2s,x) - e^{2i\pi r(1-x)/\alpha_j}L(e^{2i\pi r/\alpha_j},2s,1-x) & \text{for } 0<x<1 \\ e^{-2i\pi r/\alpha_j}L(e^{-2i\pi r/\alpha_j},2s,1) - e^{2i\pi r/\alpha_j}L(e^{2i\pi r/\alpha_j},2s,1) & \text{for } x=1\,. \end{cases}$$

where

$$L(z,s,x)=\sum_{n\geq 0}\frac{z^n}{(n+x)^s}$$

is Lerch zeta function. In our case, $z=e^{\pm 2i\pi r/\alpha_j}$ is a root of unity and

$$L(z,s,x)=\frac{1}{\alpha_j^s}\sum_{m=0}^{\alpha_j-1} z^m\zeta(s,\frac{m+x}{\alpha_j}) \tag{53}$$

splits into a sum of Hurwitz zeta functions. From [2, p. 255], $\zeta(s,x)$ is analytic with a simple pole at $s=1$ and residue 1. Therefore $\mathrm{Res}_{s=1/2}L(e^{\pm 2i\pi r/\alpha_j},2s,0)=0$ and the functions $Z(f_j^r,s,x)$ are entire.

This gives an explicit geometric expression for $\mathrm{ch}(V)^{\zeta}_{odd}(s)$ that leads with (45) and (48) to the formula for $\eta(S_Q)(s)$ from

$$\begin{aligned}-\eta(S_Q)(s) &= \langle \mathrm{ch}(V)^{\zeta}_{odd}(s)\wedge\mathcal{L}(N)_{orb},[N]\rangle_{orb}\,. \\ &= \langle \mathrm{ch}(V)^{\zeta}_{odd}(s)\wedge\mathcal{L}(N),[N_{smooth}]\rangle+\sum_i\frac{1}{\alpha_i}\sum_{r=1}^{\alpha_i-1}\mathrm{ch}(V)^{\zeta}_{odd}(f_i^r)(s)\nu(f_i^r)\,.\end{aligned} \tag{54}$$

Specialising at the regular value $s=0$ gives an explicit formula for the contact eta invariant $\eta(S_Q)(0)$. Let

$$B(t,x)=\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{+\infty}B_n(x)\frac{t^n}{n!}$$

be the generating function of Bernoulli polynomials; see e.g. [2], and

$$B_{ev}(t,x)=t\frac{\cosh(t(x-\frac12))}{2\sinh(\frac t2)}=\sum_{n=0}^{+\infty}B_{2n}(x)\frac{t^{2n}}{(2n)!}$$

its even part in t. We consider the function

$$\begin{aligned}(\Delta B_{ev})(t,x) &= \frac1t(B_{ev}(t,x)-1)=\sum_{n=1}^{+\infty}B_{2n}(x)\frac{t^{2n-1}}{(2n)!} \\ &= \frac{\cosh(t(x-\frac12))}{2\sinh(\frac t2)}-\frac1t\,.\end{aligned}$$

We shall also need

$$\widetilde{B}(t,x)=\begin{cases}B(t,x) & \text{for } 0<x<1\\ \frac12(B(t,1)+B(t,0)) & \text{for } x=1\,.\end{cases}$$

In the following statement, we recall that $V^x=\ker(\rho(f)-e^{2i\pi x}\,\mathrm{id})$ for $0<x\leq 1$.

Theorem 4.8. *On a CR Seifert manifold, it holds that*

$$\eta(S_Q)(0) = 2 \sum_{e^{2i\pi x}\in \mathrm{Spec}(\rho(f))} \dim V^x \langle (\Delta B_{ev})(\mathbf{c},x)\wedge \mathcal{L}(N), [N]_{smooth}\rangle$$

$$+ 2 \sum_{e^{2i\pi x}\in \mathrm{Spec}(\rho(f))} \sum_j \frac{1}{\alpha_j} \sum_{r=1}^{\alpha_j-1} \chi_\rho(f_j^r \mid V^x) \frac{\widetilde{B}(-2i\pi r/\alpha_j, x)}{-2i\pi r/\alpha_j} \nu(f_j^r) \, .$$

Proof.

We have to evaluate $Z(0,p,x)$ as given in (49). From e.g. [2, p. 264], we know that

$$\zeta(-p,x) = -\frac{B_{p+1}(x)}{p+1} \quad \text{for } p\in\mathbb{N}\, .$$

Moreover $B_n(1-x) = (-1)^n B_n(x)$, so that $Z(0,x,p)=0$ for p even, while for p odd

$$Z(0,x,p) = -2\dim V^x \frac{B_{p+1}(x)}{p+1}\, .$$

Hence from (50)

$$\begin{aligned}\mathrm{ch}(V)^\zeta_{odd}(0) &= -2 \sum_{e^{2i\pi x}\in \mathrm{Spec}(f)} \dim V^x \sum_{p=1\, p\, odd}^{2k-1} B_{p+1}(x) \frac{\mathbf{c}^p}{(p+1)!} \\ &= -2 \sum_{e^{2i\pi x}\in \mathrm{Spec}(f)} \dim V^x (\Delta B_{ev})(\mathbf{c},x)\, ,\end{aligned}$$

as needed. Note that one can replace ΔB_{ev} by ΔB in the formula for $\eta(S_Q)(0)$ since $\langle (\Delta B_{odd}(\mathbf{c},x)\wedge\mathcal{L}(N), [N]_{smooth}\rangle = 0$ for dimensional reasons.

We compute the discrete contribution of f_j^r. Let $z = e^{\pm 2i\pi r/\alpha_j}$. From $\zeta(0,x) = 1/2 - x$ and (53), one gets

$$\begin{aligned}L(z,0,x) &= \sum_{m=0}^{\alpha_j-1} z^m\left(\frac{1}{2} - \frac{m+x}{\alpha_j}\right) = -\frac{1}{\alpha_j}\sum_{m=0}^{\alpha_j-1} m z^m \\ &= \frac{1}{1-z}\, ,\end{aligned}$$

by deriving the identity $\sum_{m=0}^{\alpha_i-1} u^{m+1} = \frac{u(1-u^{\alpha_j})}{1-u}$ at $u=z$. Inserting it in (52) gives the result.

□

Theorem 4.8 generalises an expression given in [6, Theorem 8.8] in the three dimensional case and with a trivial representation. (Note that in [6], $\zeta(P)(s)$ includes $\dim\ker P$ by convention.) There one has

$$\eta(S_Q)(0) = \eta(*D + d_Q\delta_Q)(0) = \eta_0(M,\theta) - 1\, ,$$

where $\eta_0(M,\theta) = \mathrm{FP}_{\varepsilon=0}\,\eta(*_\varepsilon d)$ is the renormalised eta invariant of $*_\varepsilon d$ for the sub-Riemannian limit of metrics $g_\varepsilon = d\theta(\cdot, J\cdot) + \varepsilon^{-1}\theta^2$ when $\varepsilon\to 0$. From [6, Section 3], this also coincides with the adiabatic limit of $\eta(*_\varepsilon d)$ for $\varepsilon\to+\infty$ in our fibrered case.

In general dimension, the formula for $\eta(S_Q)(0)$ no longer shows up global (over M) cohomological terms, in contrast to $\eta(*D)(0)$ by Proposition 4.2. This is due to the fact that the spectrum of S_Q is more symmetric than the one of $*D$ by Proposition 4.1. While $*D$ acts on a smaller space than S_Q, it contains un-symmetrized copies of zeta functions of Laplacians, leading to additional cohomological terms at $s = 0$.

Also, from [1], one knows that $\eta(*D)(0)$ compares on general contact manifolds with the renormalised sub-Riemanian limit $\eta_0(M, \theta) = FP_{\varepsilon=0}(*_\varepsilon d)$, up to local terms. It coincides with the adiabatic limit of the eta invariant in our CR Seifert situation. This limit has been studied in depth by other means. Bilding on previous works by Bismut and Cheeger [7], Dai [9] and Zhang [22] expressed $\eta_0(M, \theta)$ for circle bundles, in the case of a trivial representation and a smooth quotient N. Taking into account that their eta invariant is twice ours, which is defined using $*D$ only instead of $*D + D*$, one sees from [9, Theorem 0.3] and [22, Theorem 1.7] that

$$\eta_0(M, \theta) = 2\langle \Delta B_{ev}(\mathbf{c}, 1) \wedge \mathcal{L}(N), [N]\rangle + \tau \tag{55}$$

where τ is the signature of the "collapsing spectrum" in the adiabatic limit. More precisely, from [9, Section 4.3], τ is computed using the Leray spectral sequence of the fibration $\mathbf{S}^1 \to M \to N$.

We briefly present this in relation to Proposition 4.2. From [9], (11) and Section 2.2, one finds that the E_2-term of Leray spectral sequence identifies with

$$\ker T \cap \ker d_H \cap \ker \delta_H = \ker \Delta_Q \,,$$

where Δ_Q is acting diagonally on horizontal, vertical and Lefschetz components of $\Omega^* M$. Then, following [9], τ is the signature of the symetric bilinear form

$$\begin{aligned} Q : E_2^{2k-1,\theta} \otimes E_2^{2k-1,\theta} &\to \mathbb{R} \\ \alpha \otimes \beta &\mapsto \int_M \alpha \wedge d_2 \beta \,. \end{aligned}$$

using the second differential of the Leray spectral sequence

$$\begin{aligned} d_2 : E_2^{2k-1,\theta} &\to E_2^{2k} \\ \theta \wedge \alpha &\mapsto L\alpha \end{aligned}$$

Now, by e.g. [20, Theorem 2], one has for $\alpha \in \mathcal{E}^{2p}$

$$*_H \alpha = \frac{1}{(2k-1-2p)!} L^{2k-1-2p} \sigma\alpha \,,$$

and therefore

$$\begin{aligned} Q(\theta \wedge L^{k-p-1}\alpha, \theta \wedge L^{k-p-1}\beta) &= \int_M \theta \wedge L^{2k-2p-1} \alpha \wedge \beta \\ &= (2k-1-2p)! \int_M (\sigma\alpha, \beta)\, \mathrm{dvol} \,, \end{aligned}$$

from which it follows that

$$\tau = \text{Signature}\,(Q) = \sum_{0 \le a+b=2p \le 2(k-1)} (-1)^p i^{a-b} \dim H^{a,b}(M) \,.$$

This is the cohomological sum in Proposition 4.2. Hence, for smooth quotient N and trivial representation, this relates (55) to the equations

$$\eta_0(M) = \eta(S_Q)(0) + \tau = \eta(*D)(0) + \text{local terms}.$$

We note in conclusion that in the adiabatic viewpoint, the term $2\langle \Delta B_{ev}(\mathbf{c},1) \wedge \mathcal{L}(N), [N]\rangle$ comes from the analysis of an eta form constructed by Bismut and Cheeger [7]. It is associated to a Dirac operator for a superconnection over the fibers. Here, in this smooth case, from our viewpoint

$$2\langle \Delta B_{ev}(\mathbf{c},1) \wedge \mathcal{L}(N), [N]\rangle = \eta(S_Q)(0)\,,$$

so that this term is interpreted as being the eta invariant of a second order hypoelliptic differential operator over the whole contact manifold M.

4.4. **The contact eta function from the dynamical viewpoint.**

We eventually work out the dynamical aspect of the eta function and eta invariant. We start with the smooth part. From Proposition 4.5 we have that for some $a > 0$

$$\mathrm{ch}(V)^{\theta}_{odd}(t) = O(e^{-at}) \quad \text{when} \quad t \to +\infty\,.$$

Due to terms of type $e^{-t(\ell(\gamma)+i\mathbf{c})^2/4t}$ in the dynamical expression of $\mathrm{ch}(V)^{\theta}_{odd}(t)$, its behaviour when $t \to 0^+$ is unclear, unless if $\|\mathbf{c}\| < m = \min(\ell(f), \ell(f_i))$. In that case, one has for some $b > 0$

$$\mathrm{ch}(V)^{\theta}_{odd}(t) - \dim V \frac{i\ell(f)\mathbf{c}}{4t\sqrt{\pi}} e^{\mathbf{c}^2/4t} = 0(e^{-b/t})\,,$$

with

$$\frac{\mathbf{c}}{4t} e^{\mathbf{c}^2/4t} = \sum_{p=0}^{k-1} \frac{\mathbf{c}^{2p+1}}{p!(4t)^{p+1}}\,.$$

To get around this, we first replace $\mathbf{c}$ by $u\mathbf{c}$ for small enough u so that the divergence when $t \to 0$ only comes from the trivial constant orbit. There previous formulae on $\mathrm{ch}(V)^{\theta}_{odd}$ hold as well with the advantage that

$$f(t) = \mathrm{ch}(V)^{\theta}_{odd}(t) - \dim V \frac{iu\mathbf{c}\ell(f)}{4t\sqrt{\pi}} e^{u^2\mathbf{c}^2/4t} \mathbf{1}_{]0,1]}(t)$$

has an entire Mellin transform over $\mathbb{C}$. Hence, proceeding as in Section 3.6 we first find that for $\Re(s) > k$

$$\begin{aligned} I(s) = \int_0^{+\infty} f(t) t^{s-1} dt &= \Gamma(s + \frac{1}{2})\, \mathrm{ch}(V)^{\zeta}_{odd}(s) \\ &\quad - \dim V \sum_{p=0}^{k-1} \frac{i\ell(f)(u\mathbf{c})^{2p+1}}{p! 4^{p+1}\sqrt{\pi}} \times \frac{1}{s-p-1}\,, \end{aligned}$$

While writing

$$f(t) = \mathrm{ch}(V)^{\theta}_{odd}(t) - \dim V \frac{i\ell(f)u\mathbf{c}}{4t\sqrt{\pi}} e^{u^2\mathbf{c}^2/4t} + \dim V \frac{i\ell(f)u\mathbf{c}}{4t\sqrt{\pi}} e^{u^2\mathbf{c}^2/4t} \mathbf{1}_{]1,+\infty]}(t)$$

and using the dynamical expression for $\mathrm{ch}(V)^{\theta}_{odd}(t)$ one has for $\Re(s) < 0$

$$I(s) = \Gamma(1-s)\Big(\sum_{n\in\mathbb{Z}^*} -\frac{i\ell(f)}{4^s\sqrt{\pi}}\chi_\rho(f^n)(\ell(f^n)+iu\mathbf{c})\big((\ell(f^n)+iu\mathbf{c})^2\big)^{s-1}\Big)$$
$$- \dim V \sum_{p=0}^{k-1} \frac{i\ell(f)(u\mathbf{c})^{2p+1}}{p!4^{p+1}\sqrt{\pi}} \times \frac{1}{s-p-1}\,.$$

Here z^s denotes the principal branch of the power function. We get then the identity of meromorphic functions through analytic continuation

$$\Gamma(s+1/2)\,\mathrm{ch}(V)^{\zeta}_{odd}(s) =$$
$$\Gamma(1-s)\sum_{n\in\mathbb{Z}^*} -\frac{i\ell(f)}{4^s\sqrt{\pi}}\chi_\rho(f^n)(\ell(f^n)+iu\mathbf{c})\big((\ell(f^n)+iu\mathbf{c})^2\big)^{s-1}$$

That extends continuously to $u = 1$ by (50).

One finds more easily the discrete dynamical contribution of f_j^r due to rapid decay of $\mathrm{ch}(V)^{\theta}_{odd}(f_j^r)$ both at 0 and $+\infty$ from Proposition 4.5. It gives that

$$\Gamma(s+1/2)\,\mathrm{ch}(V)^{\zeta}_{odd}(f_j^r)(s) =$$
$$\Gamma(1-s)\sum_{n\in\mathbb{Z}} -\frac{i\alpha_j\ell(f_j)}{4^s\sqrt{\pi}}\chi_\rho(f_j^{r+n\alpha_j})\ell(f_j^{r+n\alpha_j})\big(\ell(f_j^{r+n\alpha_j})^2\big)^{s-1}$$

is an entire function. Gathering this with (54) we get the following.

Theorem 4.9. *The following identity defines a meromorphic function φ with simple poles at $s = 1, \cdots, k$*

$$\varphi(s) = \Gamma(s+\frac{1}{2})\eta(S_Q)(s) = \frac{\Gamma(1-s)}{4^s\sqrt{\pi}}\sum_{\gamma\neq 0}\chi_\rho(\gamma)\eta(\gamma)(s)$$

with

$$\eta(\gamma)(s) = i\ell(f)\langle(\ell(\gamma)+i\mathbf{c})\big((\ell(\gamma)+i\mathbf{c})^2\big)^{s-1}\wedge\mathcal{L}(N),[N_{smooth}]\rangle \text{ if } \gamma = f^n\,,$$

and

$$\eta(\gamma)(s) = i\ell(f_j)\ell(\gamma)(\ell(\gamma)^2)^{s-1}\nu(\gamma) \text{ if } \gamma = f_j^n,\ n\not\equiv 0 \mod \alpha_j\,.$$

Moreover

$$\mathrm{Res}_{s=p}(\varphi) = \dim V\frac{i\ell(f)}{\sqrt{\pi}(p-1)!4^p}\langle\mathbf{c}^{2p-1}\wedge\mathcal{L}(N),[N_{smooth}]\rangle\,.$$

The function being regular at $s = 0$ we get the dynamical formula for the eta invariant.

Corollary 4.10. *On a CR Seifert manifold, one has*

$$\eta(S_Q)(0) = \sum_{\gamma\neq 0}\chi_\rho(\gamma)\eta(\gamma)(0)$$

with

$$\eta(\gamma)(0) = \frac{i\ell(f)}{\pi}\langle\frac{1}{\ell(\gamma)+i\mathbf{c}}\wedge\mathcal{L}(N),[N_{smooth}]\rangle \quad \text{if } \gamma = f^n\,,$$

and

$$\eta(\gamma)(0) = \frac{i\ell(f_j)}{\pi\ell(\gamma)}\nu(\gamma) \quad \text{if } \gamma = f_j^n,\ n \not\equiv 0 \mod \alpha_j\,.$$

This is a decomposition of the eta invariant into its dynamical "atoms". These dynamical series are *a priori* formal expressions coming from analytic continuation. However they can be turned into convergent ones. The smooth contribution is actually an absolutely convergent series using the $\mathbf{c} \leftrightarrow -\mathbf{c}$ symmetry; see Remark 4.7. It comes as a limit of the smooth dynamical expression in Theorem 4.9 when $s \to 0^-$. The discrete contribution are semi-convergent series when gathering the orbit contributions of $\gamma = f_j^{r+n\alpha_j}$ and $\overline{\gamma} = f_j^{r-n\alpha_j}$. Using Abel's lemma one sees that it is also the limit coming from the corresponding dynamical expression of Theorem 4.9 when $s \to 0^-$.

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Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, 91405 Orsay Cedex, France
Email address: `michel.rumin@universite-paris-saclay.fr`